Generalized Pohozhaev's identity for radial solutions of p-Laplace equations

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Abstract

We derive a generalized Pohozhaev's identity for radial solutions of p-Laplace equations, by using the approach in [5], thus extending the work of H. Brézis and L. Nirenberg [2], where this identity was implicitly used for the Laplace equation.

Key words: Generalized Pohozhaev's identity, radial solutions.

AMS subject classification: 35J25, 35J65.

1 Introduction

Any solution $u(x)$ of semilinear Dirichlet problem on a bounded smooth domain $D \subset R^n$

$$\Delta u + f(u) = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D \tag{1.1}$$

satisfies the well-known Pohozhaev's identity

$$\int_D [2nF(u) + (2-n)uf(u)] \, dx = \int_{\partial D} (x \cdot \nu) |\nabla u|^2 \, dS \,. \tag{1.2}$$

Here $F(u) = \int_0^u f(t)\, dt$, and ν is the unit normal vector on ∂D, pointing outside. A standard proof involves multiplication of the equation (1.1) by $x \cdot \nabla u$ and repeated integration by parts, see e.g., K. Schmitt [11]. In our book [5] we observed that a more straightforward derivation is to show first that $z \equiv x \cdot \nabla u$ satisfies

$$\Delta z + f'(u)z = -2f(u) \text{ in } D, \quad z = 0 \text{ on } \partial D \,, \tag{1.3}$$

and then from the equation (1.1) multiplied by z subtract the equation (1.3) multiplied by u, followed by integration over D. We used a similar approach for non-autonomous elliptic systems of Hamiltonian type in [5] and [6], including systems with power nonlinearities, obtaining an easy derivation of the critical hyperbola, see [5] for details.

For radial solutions on balls in R^n there is a more general Pohozhaev's identity. It was used implicitly in the classical paper of H. Brezis and L. Nirenberg [2], but it was not written down in the general form, as presented next. (As above $F(u) = \int_0^u f(t)\,dt$.)

Theorem 1.1 *Let $u(r) \in C^2[0,1]$ be a solution of*

$$u'' + \frac{n-1}{r}u' + f(u) = 0\,, \quad 0 < r < 1\,, \quad u'(0) = u(1) = 0\,, \tag{1.4}$$

and let $\psi(r) \in C^2[0,1]$. Then

$$\int_0^1 \left[2(\psi r^{n-1})' F(u) + \left(2\psi' r^{n-1} - (\psi r^{n-1})'\right) u f(u) - u u' L[\psi] r^{n-3}\right] dr = \psi(1) u'^2(1)\,, \tag{1.5}$$

where $L[\psi] = r^2\psi'' - (n-1)r\psi' + (n-1)\psi$.

We shall prove a more general p-Laplace version of this result, by using the approach described above, and present an application based on [2]. Similarly to [5] and [6] it appears possible to extend these results in two directions: to allow $f(r,u)$ with r dependence, and to consider systems.

Another generalization of radial Pohozhaev's identity, also stimulated by H. Brezis and L. Nirenberg [2], was found by F. Catrina [4].

2 An application

The generalized Pohozhaev's identity (1.5) appears to be too involved to use, except in the following three cases: when $n = 3$, or when $\psi(r) = r$, or in case $\psi(r) = r^{n-1}$.

In case $n = 3$, assuming that $\psi(r) \in C^3[0,1]$ satisfies $\psi(0) = 0$, we have $L[\psi] = r^2\psi'' - 2r\psi' + 2\psi$, $L[\psi](0) = 0$, and then

$$-\int_0^1 u u' L[\psi] r^{n-3}\,dr = \frac{1}{2}\int_0^1 u^2 \frac{d}{dr} L[\psi]\,dr = \frac{1}{2}\int_0^1 u^2 \psi''' r^2\,dr\,,$$

and (1.5) simplifies to become

$$\int_0^1 \left[2(\psi r^2)' F(u) + \left(2\psi' r^2 - (\psi r^2)' \right) u f(u) + \frac{1}{2} u^2 \psi''' r^2 \right] dr = \psi(1) u'^2(1) \, .$$

Example 1 $f(u) = \lambda u + u|u|^{p-1}$, with $p \geq 5$. (5 is the critical exponent $\frac{n+2}{n-2}$ for $n = 3$). Then $uf(u) = \lambda u^2 + |u|^{p+1}$, $F(u) = \frac{1}{2}\lambda u^2 + \frac{1}{p+1}|u|^{p+1}$, and the last identity becomes

$$\int_0^1 \left[\frac{p+3}{p+1} \psi' r^2 - \frac{2(p-1)}{p+1} \psi r \right] |u|^{p+1} \, dr + \frac{1}{2} \int_0^1 \left(\psi''' + 4\lambda \psi' \right) u^2 r^2 \, dr = \psi(1) u'^2(1) \, .$$

This formula results in a contradiction (proving non-existence of solutions) provided that

$$\begin{gathered} \psi(0) = 0, \;\; \psi(1) \geq 0 \\ \psi''' + 4\lambda \psi' = 0 \\ 2(p-1)\psi r - (p+3)\psi' r^2 > 0 \, . \end{gathered} \tag{2.1}$$

The equation in the second line, and the boundary conditions in line one, are satisfied by $\psi(r) = \sin \sqrt{4\lambda} r$, with $\lambda \in (0, \frac{\pi^2}{4}]$. The last inequality requires that

$$\sin \sqrt{4\lambda} r > \frac{p+3}{2(p-1)} \sqrt{4\lambda} r \cos \sqrt{4\lambda} r \, ,$$

or

$$\sin \theta - \gamma \theta \cos \theta > 0 \, ,$$

if we denote $\gamma = \frac{p+3}{2(p-1)}$, and $\theta = \sqrt{4\lambda} r$. Observe that $\gamma \in (0, 1]$, provided that $p \geq 5$, and $\theta \in (0, \pi)$ for $\lambda \in (0, \frac{\pi^2}{4})$. Then

$$\sin \theta - \gamma \theta \cos \theta \geq \gamma \left(\sin \theta - \theta \cos \theta \right) > 0 \, .$$

Conclusion: for $p \geq 5$, and $\lambda \in (0, \frac{\pi^2}{4}]$ the problem $(n = 3)$

$$u'' + \frac{2}{r} u' + \lambda u + u|u|^{p-1} = 0 \, , \;\; 0 < r < 1 \, , \;\; u'(0) = u(1) = 0$$

has no non-trivial solutions.

Remarks

1. The same conclusion holds for other $f(u)$, e.g., for $f(u) = \lambda u + u|u|^{p-1} + u|u|^{q-1}$, with $q > p \geq 5$.

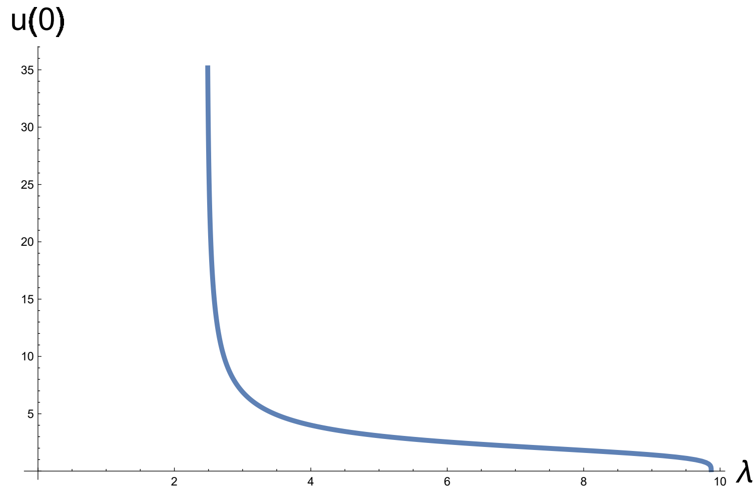

Figure 1: Solution curve of the Brezis-Nirenberg problem (2.2)

2. In case $p > 5$ non-existence of solutions for λ small was proved in the same paper of H. Brezis and L. Nirenberg [2], and in C. Budd and J. Norbury [3], see also Proposition 1.1 in [5].

In case $p = 5$, this example is a part of the classical result of H. Brezis and L. Nirenberg [2], who also proved the existence of solutions for $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ (observe that $\lambda_1 = \pi^2$ for the unit ball in R^3). It is remarkable that their non-existence result is sharp for $p = 5$. Let us recall this theorem of H. Brezis and L. Nirenberg [2] (an extension to sign-changing solutions was later given in F.V. Atkinson, H. Brezis and L.A. Peletier [1]).

Theorem 2.1 *([2]) The problem*

$$u'' + \frac{2}{r}u' + \lambda u + u^5 = 0\,, \quad 0 < r < 1\,, \quad u'(0) = u(1) = 0 \tag{2.2}$$

has a positive solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$.

We used *Mathematica* to compute the solution curve in the $(\lambda, u(0))$ plane of the Brezis-Nirenberg problem (2.2), presented in Figure 1, with $u(0)$ giving the maximum value of solutions. (We used the scaling $u = \sqrt[4]{\lambda}\, z$, to convert this equation into $z'' + \frac{2}{r}z' + \lambda\left(z + z^5\right) = 0$, to which the shoot-and-scale method, described in detail in P. Korman and D.S. Schmidt [7] applies. A program in *Mathematica* can be downloaded from [8].) The picture in Figure 1 indicates that the solution is unique at each λ, and in fact the uniqueness follows from the results of M.K. Kwong and Y. Li [9].

Our numerical computations indicate that the non-existence result on $(0, \frac{\pi^2}{4})$ in Example 1 is not sharp for $p > 5$, with solution curves tending to infinity at λ larger than $\frac{\pi^2}{4}$. In Figure 2 we present the solution curve of

$$u'' + \frac{2}{r}u' + \lambda u + u^6 = 0\,, \quad 0 < r < 1\,, \quad u'(0) = u(1) = 0\,. \tag{2.3}$$

The solution curve has a completely different shape (see [3] for the asymptotic behavior of this curve), and the smallest value of λ occurs at the first turning point, $\lambda \approx 5.91 > \frac{\pi^2}{4}$.

The identity (1.5) also simplifies in case

$$L[\psi] = r^2\psi'' - (n-1)r\psi' + (n-1)\psi = 0\,. \tag{2.4}$$

For $n \geq 3$, one solution of this Euler's equation is $\psi = r$, for which (1.5) is the classical Pohozhaev's identity:

$$\int_0^1 [2nF(u) + (2-n)uf(u)]\, r^{n-1}\, dr = {u'}^2(1)\,.$$

The other solution of (2.4) is $\psi = r^{n-1}$, giving

$$(4n-4)\int_0^1 F(u(r))r^{2n-3}\, dr = {u'}^2(1)\,.$$

This identity was used by L.A. Peletier and J. Serrin [10].

In case $n = 2$, the solutions of (2.4) are $\psi = r$ and $\psi = r\ln r$, leading to similar identities.

3 The p-Laplace case

We present the proof of generalized Pohozhaev's identity next.

Theorem 3.1 *Let $u(r) \in C^2[0,1]$ be a solution of*

$$\varphi\left(u'(r)\right)' + \frac{n-1}{r}\varphi\left(u'(r)\right) + f(u) = 0 \;\; 0 < r < 1\,, \quad u'(0) = u(1) = 0\,, \tag{3.1}$$

with $\varphi(t) = t|t|^{p-2}$, $p > 1$, and let $\psi(r) \in C^2[0,1]$. Then

$$\int_0^1 \left[(pF(u) - uf(u))\,(\psi r^{n-1})' + p\psi' uf(u)r^{n-1} - \varphi(u')uL[\psi]r^{n-3}\right]\, dr \tag{3.2}$$

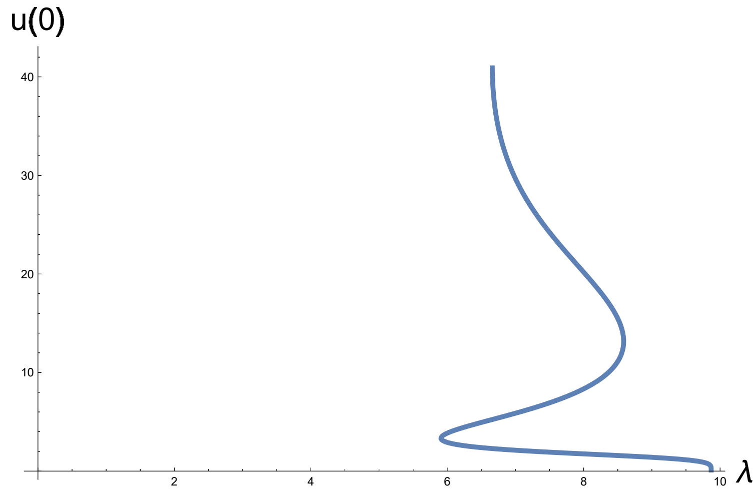

Figure 2: Solution curve of the supercritical problem (2.3)

$$= (p-1)\varphi(u'(1))\psi(1)u'(1)\,,$$

where $L[\psi] = (p-1)r^2\psi'' - (n-1)r\psi' + (n-1)\psi$.

Proof: Observe that the function $\varphi(t)$ satisfies

$$t\varphi'(t) = (p-1)\varphi(t)\,. \tag{3.3}$$

We claim that the function $v(r) = \psi(r)u'(r)$ satisfies

$$(\varphi'(u')v')' + \frac{n-1}{r}\varphi'(u')v' + f'(u)v = -p\psi' f(u) + \frac{\varphi(u')L[\psi]}{r^2}\,. \tag{3.4}$$

Indeed, using (3.3) and expressing $\varphi(u')'$ from the equation (3.1)

$$\begin{aligned}\varphi'(u')v' = \psi'\varphi' u' + \psi\varphi'(u')u'' = (p-1)\psi'\varphi(u') + \psi\varphi(u')' \\ = (p-1)\psi'\varphi(u') - \tfrac{n-1}{r}\psi\varphi(u') - \psi f(u)\,.\end{aligned}$$

Then

$$\begin{aligned}(\varphi'(u')v')' = (p-1)\psi''\varphi(u') + (p-1)\psi'\phi(u')' + \tfrac{n-1}{r^2}\psi\varphi(u') - \tfrac{n-1}{r}\psi'\varphi(u') \\ - \tfrac{n-1}{r}\psi\varphi(u')' - \psi' f(u) - f'(u)v\,.\end{aligned}$$

Also, using (3.3) again,

$$\begin{aligned}\tfrac{n-1}{r}\varphi'(u')v' &= \tfrac{n-1}{r}\varphi'(u')\left(\psi'u' + \psi u''\right)\\ &= \tfrac{(n-1)(p-1)}{r}\psi'\varphi(u') + \tfrac{n-1}{r}\psi\varphi(u')'\,.\end{aligned}$$

It follows that

$$\begin{aligned}&(\varphi'(u')v')' + \tfrac{n-1}{r}\varphi'(u')v' + f'(u)v\\ &= (p-1)\psi''\varphi + (p-1)\psi'\left(\varphi(u')' + \tfrac{n-1}{r}\varphi(u')\right) + \tfrac{n-1}{r^2}\psi\varphi - \tfrac{n-1}{r}\psi'\varphi\\ &= -p\psi'f(u) + \varphi\left[(p-1)\psi'' - \tfrac{n-1}{r}\psi' + \tfrac{n-1}{r^2}\psi\right]\,,\end{aligned}$$

which implies (3.4).

Multiplying the equation (3.1) by $(p-1)v$, and subtracting the equation (3.4) multiplied by u gives, in view of (3.3),

$$\begin{aligned}(3.5)\ \left[r^{n-1}\left((p-1)\varphi(u')v - u\varphi'(u')v'\right)\right]' + r^{n-1}v\left[(p-1)f(u) - uf'(u)\right]\\ = pr^{n-1}\psi' uf(u) - r^{n-3}u\varphi(u')L[\psi]\,.\end{aligned}$$

The second term on the left is equal to

$$[pF(u) - uf(u)]'\,\psi r^{n-1} = \left[(pF(u) - uf(u))\,\psi r^{n-1}\right]' - (pF(u) - uf(u))\left(\psi r^{n-1}\right)'\,.$$

Using this identity in (3.5), and integrating over $(0,1)$, we conclude the proof. $\diamondsuit$

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