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Exact Multiplicity of Solutions for a Class of Semilinear Neumann Problems

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Dedicated to Dieter S. Schmidt on the occasion of a significant birthday

Abstract

We use bifurcation theory to obtain the exact multiplicity of solutions for a class of Neumann problems, where the nonlinearity changes concavity at most once on some interval. Our results cover both the cubics and the Euler's problem of buckling of an elastic rod. We also study the direction of bifurcation from the trivial solution.

Key words: Bifurcation of solutions, global solution curves.

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1 Introduction

We use bifurcation theory to study the exact multiplicity of solutions for a class of Neumann problems

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \quad u'(0) = u'(1) = 0. \tag{1}$$

While a number of exact multiplicity results are known for the corresponding Dirichlet problem, see e.g. [8], [10], [5], [4] and the numerous references in those papers, relatively little seems to be known on the exact multiplicity of the Neumann problem, see e.g. J. Smoller and A. Wasserman [8] or R. Schaaf [7] for the previous works.

We investigate the problem (1) in several directions. Our main result deals with the *exact* multiplicity of solutions, which behave like polynomials with simple real roots. We begin by considering nonlinearities $f(u)$, which behave like cubics with three distinct roots, and then generalize. Our model case is a cubic $f(u) = (u-a)(u-b)(c-u)$, with the roots $a < b < c$, which changes it concavity only once at some $r \in (a,c)$. Without loss of generality we shall assume that $b = 0$, and so $a < 0 < c$ (otherwise replace $u - b \to u$), and that $r \in [0, c)$, since otherwise we may replace u by $-u$ in (1). So that we assume that $f(u) \in C^3[a,c]$ satisfies

$$\begin{gathered} f(a) = f(0) = f(c) = 0, \quad a < 0 < c \\ f(u) < 0 \quad \text{for } u \in (a,0) \text{ and } \quad f(u) > 0 \quad \text{for } u \in (0,c) \end{gathered} \tag{2}$$

$$\begin{gathered} f''(u) > 0 \text{ for } u \in (a,r) \text{ and } \quad f''(u) < 0 \text{ for } u \in (r,c), \\ \text{for some } r \in [0,c]. \end{gathered} \tag{3}$$

A very particular example is given by the Euler's problem of buckling of an elastic rod

$$u'' + \lambda \sin u = 0, \quad \text{on } (0,1), \quad u'(0) = u'(1) = 0. \tag{4}$$

Here $r = 0$, and we obtain an exact global diagram. The standard proof of this result as on page 8 in S.-N. Chow and J. Hale [1] uses time map analysis, which is relatively easy only if one uses the particular form of the nonlinearity.

Notice that the above conditions imply that $f'(0) > 0$, and then the Crandall-Rabinowitz theorem on bifurcation from simple eigenvalues [2] implies that there are infinitely many solution curves bifurcating from the trivial solution at the eigenvalues λ_k of the linearized (at zero) problem. Under some additional technical assumptions listed below (conditions (5) and (6)) we show that all these curves continue for all λ_k without any turns, and that these curves exhaust the solution set of (1). We thus obtain an *exact* count of all solutions of (1) at any value of the parameter λ. Similar result, using time map analysis, was given previously by J. Smoller and A. Wasserman [8], but only for cubic nonlinearities. We conjecture that the conditions (5) and (6) can be dropped.

Crucial to our study will be to show that any non-constant solution of (1) is non-degenerate, i.e. the corresponding linearized problem has only the trivial solution. For that we need two technical conditions. In case $\int_a^c f(u)\,du > 0$, we define the number $\theta \in (0,c)$ by the condition $\int_a^\theta f(u)\,du = 0$. We shall sometimes assume that

$$f'''(u) < 0 \quad \text{for } 0 < u < r, \tag{5}$$

and that

$$\theta < r. \tag{6}$$

Observe that in case of a cubic $f(u) = (u+a)u(c-u)$ we have $r = \frac{c-a}{3}$, $\beta = \frac{c-a}{2}$, and also that the condition (6) is satisfied if $c > (1+\sqrt{3})a$.

In another direction we study the direction of bifurcation from the trivial solution $u = 0$ for rather general nonlinearities $f(u)$. For the Dirichlet problem it is well-known that the direction of bifurcation is governed by the sign of $f''(0)$. For Neumann problem this sign is irrelevant, as it can be always switched by a change of variables $-u \to u$. In Section 4 we show that bifurcation at the odd-numbered eigenvalues is to the right, provided that $f'''(0) < 0$.

In Section 5 we show that one can use exact multiplicity results for the Dirichlet problem to derive similar results for the Neumann problem. A number of exact multiplicity results for the Dirichlet problem has been given in recent years, see e.g. [4], [5] and the references given there. In particular we obtain another approach to the Euler's problem of buckling of an elastic rod. In Section 6 we work in the opposite direction, and show that our results on the Neumann problem imply some new results for the Dirichlet problem.

We record the following simple observations as a lemma. What it basically says is that solution of any Neumann problem (1) looks like $\cos x$ on the interval $[0, n\pi]$.

Lemma 1 *Let $f \in C(R)$. The problem (1) has only one minimal and only one maximal values. Moreover, there is an integer $n \geq 1$ so that $u'(i/n) = 0$ for $i = 1, \ldots, n-1$, and either $u' > 0$ or $u' < 0$ holds on every interval $((i-1)/n, i/n)$.*

Proof: Follows from the fact that any solution of the equation in (1) is symmetric with respect to any of its critical points, which is easily seen from the uniqueness for initial value problems.

This lemma implies that it suffices to study monotone on $(0,1)$ solutions of (1). In case $u(x)$ is increasing on $(0,1)$, observe that $f(u(0)) < 0$ and $f(u(1)) > 0$.

2 Non-degeneracy of solutions

The following lemma is a variation of Sturm comparison result, similar to the one in [6].

Lemma 2 *Assume that on some interval (α,γ) we have $z'(\alpha) \le 0$, and*

$$z'' + a(x)z \le 0, \quad \textit{and} \quad z > 0, \tag{7}$$

while

$$w'' + a(x)w = 0, \quad w(\alpha) > 0 \ \textit{and} \ w'(\alpha) \ge 0, \tag{8}$$

with $a(x)$ continuous on (α,γ). Then the function $w(x)$ cannot vanish on (α,γ). If the inequality (7) is strict on a set of positive measure then $w(x)$ cannot vanish on $(\alpha,\gamma]$. Similarly, assume that (7) holds with both signs reversed and also $z'(\gamma) \le 0$, while

$$w'' + a(x)w = 0 \ \textit{on} \ (\alpha,\gamma), \quad w(\gamma) < 0 \ \textit{and} \ w'(\gamma) \ge 0. \tag{9}$$

Then the function $w(x)$ cannot vanish on (α,γ). If the inequality (7) (with the sign reversed) is strict on a set of positive measure then $w(x)$ cannot vanish on $[\alpha,\gamma)$.

Proof: Assuming otherwise, let $\xi \in (\alpha,\gamma)$ be the smallest root of $w(x)$. Since $w(\alpha) > 0$, it follows that $w(x) > 0$ on $[\alpha,\xi)$. Multiplying (7) by w, and subtracting (8) multiplied by z, then integrating over (α,ξ), we obtain

$$-z(\xi)w'(\xi) - z'(\alpha)w(\alpha) + z(\alpha)w'(\alpha) \le 0,$$

which is a contradiction, since $w'(\xi) < 0$, and so the first term is positive, while the other two are non-negative.

The second part of the lemma is proved similarly.

Remark. We may switch the signs of the inequalities in both (8) and (9).

Lemma 3 *Let $u(x)$ be any increasing solution of the Neumann problem (1) with values in the interval (a,c). Then $u(1) < \theta$.*

Proof: Just multiply the equation (1) by u' and integrate over $(0,1)$.

We shall also need the following elementary lemma. We define $\beta \in (0,c)$ by $f'(\beta) = \frac{f(\beta)}{\beta}$.

Lemma 4 *Under the conditions (2) and (3) the function $h(u) \equiv f(u) - f'(u)u$ has the following properties: $h(0) = h'(0) = h(\beta) = 0$, $h(u) \le 0$ for $u \in (a,\beta)$, $h(u) \ge 0$ for $u \in (\beta,c)$.*

Proof: Clearly $r < \beta$, and since $h'(u) = -f''(u)u$, the proof easily follows.

The following lemma will be our main tool. We begin with the increasing solutions.

Lemma 5 *Let $f(u)$ satisfy the conditions (2), (3) and (5). Then in case $r > 0$ any solution of the Neumann problem*

$$(10) \qquad u'' + f(u) = 0, \quad u' > 0 \quad \text{on } (0,1), \quad u'(0) = u'(1) = 0,$$

satisfying $a < u \le r$, is non-degenerate, i.e. the problem

$$(11) \qquad L[w] \equiv w'' + f'(u)w = 0, \quad \text{on } (0,1), \quad w'(0) = w'(1) = 0$$

has no nontrivial solutions. In case $r = 0$ the same conclusion holds for any solution of (10) satisfying $a < u < c$.

Proof: Since $u(x)$ is increasing, clearly we have

$$(12) \qquad f(u(0)) < 0 \quad \text{and} \quad f(u(1)) > 0,$$

so that $u(0) \in (a,0)$, and $u(1) \in (0,c)$. Differentiate the equation (10)

$$(13) \qquad u_x'' + f'(u)u_x = 0.$$

(We mix two notations for derivatives to make the proofs more transparent.) It follows that the functions $w(x)$ and $u_x(x)$ satisfy the same linear equation, and since they are not multiple of each other ($w'(0) = 0$, but $u_x'(0) = -f(u(0)) > 0$), it follows that $w(x)$ must vanish on $(0,1)$. We shall show that this is impossible.

Differentiating the equation (13), and using (12), we have

$$(14) \qquad u_{xx}'' + f'(u)u_{xx} + f''(u)u_x^2 = 0, \quad u_{xx}(0) > 0, \; u_{xx}(1) < 0.$$

Denote by ξ and η the points where $u(\xi) = 0$ and $u(\eta) = r$. In case $r > 0$ we clearly have $0 < \xi < 1 \le \eta$. In case $r = 0$ we have $0 < \xi = \eta < 1$. The function $z = u''(x)$ (which is positive to the left of $x = \xi$ and negative to its right) will be our first test function. Observe that $L[u''] < 0$ and $u'' > 0$ on $(0, \xi)$, and also $u''_x(0) = -f(u)u_x(0) = 0$. It follows by the first part of Lemma 2 that $w(x)$ cannot vanish on $(0, \xi]$. In case $\xi = \eta$, i.e. $r = 0$, by a similar argument $w(x)$ cannot vanish on $(\xi, 1)$ and the proof ends here. Hence we assume for the rest of the proof that $r > 0$ and $a < u \le r$. By the above $w(x)$ can vanish only on $(\xi, 1)$. It is easy to see that $w(x)$ would have to vanish exactly once on $(\xi, 1)$. This follows from the identity $u'w' - u''w = constant$, which is obtained from the equations (9) and (13). Hence, we may assume that

$$w(0) < 0 < w(1),$$

and that $w(\nu) = 0$ at some $\nu \in (\xi, 1)$.

To cover the interval $(\xi, 1)$ we shall use another test function

$$z(x) = u'' - 2f'(0)u + \gamma, \tag{15}$$

with the positive constant γ to be chosen. We have

$$L[z] = -f''(u)u_x^2 + 2f'(0)\left[f(u) - f'(u)u\right] + \gamma f'(u) \equiv g_\gamma(x). \tag{16}$$

Let $a < u_1 < u_2 < c$ be the roots of $f'(u)$. Define the point $0 < \alpha < \xi$ by $u(\alpha) = u_1$ in case $u(0) < u_1$, otherwise set $\alpha = 0$. Since by our assumptions $u(1) \le r < u_2$, we conclude that

$$f'(u(x)) > 0 \quad \text{for} \quad \alpha < x < 1. \tag{17}$$

Observe that

$$\begin{aligned} g'_\gamma(x) = -f'''(u)u_x^3 + 2f''(u)u_x\left[f(u) - f'(0)u\right] \\ +\gamma f''(u)u_x > 0 \quad \text{for } x \in (\alpha, 1), \end{aligned} \tag{18}$$

i.e. $g_\gamma(x)$ is increasing on $(\alpha, 1)$. Observe also that $z' = -f'(u)u' - 2f'(0)u' < 0$, i.e. $z(x)$ is a decreasing function on $(\alpha, 1)$, and since $z(0) > 0$, it follows that $z(x)$ is positive to the left of its root, and negative to the right. When $\gamma = 0$, by Lemma 4, $g_0(x) < 0$ for all $x \in (0, 1)$, while $z(x)$ changes sign exactly once at $x = \xi$. We now increase $\gamma > 0$. Then the root of $z(x)$ moves continuously to the right, becoming equal to ν at some $\bar{\gamma}$, and then becoming equal to 1 if γ is sufficiently large.

If $g_{\bar{\gamma}}(x) < 0$ for all $x \in (0,1)$ then by Lemma 2 $w(x)$ cannot vanish on $(0,\nu]$, a contradiction. (Notice that $z'(0) = 0$.) Otherwise, at some $\gamma_0 < \bar{\gamma}$ we have $g_{\gamma_0}(1) = 0$, and then for $\gamma > \gamma_0$ the increasing function $g_\gamma(x)$ will assume both positive and negative values on $(0,1)$. As we increase γ, the root of $g_\gamma(x)$ moves continuously to the left, becoming equal to ξ if γ is sufficiently large. We now increase γ until the roots of $z(x)$ and $g_\gamma(x)$ meet, say at some $\mu \in (\xi, 1)$. We have

$$\begin{aligned} g_\gamma(x) < 0 \quad &\text{on } (\alpha,\mu), \quad g_\gamma(x) > 0 \quad \text{on } (\mu,1). \\ z > 0 \quad &\text{on } (\alpha,\mu), \quad z < 0 \quad \text{on } (\mu,1). \end{aligned} \tag{19}$$

We show next that $w(x)$ cannot vanish on either $(0,\mu]$ or $[\mu,1)$, which will result in a contradiction.

Case I. Assume that $w(x)$ vanishes on $(0,\mu]$.
(i) Assume first that $\alpha = 0$. Then on $(0,\mu)$ we have $L[z] < 0$, while $z > 0$ and $z'(0) = 0$. By Lemma 2 $w(x)$ cannot vanish on $(0,\mu]$, a contradiction.
(ii) Assume that $\alpha > 0$. Since $w(0) < 0$, $w'(0) = 0$ and $w''(x) = -f'(u)w < 0$ for $x \in (0,\alpha)$, it follows that $w(x)$ is decreasing on $(0,\alpha)$, and in particular $w'(\alpha) \le 0$. By Lemma 2 $w(x)$ cannot vanish on $(\alpha,\mu]$, a contradiction.

Case II. Assume that $w(x)$ vanishes on $[\mu,1)$.
On $(\mu,1)$ we have $L[z] > 0$, while $z < 0$ and $z'(1) = 0$. By Lemma 2 $w(x)$ cannot vanish on $[\mu,1)$, a contradiction.

Remarks.

1. Examining the proof, we see that under the conditions of the lemma the problem (more general than the linearized problem (11))

$$L[w] \equiv w'' + f'(u)w = 0, \quad \text{on } (0,1), \quad w'(0) \ge 0, \;\; w'(1) \ge 0$$

   has no nontrivial solutions. Here we assume that $w(0) > 0 > w(1)$.

2. Clearly the same result holds in case $u'(x) < 0$ on $(0,1)$.

3. In the proof of this lemma we did not use that $f(c) = 0$. Hence the lemma holds for the functions $f(u)$ which are positive for all $u > 0$, provided they satisfy all other conditions.

We now remove the assumption that solution is a monotone function.

Lemma 6 *Let $f(u)$ satisfy the conditions (2), (3) and (5). Then any solution of the Neumann problem (1) satisfying $a < u \le r$, is non-degenerate. In case $r = 0$ the same conclusion holds for any solution of (1) satisfying $a < u < c$.*

Proof: Since we have already proved this lemma for monotone solutions, let us assume that the solution of (10) $u(x)$ changes monotonicity exactly once, with say $u'(x) > 0$ on $(0, 1/2)$, $u'(1/2) = 0$ and $u'(x) < 0$ on $(1/2, 1)$. Arguing as in the previous lemma, we conclude that $w(x)$ must vanish on both $(0, 1/2)$ and $(1/2, 1)$. We may assume that $w(0) < 0$. Observe that $w(x)$ may vanish only once on $(0, 1/2)$, and that $w(1/2) \neq 0$ (since otherwise u_x would have to vanish inside $(0, 1/2)$, which is impossible). It follows that $w(1/2) > 0$. If $w'(1/2) \le 0$, then we obtain a contradiction on the interval $(0, 1/2)$, by Lemma 5. Otherwise, we obtain a contradiction on the interval $(1/2, 1)$, completing the proof for this case.

For the general case, assume that $u(x)$ has n intervals of monotonicity, with $u'(i/n) = 0$ for $i = 1, \ldots, n-1$. As above, $w(x)$ must vanish exactly once inside each interval of monotonicity. We may assume that $w(0) < 0$, and then $w(1/n) > 0$. To avoid a contradiction on the first interval of monotonicity we must have $w'(1/n) \ge 0$. To avoid a contradiction on the second interval, we must have $w(1/2) < 0$ and $w'(2/n) \le 0$, and arguing inductively $w((n-1)/n)w'((n-1)/n) \ge 0$, and $w((n-1)/n) \neq 0$. We then obtain a contradiction on the last interval.

3 Exact multiplicity of solutions

We now consider a class of Neumann problems, depending on a positive parameter λ

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \quad u'(0) = u'(1) = 0. \tag{20}$$

Lemma 7 *For any continuous solution curve the number of maximum points of the solution $u(x)$ stays constant in λ. Moreover any solution of (20) with a given number of intervals of monotonicity is uniquely determined by its maximum (or minimum) value (i.e. the maximum value of solution uniquely identifies both λ and the corresponding solution $u(x)$).*

Proof: By Lemma 1 the critical points of solutions can occur only at $x = i/n$. Since solution varies continuously in λ, these critical points must stay in the same place, and hence their number stays constant on a branch. Turning to the second part, let $(\lambda, u(x))$ and$(\mu v(x))$ be two solution pairs of (20), with both $u(x)$ and $v(x)$ monotone on $(0,1)$. Then $u(\sqrt{\lambda}x)$ and $v(\sqrt{\mu}x)$ are two different solutions of the same initial value problem, a contradiction. If $u(x)$ and $v(x)$ have $n > 1$ intervals of monotonicity, we obtain a similar contradiction on the first interval.

The following lemma says that for large λ solutions of (20) concentrate their values near either a or c (or both) except for narrow transition layers.

Lemma 8 *Consider solutions of (20) satisfying $a < u < c$, and assume that the condition (2) holds. If $\int_a^c f(u)\,du > 0$ then for large λ any solution of (20) tends to a, except on some subintervals of $(0,1)$, whose total length tends to zero. If $\int_a^c f(u)\,du < 0$ then for large λ any solution of (20) tends to c, except on some subintervals of $(0,1)$, whose total length tends to zero. If $\int_a^c f(u)\,du = 0$ then for large λ any solution of (20) tends to either a or c, or both, except on some subintervals of $(0,1)$, whose total length tends to zero.*

Proof: Let us consider the first case, $\int_a^c f(u)\,du > 0$, with the other ones being similar. We know by the previous lemma that as $\lambda \to \infty$ solution keeps the same number of maximums and minimums, and also $a < u(x) < c$. If $u(x)$ failed to approach one of the roots of $f(u)$ on some subinterval of $(0,1)$, we would have large values of $u''(x)$ on that interval, which is impossible without the solution getting large or more oscillatory. Solution cannot approach 0 on some subinterval, since it is convex for $u < 0$, and concave for $u > 0$. So it can only approach either a or c. If m, and M denote respectively the minimum and the maximum values of the solution, then from (20) we obtain $\int_m^M f(u)\,du = 0$. It follows that solution must approach a except for transition intervals, whose length is $o(\lambda)$. (And the maximum value approaches θ, defined by $\int_a^\theta f(u)\,du = 0$.)

Theorem 1 *Assume that $\int_a^c f(u)\,du > 0$, and moreover $f(u)$ satisfies the conditions (2), (3), (5) and (6), which imply in particular that*

$$f'(0) > 0. \tag{21}$$

Then at $\lambda_n = \frac{n^2\pi^2}{f'(0)}$ the Neumann problem (20) has a curve of solutions bifurcating off a trivial solution $u = 0$. These curves continue without any *turns for all $\lambda > \lambda_n$, and they exhaust the set of non-constant solutions of (20), satisfying $a < u < c$. (See Figure 1.)*

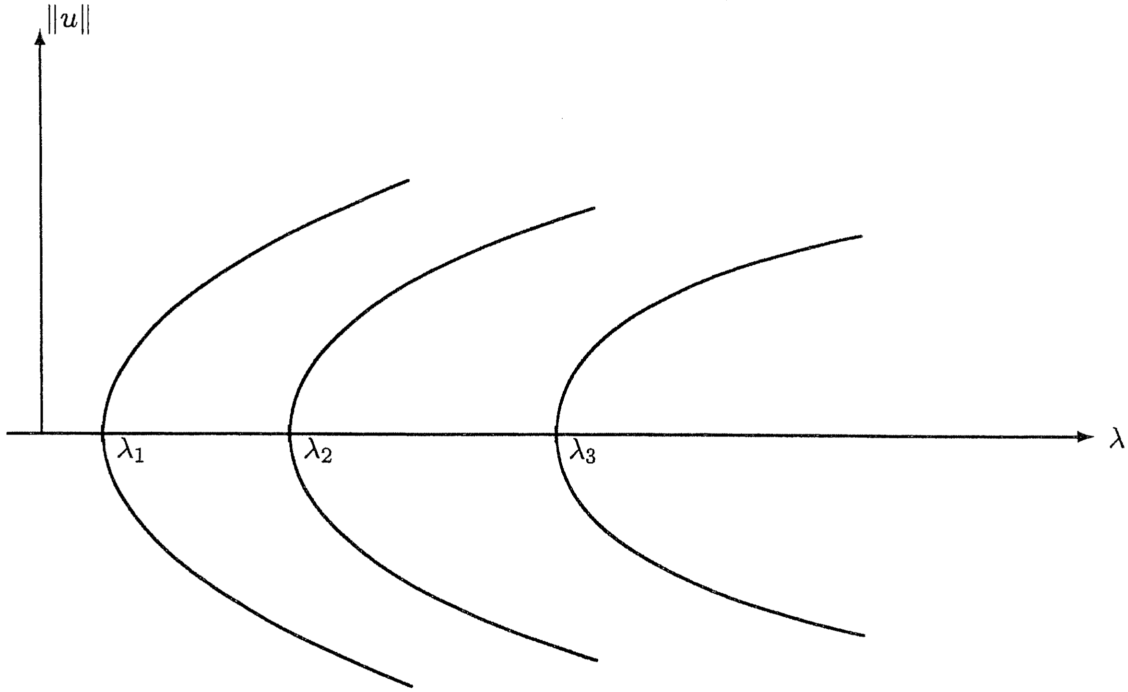

Figure 1

Proof: Multiplying the equation (20) by u', and integrating between any two consecutive points of negative minimum and positive maximum, we conclude that $\int_{u_1}^{u_2} f(u)\,du = 0$, where $u_1 < 0$ and $u_2 > 0$ are respectively the minimum and the maximum values. Since $u_1 > a$, it follows that $u_2 < \theta$. Since by our assumptions $\theta < r$, it follows that $u(x) < r$ for all x, so that by Lemma 5 all solutions of the problem (20) are non-degenerate.

It is well-known that bifurcation from zero occurs from the problem (20) at $\lambda = \lambda_n$, see [2] (see also [9] for a recent nice presentation, which develops the Crandall-Rabinowitz result in R^2, avoiding the use of Banach spaces). The direction of bifurcation is necessarily to the right, since if a curve were to bifurcate to the left, it would have no place to go as $\lambda \to 0$ (it cannot turn, cannot enter another λ_k, and it has to dissappear by $\lambda = 0$). By Lemma 8 as $\lambda \to \infty$ on any curve either minimum value goes to a or the maximum value goes to c (or both). By Lemma 7 we see that these curves exhaust the set of non-constant solutions.

Remark. In case the condition (6) does not hold, we can assert non-degeneracy only of small solutions, i.e. solutions satisfying $u(x) \in (a, r)$.

Example 1. $f(u) = (u-a)(u-b)(c-u)$, a cubic with three distinct real roots $a < b < c$. By a shift $u \to u - b$ we may assume that $f(u) = (u+a)u(c-u)$. In case $c > (1+\sqrt{3})a$ the theorem 1 applies. In particular, we obtain uniqueness of solution with a given number of maxima, and satisfying $-a < u < c$. We see that our theorem 1 provides an extension of the corresponding result in J. Smoller and A. Wasserman [8].

Example 2. $f(u) = \sin u$, with $-\pi < u < \pi$. This is a classical problem of buckling of an elastic rod, going back to Euler. Using quadrature methods it is shown that there are solutions curves bifurcating off the trivial solution at $\lambda_n = n^2\pi^2$, and continuing for all $\lambda > \lambda_n$, see e.g. [1]. We recover that, without using the particular form of the nonlinearity. (Here $r = 0$.)

Our results apply to rather general nonlinearities with arbitrary number of roots, $u_1, u_2 \ldots$. In particular if we assume that $f(u)$ changes concavity exactly once on any interval (u_k, u_{k+2}), and the area of each positive hump is less or equal to that of the adjacent to its left negative hump, then we can obtain the exact number of solutions of (20). Indeed, we may consider our problem on each interval (u_k, u_{k+2}) separately, since the above assumption on the areas of the humps excludes the possibility of solutions with values in more than two humps.

We conclude this section by remarking that solutions of the Neumann problem can be continued into periodic solutions. For example, for the elastic rod problem

$$u'' + \lambda \sin u = 0 \tag{22}$$

we can give an *exact* count of non-constant periodic solutions of any period for any value of the parameter λ. In particular, the number of periodic solutions of period 2 satisfying $-\pi < u(x) < \pi$ is equal to twice the largest n such that $n^2\pi^2 < \lambda$. Similarly we count the periodic solutions of (22) with values in $(\pi, 3\pi)$, and so on.

4 Direction of bifurcation from the trivial solution

We study the direction of bifurcation from the trivial solution for the problem

$$u'' + \lambda f(u) = 0, \quad \text{on } (0, L), \quad u'(0) = u'(L) = 0. \tag{23}$$

We consider more general $f(u) \in C^3(R)$ than in the rest of the paper. Namely, we assume that $f(0) = 0$, while $f'(0) > 0$, taking $f'(0) = 1$ without restricting the generality. Then if $f(u) = O(u^2)$ as $u \to 0$, the classical theorem of Crandall-Rabinowitz [2] implies that non-trivial solutions bifurcate off $u = 0$ at the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$. For the much studied Dirichlet problem the direction of bifurcation is determined by the sign of $f''(0)$. For the Neumann problem (23) the sign of $f''(0)$ is irrelevant, as it can always be switched by making a change $-u \to u$. Instead, the direction of bifurcation is governed by $f'''(0)$. We begin with the principal eigenvalue, and then generalize.

Theorem 2 *Assume that $f(u) \in C^3(R)$ satisfies*

$$f(0) = 0, \;\; f'(0) = 1, \;\; f'''(0) < 0. \tag{24}$$

Then bifurcation from zero at $\lambda_1 = \frac{\pi^2}{L^2}$ is to the right (i.e. in the direction of increasing λ).

Proof: According to the Crandall-Rabinowitz theorem the solution set of (23) near $(u = 0, \lambda = \lambda_1)$ is exhausted by a solution curve $(\lambda(s), s\phi_1(x) + s\psi(s,x))$, where s is a parameter defined near $s = 0$, and $\lambda(0) = \lambda_1$, $\psi(0,x) \equiv 0$, and $\phi_1(x) = \cos\frac{\pi}{L}x$. Observe that both branches of the solution curve must bifurcate to the same side. Indeed, if $u(x)$ is a solution of (23), $u(L-x)$ gives another solution of (23) at the same λ. We conclude that

$$\lambda'(0) = 0. \tag{25}$$

(Otherwise, the solution set near $(\lambda_1, 0)$ would consist of two curves, contradicting the Crandall-Rabinowitz theorem.) We differentiate the equation (23) twice in s, then set $s = 0$. Using that $u_s(0,x) = \phi_1(x)$, (24) and (25), we obtain

$$\begin{gathered} u''_{ss} + \lambda_1 u_{ss} = -\lambda_1 f''(0)\cos^2\tfrac{\pi}{L}x, \quad \text{on } (0,L), \\ u'_{ss}(0) = u'_{ss}(L) = 0. \end{gathered} \tag{26}$$

The problem (26) is a linear equation at a resonance, with the right hand side orthogonal to the kernel. Its solution is given by

$$(27)\, u_{ss} = -f''(0)\left(\frac{1}{2} - \frac{1}{6}\cos\frac{2\pi}{L}x\right) + A\cos\frac{\pi}{L}x \equiv -f''(0)\bar{u}_{ss} + A\cos\frac{\pi}{L}x,$$

where A is an arbitrary constant. Observe that

$$(28) \qquad \bar{u}_{ss} = \frac{1}{2} - \frac{1}{6}\cos\frac{2\pi}{L}x > 0 \quad \text{for all } x \in (0, L).$$

For $|s|$ small, i.e. near the bifurcation point, we have by the Taylor's formula

$$(29) \qquad u = s\cos\frac{\pi}{L}x + \frac{1}{2}s^2 u_{ss} + o(s^2).$$

Similarly, for small u we have

$$(30) \qquad f(u) = u + \frac{1}{2}f''(0)u^2 + \frac{1}{6}f'''(0)u^3 + o(u^3).$$

We now multiply the equation (23) by $\phi_1(x) = \cos\frac{\pi}{L}x$, and integrate. Using (30), we have for small u

$$(31) \qquad \begin{aligned} &(\lambda - \lambda_1)\int_0^L u\cos\tfrac{\pi}{L}x\,dx + \tfrac{1}{2}f''(0)\int_0^L u^2\cos\tfrac{\pi}{L}x\,dx \\ &+\tfrac{1}{6}f'''(0)\int_0^L u^3\cos\tfrac{\pi}{L}x\,dx + \ldots = 0, \end{aligned}$$

where we denote by $\ldots$ the higher order terms. Using (29), we have

$$(32) \int_0^L u\cos\frac{\pi}{L}x\,dx = s\int_0^L \cos^2\frac{\pi}{L}x\,dx + o(s) > 0 \quad \text{for } s > 0 \text{ and small.}$$

Observe that $\int_0^L \cos^3\frac{\pi}{L}x\,dx = 0$. Using (27) and (28), it follows that

$$(33) \qquad \begin{aligned} &\tfrac{1}{2}f''(0)\int_0^L u^2\cos\tfrac{\pi}{L}x\,dx = \tfrac{1}{2}s^3 f''(0)\int_0^L u_{ss}\cos^2\tfrac{\pi}{L}x\,dx + o(s^3) \\ &= -\tfrac{1}{2}s^3 f''(0)^2\int_0^L \bar{u}_{ss}\cos^2\tfrac{\pi}{L}x\,dx + o(s^3) \le 0 \quad \text{for } s > 0 \text{ and small.} \end{aligned}$$

Similarly,

$$(34) \qquad \begin{aligned} &\int_0^L u^3\cos\tfrac{\pi}{L}x\,dx = s^3\int_0^L \cos^4\tfrac{\pi}{L}x\,dx + o(s^3) > 0 \\ &\qquad \text{for } s > 0 \text{ and small.} \end{aligned}$$

Using (32), (33) and (34), we see from (31) that $\lambda > \lambda_1$ for $s > 0$ and small, i.e. the direction of bifurcation is to the right.

Remark. Similar analysis shows that under the conditions (24) bifurcation from zero from higher eigenvalues λ_k is to the right, provided k is odd. The oddness of k is used to show that $u(L-x)$ gives a different from $u(x)$ solution, implying that $\lambda'(0) = 0$.

5 Reduction to the Dirichlet problem

We now show that it is sometimes possible to use exact multiplicity results for the Dirichlet problem to derive similar results for the Neumann problem. Again, we begin with increasing solutions, and then generalize. For the increasing solution of (1) we have $u(0) < 0$ and $u(1) > 0$. Since $\int_{u(0)}^{u(1)} f(u)\, du = 0$, we observe that when λ is varied, $u(1)$ is an increasing function of $-u(0)$. We consider the Dirichlet problem corresponding to (1)

$$u'' + \lambda f(u) = 0, \quad \text{on } (0,1), \quad u(0) = u(1) = 0. \tag{35}$$

It is well-known that any positive solution of (35) is even with respect to the unique point of maximum at $x = 1/2$, while any negative solution is even with respect to the unique point of minimum at $x = 1/2$.

In the following lemma we make no convexity assumptions on $f(u)$.

Lemma 9 *Assume that $f(u) \in C[a,c]$ satisfies (2). Assume that for $\lambda > 0$ the problem (35) has a unique positive solution, whose maximum value lies in $(0,c)$ and is increasing in λ, and a unique negative solution, whose minimum value lies in $(a,0)$ and is decreasing in λ. The Neumann problem (1) has for $\lambda > 0$ at most one increasing solution, with values in (a,c).*

Proof: Let $u(x)$ and $v(x)$ be two different increasing solutions. Let $u(\xi) = 0$ and $v(\eta) = 0$. If $\xi = \eta$ then by stretching we will get two positive solutions of (35), corresponding to the same λ, which is impossible. So we may assume $\xi < \eta$. Observe that by reflecting the solutions across the line $x = 1$, we see that $u(x)$ satisfies the Dirichlet problem on the interval $(\xi, 2-\xi)$, and $v(x)$ on $(\eta, 2-\eta)$. If $v(1) \geq u(1)$, then by stretching the intervals $(\xi, 2-\xi)$ and $(\eta, 2-\eta)$ to the interval $(0,1)$, we obtain two positive solutions of the problem (35), for two different values of the parameter, so that for $u(x)$ the parameter value is greater than the one for $v(x)$, while the maximum values of the solutions satisfy the reverse inequality, a contradiction. In case $u(1) \geq v(1)$ we observe that $|u(0)| \geq |v(0)|$, and obtain a similar contradiction for the negative solutions of (35).

The following result follows immediately.

Theorem 3 *Under the conditions of Lemma 9 the Neumann problem (1) has for $\lambda > 0$ at most one solution, with any given number of intervals of monotonicity, and with values in (a,c).*

Examples. Assume $f(u) = ug(u)$, where $g(u)$ is a positive continuous function on some interval $(-R, R)$, $R \leq \infty$, such that $g(u)$ is decreasing for $u > 0$ and increasing for $u < 0$. Uniqueness of both positive and negative solution of (35) is well-known, and hence we conclude that the Neumann problem (1) has at most one increasing solution. Observe that in contrast to our main result, here $f(u)$ may change concavity arbitrary many times.

If we now consider $f(u) = \sin u$ on the interval $(-\pi, \pi)$, then $g(u) = \frac{f(u)}{u}$ is decreasing for $u > 0$ and increasing for $u < 0$. We then obtain another easy approach to the Euler's problem of buckling of an elastic rod. Indeed, the curves bifurcating from the trivial solutions at the eigenvalues λ_k cannot turn in view of the Lemma 6. Since by [3] all solution curves continue globally, we recover the classical bifurcation diagram, see e.g. page 8 in S.-N. Chow and J.K. Hale [1].

6 Dirichlet from Neumann

We now consider the Dirichlet problem (35), with $f(u)$ satisfying the conditions (2) and (3). Observe that these conditions imply in particular that $f'(0) > 0$, and hence bifurcation from zero occurs at each eigenvalue $\lambda_k = k^2\pi^2$. Using our results for the corresponding Neumann problem we can give a complete global picture for Dirichlet problem (35).

Theorem 4 *Assume that $f(u)$ satisfies the conditions (2), (3), (5) and (6). Then the solution set of the Dirichlet problem (35) consists of infinitely many curves bifurcating from the trivial solution at $\lambda = \lambda_k$. At $\lambda = \lambda_1$ there is a curve of negative solutions, which bifurcates forward and continues without any turns for all $\lambda > \lambda_1$. There is also a curve of positive solutions, which bifurcates backward at $\lambda = \lambda_1$. This curve makes exactly one turn at some $\bar{\lambda} < \lambda_1$, and then continues without any turns for all $\lambda > \bar{\lambda}$. At each $\lambda = \lambda_k$, for $k \geq 2$ and even, there is a curve of solutions, with both branches bifurcating forward from the trivial one. Solutions on this curve have exactly $k-1$ interior roots on $(0, 1)$. Both branches of this curve continue without any turns for all $\lambda > \lambda_k$. (See Figure 2.)*

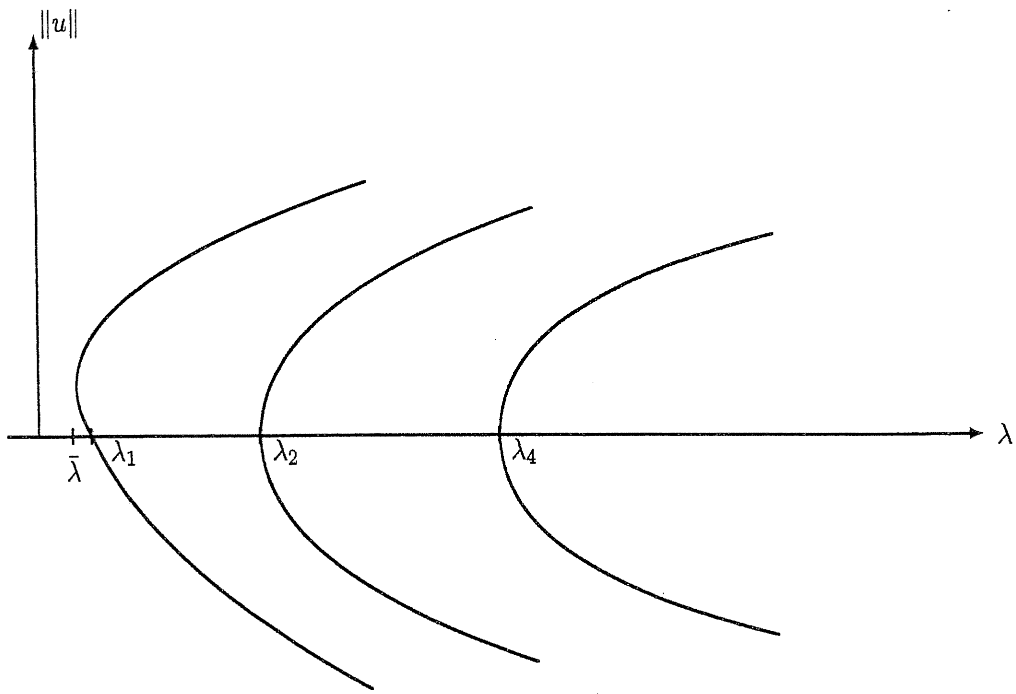

Figure 2

Proof: Our claims about the positive and negative branches through $(\lambda_1, 0)$ follow easily from the results of [5]. We now turn to the sign-changing solutions. It follows from [3] that the curves of sign-changing solutions, bifurcating at $(\lambda_k, 0)$, continue globally. The curve through $(\lambda_2, 0)$ is asymptotic to $\phi_2 = \sin 2\pi x$, and hence solutions on the curve have exactly one interior zero. Consider the branch of this curve where solutions have a negative minimum followed by a positive maximum (on the other branch the order is reversed). Denote ξ the interior root of any solution on this branch, $\xi = \xi(\lambda)$. Since solutions of autonomous equations are symmetric with respect to the critical points, it follows that the negative minimum is taken at the point $\alpha = \frac{\xi}{2}$, and the positive maximum at the point $\beta = \frac{1}{2} + \frac{\xi}{2}$. We see that $u(x)$ is an increasing solution of the Neumann problem on the interval (α, β) of length $\frac{1}{2}$. But then this Dirichlet branch cannot turn, since otherwise we will have two increasing solutions of the Neumann problem on the interval $(0, \frac{1}{2})$, and hence on the interval $(0, 1)$ by scaling, contradicting our uniqueness result for the Neumann problem. The other branch through λ_2 is treated the same way. The branches through other even eigenvalues are examined similarly.

Remark We conjecture that the solution curves through the odd eigenvalues are similar to the one through λ_1.
Example For the problem

$$u'' + \lambda u(u + a)(c - u) = 0 \quad \text{on } (0,1), \;\; u(0) = u(1) = 0,$$

with $a > 0$ and $c > (1 + \sqrt{3})a$ the Theorem 4 applies.

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