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Stability and Morse Indices of Solutions for Two Classes of Semilinear Problems

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Abstract

We study stability and exact multiplicity of positive solutions for the Dirichlet problem

$$\Delta u + \lambda f(u) = 0 \quad \text{in } |x| < 1, \quad u = 0 \text{ on } |x| = 1,$$

for large values of the parameter λ. Under our assumptions all solutions lie on a unique curve of solutions, with upper branch tending to the larger root of $f(u)$, and lower branch tending to zero, as $\lambda \to \infty$. We show that solutions on the upper branch are stable for large λ, considerably extending the earlier results of R. Gardner and L. Peletier [4]. For the lower branch, solutions were proved unstable in the same paper [4]. In the case of one space dimension we extend that result.

In another direction, we use global analysis to study the global curve of positive solutions of a class of Dirichlet problems on balls in R^n with explicit dependence on $r = |x|$

$$\Delta u - \lambda a(r)u + b(r)u^p = 0 \quad r \in (0,1), \;\; u = 0 \text{ when } r = 1,$$

for subcritical constants $p > 1$.

Key words: Stability and uniqueness of solutions.

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1 Introduction

In this paper we study in two directions positive solutions of semilinear Dirichlet problems on balls in R^n. The first direction is the study of exact multiplicity and stability of positive solutions of the problem

$$(1.1) \qquad \Delta u + \lambda f(u) = 0 \quad \text{in } |x| < 1, \quad u = 0 \text{ on } |x| = 1.$$

for the large values of the positive parameter λ. Here $x \in R^n$, $n \geq 1$, and we assumed without restricting generality the radius of the ball to be one. We assume that $f(u) \in C^2(\bar{R}_+)$, and it has the following properties

$$(1.2) \qquad f(0) = f(b) = f(c) = 0 \text{ for some constants } 0 < b < c,$$

$$(1.3) \qquad \begin{aligned} f(u) < 0 \text{ for } u \in (0,b) \cup (c, \infty), \\ f(u) > 0 \text{ for } u \in (b,c), \end{aligned}$$

$$(1.4) \qquad \int_0^c f(u)du > 0.$$

We shall denote by $\gamma \in (b,c)$ the point where $\int_0^\gamma f(u)\,du = 0$.

In view of the classical results of B. Gidas, W.-M. Ni and L. Nirenberg any positive solution of (1.1) is radially symmetric, i.e. $u = u(r)$, where $r = |x|$. Using the techniques of bifurcation theory one shows that all positive solutions of (1.1) lie on a unique curve of solutions, with the upper branch tending to c, and the lower branch tending to zero for $r \neq 0$, as $\lambda \to \infty$, see P. Korman, T. Ouyang and Y. Li [9] or [8], [15]. Under the above general conditions on $f(u)$ the solution curve may have many turns, thus producing multiple solutions at some λ's. However, it is natural to expect that for large λ both branches will stop turning, and we will have exactly two solutions. Similarly one expects that solutions on the upper branch are stable, while the ones on the lower branch are unstable for large λ. We recall the notion of stability next. For any solution u of (1.1) we consider an eigenvalue problem

$$(1.5) \qquad \Delta w + \lambda f'(u)w + \mu w = 0 \quad \text{in } |x| < 1, \quad w = 0 \text{ on } |x| = 1.$$

The Morse index of u is defined to be the number of negative eigenvalues of (1.5). Solution u is called stable if all eigenvalues of (1.5) are positive (and so the Morse index is zero).

The questions of stability and uniqueness at large λ were studied by R. Gardner and L.A. Peletier [4]. Assuming that

$$\frac{f(u)}{u-\gamma} \quad \text{is non-increasing for all } u > \gamma, \tag{1.6}$$

and $n \geq 2$ they proved the above mentioned stability properties of both branches for large λ. (The set up in [4] was a little more general. Also, surprisingly, for $n = 1$ a more restrictive condition was needed: $\frac{f(u)}{u-b}$ is non-increasing for all $u > b$.)

In Section 2 we prove stability of the upper branch, replacing the condition (1.6) by a more natural and local condition: there is an $\alpha > 0$, such that $f'(u) < 0$ for all $u \in (c-\alpha, c)$. Moreover, our proof appears to be considerably simpler than the one in [4]. Stability of the upper branch implies in particular uniqueness of solutions with $u(0)$ close to c, for large values of the parameter λ. The uniqueness part was proved previously by E.N. Dancer [2], [3] even for more general domains, although these papers had a slight extra assumption of technical nature. We outline the idea of our proof next. For large λ solutions on the upper branch are close to c except near the boundary $r = 1$. Hence $f'(u) < 0$ except near $r = 1$. If we now assume that solution is not stable, i.e. there is a negative eigenvalue μ of (1.5), then one sees that the corresponding eigenfunction can be assume to be positive and increasing except near $r = 1$. If we normalize this eigenfunction $\int_0^1 w^2(r,\lambda)\,dr = 1$, then its maximum value must be assumed increasingly close to $r = 1$, and grow with λ. We then derive some differential inequalities, which show that this is impossible. This line of reasoning was developed in [9].

The lower branch turned out to be harder to handle. Instead of proving instability of lower branch directly, we had to settle on proving that for large λ solutions on the lower branch are non-degenerate (i.e. $\mu = 0$ is not an eigenvalue of (1.5)), and then use the degree theory and stability of the upper branch to show that solutions on the lower branch have index -1, i.e. they are unstable. In one dimensional case we were able to carry this out, replacing the condition of non-increasing $\frac{f(u)}{u-b}$ by a one similar to what we had before: there is an $\alpha > 0$, such that $f'(u) < 0$ for all $u \in (0, \alpha)$. For the case of $n > 1$ we state a conjecture, whose proof would generalize the result of [4]. We remark that E.N. Dancer [2], [2] proved uniqueness of solutions on the lower branch for a class of symmetric domains under the same assumption plus a small technical assumption. He also proved non-degeneracy of the lower branch, but under a stronger condition (see assumption (iii) of the Theorem 2 in [3]).

In another direction, we study the global curve of positive solutions of a class of Dirichlet problems on balls in R^n, with explicit dependence on $r = |x|$

$$\Delta u - \lambda a(r)u + b(r)u^p = 0 \quad r \in (0,1), \;\; u = 0 \text{ when } r = 1, \tag{1.7}$$

for subcritical constants $p < \frac{n+2}{n-2}$ for $n > 2$, and $p < \infty$ for $n = 1, 2$. We show that the problem (1.7) has a smooth curve of positive solutions, which extends without any turns for all $\lambda \geq 0$. Allowing coefficients to depend on r considerably complicates the problem, and so the problems of this type are relatively little studied. We recall that various uniqueness results for the problem (1.7) have been given in the well-known paper of M.K. Kwong and Y. Li [11].

Throughout the paper we consider only the classical solutions. We denote by c various positive constants independent of λ. Sometimes we will use subscripts if we wish to distinguish such a positive constant. We shall use both notations $u'(r)$ and $u_r(r)$ for derivatives, and mix them to make the proofs more transparent.

2 Stability of solutions for large values of the parameter λ

Positive solutions of our problem satisfy

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \;\; r \in (0,1), \;\; u'(0) = u(1) = 0. \tag{2.1}$$

Theorem 2.1 *Assume that $f(u)$ satisfies conditions (1.2), (1.3) and (1.4), and there an $\alpha > 0$ such that $f'(u) < 0$ for $u \in (c - \alpha, c)$. Then solutions on the branch tending to c are stable for large λ.*

Proof: If $u(r, \lambda)$ is not stable, we can find $\mu \geq 0$ and $w(r, \lambda)$ such that

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = \mu w \;\; r \in (0,1), \;\; w'(0) = w(1) = 0. \tag{2.2}$$

Define $\eta = \eta(\lambda)$ by $u(\eta) = c - \alpha$. Clearly, $\eta(\lambda) \to 1$ as $\lambda \to \infty$. We have by our conditions on $f(u)$

$$f'(u) < 0 \quad \text{for } 0 \leq r < \eta. \tag{2.3}$$

Using (2.3), we see from (2.2) that $w(r)$ cannot have positive maximums or negative minimums on the interval $(0, \eta)$. It follows that $w(r)$ cannot vanish on $(0, \eta]$, and if w is positive (negative) on this interval, it is increasing (decreasing) function. We normalize $\int_0^1 w^2(r, \lambda)\, dr = 1$. Multiplying the PDE version of (2.2) by w and integrating, we conclude

$$-\int_{r<1} |\nabla w|^2 \, dx + \lambda \int_{r<1} f'(u) w^2 \, dx - \mu = 0. \tag{2.4}$$

From here we see that there exists a constant $c > 0$, independent of λ, so that

$$\mu < c\lambda \quad \text{for all } \lambda \text{ large.} \tag{2.5}$$

We can now outline the idea of the proof of the theorem. In (2.4) the first and the third terms are negative, and the integrand in the second term is negative, except on the interval $(\eta, 1)$. It follows that $w(r)$ has to concentrate on this interval. But we shall show that $|w'|$ cannot grow in λ at $r = 1$, and at any other root of $w(r)$ in $(\eta, 1)$, while the length of the interval $(\eta, 1)$ decreases with λ. This will contradict $\|w\|_{L^2} = 1$.

By (2.3) $w(\eta) \neq 0$, so that we may and will assume that $w(\eta) > 0$. By the above remarks $w(0) > 0$, and $w(r)$ is increasing on $(0, \eta)$. Let $\theta \in (\eta, 1)$ be the smallest root of w. Since $u(x, \lambda) \to c$, and $u_x(x, \lambda) < 0$, we can find a constant A independent of λ, and a point $\xi = \xi(\lambda)$ near $r = 0$ (say on the interval $(\frac{1}{8}, \frac{1}{4})$), such that $|u''(r)| < A$. Multiplying the equation (3.3) by $r^{n-1}u'$, and integrating over $(0, r)$, we conclude

$$\frac{1}{2} r^{2(n-1)} u'^2(r) + \lambda \int_{u(0)}^{u(r)} r^{2(n-1)} f(u) \, du = 0. \tag{2.6}$$

Define the point $\beta \in (0, \eta)$ by $u(\beta) = b$. Since $\int_{u(r)}^{u(0)} r^{2(n-1)} f(u)\, du < b^{2(n-1)} \int_{u(r)}^{u(0)} f(u)\, du$, it follows there is a constant $c_1 > 0$, such that

$$|u'(r)| \le c_1 \sqrt{\lambda} \quad \text{for any } r \in [0, 1]. \tag{2.7}$$

Similarly, we see from (2.6) that

$$|u'(r)| \ge c_2 \sqrt{\lambda} \quad \text{for any } r \in [\eta, 1], \tag{2.8}$$

for some constant $c_2 > 0$.

Differentiate the equation (2.1)

$$u_r'' + \frac{n-1}{r} u_r' + \lambda f(u) - \frac{n-1}{r^2} u' = 0. \tag{2.9}$$

From the equations (2.9) and (2.2) we conclude

$$\left[r^{n-1}\left(u''w - u'w'\right)\right]' - (n-1) r^{n-3} u'w + \mu r^{n-1} u'w = 0.$$

Integrate this formula over (ξ, θ), obtaining

$$\begin{aligned} -\theta^{n-1} u'(\theta) w'(\theta) + \xi^{n-1} u'(\xi) w'(\xi) - \xi^{n-1} u''(\xi) w(\xi) \\ -(n-1) \textstyle\int_\xi^\theta r^{n-3} u'w \, dr + \mu \int_\xi^\theta r^{n-1} u'w \, dr = 0. \end{aligned} \tag{2.10}$$

In (2.10) the first, second and the fifth terms are negative. ($w(r)$ was assumed to be positive before its first root at θ.) We can therefore estimate the absolute value of the first term by the sum of absolute values of the third and the fourth terms:

$$\theta^{n-1} u'(\theta) w'(\theta) < \xi^{n-1} |u''(\xi)| w(\xi) + (n-1) \int_\xi^\theta r^{n-3} |u'| w \, dr. \tag{2.11}$$

The first term on the right is bounded by $Aw(\xi)$, and hence it is bounded uniformly in λ. Indeed, since w is increasing on (ξ, η), if $w(\xi)$ were to become large at some λ, we would have $w(r)$ large over the entire interval (ξ, η), contradicting the normalization of w. The second term on the right in (2.11) we estimate as follows, using (2.7),

$$\int_\xi^\theta r^{n-3} |u'| w \, dr \le 8 \left(\int_0^1 u_r^2 \, dr \right)^{\frac{1}{2}} \left(\int_0^1 w^2 \, dr \right)^{\frac{1}{2}} \le c\sqrt{\lambda},$$

for some constant $c > 0$. We conclude from (2.11) and (2.8) that for some constant $c > 0$ we have

$$|w'(\theta)| < c \quad \text{uniformly in } \lambda. \tag{2.12}$$

In view of (2.8) we also have

$$\theta - \eta = O\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{as } \lambda \to \infty. \tag{2.13}$$

Case (i): w has only one local maximum on (η, θ). Let $\bar{\eta}$ be the point of global maximum of w on (η, θ). On $(\bar{\eta}, \theta)$ we have $w' \le 0$. On $(\bar{\eta}, \theta)$

we introduce a new variable $t = \theta - r$. In view of (2.5) we then have for $0 < t \le \theta - \bar{\eta} < c\frac{1}{\sqrt{\lambda}}$

$$(2.14)\ w''(t) = w''(r) = -\frac{n-1}{r}w'(r) - \lambda f'(u)w + \mu w \le cw + \bar{c}w'(t),$$

for some constant $c > 0$ independent of λ, and $\bar{c} = \frac{n-1}{\bar{\eta}} > 0$. We also have

$$(2.15)\qquad w(0) = 0,$$

and in view of (2.12)

$$(2.16)\qquad w'(0) \le c.$$

We now integrate twice (2.14), using the boundary conditions (2.15) and (2.16). We have for $0 < t < c\frac{1}{\sqrt{\lambda}}$

$$(2.17)\qquad w(t) \le c\lambda \textstyle\int_0^t (t-s)w(s)\,ds + w'(0)t + \bar{c}\int_0^t w(s)\,ds$$

$$(2.18)\qquad \le c\sqrt{\lambda}\textstyle\int_0^t w(s)\,ds + c\frac{1}{\sqrt{\lambda}}.$$

Applying the Gronwall's inequality for $0 < t < c\frac{1}{\sqrt{\lambda}}$,

$$w(t) \le c\frac{1}{\sqrt{\lambda}}e^{c\sqrt{\lambda}t} \le c\frac{1}{\sqrt{\lambda}},$$

i.e.

$$(2.19)\qquad w(\bar{\eta}) < c\frac{1}{\sqrt{\lambda}}.$$

This shows that the maximal value on (η, θ) decreases in λ.

Case (ii): $w(r)$ has more than one local maximum on (η, θ). As we continue the function $w(r)$ leftwards from $\bar{\eta}$, its largest point of local maximum on (η, θ), we shall encounter a point of local minimum θ_1, and then the next point of local maximum at $\bar{\eta}_1$, $\theta < \bar{\eta}_1 < \theta_1$. We now set $t = \theta_1 - r$ on the interval $(\bar{\eta}_1, \theta_1)$. As before inequality (2.14) holds, and this time by what we have just proved $w(0) \le c\frac{1}{\sqrt{\lambda}}$, and also $w'(0) = 0$. Integrating (2.14) we obtain (with $\bar{c}_1 = \frac{n-1}{\bar{\eta}_1}$)

$$w(t) \le c\lambda \textstyle\int_0^t (t-s)w(s)\,ds + w(0) + \bar{c}_1\int_0^t w(s)\,ds - \bar{c}_1 w(0)t$$
$$\le c\sqrt{\lambda}\textstyle\int_0^t w(s)\,ds + c\frac{1}{\sqrt{\lambda}},$$

and proceeding as before we conclude that $w(\bar{\eta}_1) < c\dfrac{1}{\sqrt{\lambda}}$. Continuing this way to other possible points of local maximum, we conclude that the global maximum of w on (η, θ) is bounded by $c\frac{1}{\sqrt{\lambda}}$, and the same is true for the global maximum of w on $(0, \theta)$, since on $(0, \eta)$ w has no points of maximum.

If now $\theta = 1$, i.e. w has only one root on $(\eta, 1]$, we obtain a contradiction, since L^2 norm of w will tend to zero as $\lambda \to \infty$ (recall that $||w||_{L^2} = 1$). Otherwise let $\bar{\theta}$ be the next root of $w(r)$, $\theta < \bar{\theta} \le 1$. We may assume that $w > 0$ on $(\theta, \bar{\theta})$. We repeat the argument leading to (2.12), by using $\bar{\theta}$ in place θ, and θ in place of ξ. We conclude that $|w'(\bar{\theta})|$ is uniformly bounded. Then arguing as before we estimate $w(r) \le c\dfrac{1}{\sqrt{\lambda}}$ on $(\theta, \bar{\theta})$. Repeating this argument between any two roots of $w(r)$, we obtain a contradiction with $||w||_{L^2} = 1$.

We now turn to the one-dimensional problem

$$u'' + \lambda f(u) = 0 \quad r \in (-1, 1), \quad u(-1) = u(1) = 0. \tag{2.20}$$

We shall need the corresponding linearized problem

$$w'' + \lambda f'(u)w = 0 \quad r \in (-1, 1), \quad w(-1) = w(1) = 0. \tag{2.21}$$

It is well known that the positive solution of (2.20) is symmetric with respect with $x = 0$, and then the same is true for any non-trivial solution of (2.21). (As we recall below any non-trivial solution of (2.21) is positive. If $w(x)$ failed to be an even function, $w(-x)$ would be a linearly independent solution of the same problem, which is impossible.) We shall therefore mostly work on the half interval $(0, 1)$.

Theorem 2.2 *Assume that $f(u)$ satisfies conditions (1.2), (1.3) and (1.4), and there is an $\alpha \in (0, b)$ such that*

$$f'(u) < 0 \quad \text{for } u \in (0, \alpha). \tag{2.22}$$

Then for large λ solutions on the branch tending to zero are non-degenerate, i.e. the problem (2.21) has only the trivial solution.

Proof: It is well-known that any non-trivial solution of (2.21) does not vanish inside $(-1, 1)$, and so it can be assumed to be positive. (Notice that w and u' satisfy the same equation, and use the Sturm comparison theorem.) Assume contrary to what we want to prove that the problem (2.21) has non-trivial solutions $w(x) = w(x, \lambda)$, for some arbitrarily large λ, which we normalize so that $\int_0^1 w^2(x,\lambda)\,dx = 1$. Define $\eta = \eta(\lambda) \in (0,1)$ by $u(\eta,\lambda) = \alpha$. Clearly, $\eta = o(1)$ as $\lambda \to \infty$, since $u(x,\lambda)$ tends to zero (we shall derive a more precise asymptotic relation below). Using condition (2.22) and equation (2.21), we conclude that

$$w''(x) > 0 \quad \text{on } (\eta, 1).$$

It follows that $w(x)$ is decreasing on $(\eta,1)$, and hence $|w'(1)|$ cannot get large, in view of normalization of w. I.e. we can find a constant c independent of λ so that

$$|w'(1)| < c. \tag{2.23}$$

Differentiating the equation (2.20), and combining the result with (2.21), we easily conclude that for all $x \in [0,1]$

$$u''w - u'w' = \text{constant} = -u'(1)w'(1) = o(1) \;\text{ as } \lambda \to \infty, \tag{2.24}$$

using (2.23) and that $|u'(1)| = o(1)$ (since $u(x,\lambda)$ is a unimodular function tending to zero for all $x \neq 0$). Evaluating (2.24) at $x = 0$, and using the equation (2.20), we see that

$$-\lambda f(u(0))w(0) = o(1) \;\text{ as } \lambda \to \infty. \tag{2.25}$$

On the lower branch $u(0,\lambda)$ is decreasing, and hence it is bounded away from the larger root of $f(u)$. Since $u(0,\lambda)$ is also bounded away from the lower root of $f(u)$ (as $\int_0^{u(0)} f(u)\,du > 0$), it follows that $f(u(0,\lambda))$ is bounded away from zero, and then we conclude from (2.25)

$$w(0) = o(\frac{1}{\lambda}) \quad \text{as } \lambda \to \infty. \tag{2.26}$$

We claim that

$$\eta = O(\frac{1}{\sqrt{\lambda}}) \quad \text{as } \lambda \to \infty. \tag{2.27}$$

Assuming this for the moment, we show how to complete the proof of the theorem. Using the equation (2.21) and (2.26), we have

$$w'' \le c\lambda w \;\text{ on } (0,\eta), \;\; w(0) = o(\frac{1}{\lambda}), \;\; w'(0) = 0.$$

We then have

$$w(x) = \textstyle\int_0^x (x-t)w''(t)\,dt + w(0) \tag{2.28}$$
$$\le c\lambda \textstyle\int_0^x (x-t)w(t)\,dt + o(\frac{1}{\lambda}).$$

Since by the claim (2.27) we have on $(0,\eta)$: $x - t < x < c\dfrac{1}{\sqrt{\lambda}}$, we conclude from (2.28)

$$w(x) \le c\sqrt{\lambda}\int_0^x w\,dt + o(\frac{1}{\lambda}).$$

By Gronwall's inequality we then have

$$w(x) \le o(\frac{1}{\lambda})e^{c\sqrt{\lambda}x} \le o(\frac{1}{\lambda}) \quad \text{for } x \in (0,\eta). \tag{2.29}$$

Combining this with the fact that $w(x)$ is decreasing on $(\eta, 1)$, we conclude that $\int_0^1 w^2\,dx$ goes to zero as $\lambda \to \infty$, contradicting our normalization, $\int_0^1 w^2\,dx = 1$.

It remains to prove the claim (2.27). Recall that we defined γ by $\int_0^\gamma f(u)\,du = 0$. Clearly we can find a β close to γ, $\beta < \gamma$ so that $\int_\alpha^\beta f(u)\,du > c_0 > 0$. Define $\xi \in (0,\eta)$ by $u(\xi) = \beta$. Since $u(0) > \gamma$, we then have for any $u \in (\alpha, b)$

$$\int_u^{u(0)} f(u)\,du > \int_\alpha^\beta f(u)\,du > c_0 > 0, \tag{2.30}$$

while for any $u \in (b, \beta)$

$$\int_u^{u(0)} f(u)\,du > \int_\beta^\gamma f(u)\,du > c_1 > 0. \tag{2.31}$$

Combining (2.30) and (2.31) we conclude that for any $u \in (\alpha, \beta)$

$$\int_u^{u(0)} f(u)\,du > c_2 > 0. \tag{2.32}$$

We now multiply our equation (2.20) by u' and integrate over $(0,x)$, obtaining

$$u'^2(x) = 2\lambda\int_{u(x)}^{u(0)} f(u)\,du. \tag{2.33}$$

In view of (2.32) we have

$$|u'(x)| \ge \sqrt{2}c_2\sqrt{\lambda} \quad \text{for any } x \in (\xi,\eta). \tag{2.34}$$

Since the function $u(x)$ travels the distance from α to β with the speed estimated from below by (2.34), we can estimate the "time"

$$\eta - \xi = O(\frac{1}{\sqrt{\lambda}}) \quad \text{as } \lambda \to \infty. \tag{2.35}$$

On the interval $(0, \xi)$ the solution $u(x)$ is bounded away from both b and c, and so $f(u)$ is bounded from zero, and hence

$$|u''(x)| \geq c\lambda \quad \text{for } x \in (0, \xi). \tag{2.36}$$

In the formula

$$u(x) = \int_0^x (x-t)u''(t)\,dt + u(0),$$

we set $x = \xi$, and estimate, using (2.36)

$$|\beta - u(0)| \geq c\lambda \frac{\xi^2}{2}. \tag{2.37}$$

Since $|\beta - u(0)| = u(0) - \beta < c - \beta$, it follows from (2.37) that

$$\xi = O(\frac{1}{\sqrt{\lambda}}) \quad \text{as } \lambda \to \infty. \tag{2.38}$$

Combining the estimates (2.38) and (2.35), we conclude the proof of the claim (2.27), and of the theorem.

As an application we have the following result for one-dimensional problems.

Theorem 2.3 *Assume that $f(u)$ satisfies the conditions of both Theorem 2.1 and 2.2. Then the problem (2.20) has exactly two solutions for λ sufficiently large. Moreover the larger solution (tending to c) is stable, and the lower one (tending to zero for $x \neq 0$) is unstable, with index -1.*

Proof: We only need to discuss the lower branch, since the upper branch is covered by Theorem 2.1. Since there is only one solution curve, and the lower branch does not turn for large λ, it follows that there are exactly two solutions for large λ. The stable larger solution has index 1, and by a standard use of degree theory, see [4], it follows that the smaller one has index -1, and is therefore unstable.

We state the following conjecture.

Conjecture. Consider the problem (2.1) with $f(u)$ satisfying the conditions (1.2), (1.3) and (1.4). Assume that any non-trivial solution of the corresponding linearized problem

$$(2.39) \quad w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0 \;\; r \in (0,1), \;\; w'(0) = w(1) = 0$$

is of one sign. Then for large λ solutions on the branch tending to zero are non-degenerate, with Morse index equal to -1. Moreover, the problem (2.1) has exactly two positive solutions.

R. Gardner and L.A. Peletier [4] have proved the above statement assuming that $\frac{f(u)}{u-\gamma}$ is non-increasing for $u > \gamma$. Using the work of M.K. Kwong and L. Zhang [12] it was shown later that this condition implies that any solution of (2.39) is of one sign, see e.g. J. Wei [15]. The value of proving the above conjecture is that there are other conditions available for $w(r) > 0$, see e.g. [8].

3 A global solution curve for a class of nonlinearities with an explicit dependence on r

We study existence and uniqueness of positive classical solution for the problem

$$(3.1) \qquad \Delta u + f(r,u) = 0 \quad r \in (0,1), \;\; u = 0 \text{ when } r = 1.$$

It follows from [5] that under the condition

$$(3.2) \qquad f_r(r,u) \le 0 \quad \text{for all } r \in [0,1) \text{ and } u > 0$$

any positive solution of (3.1) is radially symmetric, and hence it satisfies

$$(3.3) \quad u''(r) + \frac{n-1}{r} u'(r) + f(r,u) = 0 \quad r \in (0,1), \;\; u'(0) = u(1) = 0.$$

We consider $f(r,u) = -a(r)u + b(r)u^p$, where $1 < p < p^*$, with $p^* = \frac{n+2}{n-2}$ for $n > 2$, and $p^* = \infty$ for $n = 1, 2$. The functions $a(r), b(r) \in C^2([0,1])$ satisfy

$$(3.4) \qquad a(r) > 0, \;\; a'(r) \ge 0, \;\; a''(r) \ge 0 \quad \text{for all } r \in [0,1],$$

$$(3.5) \qquad b(r) > 0, \;\; b'(r) \le 0, \;\; b''(r) \le 0 \quad \text{for all } r \in [0,1].$$

Notice that these conditions imply (3.2), and so all solutions of (3.1) are radially symmetric. In [7] we considered the same problem in case of constant $a(r)$ and $b(r)$.

We shall need the following form of Sturm's comparison theorem, taken from [12]. We consider a differential operator, defined on functions $u = u(r)$ of class C^2

$$L[u] \equiv p(r)u'' + q(r)u' + z(r)u,$$

with continuous coefficients $p(r)$, $q(r)$ and $z(r)$.

Lemma 3.1 *Assume that on some interval $I \subseteq (-\infty, \infty)$ we have $p(r) > 0$ and*

$$L[u] \geq 0,$$

while

$$L[v] \leq 0,$$

with at least one of the inequalities being strict on a set of positive measure. Then the function $v(r)$ oscillates faster than $u(r)$, provided that they are both non-negative. More precisely, assume that $u(\alpha) = u(\beta) = 0$ for some $\alpha, \beta \in I$, $u(r) > 0$ on (α, β), while $v(\alpha) \geq 0$. Then $v(r)$ must vanish on (α, β).

Crucial role will be played by the linearized problem for (3.1)

$$\begin{aligned} L[w] \equiv w''(r) + \tfrac{n-1}{r}w'(r) + f_u(r,u)w = 0 \quad & r \in (0,1), \\ w'(0) = w(1) = 0. & \end{aligned} \tag{3.6}$$

Lemma 3.2 *Under the conditions (3.4) and (3.5) any non-trivial solution of (3.6) cannot vanish exactly once on $(0,1)$.*

Proof: Let $u(r)$ be a degenerate solution of (3.3), i.e. (3.6) has a non-trivial solution $w(r)$. We shall use a test function $v = ru_r + \mu u$ with a constant μ to be specified. One easily checks that $v(r)$ satisfies the equation

$$L[v] = \mu\,(f_u u - f) - 2f - rf_r. \tag{3.7}$$

The sign of the function $L[v]$ is governed by the function $\mu = j(r)$, where

$$j(r) = \frac{2f}{f_u u - f} + \frac{rf_r}{f_u u - f}. \tag{3.8}$$

Indeed, $L[v] > 0$ (< 0), provided that $j(r) < \mu$ ($> \mu$). (Observe that $f_u u - f > 0$.) For our nonlinearity $f(r,u)$ one easily computes

$$(p-1)j(r) = -2\frac{a(r)}{b(r)}u^{1-p} + 2 - \frac{ra'(r)}{b(r)}u^{1-p} + \frac{rb'(r)}{b(r)}. \tag{3.9}$$

One checks that under our conditions $j'(r) < 0$ for all $r \in (0,1)$, and hence $j(r)$ is a decreasing function. We claim next that

$$f(0, u(0)) > 0. \tag{3.10}$$

Indeed, writing $f(r,u) = u\left(-a(r) + b(r)u^{p-1}\right)$, we have in the brackets a decreasing function. If inequality (3.10) was violated, we would have $f(r,u(r)) \leq 0$ for all r. Then multiplying the PDE version of our equation (3.3) by u and integrating over the unit ball, we obtain a contradiction. It follows that

$$j(0) = \frac{2f(0,u(0))}{(p-1)b(0)u^p(0)} > 0.$$

Since $j(r)$ tends to $-\infty$ near $r = 1$, it follows that $j(r)$ changes sign exactly once on $(0,1)$, say at $r = \bar{r}$. Clearly, $j(r) > 0$ for $r \in [0, \bar{r})$, and $j(r) < 0$ for $r \in (\bar{r}, 1)$.

Assume contrary to what we wish to prove that the function $w(r)$ vanishes exactly once inside $(0,1)$, and denote by τ the root of $w(r)$. We claim that

$$\tau < \bar{r}. \tag{3.11}$$

Indeed, assuming that on the contrary $\tau \geq \bar{r}$, we select a test function $v(r) = ru_r$, i.e. we are choosing $\mu = 0$. Then $v(r) < 0$ for all r, while $L[v] > 0$ for $r > \bar{r}$ (from (3.7) and (3.8) we see that $L[v]$ equals $j(r)$ times a negative factor). It follows that v oscillates faster than w on the interval $(\tau, 1)$, and hence it has to vanish on this interval, a contradiction, which proves (3.11). We now fix our test function $v(r) = ru_r + \mu_0 u$, by choosing $\mu_0 = j(\tau) > 0$. On $(0, \tau)$ we have $j(r) > \mu_0$, and hence $L[v] < 0$. Similarly on $(\tau, 1)$ we have $L[v] > 0$.

Arguing similarly to M.K. Kwong and L. Zhang [12], we now examine the function $v(r)$. On the interval $(0, \tau)$ $v(r)$ starts positive, and while it stays positive it oscillates faster than $w(r)$. Hence v must have its first root at some $\sigma < \tau$. After passing σ, v becomes negative, and hence it oscillates slower that w. Hence v can have no roots on $(\sigma, \tau]$, and so $v(\tau) < 0$. On $(\tau, 1)$ v oscillates faster than w, while it stays negative. Since $w(\tau) = w(1) = 0$, $v(r)$ cannot stay negative on $(\tau, 1)$, it has to have its second root on that interval, say at ξ. Across ξ $v(r)$ has to change sign, since otherwise for the equation $L[u] = 0$ we would have a subsolution $v(r)$ touching from below a zero solution, which is impossible. After ξ $v(r)$ is positive, it oscillates slower than w, and so v cannot have any more roots. Hence $v(1) > 0$. On the other hand, $v(1) = u_r(1) < 0$, a contradiction, proving that $w(r)$ cannot vanish exactly once inside $(0, 1)$.

Remark. The lemma still holds if $a(r) \equiv 0$. This time $j(r)$ does not necessarily change sign. If it does not, i.e. $j(r)$ stays positive on $(0, 1)$, we set $\mu_0 = j(\tau)$, and proceed exactly as before.

Lemma 3.3 *Under the conditions (3.4) and (3.5) any non-trivial solution the problem (3.6) has to vanish at least once.*

Proof: Since $f_u u - f > 0$ for all $u > 0$ and $r \in (0, 1)$, this follows by Sturm comparison theorem, when comparing the equations (3.3) and (3.6).

Combining Lemmas 3.2 and 3.3, we conclude that any non-trivial solution of (3.6) has to vanish at least twice.

Lemma 3.4 *Assume* $1 < p < p^*$. *Under the conditions (3.5) the problem*

$$\Delta u + b(r)u^p = 0, \quad r \in (0, 1), \quad u = 0 \ \ \textit{for } r = 1 \tag{3.12}$$

has a positive solution, with the Morse index equal to one.

Proof: In case $b(r)$ is equal to a positive constant this lemma is known, see e.g. Korman [7]. We now consider a family of problems

$$\begin{gathered} \Delta u + [\nu b(r) + (1 - \nu)b(1)]\,|u|^p = 0, \quad r \in (0, 1), \\ u = 0 \ \text{ for } r = 1, \end{gathered} \tag{3.13}$$

depending on a parameter ν, $0 \leq \nu \leq 1$. By maximum principle any non-trivial solution of (3.13) is positive, and hence radially symmetric. When $\nu = 0$ we have a unique positive solution, whose Morse index is one, i.e. the corresponding linearized problem has exactly one negative eigenvalue. It is also known that this solution is non-degenerate, i.e. the second eigenvalue of the linearized problem is positive, see [13]. As we vary ν, solution of (3.13) varies continuously, and so do the principal eigenvalue and eigenvector of the corresponding linearized problem, until we reach a degenerate solution. At a degenerate solution zero is an eigenvalue of the corresponding linearized problem, and by continuity it has to be either first or second eigenvalue. If it is first, then the corresponding eigenfunction is of one sign, which is impossible by Lemma 3.3. If zero is the second eigenvalue, corresponding eigenfunction vanishes exactly once on $(0,1)$, which is impossible by Lemma 3.2 (see the remark after it). We conclude that no degenerate solutions are encountered, and at $\nu = 1$ we conclude the lemma.

We are now ready for the main result of this section.

Theorem 3.1 *Assume $1 < p < p^*$. Under the conditions (3.4) and (3.5) the problem*

$$\Delta u - \lambda a(r) u + b(r) u^p = 0, \quad r \in (0,1), \; u = 0 \text{ for } r = 1, \tag{3.14}$$

has a smooth curve of positive solutions for all $\lambda \geq 0$. This curve admits no turns, and the Morse index of any solution on this curve is one.

Proof: We begin with the unique positive solution at $\lambda = 0$. Since this solution is non-singular, we can continue it for small $\lambda > 0$ using the implicit function theorem. We claim that solutions on this curve stay non-singular. Indeed, by Lemma 3.4 at $\lambda = 0$, zero lies between the first and second eigenvalues of the corresponding linearized problem. If a singular solution appears on the curve, it means that zero is now an eigenvalue, and it is either the first or the second one. It cannot be the second one, since then the corresponding eigenfunction satisfies the linearized equation (3.6), and it vanishes exactly once on $(0,1)$, which is excluded by Lemma 3.2. If on the other hand, zero is the first or principal eigenvalue, then the nontrivial solution of (3.6) is of one sign, and we may assume that in fact $w(r) > 0$ on $(0,1)$. But this leads to a contradiction by Lemma 3.3.

Since solutions on this curve cannot go to infinity at a finite $\lambda \geq 0$ by a routine modification of the result of B. Gidas and J. Spruck [6], it follows that this curve continues without turns for all $\lambda \geq 0$.

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References

[1] S.B. Angenent, Uniqueness of the solution of a semilinear boundary value problem, *Math. Ann* **272**, 129-138 (1985).

[2] E.N. Dancer, On the uniqueness of the positive solution of a singularly perturbed problem, *Rocky Mountain J. of Math.* **25(3)**, 957-975 (1995).

[3] E.N. Dancer, On positive solutions of some singularly perturbed problems where the nonlinearity changes sign, *Topological Methods in Nonlinear Analysis* **5**, 141-175 (1995).

[4] R. Gardner and L.A. Peletier, The set of positive solutions of semilinear equations in large balls, *Proc. Royal Soc. Edinburgh* **104A**, 53-72 (1986).

[5] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.* **68**, 209-243 (1979).

[6] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. in PDE* **6(8)**, 883-901 (1981).

[7] P. Korman, A global solution curve for a class of semilinear equations, *Differential Equations and Computational Simulations III*, J. Graef, R. Shivaji, B. Soni & J. Zhu, Eds, *EJDE, Conf. 01* 119-127 (1997).

[8] P. Korman, Curves of sign-changing solutions for semilinear equations, Preprint.

[9] P. Korman, Y. Li and T. Ouyang, An exact multiplicity result for a class of semilinear equations, *Commun. PDE.* **22** (3&4), 661-684 (1997).

[10] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n, *Arch. Rational Mech. Anal.* **105**, 243-266 (1989).

[11] M.K. Kwong and Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.* **333(1)**, 339-363, (1992).

[12] M.K. Kwong and L. Zhang, Uniqueness of the positive solutions of $\Delta u + f(u) = 0$ in an annulus, *Differential and Integral Equations* **4**, 583-599 (1991).

[13] M. Ramaswamy and P.N. Srikanth, Symmetry breaking for a class of semilinear elliptic problems, *Trans. Amer. Math. Soc.* **304**, 839-845 (1987).

[14] P.N. Srikanth, Uniqueness of solutions of nonlinear Dirichlet problems, *Differential and Integral Equations* **6(3)**, 663-670 (1993).

[15] J. Wei, Exact multiplicity for some nonlinear elliptic equations in balls. *Proc. Amer. Math. Soc.* **125** (1997), no. 11, 3235–3242.

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