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Global solution curves for boundary value problems, with linear part at resonance

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ABSTRACT

We study existence and multiplicity of solutions for both Dirichlet and Neumann two-point boundary value problems at resonance. We obtain a detailed picture of the solution set, which, in particular, provides an effective way to compute all of the solutions. Our multiplicity results range from uniqueness to infinite multiplicity. Our approach can be seen as a dynamical version of the classical Liapunov–Schmidt procedure. After decomposing the space, we perform continuation in the subspace orthogonal to the kernel.

1. Introduction

We are interested in the exact number of solutions, as well as in the precise structure of the solution set, when parameters are varied, for both the Dirichlet problem (k is a positive parameter, the functions $g(u)$ and $f(x)$ are given)

$$u''(x) + u(x) + kg(u(x)) = f(x), \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0, \tag{1.1}$$

and the Neumann problem

$$u''(x) + kg(u(x)) = f(x), \quad x \in (0, \pi), \quad u'(0) = u'(\pi) = 0. \tag{1.2}$$

The common feature of both the problems is that the linear part is at resonance, i.e. when $k = 0$ and $f(x) \equiv 0$ both the problems have non-trivial solutions. There is an enormous literature devoted to problems of this type, going back to the classical papers of A.C. Lazer and D.E. Leach [1] and E.M. Landesman and A.C. Lazer [2] (with the important early contributions of A. Ambrosetti and G. Prodi [3], M.S. Berger and E. Podolak [4] and J. Kazdan, and F.W. Warner [5]). Similarly to these papers, we will obtain necessary and sufficient conditions for existence of solutions. In addition, we establish some detailed properties of the solution curves, when parameters are varied. What are the natural parameters? For the problem (1.1) it is natural to decompose (in L^2) the forcing term $f(x) = \mu \sin x + e(x)$, and the solution $u(x) = \xi \sin x + v(x)$, with $\int_0^\pi e(x) \sin x \, dx = 0$ and $\int_0^\pi v(x) \sin x \, dx = 0$. Then the relevant parameters are k, μ and ξ. When $k = 0$ and $\mu = 0$ the problem (1.1) is linear, and it has a unique solution, provided that the first harmonic ξ is fixed. We then use the implicit function theorem to solve for both u and μ as functions of k, and show that we can continue the solution with fixed first harmonic ξ when parameter k is varied, assuming that k stays below the second eigenvalue. (That is we produce a solution curve $(u(k), \mu(k))$, with the first harmonic of $u(k)$ equal to ξ.) Hence, for every fixed k, $k < \lambda_2$, the problem (1.1) has a solution with any $\xi \in R$ fixed. We then show that all these solutions are connected, when ξ is varied, i.e. they lie on the same solution curve. Namely, we show that $\mu = \phi(\xi)$, with some smooth function $\phi(\xi)$ (with fixed $k < \lambda_2$). We then study the properties of the function $\mu = \phi(\xi)$, to show that its range includes all admissible μ's.

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We use a similar approach for the Neumann problem (1.2), obtaining similar results, and even more detailed results are proved for a class of oscillating $g(u)$. Neumann boundary conditions are similar to the periodic ones, and so our results are similar to our results on pendulum-like equations [6]. That paper was motivated by the nice work G. Tarantello [7], which of course influenced the present paper as well. Among other papers on periodic solutions of pendulum-like equations we mention A. Castro [8], J. Cepicka, P. Drabek and J. Jensikova [9], G. Fournier and J. Mawhin [10].

Our approach can be seen as a dynamical version of the classical Liapunov–Schmidt procedure. After decomposing the space, we do not seek to solve the equation off the kernel in one step, but rather perform continuation in that space. In addition to the conceptual clarity, our approach opens a way for efficient numerical computation of solutions. *All* of the solutions for problems (1.1) and (1.2) can be computed by continuation first in k, and then in the first harmonic ξ. (One can usually skip continuation in k, since it is typically easy to jump on the solution curve $\mu = \phi(\xi)$ for any fixed k.) Observe that our results imply that there are no turns back on the solution curves, when one continues in either k or ξ. This makes it particularly easy to numerically implement the continuation process.

As we mentioned, there is a large body of literature on problems at resonance. The book by A. Ambrosetti and G. Prodi [11] provides a nice introduction. A recent paper by A. Cañada [12] has a number of additional references.

2. Preliminary results

Recall that L^2 functions defined on the interval $[0, \pi]$ can be represented by the Fourier sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, or by the Fourier cosine series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, with both the series convergent in L^2. Any function in $L^2(0, \pi)$ can also be represented by the "full" Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, and then $H^1(0, \pi)$ is the subspace of $L^2(0, \pi)$, for which $\sum_{n=1}^{\infty}(n^2 + 1)(a_n^2 + b_n^2)$ is finite. Similarly, $H_0^1(0, \pi)$ is the subspace of $L^2(0, \pi)$ functions $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, with $\sum_{n=1}^{\infty}(n^2 + 1)b_n^2 < \infty$. Higher order Sobolev spaces can be similarly described through the Fourier coefficients. Throughout the paper we will consider only strong solutions. For the Dirichlet problem, strong solutions are of class $H^2(0, \pi) \cap H_0^1(0, \pi)$, i.e. functions $\sum_{n=1}^{\infty} b_n \sin nx$, with $\sum_{n=1}^{\infty}(n^4 + 1)b_n^2 < \infty$. For the Neumann problem u is a strong solution if $u \in H^2(0, \pi)$ and $u' \in H_0^1(0, \pi)$, i.e. functions $u = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, with $\sum_{n=1}^{\infty}(n^4 + 1)a_n^2 < \infty$. By Sobolev imbedding, strong solutions are in $C^1[0, \pi]$.

The following simple lemma is similar to Wirtinger's inequality for periodic functions.

Lemma 2.1. *If $f(x) \in H_0^1(0, \pi)$, and $b_1 = 0$, then*

$$\int_0^{\pi} f'^2(x)\,dx \geq 4 \int_0^{\pi} f^2(x)\,dx.$$

If $f(x) \in H_0^1(0, \pi)$, then

$$\int_0^{\pi} f'^2(x)\,dx \geq \int_0^{\pi} f^2(x)\,dx.$$

Proof. By Parseval identities $\int_0^{\pi} f^2(x)\,dx = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2$ and $\int_0^{\pi} f'^2(x)\,dx = \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 b_n^2$, and the proof follows. ◇

Lemma 2.2. *Let $f(x) \in H^1(0, \pi)$ be represented by Fourier cosine series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, convergent in $H^1(0, \pi)$. If $a_0 = a_1 = 0$, then*

$$\int_0^{\pi} f'^2(x)\,dx \geq 4 \int_0^{\pi} f^2(x)\,dx.$$

If $a_0 = 0$, then

$$\int_0^{\pi} f'^2(x)\,dx \geq \int_0^{\pi} f^2(x)\,dx.$$

Next we consider a linear problem in the class of functions with vanishing first harmonic

$$w''(x) + w(x) + a(x)w(x) = \mu \sin x, \quad \text{for } x \in (0, \pi), \tag{2.1}$$
$$w(0) = w(\pi) = 0, \quad \int_0^{\pi} w(x) \sin x\,dx = 0.$$

Here $a(x)$ is a given continuous function, and we are looking for a solution pair $(w(x), \mu)$, where $w \in H^2(0, \pi) \cap H_0^1(0, \pi)$, and μ is a constant.

Lemma 2.3. *If $|a(x)| < 3$, then the only solution of (2.1) is $\mu = 0$ and $w(x) \equiv 0$.*

Proof. We can represent $w(x) = \sum_{n=2}^{\infty} b_n \sin nx$. Multiply the equation in (2.1) by $w''(x)$ and integrate. Since $\int_0^{\pi} w''(x) \sin x \, dx = 0$, we have

$$\int_0^{\pi} w''^2(x) \, dx = \int_0^{\pi} w'^2(x) \, dx - \int_0^{\pi} a(x) w w'' \, dx. \tag{2.2}$$

The function $w'(x)$ satisfies the conditions of Lemma 2.2, and hence

$$\int_0^{\pi} w'^2(x) \, dx - \int_0^{\pi} a(x) w w'' \, dx < \frac{1}{4} \int_0^{\pi} w''^2(x) \, dx + 3 \left(\int_0^{\pi} w^2(x) \, dx \right)^{1/2} \left(\int_0^{\pi} w''^2(x) \, dx \right)^{1/2}.$$

Combining this with (2.2), we conclude that

$$\int_0^{\pi} w''^2(x) \, dx < 16 \int_0^{\pi} w^2(x) \, dx.$$

On the other hand, applying Lemmas 2.1 and 2.2,

$$\int_0^{\pi} w''^2(x) \, dx \geq 16 \int_0^{\pi} w^2(x) \, dx,$$

a contradiction, unless $w \equiv 0$. ⋄

We now consider another linear problem

$$\begin{aligned} & w''(x) + w(x) + a(x) w(x) = 0, \quad \text{for } x \in (0, \pi), \\ & w(0) = w(\pi) = 0, \end{aligned} \tag{2.3}$$

where $a(x)$ is a given continuous function.

Lemma 2.4. *If $a(x) < 3$, then any non-trivial solution of (2.3) is of one sign, i.e. we may assume that $w(x) > 0$ on $(0, \pi)$.*

Proof. If $w(x)$ was sign changing, it would be the second or higher eigenfunction of the problem

$$w''(x) + w(x) + a(x) w(x) = \lambda w(x), \quad w(0) = w(\pi) = 0, \tag{2.4}$$

corresponding to the eigenvalue $\lambda = 0$. But the eigenvalue problem

$$w''(x) + w(x) + 3w(x) = \lambda w(x), \quad w(0) = w(\pi) = 0 \tag{2.5}$$

has all eigenvalues larger than the corresponding ones of (2.4). The eigenvalues of (2.5) are $\lambda_k = 4 - k^2$, $k = 1, 2, \ldots$, with $\lambda_2 = 0$. It follows that the eigenvalues of (2.4), from the second one onwards, must be negative, a contradiction. ⋄

3. Continuation of solutions with a fixed first harmonic

As we mentioned, we shall consider strong solutions, which for Dirichlet problem means solutions of class $H^2(0, \pi) \cap H_0^1(0, \pi)$. We further denote by H_1^2 the subspace, with vanishing first coefficient of the Fourier sine series, i.e. $H_1^2 = \{u(x) \in H^2(0, \pi) \cap H_0^1(0, \pi) \mid \int_0^{\pi} u(x) \sin x \, dx = 0\}$. Similarly, we denote by L_1^2 the subspace of $L^2(0, \pi)$, consisting of functions with vanishing first coefficient of the Fourier sine series, i.e. $L_1^2(0, \pi) = \{u(x) \in L^2(0, \pi) \mid \int_0^{\pi} u(x) \sin x \, dx = 0\}$.

Lemma 3.1. *Consider the problem*

$$\begin{aligned} & u''(x) + u(x) + kg(u(x)) = \mu \sin x + e(x), \quad \text{for } x \in (0, \pi), \\ & u(0) = u(\pi) = 0, \quad \int_0^{\pi} u(x) \sin x \, dx = \xi, \end{aligned} \tag{3.1}$$

where k, μ and ξ are parameters, and $e(x) \in L_1^2$ is a given function. Assume that $g(u) \in C^1(R)$ satisfies

$$|g(u)| \leq M_0, \quad \text{for all } u \in R, \tag{3.2}$$

$$|g'(u)| \leq M, \quad \text{for all } u \in R, \tag{3.3}$$

with some positive constants M_0 and M. Assume finally that

$$k \leq \frac{3}{M}. \tag{3.4}$$

Then for any $\xi \in R$ one can find a unique pair of $\mu = \mu(\xi, k)$ and $u = u(x, \xi, k)$, solving (3.1).

Proof. We begin by assuming that $\xi = 0$. We wish to prove that there is a unique $\mu = \mu(k)$ for which the problem (3.1) has a solution, and the uniqueness of that solution. We recast the Eq. (3.1) in the operator form

$$F(u, \mu, k) = e(x), \tag{3.5}$$

where $F\ : H_1^2 \times R \times R \to L_1^2$ is defined by

$$F(u, \mu, k) = u''(x) + u(x) + kg(u(x)) - \mu \sin x.$$

When $k = 0$ and $\mu = 0$, the problem (3.1), with $\xi = 0$, has a unique solution, as one can see using Fourier sine series. We now continue this solution for the increasing k, i.e. we solve (3.1) for the pair (u, μ) as a function of k. Compute the Frechet derivative

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = w''(x) + w(x) + kg'(u(x))w(x) - \mu^* \sin x.$$

In view of the assumptions (3.3) and (3.4), Lemma 2.3 applies, and hence the only solution of the linearized problem

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = 0,$$
$$w(0) = w(\pi) = 0, \quad \int_0^\pi w(x) \sin x \, dx = 0$$

is $(w, \mu^*) = (0, 0)$, i.e. the operator $F_{(u,\mu)}(u, \mu, k)(w, \mu^*)$ is injective.

We next verify that this operator is also surjective, i.e. for any $e^*(x) \in L_1^2$ the problem

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = e^*(x), \tag{3.6}$$
$$w(0) = w(\pi) = 0, \quad \int_0^\pi w(x) \sin x \, dx = 0$$

has a solution (w, μ^*). Let us consider an auxiliary problem

$$L[w] \equiv w''(x) + w(x) + kg'(u(x))w(x) = \mu^* \sin x + e^*(x), \tag{3.7}$$
$$w(0) = w(\pi) = 0,$$

which differs from (3.6) only in that we do not require the solution to have a vanishing first harmonic. We distinguish between the two cases.

Case 1. The operator $L[w]$, subject to the boundary conditions $w(0) = w(\pi) = 0$, is invertible. We then write the solution of (3.7) as

$$w(x) = \mu^* L^{-1}(\sin x) + L^{-1}(e^*(x)). \tag{3.8}$$

In view of Lemma 2.3, $\int_0^\pi L^{-1}(\sin x) \sin x \, dx \neq 0$, and then we can choose μ^* in (3.8), so that $\int_0^\pi w(x) \sin x \, dx = 0$, i.e. we obtain the solution of (3.6).

Case 2. The operator $L[w]$, subject to the boundary conditions $w(0) = w(\pi) = 0$, is not invertible. Hence there is a kernel. The kernel is one dimensional (since the value of $w'(\pi)$, together with boundary condition $w(\pi) = 0$, uniquely determines the solution), and in view of Lemma 2.4, it is spanned by a positive function $\bar{w}(x)$. Since $\bar{w}(x) > 0$, we can now choose a constant μ^*, so that the right-hand side of (3.7) is orthogonal to the kernel, and hence that equation has infinitely many solutions of the form $w(x) = w_0(x) + t\bar{w}(x)$, where $w_0(x)$ is one of the solutions. We now select the constant t, so that $\int_0^\pi w(x) \sin x \, dx = 0$, obtaining a solution of (3.6).

Hence the implicit function theorem applies, and locally we have a curve of solutions $u = u(k)$ and $\mu = \mu(k)$. To show that this curve continues for all k satisfying our condition (3.4), we only need to show that this curve cannot go to infinity at some k, i.e. we need an a priori estimate. We multiply the equation in (3.1) by $u''(x)$ (with vanishing first Fourier sine coefficient), and integrate

$$\int_0^\pi u''^2 \, dx - \int_0^\pi u'^2 \, dx + k \int_0^\pi g(u)u'' \, dx = \int_0^\pi e(x)u'' \, dx. \tag{3.9}$$

Since $u(x) = \sum_{k=2}^\infty u_k \sin kx$, and $u'(x) = \sum_{k=2}^\infty ku_k \cos kx$, Lemma 2.2 applies to u', implying that

$$\int_0^\pi u'^2 \, dx \le \frac{1}{4} \int_0^\pi u''^2 \, dx.$$

Since $g(u)$ and $e(x)$ are bounded, we conclude from (3.9) that $\int_0^\pi u''^2 \, dx$ is bounded, which implies a bound on $\int_0^\pi u'^2 \, dx$, and then on $\int_0^\pi u^2 \, dx$ by Lemmas 2.1 and 2.2. This gives us a H^2 bound (and also a uniform bound by Sobolev embedding theorem). It follows that we can continue the solution for all k, satisfying (3.4).

Turning to the case of general ξ, we reduce it to the $\xi = 0$ case, by setting $v(x) = u(x) - \frac{2}{\pi}\xi \sin x$. Observe that $v(x)$ satisfies

$$v''(x) + v(x) + kg\left(v(x) + \frac{2}{\pi}\xi \sin x\right) = \mu \sin x + e(x),\ x \in (0, \pi), \tag{3.10}$$
$$v(0) = v(\pi) = 0, \quad \int_0^\pi v(x) \sin x \, dx = 0.$$

Even though this problem is slightly different from (3.1), with $\xi = 0$, it is clear that we can repeat the above argument, and obtain a curve of solutions of (3.10), with zero first harmonic. Then $u = u(x, k)$ is a solution curve of the problem (3.1). ◇

We shall need the following simple corollary.

Lemma 3.2. *Assume that the conditions of Lemma 3.1 hold, and think of k as fixed. Then for any $\xi \in R$ there is a unique pair $(\mu, u(x))$ solving (3.1).*

Proof. Let $(\mu_1, u_1(x))$ be another solution of (3.1), corresponding to the same ξ. We now continue it backwards in k on the curve of solutions of (3.1). Similarly, we continue $(\mu, u(x))$ for decreasing k on another curve of solutions of (3.1). These solution curves do not intersect, since the implicit function theorem applies at each point. At $k = 0$ we obtain two solutions with the same ξ, a contradiction. ◇

4. Continuation in ξ for fixed k

We now consider a Dirichlet problem at resonance

$$u''(x) + u(x) + kg(u(x)) = \mu \sin x + e(x), \quad \text{for } x \in (0, \pi), \tag{4.1}$$
$$u(0) = u(\pi) = 0,$$

where, as before, $e(x) \in L^2_1$ is a given function, with $\int_0^\pi e(x) \sin x \, dx = 0$. Let us assume that the function $g(u)$ has finite limits at infinity, $g(\infty) = \lim_{u\to\infty} g(u)$ and $g(-\infty) = \lim_{u\to-\infty} g(u)$, and assume that

$$g(-\infty) < g(u) < g(\infty) \quad \text{for all } u \in R. \tag{4.2}$$

Multiplying the Eq. (4.1) by $\sin x$, and integrating over $(0, \pi)$, we see that

$$\frac{4}{\pi}kg(-\infty) < \mu < \frac{4}{\pi}kg(\infty). \tag{4.3}$$

This is obviously a necessary condition for the solvability of (4.1). Following the classical paper of E.M. Landesman and A.C. Lazer [2], we wish to show that this condition is also sufficient, and derive some extra information on the solution curves.

We can decompose the solution of (4.1) in the form

$$u(x) = \xi \sin x + U_\xi(x), \tag{4.4}$$

with $\int_0^\pi U_\xi(x) \sin x \, dx = 0$.

Lemma 4.1. *There is a constant $m > 0$, so that for any $\xi \in R$ and any $\mu \in R$ we have*

$$\|U_\xi\|_{L^\infty[0,\pi]} < m.$$

Proof. Substituting the ansatz (4.4) into (4.3), we obtain the problem (3.10), with $U_\xi(x)$ in place of $v(x)$, and ξ in place of $\frac{2}{\pi}\xi$. This means that we can estimate $\|U_\xi\|_{H^2[0,\pi]}$ as before, and then use the Sobolev embedding. ◇

Recall that for any fixed k, with $k < \frac{3}{M}$, the value of ξ uniquely identifies the solution pair $(\mu, u(x))$. Hence, the solution set of (4.1) can be faithfully depicted by planar curves in the (ξ, μ) plane, which we call *solution curves*.

Theorem 4.1. *Assume that the conditions of Lemma 3.1 hold, and in addition assume that μ satisfies the necessary conditions (4.3). Then the problem (4.1) has a solution. Moreover, all solutions of (4.1) lie on a unique solution curve, which is given by an equation $\mu = \phi(\xi)$, where $\phi(\xi)$ is a bounded continuous function, having limits at infinity: $\phi(-\infty) = \frac{4}{\pi}kg(-\infty)$, and $\phi(\infty) = \frac{4}{\pi}kg(\infty)$.*

Proof. By Lemma 3.1, for any $\xi \in R$ one can find a unique solution pair $\mu = \mu(\xi)$ and $u = u(x, \xi)$ (k is now fixed). We show that when one varies ξ, all these solutions link up, to form a unique solution curve.

We claim that any solution of (4.1) can be continued in ξ, i.e. the implicit function theorem applies. The proof is essentially the same as that for the continuation in k above. Letting $u(x) = \xi \sin x + v(x)$, we recast the problem (4.1) in the operator form

$$F(v, \mu, \xi) = e(x), \tag{4.5}$$

where $F : H_1^2 \times R \times R \to L_1^2$ is defined by

$$F(v, \mu, \xi) = v''(x) + v(x) + kg(\xi \sin x + v(x)) - \mu \sin x.$$

The Frechet derivative this time is

$$F_{(v,\mu)}(v, \mu, \xi)(w, \mu^*) = w''(x) + w(x) + kg'(\xi \sin x + v(x))w(x) - \mu^* \sin x.$$

By Lemma 2.2 the only solution of the linearized problem

$$F_{(v,\mu)}(v, \xi, k)(w, \mu^*) = 0,$$
$$w(0) = w(\pi) = 0, \quad \int_0^\pi w(x) \sin x \, dx = 0$$

is $(w, \mu^*) = (0, 0)$. As before, we verify the surjectivity of $F_{(v,\mu)}(v, \xi, k)(w, \mu^*)$. That is the implicit function theorem applies, giving us locally a curve of solutions $u = u(x, \xi)$ and $\mu = \mu(\xi)$. Exactly as before, we see that solutions on this curve remain bounded, and hence we can continue the curve for all $\xi \in R$. Since the value of ξ uniquely identifies the solution pair $(\mu, u(x))$, and our solution curve exhausts all possible ξ's, it follows that there are no other solutions.

Turning to the limits at infinity, observe that the decomposition (4.4) and Lemma 4.1 imply that $u(x) \to \infty \, (-\infty)$ as $\xi \to \infty \, (-\infty)$ on any closed subinterval of $(0, \pi)$. Multiplying the Eq. (4.1) by $\sin x$, and integrating over $(0, \pi)$ we conclude that $\mu(\pm\infty) = \frac{4}{\pi} kg(\pm\infty)$. $\diamond$

For monotone $g(u)$ we have an uniqueness result.

Theorem 4.2. *In addition to the conditions of the preceding theorem, assume that $g'(u) > 0$ for all $u \in R$. Then $\phi'(\xi) > 0$ for all $\xi \in R$, which implies that for any μ satisfying the necessary conditions (4.3), the problem (4.1) has a unique solution.*

Proof. Assume on the contrary that $\mu'(\xi_0) = 0$ for some ξ_0. Differentiating the Eq. (4.1) in ξ, setting $\xi = \xi_0$, and denoting $w = u_\xi|_{\xi=\xi_0}$, we have

$$w''(x) + w(x) + kg'(u(x, \xi_0))\, w(x) = 0, \quad w(0) = w(\pi) = 0. \tag{4.6}$$

By Lemma 2.3 $w(x) > 0$. Multiplying the Eq. (4.6) by $\sin x$, and integrating over $(0, \pi)$

$$\int_0^\pi g'(u(x, \xi_0))\, w(x) \sin x \, dx = 0,$$

which is impossible, since the integrand is positive. $\diamond$

5. The Neumann problem

We shall consider strong solutions of all Neumann type problems in this and the following sections, i.e. $u \in H^2(0, \pi)$ and $u' \in H_0^1(0, \pi)$, which can be represented by the Fourier cosine series $u(x) = a_0 + \sum_{n=1}^\infty a_n \cos nx$. We shall denote this space by $\bar{H}^2$, and by $\bar{H}_1^2$ we denote its subspace with $a_0 = 0$. We shall write $\bar{L}_1^2$ for the subspace of $L^2(0, \pi)$, with $\int_0^\pi w(x)\, dx = 0$.

Following the scheme used for the Dirichlet problem, we will do continuation of solutions with the first harmonic, $a_0 = \frac{1}{\pi} \int_0^\pi u(x)\, dx$, being fixed, which will lead us to consider the following linear problem:

$$w''(x) + a(x)w(x) = \mu, \quad \text{for } x \in (0, \pi), \tag{5.1}$$
$$w'(0) = w'(\pi) = 0, \quad \int_0^\pi w(x)\, dx = 0.$$

Lemma 5.1. *If $|a(x)| < 1$, then the only solution of (5.1) is $\mu = 0$ and $w(x) \equiv 0$.*

Proof. Multiply the equation in (5.1) by w'', and integrate over $(0, \pi)$. After applying Schwarz inequality to the second term, we have

$$\int_0^\pi w''^2 \, dx < \int_0^\pi w^2 \, dx.$$

However, Lemmas 2.1 and 2.2 imply the opposite inequality. It follows that $w(x) \equiv 0$. $\diamond$

The following lemma establishes positivity of solution of the linearized problem:

$$w''(x) + a(x)w(x) = 0, \quad \text{for } x \in (0, \pi),$$
$$w'(0) = w'(\pi) = 0. \tag{5.2}$$

Lemma 5.2. *If $a(x) < 1$, then any non-trivial solution of (5.2) is of one sign, i.e. we may assume that $w(x) > 0$ on $(0, \pi)$.*

Proof. Assume not. Then either $w(x)$ vanishes on $(0, \pi/2]$, or on $[\pi/2, \pi)$. In the first case, let $\xi \in (0, \pi/2]$ be the smallest root of $w(x)$. We may assume that $w(x) > 0$ on $(0, \xi)$. Multiply (5.2) by $\cos x$, and integrate over $(0, \xi)$

$$w'(\xi)\cos\xi + \int_0^\xi (a(x) - 1)\, w(x) \cos x \, dx = 0.$$

The first term on the left is non-positive, and the second one is negative, a contradiction. In case $w(x)$ vanishes on $[\pi/2, \pi)$, let $\eta \in [\pi/2, \pi)$ denote the largest root of $w(x)$, and we may assume that $w(x) > 0$ on (η, π). Multiply (5.2) by $\cos x$, and integrate over (η, π)

$$-w'(\eta)\cos\eta + \int_\eta^\pi (a(x) - 1)\, w(x) \cos x \, dx = 0.$$

The first term on the left is non-negative, and the second one is positive, again we reach a contradiction. ◇

Our goal is to describe the solution curve of the problem

$$u''(x) + kg(u(x)) = \mu + e(x), \quad \text{for } x \in (0, \pi),$$
$$u'(0) = u'(\pi) = 0. \tag{5.3}$$

We shall decompose the solution of (5.3) in the form

$$u(x) = \xi + U_\xi(x), \tag{5.4}$$

with $\int_0^\pi U_\xi(x)\, dx = 0$. Substituting this into (5.3), we obtain

$$U_\xi'' + kg\left(\xi + U_\xi\right) = \mu + e(x), \quad U_\xi'(0) = U_\xi'(\pi) = 0. \tag{5.5}$$

It turns out that in the Neumann case one can get a uniform bound on U_ξ without assuming that $g(u)$ is bounded. This, in particular, will allow us to consider the case when $g(u)$ has infinite limits at $\pm\infty$. We present the a priori bound next, after introducing some notation.

Given $e(x) \in L^2(0, \pi)$, we define

$$\|e\| = \min\left\{\pi |e|_{L^1}, \sqrt{\pi}|e|_{L^2}\right\},$$

where both L^1 and L^2 norms (and L^∞ norm below) are taken over $(0, \pi)$. The following lemma, is similar to G. Tarantello [7], shows that for small $e(x)$ solution of (5.3) has small oscillation, uniformly in μ and ξ.

Lemma 5.3. *Assume that $g(u)$ satisfies the condition (3.3), and let $kM < 1$. Then for any $\mu \in R$ and any $\xi \in R$, we have*

$$|U_\xi(x)|_{L^\infty} < \frac{1}{1 - kM}\|e\|. \tag{5.6}$$

Proof. Multiply the Eq. (5.5) by U_ξ, and integrate

$$-\int_0^\pi U_\xi'^2\, dx + k\int_0^\pi \left[g(\xi + U_\xi) - g(\xi)\right] U_\xi\, dx = \int_0^\pi U_\xi e\, dx.$$

By mean value theorem we estimate the second term on the left by

$$kM \int_0^\pi U_\xi^2\, dx \le kM \int_0^\pi U_\xi'^2\, dx,$$

with the second estimate following in view of Lemma 2.2. Hence

$$(1 - kM)\int_0^\pi U_\xi'^2\, dx \le \int_0^\pi |U_\xi e|\, dx \le |U_\xi|_{L^\infty} |e|_{L^1}. \tag{5.7}$$

Writing $U_\xi(x) = \int_{x_0}^x U'_\xi(x)\,dx$, where x_0 is any root of $U_\xi(x)$, and using (5.7)

$$|U_\xi|_{L^\infty} \le \int_0^\pi |U'_\xi|\,dx \le \sqrt{\pi}\left(\int_0^\pi {U'_\xi}^2\,dx\right)^{1/2} \le \frac{\sqrt{\pi}}{\sqrt{1-kM}}\,|U_\xi|_{L^\infty}^{1/2}|e|_{L^1}^{1/2}, \tag{5.8}$$

which gives us the first estimate (see the definition of $\|e\|$).

From (5.7) we also conclude (using Lemma 2.2)

$$(1-kM)\int_0^\pi U_\xi^2\,dx \le |U_\xi|_{L^2}|e|_{L^2},$$

which implies

$$|U_\xi|_{L^2} \le \frac{1}{1-kM}|e|_{L^2}.$$

Then returning to (5.7),

$$(1-kM)\int_0^\pi {U'_\xi}^2\,dx \le |U_\xi|_{L^2}|e|_{L^2} \le \frac{1}{1-kM}|e|_{L^2}^2,$$

and from (5.8) we obtain the second estimate

$$|U_\xi|_{L^\infty} \le \frac{\sqrt{\pi}}{1-kM}|e|_{L^2}. \quad \diamond$$

Lemma 5.4. *Consider the problem*

$$u''(x) + kg(u(x)) = \mu + e(x), \quad \text{for } x \in (0,\pi), \qquad u'(0) = u'(\pi) = 0, \quad \frac{1}{\pi}\int_0^\pi u(x)\,dx = \xi, \tag{5.9}$$

where k, μ and ξ are parameters, and $e(x) \in \bar{L}_1^2$ is a given function. Assume that $g(u) \in C^1(R)$ satisfies the conditions (3.2) and (3.3), and assume finally that

$$kM \le 1. \tag{5.10}$$

Then for any $\xi \in R$ one can find a unique pair of $\mu = \mu(\xi,k)$ and $u = u(x,\xi,k)$, solving (5.9).

Proof. As before, we begin with the case $\xi = 0$. We prove that there is a unique $\mu = \mu(k)$ for which the problem (5.9) has a solution, and uniqueness of that solution. We recast the Eq. (5.9) in the operator form

$$F(u,\mu,k) = e(x),$$

where $F : \bar{H}_1^2 \times R \times R \to \bar{L}_1^2(0,\pi)$ is defined by

$$F(u,\mu,k) = u''(x) + kg(u(x)) - \mu.$$

When $k = 0$ and $\mu = 0$, the problem (5.9), with $\xi = 0$, has a unique solution, as one can see by using Fourier cosine series. We now continue this solution for increasing k, i.e. we solve (5.9) for the pair (u,μ) as a function of k. As before we show that the implicit function theorem applies. Compute the Frechet derivative

$$F_{(u,\mu)}(u,\mu,k)(w,\mu^*) = w''(x) + w(x) + kg'(u(x))w(x) - \mu^*.$$

In view of the assumptions (3.4) and (5.10), Lemma 5.1 applies, and hence the only solution of the linearized problem

$$F_{(u,\mu)}(u,\mu,k)(w,\mu^*) = 0, \qquad w(0) = w(\pi) = 0, \quad \int_0^\pi w(x)\,dx = 0$$

is $(w,\mu^*) = (0,0)$, i.e. the operator $F_{(u,\mu)}(u,\mu,k)(w,\mu^*)$ is injective. One verifies as before that the operator $F_{(u,\mu)}(u,\mu,k)(w,\mu^*)$ is surjective. It is here that we use positivity for the linearized problem, given by Lemma 5.2. As we continue in k, solutions remain bounded, since ξ is fixed, while U_ξ is bounded by Lemma 5.3.

As before the case of general ξ is reduced to that of $\xi = 0$. $\diamond$

As in the Dirichlet case, we prove the following corollary.

Corollary. *Assume that the conditions of Lemma 5.4 hold, and think of k as fixed. Then for any $\xi \in R$ there is a unique pair $(\mu, u(x))$ solving (5.3).*

We assume that $g(u)$ has finite or infinite limits at infinity, and that condition (4.2) holds. Integrating the Eq. (5.3), we obtain a necessary condition for its solvability

$$g(-\infty) < \mu/k < g(\infty). \tag{5.11}$$

As before, we prove the following result.

Theorem 5.1. *Assume that the conditions of Lemma 5.4 hold, and in addition assume that μ satisfies the necessary conditions (5.11). Then the problem (5.3) has a solution. Moreover, all solutions of (5.3) lie on a unique solution curve, which is given by an equation $\mu = \phi(\xi)$, where $\phi(\xi)$ is a bounded continuous function, having limits at infinity: $\phi(-\infty) = kg(-\infty)$, and $\phi(\infty) = kg(\infty)$.*

We observe that this theorem also provides a solution to the following inverse problem: given $\xi \in R$ find μ, so that the problem (5.3) has a solution of average ξ. By Theorem 5.1, if $kM < 1$, this inverse problem is solvable for all $\xi \in R$, and the solution can be computed by continuation in k, starting with $k = 0$.

6. Infinitely many solutions of Neumann problem

We consider the problem (5.3) again, but this time we assume the function $g(u)$ to be oscillating at infinity, instead of having limits at infinity. We will consider the following class of $g(u)$, which is motivated by G. Tarantello [7].

Definition. We say that the function $g(u) \in C^2(R)$ is of class $\mathbf{T}_m$ if
(i) $g(u)$ changes sign infinitely many times;
(ii) let ρ denote any point of local maximum (minimum) of $g(u)$, then on the interval $|u - \rho| < m$ we have

$$g(u) > 0 \quad (< 0) \quad \text{and} \quad g''(u) < 0 \quad (> 0). \tag{6.1}$$

In other words, $g(u)$ has only positive maximums and negative minimums, and they are spaced out. For example, $\sin u \in \mathbf{T}_{\pi/2}$.

Definition. We say that the function $u(x) \in C[0, \pi]$ is of class $\mathbf{t}_m$ if, when writing $u(x) = \xi + U_\xi(x)$, with $\int_0^\pi U_\xi(x)\,dx = 0$, we have

$$|U_\xi(x)| < m/2 \quad \text{for all } x \in [0, \pi]. \tag{6.2}$$

If $u(x)$ is of class $\mathbf{t}_m$, it follows that the range of $u(x)$ belongs to an interval of length less than m. Hence, if we assume that the range of $u(x)$ includes any point of local maximum (minimum) of $g(u)$, and that $g(u)$ is of class $\mathbf{T}_m$, then (6.1) holds, which implies that $g''(u(x))$ is of one sign. This will allow us to compute the direction of turn of the curve $\mu = \mu(\xi)$.

Theorem 6.1. *Assume that conditions of the Theorem 5.1 are satisfied, and in addition $g(u) \in \mathbf{T}_m$, and*

$$\frac{1}{1 - kM}\|e\| < \frac{m}{2}. \tag{6.3}$$

Then, in addition to the conclusions of Theorem 5.1, the function $\mu = \phi(\xi)$ changes sign infinitely many times. Moreover, it is positive at any of its local maximums, and negative at its local minimums, and it has no points of inflection. If, in addition, $g(u)$ is a periodic function, of period say 2π, then the same is true for the function $\phi(\xi)$.

Proof. The condition (6.3) and Lemma 5.3 imply that $u(x) \in \mathbf{t}_m$. Integrate the Eq. (5.3),

$$k \int_0^\pi g(u(x))\,dx = \mu\pi. \tag{6.4}$$

Along the solution curve ξ changes continuously from $-\infty$ to ∞. When ξ equals to a point of local maximum of $g(u)$, then by our assumptions the integrand on the left in (6.4) is positive, and hence $\mu > 0$. Similarly $\mu < 0$, when ξ passes a point of local minimum of $g(u)$. Since $g(u)$ changes sign infinitely many times, the same is true for $\phi(\xi)$.

Turning to the critical points of $\mu = \phi(\xi)$, we differentiate the Eq. (5.3) in ξ

$$u_\xi'' + kg'(u)u_\xi = \mu'(\xi). \tag{6.5}$$

At a critical point ξ_0 we have $\mu'(\xi_0) = 0$. We now set $\xi = \xi_0$ in (6.5). Then $w(x) \equiv u_\xi |_{\xi=\xi_0}$ satisfies the linearized problem (5.2), and hence by Lemma 5.2, $w(x) > 0$. Integrating (6.5) at $\xi = \xi_0$

$$k \int_0^\pi g'(u(x))w(x)\,dx = 0. \tag{6.6}$$

We differentiate (5.3) in ξ again, and set $\xi = \xi_0$

$$u''_{\xi\xi} + kg'(u)u_{\xi\xi} + kg''(u)w^2 = \mu''(\xi_0).$$

Combining this with the linearized problem (5.2), we have

$$k \int_0^\pi g''(u(x))w^3(x)\,dx = \mu''(\xi_0) \int_0^\pi w(x)\,dx. \tag{6.7}$$

At a critical point ξ_0, we see from (6.6) that $g'(u)$ changes sign on the range of $u(x)$, i.e. the range of $u(x)$ includes a critical point of $g(u)$. Since $u(x) \in \mathbf{t}_m$, the range of $u(x)$ includes exactly one critical point of $g(u)$. If it is a point of maximum of $g(u)$, then the first set of inequalities in (6.1) holds, and then we see from (6.4) and (6.7) that $\mu(\xi_0) > 0$ and $\mu''(\xi_0) < 0$, while the opposite inequalities hold in the case of a point of minimum.

Finally, assume that $g(u)$ is 2π-periodic. If $u(x)$ is a solution with first harmonic equal to ξ, then $u(x) + 2\pi$ is a solution with first harmonic equal to $\xi + 2\pi$, corresponding to the same μ. Hence, we have $\phi(\xi + 2\pi) = \phi(\xi)$. ◇

Remark. Since μ vanishes infinitely many times in the above theorem, it follows that the problem (with $\int_0^\pi e(x)\,dx = 0$)

$$u'' + kg(u) = e(x), \quad 0 < x < \pi, \quad u'(0) = u'(\pi) = 0 \tag{6.8}$$

has infinitely many solutions.

7. Numerical computation of solutions

We consider again the Dirichlet problem (4.1), which we wish to solve for all possible values of the parameter μ (or for all possible values of the first harmonic ξ). We will do continuation in ξ, using Newton's method, which will require us to solve the following linear problem: given $a(x) \in C(0, \pi)$, $f(x) \in L^2(0, \pi)$, and $\xi \in R$, find the pair $(\mu, w(x))$ solving

$$\begin{aligned} &w''(x) + w(x) + a(x)w(x) = \mu \sin x + f(x), \quad \text{for } x \in (0, \pi), \\ &w(0) = w(\pi) = 0, \quad \frac{2}{\pi} \int_0^\pi w(x) \sin x\,dx = \xi. \end{aligned} \tag{7.1}$$

Using Green's function $G(x, \xi)$ we express the solution of the first two equations in (7.1) as

$$w(x) = \int_0^\pi G(x, \xi)\,(\mu \sin \xi + f(\xi))\,d\xi, \tag{7.2}$$

and then select μ, so that the third equation in (7.1) is also satisfied. Namely,

$$\mu = \frac{\frac{\pi}{2}\xi - \int_0^\pi \int_0^\pi G(x, \xi) f(\xi) \sin x\,d\xi\,dx}{\int_0^\pi \int_0^\pi G(x, \xi) \sin \xi \sin x\,d\xi\,dx}. \tag{7.3}$$

(Find μ from (7.3), plug it into (7.2) to obtain the solution of (7.1).) Recall that

$$G(x, \xi) = -\frac{1}{W} \begin{cases} y_1(x)y_2(\xi), & \text{for } 0 \le x \le \xi \\ y_1(\xi)y_2(x), & \text{for } \xi \le x \le \pi, \end{cases}$$

where $y_1(x)$ solves the equation

$$w''(x) + w(x) + a(x)w(x) = 0, \quad x \in (0, \pi), \tag{7.4}$$

subject to the initial conditions $w(0) = 0$ and $w'(0) = 1$, while $y_2(x)$ solves the same equation with the initial conditions $w(\pi) = 0$ and $w'(\pi) = -1$. The constant W denotes the Wronskian $W = y_1'(x)y_2(x) - y_2'(x)y_1(x)$. While in general one cannot write down $y_i(x)$'s by hand, the *Mathematica* command **NDSolve** gives their accurate approximations by interpolating functions (one can even differentiate these approximations, and verify that they give accurate solutions of (7.4)).

Turning to the nonlinear problem, our goal is to solve

$$\begin{aligned} &u''(x) + u(x) + kg(u(x)) = \mu \sin x + e(x), \quad \text{for } x \in (0, \pi), \\ &u(0) = u(\pi) = 0, \end{aligned} \tag{7.5}$$

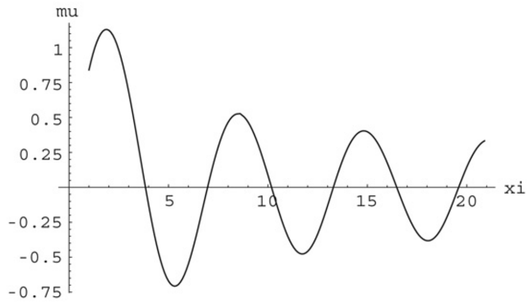

Fig. 1. Solution curve for the problem (7.8).

for any constant $\mu = \mu^*$ and any $e(x) \in L^2(0, \pi)$, satisfying $\int_0^\pi e(x) \sin x \, dx = 0$. The constant k is fixed, satisfying $k \le \frac{3}{M}$ (see Lemma 3.1). According to the Theorem 4.1, for any $\xi \in R$ there exists a solution pair $(\mu, u(x))$ of (7.5). That is for each ξ we wish to compute a pair $(\mu, v(x))$, satisfying

$$u''(x) + u(x) + kg(u(x)) = \mu \sin x + e(x), \quad \text{for } x \in (0, \pi), \tag{7.6}$$
$$u(0) = u(\pi) = 0, \quad \frac{2}{\pi} \int_0^\pi u(x) \sin x \, dx = \xi.$$

To solve (7.6), we apply Newton's method, which involves solving a sequence of linear problems of type (7.1) above

$$u''_{n+1}(x) + u_{n+1}(x) + kg(u_n(x)) + kg'(u_n(x))(u_{n+1} - u_n) = \mu_{n+1} \sin x + e(x), \tag{7.7}$$
$$u_{n+1}(0) = u_{n+1}(\pi) = 0, \quad \frac{2}{\pi} \int_0^\pi u_{n+1}(x) \sin x \, dx = \xi,$$

starting with some (μ_0, u_0).

To summarize, one starts the solution process with some fixed ξ, say $\xi = 0$. To compute the corresponding pair $(\mu, v(x))$, we use Newton's iterations (7.7), starting with some initial guess $(\mu_0, v_0(x))$ (or alternatively $(\mu, v(x))$ may be computed by continuation in k, starting with $k = 0$, where the solution can be written down through Fourier series). Once we have a solution pair $(\mu(\xi), v(x, \xi))$, we proceed to compute the solution pair at $\xi + \Delta\xi$, by using $(\mu(\xi), v(x, \xi))$ as initial guess for Newton's scheme (7.7). To obtain a solution of the problem (7.5) at a fixed $\mu = \mu^*$, just select a ξ^*, so that $\mu^* = \mu(\xi^*)$, and then $u(x) \equiv v(x, \xi^*)$ is a desired solution.

Example. We have numerically solved the problem

$$u''(x) + u(x) + \sin(u(x)) = \mu \sin x + x - \pi/2, \quad \text{for } x \in (0, \pi), \tag{7.8}$$
$$u(0) = u(\pi) = 0.$$

We look for solution of the form $u(x) = \xi \sin x + v(x)$. We had started with $\xi = 1$, and calculated the solution at $\xi = 1$, using the scheme (7.7) with the initial guess $u_0(x) = 1 + x$, and then continued in ξ, as described above. In Fig. 1 we plot μ versus ξ. Our computations suggest that the problem has infinitely many solutions at $\mu = 0$, similarly to the result we proved above for the Neumann problem. We see that the roots of $\mu(\xi)$ are about equally spaced, while oscillations are dumped, with a picture similar to that of Bessel's functions. (Compare with the corresponding Neumann problem, where $\mu(\xi)$ is 2π-periodic.)

We conclude this example with a remark on our *Mathematica* program. When we solve the linear problem (7.1), by using formulas (7.2) and (7.3), *Mathematica* returns the answer as an interpolating function. When we use Newton's iterations (7.7) to solve the nonlinear problem (7.8) one needs to solve a linear problem, whose coefficients are interpolating functions. It seems that *Mathematica* is trying to produce a highly accurate solution at this point, and it becomes too slow. Instead, each time we solved linear problem, we evaluated its solution on a mesh (dividing the interval $(0, \pi)$ into 200 pieces), and then used **Interpolation** command to reconstruct the solution. As a result, our program computes quite accurate solutions, reasonably fast. (Apparently, *Mathematica* uses way more than 200 points for interpolation.)

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