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EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR PROBLEMS

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1. INTRODUCTION

WE STUDY existence of periodic solutions for a nonlinear non-coercive boundary value problem

$$\begin{aligned} u_y &= \rho(u_{xx}) && y = 1, \\ \Delta u &= f(x, y, z, u, Du) && 0 < y < 1, \\ u &= 0 && y = 0. \end{aligned} \tag{1}$$

Here f is 2π periodic in x and z and also depends on u and its first derivatives, we are looking for 2π periodic in x and z solution $u(x, y, z)$.

The case $\rho(u_{xx}) = -Fu_{xx}$, $F = \text{const} > 0$ corresponds to the boundary condition for the water waves on a running stream (see [6]). We show in Section 5 that the problem (1) is ill-posed in this case. This fact supports the belief that the water wave problem in three dimensions without surface tension (which differs from (1) basically in that the upper boundary is free) is ill-posed, see [5].

Our interest in the problem (1) lies primarily in the fact that the problem is non-coercive, and so one cannot use the general theory to get *a priori* estimates, see [2], [3] for discussion. However, if we assume $F < 0$ (or the gravity to be pointing up) we can prove existence of periodic solutions for (1). Namely, we have the following

THEOREM 1. Assume that for the problem (1) with f 2π periodic in x and z we have:

(i) $\rho(0) = 0$, $\dot{\rho}(0) > 0$
(ii) $\rho \in C^4$, $f, f_u, f_{u_x}, f_{u_y}, f_{u_z} \in C^3$ (in all arguments).

Then for $\|f(x, y, z, 0, 0)\|_3$, $\|f_u(x, y, z, 0, 0)\|_3$, $\|f_{u_x}(x, y, z, 0, 0)\|_3$, $\|f_{u_y}(x, y, z, 0, 0)\|_3$, $\|f_{u_z}(x, y, z, 0, 0)\|_3$ sufficiently small, the problem (1) has a C^2 solution, 2π periodic in x and z.

The proof by a contractive mapping argument is carried out in Section 4. It uses *a priori* estimates for a linear problem which are derived in Section 3 by Fourier analysis. To simplify the presentation we shall assume $\dot{\rho}(0) = 1$.

Remark. Using Nash's implicit function theorem it is possible to prove existence for $\rho =$

$\rho(u_{xx}, u_{xz}, u_{zz})$ and $f = f(x, y, z, u, Du, D^2u)$, but it requires more differentiability of ρ and f, see [2], [3] for details.

2. NOTATION AND THE PRELIMINARIES

By V we shall denote the domain $0 \le x,\ z \le 2\pi,\ 0 \le y \le 1$; the upper part of its boundary $(y = 1)$ we denote by V_t. We shall write $\|\cdot\|_m$ for the m^{th} Sobolev norm for functions in V and $\overline{\|\cdot\|}_m$ for functions on V_t. Corresponding Sobolev spaces we denote by H^m and $\bar{H}^m$. By $G^m\,(m \ge 3)$ we denote the subspace of H^m consisting of functions $u \in H^m$ such that $\overline{\|u_{xx}\|}_{m-1} < \infty$. The norm in G^m is denoted by $|\cdot|_m$, $|u|_m = \|u\|_m + \overline{\|u_{xx}\|}_{m-1}$. Clearly G^m is a Banach space.

We shall write u_i, $i = 0, 1, 2, 3$ for u, u_x, u_y, u_z correspondingly; $\mathrm{D}\,u = \operatorname{grad} u$. We shall write $\delta(t)$ for any function which tends to 0 as $t \to 0$. All positive constants independent of unknown functions we denote by c.

We need the following standard lemma, see for example [2].

LEMMA 1. Suppose that the functions $w_1, \dots, w_s \in C^m(V)$ or $C^m(V_t)$. Suppose that $\varphi = \varphi(w_1, \dots, w_s)$ possesses continuous derivatives up to order $m \ge 1$ bounded by c on $\max_i |w_i| < 1$. Then

$$\|\varphi(w_1, \dots, w_s)\|_m \le C\|w\|_m + 1) \quad \text{for} \quad \max_i \|w_i\|_{L^\infty} < 1. \tag{2}$$

(We denote $\|w\|_m = \max_i \|w_i\|_m$). If in addition we assume $\varphi(0, \dots, 0) = 0$, $m \ge 1$ then

$$\|\varphi(w_1, \dots, w_s)\|_m = \delta(\|w\|_m), \tag{3}$$

where $\delta(t) \to 0$ as $t \to 0$.

Remark. The lemma is also true for functions $\varphi = \varphi(x, y, z, w_1, \dots, w_s)$ with $\varphi \in C^m$ in all variables. Conclusion (2) is as before, and for (3) the corresponding assumption is $\varphi(x, y, z, 0, \dots, 0) = 0$ for all $(x, y, z) \in V$ or V_t.

To prove theorem 1 we define the map $T: u \to v$ by solving the following linear problem

$$\begin{aligned} v_y - v_{xx} &= \rho(u_{xx}) - u_{xx} && y = 1, \\ \Delta v &= f(x, y, z, u, \mathrm{D}\,u) && 0 < y < 1, \\ v &= 0 && y = 0. \end{aligned} \tag{4}$$

Clearly, the fixed points of T are solutions of (1).

3. *A PRIORI* ESTIMATES FOR THE LINEAR PROBLEM (4)

To simplify notation we rewrite (4) as

$$v_y - v_{xx} = g(x, z) \qquad y = 1, \tag{5a}$$

$$\Delta v = f(x, y, z) \qquad 0 < y < 1, \tag{5b}$$

$$v = 0 \qquad y = 0, \tag{5c}$$

where f and g are given functions, 2π periodic in x and z.

LEMMA 2. Assume $g, f \in H^m$. Then the problem (5a, b, c) has a unique, 2π periodic in x and z, solution, and

$$|v|_{m+1} \leqslant c(\overline{\|g\|}_m + \|f\|_m) \qquad (m \geqslant 0 \quad \text{an integer}). \tag{6}$$

Proof. Represent v, f, g by their double Fourier series; use summation convention:

$$v(x, y, z) = v_{jk}(y)\, e^{ijx+ikz}, f(x, y, z) = f_{jk}(y) e^{ijx-ikz}, g = g_{jk}\, e^{ijx+ikz}, \ (i = \sqrt{-1}).$$

Substituting these expressions into (5) and suppressing the indices j, k (i.e. writing v for v_{jk}, f for f_{jk} and g for g_{jk}) we get:

$$v'(1) + j^2 v(1) = g \tag{7.a}$$

$$v''(y) - (j^2 + k^2) v(y) = f(y) \tag{7.b}$$

$$v(0) = 0. \tag{7.c}$$

Set $\rho = \sqrt{(j^2 + k^2)}$. For $\rho \neq 0$ solution of (7b), (7c) is

$$v(y) = \gamma \sinh \rho y + \frac{1}{\rho} \int_0^y f(t) \sinh \rho(y - t)\, dt. \tag{8}$$

To find γ we substitute this into (7.a):

$$\gamma(\rho \cosh \rho + j^2 \sinh \rho) + \int_0^1 f(t)[\cosh \rho(1 - t) + \frac{j^2}{\rho} \sinh \rho(1 - t)]\, dt = g. \tag{9}$$

Denote $A = \rho \cosh \rho + j^2 \sinh \rho$ and notice that

$$A \geqslant c(\rho + j^2)\, e^{\rho}. \tag{10}$$

Using (8) and (9) we get:

$$\begin{aligned} Av(y) &= g \sinh \rho y - \int_0^1 f(t)[\cosh \rho(1 - t) \sinh \rho y + \frac{j^2}{\rho} \sinh \rho(1 - t) \sinh \rho y]\, dt \\ &\quad + \frac{1}{\rho} \int_0^y f(t)[\rho \sinh \rho(y - t) \cosh \rho + j^2 \sinh \rho(y - t) \sinh \rho]\, dt \\ &= g \sinh \rho y - \int_y^1 f(t)\, [\cosh \rho(1 - t) \sinh \rho y + \frac{j^2}{\rho} \sinh \rho(1 - t) \sinh \rho y]\, dt \\ &\quad - \int_0^y f(t)[A_1 + \frac{j^2}{\rho} A_2]\, dt, \end{aligned}$$

where

$$A_1 = \cosh \rho(1 - t) \sinh \rho y - \sinh \rho(y - t) \cosh \rho,$$

$$A_2 = \sinh \rho(1 - t) \sinh \rho y - \sinh \rho(y - t) \sinh \rho.$$

Using standard identities for hyperbolic functions we compute:

$$A_1 = \sinh \rho t \cosh \rho(y - 1),$$

$$A_2 = \sinh \rho t \sinh \rho(1 - y),$$

so that finally

$$Av(y) = g \sinh \rho y - \int_y^1 f(t)[\cosh \rho(1-t) \sinh \rho y + \frac{j^2}{\rho} \sinh \rho(1-t) \sinh \rho y] \, dt$$

$$- \int_0^y f(t)[\sinh \rho t \cosh p(y-1) + \frac{j^2}{\rho} \sinh \rho t \sinh \rho(1-y)] \, dt. \tag{11}$$

Since $\sinh x$, $\cosh x \sim e^x/2$ as $x \to +\infty$, we easily estimate:

$$A|v(y)| \leq c\left[|g| \, e^{\rho y} + e^{\rho}\left(1 + \frac{j^2}{\rho}\right) \int_y^1 |f(t)| \, e^{\rho(y-t)} \, dt + e^{\rho}\left(1 + \frac{j^2}{\rho}\right) \int_0^y |f(t)| \, e^{\rho(t-y)} \, dt\right]$$

$$\leq c\left[|g| \, e^{\rho y} + e^{\rho}\left(1 + \frac{j^2}{\rho}\right)\left(\int_y^1 |f(t)|^2 \, dt\right)^{1/2} \left(\int_y^1 e^{2\rho(y-t)} \, dt\right)^{1/2}\right.$$

$$\left. + e^{\rho}\left(1 + \frac{j^2}{\rho}\right)\left(\int_0^y |f(t)|^2 \, dt\right)^{1/2} \left(\int_0^y e^{2\rho(t-y)} \, dt\right)^{1/2}\right]$$

$$\leq c\left[|g| \, e^{\rho y} + \frac{e^{\rho}}{\sqrt{\rho}}\left(1 + \frac{j^2}{\rho}\right)\left(\int_0^1 |f(t)|^2 \, dt\right)^{1/2}\right]. \tag{12}$$

Combining the estimates (10) and (12) we derive:

$$|v(y)| \leq c\left[\frac{|g|}{\rho + j^2} + \frac{1}{\rho^{3/2}}\left(\int_0^1 |f(t)|^2 \, dt\right)^{1/2}\right].$$

Using the Schwarz inequality and restoring the subscripts we have:

$$\int_0^1 |v_{jk}(y)|^2 \, dy \leq \frac{c}{j^2 + k^2}\left[|g_{jk}|^2 + \int_0^1 |f_{jk}(t)|^2 \, dt\right]. \tag{13}$$

In the case $j = k = 0$ we easily obtain from (7)

$$\int_0^1 |v_{00}(y)|^2 \, dy \leq c\left[|g_{00}|^2 + \int_0^1 |f_{00}(t)|^2 \, dt\right]. \tag{14}$$

Differentiating (11), and going through the same steps as in derivation of (13), we estimate:

$$|v'_{jk}|^2 \leq c\left[|g_{jk}|^2 + \int_0^1 |f_{jk}(t)|^2 \, dt\right] \quad (\forall j, k). \tag{15}$$

Combining (13), (14), (15) we get:

$$\|v\|_1^2 \leq c[\overline{\|g\|}_0^2 + \|f\|_0^2]. \tag{16}$$

Also using (15) we have:

$$\overline{\|v_y\|}_0^2 = \sum_{j,k} |v'_{jk}(1)|^2 \leq c(\overline{\|g\|}_0^2 + \|f\|_0^2),$$

and hence by (5.a)

$$\overline{\|v_{xx}\|}_0^2 \leq 2(\overline{\|v_y\|}_0^2 + \overline{\|g\|}_0^2) \leq c\overline{\|g\|}_0^2 + \|f\|_0^2). \tag{17}$$

Adding (16) and (17)

$$|v|_1 \leqslant c(\overline{\|g\|}_0^2 + \|f\|_0^2), \tag{18}$$

which proves lemma 2 for $m = 0$.

To get higher order estimates let α denote any derivative in x and z variables only. Differentiating (5) we get:

$$\begin{aligned} v_y^\alpha - v_{xx}^\alpha &= g^\alpha & y &= 1, \\ \Delta v^\alpha &= f^\alpha & 0 &< y < 1, \\ v^\alpha &= 0 & y &= 0. \end{aligned} \tag{19}$$

By (18)

$$\|v^\alpha\|_1^2 + \overline{\|v_{xx}^\alpha\|}_0^2 \leqslant c(\overline{\|g^\alpha\|}_0^2 + \|f^\alpha\|_0^2).$$

Summing over all such α, $|\alpha| \leqslant m$

$$\sum_{|\alpha| \leqslant m} \|v^\alpha\|_1^2 + \overline{\|v_{xx}\|}_m^2 \leqslant c(\overline{\|g\|}_m^2 + \|f\|_m^2). \tag{20}$$

By the definition of Sobolev norms

$$\begin{aligned} \sum_{|\alpha| \leqslant m} \|v^\alpha\|_1^2 = \|v\|_{m+1}^2 - \sum_{|\alpha| \leqslant m-1} \|v_{yy}^\alpha\|_0^2 - \sum_{|\alpha| \leqslant m-2} \|v_{yyy}^\alpha\|_0^2 \\ - \dots - \|D_y^{m+1} v\|_0^2 \geqslant \|v\|_{m-1}^2 - c\|v_{yy}\|_{m-1}^2. \end{aligned} \tag{21}$$

From the equation (5.b) $v_{yy} = f - v_{xx} - v_{zz}$, so that using (20)

$$\|v_{yy}\|_{m-1}^2 \leqslant c[\|f\|_{m-1}^2 + \|v_{xx}\|_{m-1}^2 + \|v_{zz}\|_{m-1}^2) \leqslant c(\overline{\|g\|}_m^2 + \|f\|_m^2). \tag{22}$$

Combining (20), (21), (22) we get:

$$\|v\|_{m+1}^2 + \overline{\|v_{xx}\|}_m^2 \leqslant c[\overline{\|g\|}_m^2 + \|f\|_m^2],$$

which finishes the proof of lemma 2.

4. PROOF OF THEOREM 1

We proceed to show that the map T defined by (4) satisfies the conditions of contractive mapping theorem in a sufficiently small ball around the origin in the space G^m, $m \geqslant 4$. Denote:

$$\theta_i = \|f_{u_i}(x, y, z, 0, 0)\|_{m-1}, i = 0, 1, 2, 3; \; \theta_4 = \|f(x, y, z, 0, 0)\|_{m-1}$$

(i) T is a contraction. Indeed, let $v^1 = Tu^1$, $v^2 = Tu^2$ with $|u^1|_m$, $|u^2|_m \leqslant r$. Set $U = u^1 - u^2$, $V = v^1 - v^2$. Define $\varphi(t) = \rho(t) - t$, $\varphi(0) = \varphi'(0) = 0$. Then from (4)

$$\begin{aligned} V_y - V_{xx} &= \varphi(u_{xx}^1) - \varphi(u_{xx}^2) & y &= 1, \\ \Delta V &= f(x, y, z, u^1, Du^1) - f(x, y, z, u^2, Du^2) & 0 &< y < 1, \\ V &= 0 & y &= 0. \end{aligned} \tag{23}$$

By the mean-value theorem we write:

$$\varphi(u_{xx}^1) - \varphi(u_{xx}^2) = \int_0^1 \varphi'(tu_{xx}^1 + (1-t)u_{xx}^2)U_{xx}\,dt,$$

$$f(x,y,z,u^1,Du^1) - f(x,y,z,u^2,Du^2)$$
$$= \int_0^1 \sum_{i=0}^{3} f_{u_i}(x,y,z,tu^1 + (1-t)u^2, tDu^1 + (1-t)Du^2)U_i\,dt.$$

By Sobolev lemma we have (independent of t):

$$\|tu_{xx}^1 + (1-t)u_{xx}^2\|_{L^\infty} \leq c(\overline{\|u_{xx}^1\|}_{m-1} + \overline{\|u_{xx}^2\|}_{m-1}) \leq cr,$$
$$\|t\,Du^1 + (1-t)\,Du^2\|_{L^\infty} \leq c(\|u^1\|_m + \|u^2\|_m) \leq cr,$$

and hence using corollary of lemma 1 we estimate:

$$\overline{\|\varphi(u_{xx}^1) - \varphi(u_{xx}^2)\|}_{m-1} \leq \sup_{0\leq t\leq 1} \overline{\|\varphi'(tu_{xx}^1 + (1-t)u_{xx}^2)\|}_{m-1}\overline{\|U_{xx}\|}_{m-1} \leq \delta(r)|U|_m, \tag{24}$$

$$\|f(x,y,z,u^1,Du^1) - f(x,y,z,u^2,Du^2)\|_{m-1}$$
$$\leq \sum_{i=0}^{3}\Big[\sup_{0\leq t\leq 1}\|f_{u_i}(x,y,z,tu^1 + (1-t)u^2, t\,Du^1 + (1-t)Du^2) - f_{u_i}(x,y,z,0,0)\|_{m-1}$$
$$+ \|f_{u_i}(x,y,z,0,0)\|_{m-1}\Big]\|U_i\|_{m-1} \leq \left(\delta(r) + \sum_{i=0}^{3}\theta_i\right)|U|_m. \tag{25}$$

Applying lemma 2 to (23) and using (24), (25) we have:

$$|V|_m \leq \left(\sum_{i=0}^{3}\theta_i + \delta(r)\right)|U|_m < 1/2\,|U|_m,$$

provided r and θ_i, $i = 0, 1, 2, 3$ are sufficiently small.

(ii) T takes sufficiently small ball into itself. Indeed, let $|u|_m \leq r$, $v = Tu$ and we want to show that $|v|_m \leq r$. We shall use part (i) setting $u^1 = u$, $v^1 = v$ and $u^2 \equiv 0$, $v^2 = Tu^2$. First, notice that by lemma 2

$$|v^2|_m \leq \|f(x,y,z,0,0)\|_{m-1} = \theta_4.$$

Hence by (i)

$$|v|_m - |v^2|_m \leq |v - v^2|_m \leq \tfrac{1}{2}|u|_m,$$

so that

$$|v|_m \leq \tfrac{1}{2}|u|_m + \theta_4 \leq r,$$

assuming $\theta_4 \leq r/2$.

Finally, fixing $m = 4$ we complete the proof of the theorem 1.

5. CASE $F > 0$

THEOREM 2. In the case $F \equiv \dot{\rho}(0) > 0$ the problem (1) is ill-posed in the following sense: consider a simplified linear homogeneous problem:

$$\begin{aligned} u_y + F u_{xx} &= 0 \quad (F > 0) \quad && y = 1, \\ \Delta u &= 0 && 0 < y < 1, \\ u &= 0 && y = 0. \end{aligned} \tag{26}$$

Then for a dense countable set of numbers F the problem (26) has nontrivial 2π periodic in x and z solutions, while for the complimentary (dense) set of F the only such solution is the trivial one.

Proof. Setting

$$u(x, y, z) = \sum_{j,k=-\infty}^{\infty} u_{jk}(y)\, e^{ijx - ikz}$$

we find that $u_{jk} = c_{jk} \sinh \sqrt{(j^2 + k^2)}y$, where the constants c_{jk} must satisfy

$$c_{jk}(\sqrt{(j^2 + k^2)} \cosh \sqrt{(j^2 + k^2)} - Fj^2 \sinh \sqrt{(j^2 + k^2)}) = 0. \tag{27}$$

Consider the set $\mathfrak{F} = \{F_{j,k}\}$, where we denote

$$F_{j,k} = \frac{\sqrt{(j^2 + k^2)}}{j^2} \coth \sqrt{(j^2 + k^2)}.$$

If $F \in \mathfrak{F}$, i.e. $F = F_{j_0, k_0}$ for some (j_0, k_0) then $u = (e^{ij_0x + ik_0z} + e^{-ij_0x - ik_0z}) \sinh\sqrt{j_0^2 + k_0^2}\, y$ is a nontrivial (real) solution of (26).

The set $\mathfrak{F}$ is clearly countable, it remains to show that it is dense. For that it suffices to approximate any rational $t = p/q$ by numbers from $\mathfrak{F}$. Consider

$$F_{q^n, q^{2n}t} = \frac{\sqrt{(q^{2n} + q^{4n}t^2)}}{q^{2n}} \coth \sqrt{(q^{2n} + q^{4n}t^2)} \to t,$$

as $n \to \infty$, completing the proof.

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REFERENCES

1. KLAINERMAN S., Global existence for nonlinear wave equations, Ph.D. dissertation, New York University (1978).
2. KORMAN P., Existence of solutions for a nonlinear boundary value problem associated with water waves, Ph.D. dissertation, New York University (1981).
3. KORMAN P., Existence of solutions for a class of nonlinear non-coercive problems, preprint.
4. MITYAGIN B., Nonlinear boundary value problem for Poisson equation without solution: an example, preprint.
5. SHINBROT, M., Water waves over periodic bottoms in three dimensions, *J. Inst. math. Applic.* **25**, 367–385 (1980).
6. STOKER J., *Water Waves*, Interscience, New York (1957).