Nonlinear Analysis, Theory, Methods & Applications, Vol. 10, No. 12, pp. 1471–1476, 1986
Printed in Great Britain.

0362–546X/86 \$3.00 + .00
Pergamon Journals Ltd.

EXISTENCE OF SOLUTIONS FOR A CLASS OF SEMILINEAR NONCOERCIVE PROBLEMS

PHILIP KORMAN
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221-0025, U.S.A.

(*Received in revised form* 13 *November* 1985; *received for publication* 20 *January* 1986)

Key words and phrases: Existence of solutions, *a priori* estimates, degree theory.

1. INTRODUCTION

WE STUDY existence of periodic solutions for a nonlinear noncoercive boundary value problem

$$\begin{aligned} u_y - u_{xx} &= g(x, z, u) && y = 1, \\ \Delta u &= f(x, y, z, u, Du) && 0 < y < 1, \\ u &= 0 && y = 0. \end{aligned} \tag{1}$$

Here the functions f and g are assumed to be 2π periodic in x and z, and we are looking for 2π periodic in x, z solution $u(x, y, z)$.

This problem arises in three-dimensional water wave theory under the assumption that the gravity is pointing up, see [7]. We are interested in the problem primarily since it represents one of the simplest possible noncoercive elliptic problems (i.e. the Lopatinski–Shapiro condition fails, see [3]). We see that this model has basically the same properties as coercive elliptic problems with respect to maximum principle, nonlinear existence results and the properties of eigenvalues and eigenfunctions.

From the results of [4] we can conclude existence of solutions for (1), provided f and g are sufficiently small (and smooth). In this paper we remove the smallness condition. (The growth of f in Du then has to be restricted by a condition of Nagumo type.)

To prove existence we follow the technique learned from Tsai [9], and which involves a combination of degree theory and the method of super- and subsolutions (a similar approach was used by Serrin in [8]). In his paper, making use of the well-known global gradient bounds for the quasilinear second order elliptic problems, Tsai shows that solution cannot escape from the set $\Lambda = \{u \in C^1 | \alpha < u < \beta, |\nabla u| < M + 1\}$, where the constants α, β are sub- and supersolutions respectively, and M is the bound for $|\nabla u|$, provided $\alpha < u < \beta$. Then using degree theory, the solution operator is deformed to the identity operator, proving existence.

For our problem we cannot quote any global estimates for $|\nabla u|$ (and also no compactness results in C^k spaces). This leads us to derive *a priori* estimates in Sobolev norms for the nonlinear problem (1). After that we use degree theory in a way similar to Tsai's.

In the last section we study the eigenvalues and eigenfunctions of the linear problem.

2. NOTATION AND PRELIMINARY RESULTS

By V we shall denote the domain $0 \leq x, z \leq 2\pi$, $0 \leq y \leq 1$; the upper part of its boundary ($y = 1$) we denote by V_t. We consider a subspace of $W^{m,p}$, consisting of functions 2π periodic in x and z; when writing $W^{m,p}(V)$, $W^{m,p}(V_t)$, $H^m(V)$, $H^m(V_t)$, we shall always refer to that subspace. By $\|\cdot\|_{m,p}$ we denote the norm in $W^{m,p}(V)$, by $\|\cdot\|_m$ the one in $W^{m,2}(V) \equiv H^m(V)$, and by $\overline{\|\cdot\|}_m$ the norm in $H^m(V_t)$, see [1] for definitions. Also $|u|_{L^\infty} = \operatorname*{ess\,sup}_V |u|$. We denote by c all positive constants independent of the unknown function u; by $\bar{c}$ all irrelevant positive constants depending continuously on $|u|_{L^\infty}$, but not on the derivatives of u; $Du = \nabla u$, $|\nabla u| = \sqrt{u_x^2 + u_y^2 + u_z^2}$. We shall write $\int h \equiv \iiint_V h \, dx \, dy \, dz$, $\int_t h = \iint_{V_t} h \, dx \, dz$. Sometimes we write x_1, x_2, x_3 for x, y, z;

$$f_{x_i} = \frac{\partial f}{\partial x_i}, u_i = \frac{\partial u}{\partial x_i}, \qquad i = 1, 2, 3.$$

We need the following lemmas, see [3, 4] for proofs or references.

LEMMA 1. For any $\varepsilon > 0$ and integer $n > 0$ one has

(i) $\overline{\|v\|}_n \leq \|v\|_{n+1}$

(ii) $\|v\|_n, \overline{\|v\|}_n \leq \varepsilon \|v\|_{n+1} + c(\varepsilon)\|v\|_0$.

LEMMA 2. Suppose $w_1, \ldots, w_s \in C^r(V)$ (or $C^r(V_t)$). Suppose that $\phi = \phi(w_1, \ldots, w_s)$ possesses continuous derivatives up to order $r \geq 1$ on $|w_i| \leq c_0$, $i = 1, \ldots, s$. Then

$$\|\phi(w_1, \ldots, w_s)\|_r \leq c\left(\sum_{i=1}^{s} \|w_i\|_r + 1\right) \qquad \text{for } |w_i|_{L^\infty} \leq c_0, \quad i = 1, \ldots, s,$$

and the same is true for $\overline{\|\cdot\|}_m$ norm.

LEMMA 3. We have the following relations between our norms and spaces.

(i) $\|u\|_{j,p} \leq c\|u\|_{m,r}^{a} \|u\|_{0,q}^{1-a}$,

for any integers $0 \leq j < m$, provided that $j/m \leq a \leq 1$, $1 \leq q$, $r \leq \infty$, $1/p = j/3 + a(1/r - m/3) + (1 - a)(1/q)$, and $m - j - n/r$ is not a nonnegative integer

(ii) $H^s(V)$ is compactly imbedded in $H^{s-\varepsilon}(V)$ for any real $s > 0$ and $\varepsilon > 0$.

(iii) For any integer $m \geq 0$

$$\overline{\|u\|}_m \leq c\|u\|_{m+1/2}.$$

Proof. The property (i) can be found in [2, p. 27], where it is stated for the domains with C^m boundary. Let V_1 be any C^m domain in the strip $0 \leq y \leq 1$, such that $V \subset V_1$. Then (i) is true for V_1. Since our functions are periodic, it is easy to see that it is also true for V.

The property (ii) can be found in [6, pp. 99–101]. The statement (iii) is standard for C^1 domains, see [6, p. 41]. We adapt it to V in the same way as (i).

The following lemma was proved in [5].

LEMMA 4. Consider the linear problem

$$\begin{aligned} u_y - u_{xx} + \varepsilon u &= g(x, z) \qquad && y = 1, \varepsilon \geq 0, \\ \Delta u &= f(x, y, z) && 0 < y < 1, \\ u &= 0 && y = 0. \end{aligned} \tag{2}$$

Let $f \in H^m$, $g \in \bar{H}^{m+1}$ ($m \geq 2$), be 2π periodic in x, z. Then the problem (2) has unique 2π periodic in x, z classical solution $u(x, y, z)$, and independent of $\varepsilon \geq 0$ the following estimate holds (with $m \geq 0$)

$$\|u\|_{m+2} \leq c(\|f\|_m + \overline{\|g\|}_{m+1}). \tag{3}$$

3. THE *A PRIORI* ESTIMATE

THEOREM 1. Let u be 2π periodic in x, z solution of

$$\begin{aligned} u_y - u_{xx} + \varepsilon u &= \lambda g(x, z, u) & y &= 1, \\ \Delta u &= \lambda f(x, y, z, u, Du) & 0 &< y < 1, \\ u &= 0 & y &= 0. \end{aligned} \tag{4}$$

Here $f \in C^3(V)$ and $g \in C^4(V_t)$ are 2π periodic in x, z; $\varepsilon \geq 0$, $0 \leq \lambda \leq 1$. We assume that (recall that $\bar{c} = \bar{c}(|u|_{L^\infty})$)

(i) $|f| \leq \bar{c}(1 + |\nabla u|^{2-\delta})$, $\quad 0 < \delta \leq 2$,
(ii) $|f_{x_i}|, |f_u| \leq \bar{c}(1 + |\nabla u|^2)$, $\quad i = 1, 2, 3$,
(iii) $|f_{u_i}| \leq \bar{c}(1 + |\nabla u|)$, $\quad i = 1, 2, 3$.

Then $\|u\|_{9/2} \leq M$, $M = M(|u|_{L^\infty})$, independent of $\varepsilon \geq 0$ and $0 \leq \lambda \leq 1$.

Proof. Using lemmas 1, 2, and 4, we estimate:

$$\begin{aligned} \|u\|_2 &\leq c(\overline{\|g\|}_1 + \|f\|_0) \leq \bar{c}(1 + \overline{\|u\|}_1 + (\textstyle\int f^2)^{1/2}) \\ &\leq \bar{c}(1 + \bar{\varepsilon}\|u\|_2 + c(\bar{\varepsilon})\|u\|_0 + (\textstyle\int |\nabla u|^{4-2\delta})^{1/2}). \end{aligned}$$

In the case $1 \leq \delta \leq 2$ this easily implies the inequalities (6) below. So let $0 < \delta < 1$. Then choosing $\bar{\varepsilon}$ small enough we get (since $\|u\|_0 \leq \bar{c}$)

$$\|u\|_2 \leq \bar{c}(1 + \|u\|_{1,4-2\delta}^{2-\delta}). \tag{5}$$

By lemma 3 we estimate ($(4 - 2\delta)/\delta > 1$)

$$\|u\|_{1,4-2\delta} \leq c\|u\|_2^{1/2}\|u\|_{0,(4-2\delta)/\delta}^{1/2} \leq \bar{c}\|u\|_2^{1/2}.$$

Combining this with (5), we obtain

$$\|u\|_2 \leq \bar{c}(1 + \|u\|_2^{1-\delta/2}),$$

which implies (using lemma 3 for the second inequality)

$$\|u\|_2 \leq \bar{c}, \qquad \|u\|_{1,4} \leq \bar{c}. \tag{6}$$

Next, by lemmas 1, 4 and the chain rule we estimate

$$\begin{aligned} \|u\|_3 &\leq c(\overline{\|g\|}_2 + \|f(x, y, z, u, Du)\|_1) \\ &\leq c\Big(\bar{\varepsilon}\|u\|_3 + c(\bar{\varepsilon})\|u\|_0 + \|f\|_0 + \sum_{i=1}^{3} \|f_{x_i}\|_0 \\ &\quad + \sum_{i=1}^{3} \|f_u u_i\|_0 + \sum_{i,j=1}^{3} \|f_{u_i} u_{ij}\|_0\Big). \end{aligned} \tag{7}$$

Using (6), it is easy to see that $\|f\|_0, \|f_{x_i}\|_0 \leq \bar{c}$. Next, using (ii), (6) and lemma 3, we estimate

$$\|f_u u_i\|_0 \leq \|f_u\|_{L^4}\|u_i\|_{L^4} \leq \bar{c}\left(1 + \int |\nabla u|^8\right)^{1/4} \|u\|_{1,4}$$
$$\leq \bar{c}(1 + \|u\|_{1,8}^2) \leq \bar{c}(1 + \|u\|_3^{5/6}\|u\|_{L^\infty}^{7/6}) \leq \bar{c}(1 + \|u\|_3^{5/6}).$$

Similarly, by (iii)

$$\|f_{u_i} u_{ij}\|_0 \leq \|f_{u_i}\|_{L^4}\|u_{ij}\|_{L^4} \leq \bar{c}(1 + \|u\|_{1,4})\|u\|_{2,4}$$
$$\leq \bar{c}\|u\|_{2,4} \leq \bar{c}\|u\|_3^{5/6}\|u\|_{L^\infty}^{1/6} \leq \bar{c}\|u\|_3^{5/6}.$$

Using these estimates in (7), and choosing $\bar{\varepsilon}$ sufficiently small, we get

$$\|u\|_3 \leq \bar{c}(1 + \|u\|_3^{5/6}),$$

which implies

$$\|u\|_3 \leq \bar{c}, \tag{8}$$

and by Sobolev's imbedding theorem $\|\nabla u\|_{L^\infty} \leq \bar{c}$. The last estimate is crucial, since now we can use lemma 2 to estimate higher derivatives:

$$\|u\|_4 \leq c(\|g\|_3 + \|f(x, y, z, u, Du)\|_2)$$
$$\leq c(\bar{\varepsilon}\|u\|_4 + c(\bar{\varepsilon})\|u\|_0 + \bar{c} + \bar{c}\|u\|_3),$$

which implies as before $\|u\|_4 \leq \bar{c}$. Similarly $\|u\|_5 \leq \bar{c}$. Since $\|u\|_{9/2} \leq \|u\|_5 \leq \bar{c} \equiv M$, the proof is complete.

Remark. It is easy to see that one can allow $\delta = 0$, provided the constant $\bar{c}$ in (i) is sufficiently small.

4. EXISTENCE OF SOLUTIONS

THEOREM 2. Let f and g in the problem (1) satisfy the conditions of theorem 1. Assume moreover that there exist constants $\alpha < 0 < \beta$ such that

$$g(x, z, \alpha) \geq 0, f(x, y, z, \alpha, 0) < 0; g(x, z, \beta) \leq 0, f(x, y, z, \beta, 0) > 0. \tag{9}$$

Then the problem (1) has a C^2 solution, 2π periodic in x, z, such that $\alpha \leq u(x, y, z) \leq \beta$.

Proof. Let $\Lambda = \{u \in H^{9/2}, \alpha < u < \beta, \|u\|_{9/2} < M + 1\}$, with M determined as in theorem 1 from α and β (i.e. for any solution of (1) satisfying $\alpha \leq u \leq \beta$, we have $\|u\|_{9/2} \leq M$). Notice that Λ is a bounded open set in $H^{9/2}$ containing zero (recall, both $H^{9/2}$ and Λ consist of 2π periodic in x, z functions). Define the map $T: u \to v$ by solving

$$\begin{aligned} v_y - v_{xx} + \varepsilon v &= g(x, z, u) & y &= 1 \\ \Delta v &= f(x, y, z, u, Du) & 0 &< y < 1, \\ v &= 0 & y &= 0. \end{aligned} \tag{10}$$

Since (by lemmas 3, 4) $\|v\|_5 \leq c(\|f\|_3 + \|g\|_4) \leq c(1 + \|u\|_{9/2})$, the map T is compact from Λ to $H^{9/2}$.

Define the map $H(u, \lambda) = u - \lambda Tu$, $u \in \bar{\Lambda}$, $0 \leq \lambda \leq 1$. We wish to show that

$$\deg(I - \lambda T, \Lambda, 0) = \deg(I, \Lambda, 0) = 0. \tag{11}$$

For this we proceed to show that $H(u, \lambda) \neq 0$ for $u \in \partial\Lambda$ and $0 \leq \lambda \leq 1$. Indeed $H(u, 0) \neq 0$ for $u \in \partial\Lambda$, since $0 \notin \partial\Lambda$. If $H(u, 1) = 0$ for some $u \in \partial\Lambda$, then u is a (desired) fixed point.

Next we show that $H(u, \lambda) \neq 0$ for $0 < \lambda < 1$ and $u \in \partial\Lambda$. Assume not, i.e. $H(\bar{u}, \bar{\lambda}) = 0$ for some $\bar{u} \in \partial\Lambda$, $0 < \bar{\lambda} < 1$. Then

$$\begin{aligned} \bar{u}_y - \bar{u}_{xx} + \varepsilon\bar{u} &= \bar{\lambda}g(x, z, \bar{u}) && y = 1, \\ \Delta\bar{u} &= \bar{\lambda}f(x, y, z, \bar{u}, D\bar{u}) && 0 < y < 1, \\ \bar{u} &= 0 && y = 0. \end{aligned}$$

By theorem 1, $\|\bar{u}\|_{9/2} \leq M < M + 1$. Hence at some point $P_0 = (x_0, y_0, z_0) \in \bar{V}$ we must have $\bar{u}(x_0, y_0, z_0) = \beta$ (or $\bar{u}(P_0) = \alpha$, which is eliminated in the same way). Consider two cases.

(i) $P_0 \in V$. Since P_0 is a point of maximum, $\nabla\bar{u}(P_0) = 0$, $\Delta\bar{u}(P_0) \leq 0$. But then by choice of β

$$0 \geq \Delta\bar{u}(P_0) = \bar{\lambda}f(P_0, \bar{u}(P_0), \nabla\bar{u}(P_0)) = \bar{\lambda}f(x_0, y_0, z_0, \beta, 0) > 0,$$

a contradiction.

(ii) $P_0 \in \partial V$. If P_0 lies on the side part of the boundary ($0 < y < 1$) argue as in (1).

Assume (x_0, y_0, z_0) is at the bottom part of ∂V, i.e. $y_0 = 0$. Then $0 = \bar{u}(x_0, 0, z_0) = \beta$, a contradiction.

If P_0 is on $V_t (y = 1)$, then since it is a point of maximum, we have $\bar{u}_y(P_0) \geq 0$, $\bar{u}_{xx}(P_0) \leq 0$, and hence at P_0

$$0 < \varepsilon\beta \leq \bar{u}_y - \bar{u}_{xx} + \varepsilon\bar{u} = g(x_0, z_0, \bar{u}(P_0)) = g(x_0, z_0, \beta) \leq 0,$$

a contradiction.

Therefore (11) is justified, and we obtain existence of solution u^ε to (10) with $\|u^\varepsilon\|_{9/2} \leq c$. Since (by Sobolev imbedding, $n = 3$) the set $\{u^\varepsilon\}$ is compact in C^2, if we now let $\varepsilon \to 0$ along some sequence, we can pass to the limit (along a subsequence) in (10), obtaining a C^2 solution of the original problem (10).

5. PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS

Let $u(x, y, z)$ be 2π periodic in x, z, nontrivial solution of

$$u_y - u_{xx} = 0 \qquad y = 1, \tag{12a}$$

$$\Delta u + \lambda u = 0 \qquad 0 < y < 1 \tag{12}$$

$$u = 0 \qquad y = 0.$$

It is easy to see that all eigenvalues λ are real, and the eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(V)$. More precisely, we have the following lemma.

LEMMA 5. The eigenvalues of (12) form a sequence $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$, $\lim_{n\to\infty} \lambda_n = \infty$. The principle eigenvalue $\lambda_1 = \pi^2/4$; it is simple, its eigenspace is spanned by $\varphi_1 = \sin(\pi/2)y \geq 0$.

Proof. Substituting

$$u(x, y, z) = \sum_{j,k=-\infty}^{\infty} u_{jk}(y)\, e^{ijx+ikz}$$

into (12), we get

$$\begin{aligned} &u''_{jk} + (\lambda - j^2 - k^2)u_{jk} = 0 \qquad 0 < y < 1, \\ &u_{jk}(0) = 0, \qquad u'_{jk}(1) + j^2 u_{jk}(1) = 0 \qquad (-\infty < j, k < \infty). \end{aligned} \tag{13}$$

Let $\nu_{n,j}$ $(n = 1, 2, \ldots)$ be nonzero solutions of $\tan \nu = -(\nu/j^2)$, $j \neq 0$. It is easy to see from (13) that the eigenvalues are $(\lambda_1 = \lambda_{0,0,0})$

$$\lambda_{n,j,k} = \nu_{n,j}^2 + j^2 + k^2 \qquad j, n = 1, 2, \ldots; \quad k = 0, 1, 2, \ldots,$$

$$\lambda_{n,0,k} = \left(\frac{\pi}{2} + n\pi\right)^2 + k^2 \qquad n, k = 0, 1, 2, \ldots.$$

Corresponding (real) eigenfunctions are

$$\phi_{n,j,k} = \sin(\lambda_{n,j,k} - j^2 - k^2)^{1/2} y \quad (c_1\, e^{ijx+ikz} + c_2\, e^{-ijx+ikz} + \bar{c}_2\, e^{ijx-ikz} + \bar{c}_1\, e^{-ijx-ikz}),$$

where c_1, c_2 are arbitrary complex constants.

Remark. Let $N(\lambda)$ denote the number of eigenvalues of (12) which are less than λ. Then using standard techniques (see e.g. Courant and Hilbert, Vol. 1), it is easy to show that $N(\lambda) \sim (1/6)\lambda^{3/2}$ for $\lambda \to \infty$. The asymptotic formula is the same for the problem obtained from (12) by changing (12a) to $u = 0$.

Notice that results of his section are parallel to the ones for coercive problems.

Acknowledgements—I wish to thank L. Nirenberg for posing the problem and his attention to my work. I am grateful to A. Leung for useful discussions.

REFERENCES

1. ADAMS R. A., *Sobolev Spaces*, Academic Press, New York (1975).
2. FRIEDMAN A., *Partial Differential Equations*, Robert E. Krieger, New York (1976).
3. KORMAN P., Existence of solutions for a class of nonlinear non-coercive problems, *Communs Partial diff. Eqns* **8**, 819–846 (1983).
4. KORMAN P., Existence of periodic solutions for a class of nonlinear problems, *Nonlinear Analysis* **7**, 873–879 (1983).
5. KORMAN P., On application of the monotone iteration scheme to noncoercive elliptic and hyperbolic problems, *Nonlinear Analysis* **8**, 97–105 (1984).
6. LIONS J. L. & MAGENES E., *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1. Springer, New York (1972).
7. SHINBROT M., Water waves over periodic bottoms in three dimensions, *J. Inst. Math. Applic.* **25**, 367–385 (1980).
8. SERRIN J., The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, *Phil. Trans. R. Soc. Lond.* **264**, 413–496 (1969).
9. TSAI L. Y., Existence of solutions of nonlinear elliptic systems, *Bull. Inst. Math. Acad. Sinica* **8**, 111–127 (1980).