

A remark on the non-degeneracy condition

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Abstract

The structure of the set of positive solutions of the semilinear elliptic boundary value problem

$$\Delta u(x) + \lambda f(u(x)) = 0 \quad \text{for } x \in D, \quad u = 0 \quad \text{on } \partial D$$

depends on a certain non-degeneracy condition, which was proved by K.J. Brown [1] and T. Ouyang and J. Shi [5]. We provide a short alternative proof of that condition.

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We consider the semilinear boundary value problem

$$(1) \quad \Delta u(x) + \lambda f(u(x)) = 0 \quad \text{for } x \in D, \quad u = 0 \quad \text{on } \partial D,$$

depending on the positive parameter λ . We assume that D is a bounded region in R^n , and $f \in C^1(\bar{R}_+)$. One considers the linearized problem for (1)

$$(2) \quad \Delta w(x) + \lambda f'(u(x))w = 0 \quad \text{for } x \in D, \quad w = 0 \quad \text{on } \partial D.$$

If the problem (2) admits a non-trivial solution, we refer to $u(x)$ as a singular solution of (1). (We consider only the classical solutions of both (1) and (2), $u, w \in C^2(\bar{D})$.) At a non-singular solution of (1) we can apply the implicit function theorem to continue the solution to nearby λ 's. At a singular solution the following non-degeneracy condition is crucial

$$(3) \quad \int_D f(u(x))w(x) dx > 0.$$

As explained in K.J. Brown [1], condition (3) (actually a more general condition $\int_D f(u(x))w(x) dx \neq 0$) ensures that all positive solutions of (1) lie on a single smooth curve in a neighborhood of (λ, u) (see also Korman [3], [4] and T. Ouyang and J. Shi [5]).

K.J. Brown [1] proved the following theorem. ($n(x)$ denotes an unit normal vector at $x \in \partial D$, pointing outside.)

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Theorem 1 Suppose that D is a bounded domain with smooth boundary such that $x \cdot n(x) > 0$ for all $x \in \partial D$. Assume that $\lambda > 0$ and $f(0) \geq 0$. Assume that at some solution $(\lambda, u(x))$ of (1) the linearized problem (2) admits a positive solution $w(x)$. Then the non-degeneracy condition (3) holds.

A similar result was proved independently by T. Ouyang and J. Shi [5]. Both papers had elegant proofs, reminiscent of that used for the Pohozaev's identity. The domain D described in the Theorem 1 is called star-shaped. As for smoothness of the boundary, it appears one only needs enough for the divergence theorem to apply (which holds for very general domains). The purpose of this note is to offer a shorter alternative proof of the above theorem, which comes at the expense of a little stricter assumption on the smoothness of the boundary. Namely, we assume that ∂D satisfies the exterior sphere condition. We remark that for balls in R^n the Theorem 1 was proved in [3], even without the assumption $w(x) > 0$.

Proof of the Theorem 1. Let $U(x)$ be the solution of

$$(4) \quad \Delta U(x) = 0 \quad \text{for } x \in D, \quad U = x \cdot \nabla u \text{ on } \partial D.$$

It is well-known that (4) has a solution $U \in C^2(D) \cap C(\partial D)$, see e.g. p. 101 in [2]. It is here that we use the smoothness of the domain D . The function $v(x) = x \cdot \nabla u - U(x)$ satisfies

$$(5) \quad \begin{aligned} \Delta v + \lambda f'(u)v &= -2\lambda f(u) - \lambda f'(u)U \quad \text{for } x \in D, \\ v &= 0 \quad \text{on } \partial D. \end{aligned}$$

Using the Fredholm alternative (or just multiplying the equation (2) by v , the equation (5) by w , integrating and subtracting) we conclude that the right hand side of (5) is orthogonal to w , i.e.

$$\lambda \int_D f(u)w dx = -\frac{1}{2}\lambda \int_D f'(u)Uw dx.$$

Using the equation (2) and the divergence theorem, we have

$$-\lambda \int_D f'(u)Uw dx = \int_D \Delta w U dx = \int_{\partial D} \frac{\partial w}{\partial n} U dS = \int_{\partial D} \frac{\partial w}{\partial n} x \cdot \nabla u dS.$$

Observe that ∂D is a level set of u ($u = 0$ on ∂D). It follows that $\nabla u = -|\nabla u|n(x)$. Since $f(0) \geq 0$, by the maximum principle $|\nabla u(x)| > 0$ for all $x \in \partial D$. It is also well-known that $\frac{\partial w}{\partial n} < 0$ on ∂D , since $w > 0$ in D . We then have

$$\int_D f(u)w dx = -\frac{1}{2\lambda} \int_{\partial D} \frac{\partial w}{\partial n} |\nabla u| x \cdot n(x) dx > 0,$$

and the proof follows.

References

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