Math. Nachr. **229** (2001), 119 – 127

On the Multiplicity of Solutions of Semilinear Equations

By PHILIP KORMAN of Cincinnati

(Received May 31, 1999; revised version March 1, 2000; accepted March 14, 2000)

Abstract. We study uniqueness and exact multiplicity of solutions of semilinear equations for both balls and annular domains. We assume the annular domains to be "thin", but allow them to be wider than in the previous works.

1. Introduction

We study exact multiplicity and uniqueness of solutions for problems on both balls and annular domains in $\mathbb{R}^n$. On an annulus $\Omega = \{x \mid A < |x| < B\}$ in $\mathbb{R}^n$, $n \geq 2$, we consider the problem

$$\Delta u + \lambda f(u) \;=\; 0 \;\text{ in }\; \Omega\,, \quad u \;=\; 0 \;\text{ on }\; \partial\Omega\,. \tag{1.1}$$

We study positive radially symmetric solutions of (1.1), depending on a positive parameter λ. This problem arises in many applications, and there is large literature on the subject, including W.–M. NI and R. NUSSBAUM [11], S. S. LIN [10], P. KORMAN [5]. Recall that the problem (1.1) may also have positive non–radial solutions, in contrast to the case when domain is a ball in $\mathbb{R}^n$, when all positive solutions are necessarily radially symmetric, in view of the well–known results of B. GIDAS, W.–M. NI and L. NIRENBERG [3]. To get exact multiplicity results, we shall restrict our attention to the case of "thin" annulus, which we define next. Set $c_n = (2n-3)^{\frac{1}{n-2}}$ for $n \geq 3$, and $c_2 = e^2$. We shall assume

$$B \;\leq\; c_n A\,. \tag{1.2}$$

The special role of "thin" annulus was recognized first by W.–M. NI and R. NUSSBAUM [11], but they had $c_n = (n-1)^{\frac{1}{n-2}}$ for $n \geq 3$, and $c_2 = e$. The same condition appeared later in S. S. LIN [10].

The crucial role in our study will be played by the linearized problem

$$\Delta w + \lambda f'(u) w \;=\; 0 \;\text{ in }\; \Omega\,, \quad w = 0 \;\text{ on }\; \partial\Omega\,. \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary: 35J60; Secondary: 34B15.
Keywords and phrases. Positivity for linearized problem, uniqueness and multiplicity.

In fact we proved in [5] that in case $f(u) \in C^2\left(\overline{\mathbb{R}_+}\right)$ satisfies

$$(1.4) \qquad f(u) \;>\; 0 \quad \text{for all} \quad u \;>\; 0$$

then any non–trivial solution of (1.3) is of one sign. We used positivity of $w(r)$ to prove several uniqueness an exact multiplicity results. In the present work we present a considerably easier proof of positivity of $w(r)$ (see Corollary to Theorem 2.4). Moreover, we observe that positive solutions on annular domains are unimodular, and "tilted" to the left. This allowed us to considerably increase the width of the annulus on which uniqueness and exact multiplicity results hold.

We also consider a class of polynomial non–linearities on balls in $\mathbb{R}^2$. Using a transformation introduced by W.–M. Ni and R. Nussbaum [11], we are able to describe the global solution structure.

2. Positivity of solution of the linearized equation

We consider a class of two–point boundary value problems for $u = u(s)$, with

$$(2.1) \qquad u'' + g(s,u) \;=\; 0 \;\text{ for }\; s \in (a,b)\,, \quad u(a) \;=\; u(b) \;=\; 0\,.$$

Here (a,b) is a bounded interval, $g(s,u) \in C^2\left((a,b) \times \overline{\mathbb{R}}_+\right)$, and we are interested in positive solutions, $u(s) > 0$ for $s \in (a,b)$. We assume throughout this paper that

$$(2.2) \qquad g_s(s,u) \;\le\; 0 \quad \text{for all} \quad s \in (a,b) \times \mathbb{R}_+\,.$$

The following lemma is included in P. Korman and T. Ouyang [7].

Lemma 2.1. *Under the condition* (2.2) *any positive solution of* (2.1) *admits only one point of local maximum* (*which is of course the point of global maximum*).

We shall denote by s_0 the point of maximum of the solution $u(s)$. The following lemma shows that solutions of (2.1) are skewed to the left.

Lemma 2.2. *Assuming* (2.2)*, we have*

$$(2.3) \qquad s_0 \in \left(a, \frac{a+b}{2}\right].$$

Proof. In view of Lemma 2.1, for any value of u_1 below the maximal value of $u(s_0)$ we can find two points $\alpha < s_0 < \beta$, so that $u(\alpha) = u(\beta) = u_1$. Multiplying the equation (2.1) by u', and integrating over (α, s_0), we have

$$-\frac{1}{2}\,{u'}^2(\alpha) + \int_{u_1}^{u(s_0)} g(s_1(u),u)\,du \;=\; 0\,,$$

where $s_1(u)$ denotes the inverse function of $u(s)$ on the interval (α, s_0). Similarly, multiplying the equation (2.1) by u', and integrating over (s_0, β),

$$\frac{1}{2}\,{u'}^2(\beta) + \int_{u(s_0)}^{u_1} g(s_2(u),u)\,du \;=\; 0\,,$$

where $s_2(u)$ denotes the inverse function of $u(s)$ on the interval (s_0, β). Adding,

$$\frac{1}{2}u'^2(\beta) - \frac{1}{2}u'^2(\alpha) + \int_{u_1}^{u(s_0)} [g(s_1(u), u) - g(s_2(u), u)]\, du \;=\; 0\,.$$

Since $s_2(u) > s_1(u)$ for all $u \in (u_1, u(s_0))$, the assumption (2.2) implies that the integral term is non–negative, and hence

$$|u'(\alpha)| \;\geq\; |u'(\beta)|\,.$$

We conclude that $u(s)$ is climbing faster to the left of s_0, and so it will reach its maximum value sooner from the left. $\square$

We shall need the following form of Sturm's comparison theorem, which we learned from [9]. We consider a differential operator, defined on functions $u = u(r)$ of class C^2

$$L[u] \;\equiv\; p(s)u'' + q(s)u' + z(s)u\,,$$

with continuous coefficients $p(s)$, $q(s)$ and $z(s)$.

Lemma 2.3. *Assume that on some interval $I \subseteq (-\infty, \infty)$ we have $p(s) > 0$ and*

$$L[u] \;\geq\; 0\,,$$

while

$$L[v] \;\leq\; 0\,,$$

with at least one of the inequalities being strict on a set of positive measure. Then the function $v(s)$ oscillates faster than $u(s)$, provided that they are both non–negative. More precisely, assume that $u(\alpha) = u(\beta) = 0$ for some $\alpha, \beta \in I$, $u(s) > 0$ on (α, β), while $v(\alpha) \geq 0$. Then $v(s)$ must vanish on (α, β).

We study next the linearization of the problem (2.1)

$$w'' + g_u(s,u)w \;=\; 0 \;\text{ for }\; s \in (a,b)\,, \quad w(a) \;=\; w(b) \;=\; 0\,. \tag{2.4}$$

We shall need a condition, originated in W.–M. Ni and R. Nussbaum [11]

$$(s-a)g_s + 2g \;\geq\; 0 \quad \text{for all} \quad s \in \left(a, \frac{a+b}{2}\right) \quad \textit{and} \quad u \;>\; 0\,. \tag{2.5}$$

Theorem 2.4. *Assume that conditions* (2.2) *and* (2.5) *hold for the problem* (2.1). *Then any non–trivial solution of the linearized problem* (2.4) *is of one sign.*

Proof. We show first that $w(s)$ cannot vanish on $(a, s_0]$. Consider a test function $z(s) = (s-a)u_s$. Then $z(a) = 0$, $z(s) > 0$ for $s \in (a, s_0)$, and by (2.5) and Lemma 2.2

$$z'' + g_u(s,u)z \;=\; -(s-a)g_s - 2g \;\leq\; 0 \quad \text{for all} \quad s \in (a, s_0)\,.$$

It follows that $z(s)$ oscillates faster than $w(s)$ on the interval $(a, s_0]$, and so $w(s)$ cannot vanish on $(a, s_0]$. (Without loss of generality we may assume $w(s)$ to be positive between any two of its roots.)

To see that $w(s)$ cannot also vanish on $[s_0, b)$, we employ a test function $v(s) = -u_s$. Then $v(s) > 0$ on (s_0, b) and

$$v'' + g_u(s,u)v \;=\; g_s \;<\; 0 \quad \text{for all} \quad s \in (s_0, b)\,.$$

It follows that $v(s)$ oscillates faster than $w(s)$ on the interval $[s_0, b)$, and so $w(s)$ cannot vanish on $[s_0, b)$. We conclude that $w(s)$ is either trivial, or else it is of one sign. □

Corollary 2.5. *Under the condition* (1.4) *any non–trivial solution of* (1.3) *is of one sign.*

Next we prove non–degeneracy of solutions to superlinear problems.

Theorem 2.6. *Assume that conditions* (2.2) *and* (2.5) *hold for the problem* (2.1), *and in addition*

$$ug_u(s,u) \;\geq\; g(s,u) \quad \text{for all} \quad s \in (a,b) \times \mathbb{R}_+\,. \tag{2.6}$$

Then the problem (2.4) *admits only the trivial solution.*

Proof. Condition (2.6) and the Sturm comparison theorem imply that $w(s)$ oscillates faster than $u(s)$, and hence $w(s)$ must have a root inside (a, b). By the Theorem 2.4 it follows that $w(s) \equiv 0$. □

3. Applications to uniqueness and exact multiplicity

We begin with a uniqueness result, extending the Theorem 2.4 in W.–M. Ni and R. Nussbaum [11].

Theorem 3.1. *Assume that conditions* (2.2), (2.5) *and* (2.6) *hold for the problem* (2.1). *Assume that $g_u(s, 0)$ is decreasing in s for all $s \in (a, b)$. Moreover, assume that*

$$\lim_{u\to\infty} \frac{g(s,u)}{u} \;=\; \infty \quad \text{uniformly in} \quad s \in (a,b)\,. \tag{3.1}$$

Then the problem (2.1) *admits at most one positive solution.*

Proof. We follow the proof of Theorem 2.4 in P. Korman [5]. We imbed the problem (2.1) into a family of problems, depending on a parameter μ, $0 \leq \mu \leq 1$

$$\begin{aligned} &u'' + \mu g(s,u) + (1-\mu)g(\bar{s},u) \;=\; 0 \quad \text{for} \quad s \in (a,b)\,,\\ &u(a) \;=\; u(b) \;=\; 0\,, \end{aligned} \tag{3.2}$$

where $\bar{s}$ is a point on the interval $[a, b]$ to be specified. When $\mu = 0$ the problem (3.2) has at most one positive solution by Lemma 2.3 in P. Korman [5]. The linearized problem for (3.2) is given by

$$\begin{aligned} &w'' + [\mu g_u(s,u) + (1-\mu)g_u(\bar{s},u)]\,w \;=\; 0 \quad \text{for} \quad s \in (a,b)\,,\\ &w(a) \;=\; w(b) \;=\; 0\,. \end{aligned} \tag{3.3}$$

By the Theorem 2.6 the problem (3.3) has only the trivial solution for any $\mu \in [0,1]$, i. e. any positive solution of the problem (3.2) is non–degenerate. When $\mu = 1$ we have the original problem (2.1). If the problem (2.1) had two positive solutions, we could continue both of them for decreasing μ on two smooth solution curves, which cannot turn or merge, obtaining two positive solutions at $\mu = 0$, which is a contradiction. Two things need to be checked: that solutions on these curves stay bounded and non–trivial. Boundness follows by Lemma 2.1 in P. Korman [5]. It is on this step that the superlinearity condition (3.1) is used.

If at some $\bar{\mu} \in [0,1]$ solution becomes zero (which may only happen in case $g(s,0) = 0$ for all $s \in (a,b)$), then we would have at $\mu = \bar{\mu}$ bifurcation of positive solution from the trivial one, which can only happen if the problem (3.3) at $u = 0$ and $\mu = \bar{\mu}$ has a positive solution. Define μ_1 to be the principal eigenvalue of the problem

$$\begin{aligned} \phi'' + g_u(s,0)\phi \ &= \ \mu_1 \phi\,, \quad \text{for} \quad a \ < \ s \ < \ b\,, \\ \phi(a) \ = \ \phi(b) \ &= \ 0\,. \end{aligned} \tag{3.4}$$

Assume first $\mu_1 < 0$. Then we choose the constant $\bar{s} = b$ in (3.2), so that $g_u(\bar{s},0) = \min_{s \in [a,b]} g_u(s,0)$. Then $\bar{\mu} g_u(s,0) + (1-\bar{\mu}) g_u(\bar{s},0) < g_u(s,0)$ for all $s \in [a,b]$. It follows that the problem (3.3) at $u = 0$ and $\mu = \bar{\mu}$ cannot have a positive solution, since the principal eigenvalue for the operator on the left would have to be smaller than μ_1, hence negative, rather than zero. In case $\mu_1 \geq 0$ we select $\bar{s} = a$ such that $g_u(\bar{s},0) = \max_{s \in [a,b]} g_u(s,0)$, and obtain a similar contradiction. $\square$

Remark 3.2. Comparing with the Theorem 2.4 in [3], we need (2.5) to hold only on the left half of the interval (a,b), compared with the whole interval in [11]. We have though two extra conditions: g_u decreasing in s, and (3.1). However, on an annulus the first additional condition follows from our other conditions, while (3.1) is not very restrictive.

We now apply our results to positive radial solutions for Dirichlet problems on an annulus $\Omega = \{x \mid A < |x| < B\}$ in $\mathbb{R}^n$, $n \geq 2$,

$$\Delta U + \lambda f(U) \ = \ 0 \ \text{ in } \ \Omega\,, \quad U \ = \ 0 \ \text{ on } \ \partial\Omega\,. \tag{3.5}$$

Here λ is a positive parameter. Recall that the problem (3.5) may also have non–radial positive solutions. Set $c_n = (2n-3)^{\frac{1}{n-2}}$ for $n \geq 3$, and $c_2 = e^2$. We shall assume

$$B \ \leq \ c_n A\,. \tag{3.6}$$

We make a standard change of variables. In case $n \geq 3$ we let $s = r^{2-n}$ and $u(s) = U(r)$, transforming (3.5) into the problem

$$u'' + \alpha(s) f(u) \ = \ 0\,, \quad \text{for} \quad a \ < \ s \ < \ b\,, \quad u(a) \ = \ u(b) \ = \ 0\,, \tag{3.7}$$

where $\alpha(s) = (n-2)^{-2} s^{-2k}$ with $k = 1 + \frac{1}{n-2}$, $a = B^{2-n}$ and $b = A^{2-n}$. In case $n = 2$ we set $s = -\log r$, and $u(s) = U(r)$, obtaining again the problem (3.7), this time with $\alpha(s) = e^{-2s}$ and $a = -\log B$, $b = -\log A$.

The following theorem extends the Theorem 2.4 in [11], allowing for a larger annulus.

Theorem 3.3. *Assume that $B \le c_n A$, $f(u) \in C^2\big(\overline{\mathbb{R}}_+\big)$, $f(u) > 0$ and $uf'(u) > f(u)$ for all $u > 0$, and assume finally that $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$. Then the problem* (3.5) *has at most one positive solution for any $\lambda > 0$.*

Proof. Observe that our conditions imply that $f(0) = 0$. One checks that the Theorem 3.1 applies, giving uniqueness of solution to the problem (3.7), which is equivalent to the original problem. □

Next we generalize an exact multiplicity result from [5], allowing for a wider annulus. This generalization is made possible by the Theorem 2.4, which showed that it suffices to assume that (2.5) holds only for the left half of our interval (a, b). The proof is exactly the same as in [5].

Theorem 3.4. *Assume that $B \le c_n A$, $f(u) \in C^2\big(\overline{\mathbb{R}}_+\big)$, $f(u) > 0$ for $u \ge 0$, and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, and assume finally that*

$$f''(u) \; > \; 0 \quad \textit{for almost all} \quad u \; > \; 0 \, .$$

Then there is a critical $\lambda_0 > 0$, such that the problem (3.5) *has exactly two positive solutions for $0 < \lambda < \lambda_0$, it has exactly one positive solution at $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda \in (0, \lambda_0)$ has two branches $u^-(r, \lambda)$ and $u^+(r, \lambda)$, with $u^-(r, \lambda) < u^-(r, \lambda)$ for all $r = |x| \in (A, B)$. The lower branch $u^-(r, \lambda)$ is strictly monotone increasing in λ, and $\lim_{\lambda\to 0^+} u(r, \lambda) = 0$ for all $r \in (A, B)$. For the upper branch $\lim_{\lambda\to 0^+} \max_r u(r, \lambda) = \infty$.*

We now turn to the positive solutions on a ball in $\mathbb{R}^2$, whose radius is assumed to be one without restricting the generality

$$\Delta u + f(u) \; = \; 0 \;\; \text{for} \;\; |x| \; < \; 1\,, \quad u \; = \; 0 \;\; \text{for} \;\; |x| \; = \; 1\,. \tag{3.8}$$

By the classical result of B. Gidas, W.–M. Ni and L. Nirenberg [3] any positive solution of (3.8) is radially symmetric, and so the problem reduces to (with $r = |x|$)

$$u'' + \frac{1}{r}\, u' + f(u) \; = \; 0 \;\; \text{for} \;\; 0 \; < \; r \; < \; 1\,, \quad u'(0) \; = \; u(1) \; = \; 0\,. \tag{3.9}$$

Following W.–M. Ni and R. Nussbaum [11] we perform a change of variables $s = r^\delta$, followed by $u(s) = \frac{z(s)}{\sqrt{s}}$, where $\delta > 0$ is a constant, to be specified. The problem (3.9) transforms into

$$z'' + g(s, z) \; = \; 0 \;\; \text{for} \;\; 0 \; < \; s \; < \; 1\,, \quad z(0) \; = \; z(1) \; = \; 0\,, \tag{3.10}$$

where

$$g(s, z) \; = \; \frac{1}{4s^2}\, z + \frac{1}{\delta^2}\, s^{\frac{2}{\delta} - \frac{3}{2}} f\big(s^{-\frac{1}{2}} z\big)\,.$$

In the linearized problem for (3.9)

$$w'' + \frac{1}{r}\, w' + f'(u) w \; = \; 0 \;\; \text{for} \;\; 0 \; < \; r \; < \; 1\,, \quad w'(0) \; = \; w(1) \; = \; 0 \tag{3.11}$$

we perform similar changes of variables $s = r^{\delta}$, $u(s) = \frac{z(s)}{\sqrt{s}}$, and then $w(s) = \frac{v(s)}{\sqrt{s}}$, obtaining

$$(3.12) \qquad \begin{aligned} & v'' + \frac{1}{4s^2}\, v + \frac{1}{\delta^2}\, s^{\frac{2}{\delta}-2} f'\big(s^{-\frac{1}{2}} z\big) v \;=\; 0 \quad \text{for } 0 < s < 1, \\ & v(0) \;=\; v(1) \;=\; 0\,, \end{aligned}$$

which is precisely the linearization of the problem (3.10). We prefer not to work with the problem (3.10) because of the singularity at $s = 0$. Instead, after proving that $v(s)$ is positive (or zero), we conclude that $w(r)$ is positive (or zero), which allows us to apply the techniques of bifurcation theory to the original problem (3.9).

We now consider a class of polynomial nonlinearities

$$f(u) \;=\; \sum_{i=1}^{k} a_i u^{p_i} + \alpha\,,$$

with constants $a_i > 0$, $1 < p_1 < p_2 < \cdots < p_k$, and $\alpha \geq 0$.

Theorem 3.5. (i) *Assume $\alpha > 0$, and $1 < p_k \leq 4$. Then any non–trivial solution of the linearized problem* (3.12) *is of one sign.*

(ii) *Assume that $\alpha = 0$, and $p_k \leq p_1 + 4$. Then the linearized problem* (3.12) *has only the trivial solution.*

Proof. (i) We select $\delta = \frac{4}{3}$, so that $\frac{2}{\delta} - \frac{3}{2} = 0$. Then one checks that the Theorem 2.4 applies. Here $a = 0$, $b = 1$, and since $sg_s + 2g \geq 0$ for all s, the condition (2.5) holds. (Notice that both the proof of the Theorem 2.4, as well as the proof of unimodality of $u(s)$ in [5], are not affected by the singularity at $s = 0$.)

(ii) This time we select δ so that $\frac{2}{\delta} - \frac{3}{2} = \frac{p_1}{2}$. One easily checks that the Theorem 2.6 applies. ☐

We now have the following exact multiplicity result. The uniqueness in its second part is due originally to W.–M. Ni and R. Nussbaum, [11].

Theorem 3.6. *Consider the problem*

$$(3.13) \qquad \Delta u + \lambda f(u) \;=\; 0 \;\; \textit{for} \;\; |x| \;<\; 1\,, \quad u \;=\; 0 \;\; \textit{for} \;\; |x| \;=\; 1\,,$$

depending on a positive parameter λ. In the conditions of the first part of the Theorem 3.5 *there is a critical $\lambda_0 > 0$, such that the problem* (3.13) *has exactly two positive solutions for $0 < \lambda < \lambda_0$, it has exactly one positive solution at $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve.*

In the conditions of the second part of the Theorem 3.5 *the problem* (3.13) *has exactly one positive solution for any λ. Moreover, all positive solutions are non–degenerate, and lie on a single smooth solution curve.*

Proof. The proof of the first part follows the arguments in [6] or [5], so we just sketch it. When $\lambda = 0$ there is a trivial solution $u = 0$, which we can continue for small λ, using the implicit function theorem. This solution curve cannot be continued indefinitely for increasing λ, since it is well–known that no solution exists for large λ, see e.g. H. AMANN [1]. Observe that the solution curve cannot go to infinity at a positive λ, as follows from the proof of the Theorem 1.4 in J. HEMPEL [4]. It follows that the solution curve must reach a turning point (λ_0, u_0). Since $f(u)$ is convex, it is easy to compute the direction of bifurcation at (λ_0, u_0), and at any other turning point. In fact the formula (2.31) in [5] shows that a turn to the right must always occur. This means that after the initial turn at (λ_0, u_0) no more turns may occur, and so the solution curve always travels to the left. By above it cannot go to infinity at a positive λ, hence it has to go infinity as $\lambda \to 0$, since at $\lambda = 0$ there are no non–trivial solutions.

Turning to the second part, we recall from [6] that solutions of (3.13) lie on smooth solution curves, and the well–known result that positive solutions of (3.13) may be parameterized by $u(0)$. It is easy to see that solutions on any curve go to zero as $\lambda \to \infty$, and they go to infinity as $\lambda \to 0$ (recall [4] again). It follows that one solution curve "takes up" all possible values of $u(0)$. We conclude that there is exactly one solution curve (existence of solution is well–known, [11]), and since the curve admits no turns, we conclude the uniqueness of solution. □

Remark 3.7. W.–M. NI and R. NUSSBAUM [11] conjectured that the condition $p_k \le p_1 + 4$ in the second part of the Theorem 3.5 is not necessary. As far as we know, this is still unknown. Similarly, it seems likely that condition $p_k \le 4$ in the first part can be dropped.

Acknowledgements

I wish to thank the referee for several useful remarks.

References

[1] AMANN, H.: Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces, SIAM Rev. **18** (1976), 620 – 709

[2] CRANDALL, M. G., and RABINOWITZ, H.: Bifurcation, Perturbation of Simple Eigenvalues and Linearized Stability, Arch. Rational Mech. Anal. **52** (1973), 161 – 180

[3] GIDAS, B., NI, W.–M., and NIRENBERG, L.: Symmetry and Related Properties via the Maximum Principle, Commun. Math. Phys. **68** (1979), 209 – 243

[4] HEMPEL, J.: On a Superlinear Differential Equation, Indiana Univ. Math. J. **26** (1977), 265 – 275

[5] KORMAN, P.: Uniqueness and Exact Multiplicity Results for Two Classes of Semilinear Problems, Nonlinear Analysis **31** (7) (1998), 849 – 865

[6] KORMAN, P., LI, Y., and OUYANG, T.: An Exact Multiplicity Result for a Class of Semilinear Equations, Commun. PDE. **22** (3&4) (1997), 661 – 684

[7] KORMAN, P., and OUYANG, T.: Solution Curves for Two Classes of Boundary–Value Problems, Nonlinear Analysis **27** (9) (1996), 1031 – 1047

[8] KWONG, M. K., and LI, Y.: Uniqueness of Radial Solutions of Semilinear Elliptic Equations, Trans. Amer. Math. Soc. **333** (1) (1992), 339 – 363

[9] KWONG, M. K., and ZHANG, L.: Uniqueness of the Positive Solutions of $\Delta u + f(u) = 0$ in an Annulus, Differential and Integral Equations **4** (1991), 583 – 599

[10] LIN, S. S.: Positive Radial Solutions and Non–Radial Bifurcation for Semilinear Elliptic Equations in Annular Domains, J. Differential Equations **86** (1990), 367 – 391

[11] Ni, W.–M., and Nussbaum, R.: Uniqueness and Non–Uniqueness for Positive Radial Solution of $\Delta u + f(u, r) = 0$, Comm. Pure Appl. Math. **38** (1985), 67 – 108

Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221–0025
USA