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A Maximum Principle for Fourth Order Ordinary Differential Equations

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Abstract

We present a maximum principle for fourth order ordinary differential equations, based on a new approach involving counting of inflection points. We use our results to compute solutions of nonlinear equations describing static displacements of a uniform beam.

KEY WORDS: Maximum principle, fourth order equations, computation of solutions.

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The purpose of this paper is to prove the following maximum principle.

Theorem 1. Let $u(x)$ be a real-valued function of class C^4 on the interval $[0,\ell]$, which satisfies the inequalities

$$Lu \equiv (a(x)u'')'' \geq 0 \quad \text{for } 0 < x < \ell \tag{1}$$

$$u(0) \geq 0,\ u'(0) \geq 0,\ u(\ell) \geq 0,\ u'(\ell) \leq 0,$$

where $a(x)$ is a real-valued function of class $C^2[0,\ell]$, $a(x) > 0$ on $[0,\ell]$. Then $u(x) \geq 0$ for $0 \leq x \leq \ell$.

Corollary. In the conditions of the above theorem assume that

$$(a(x)u'')'' \geq 0 \quad \text{for } 0 < x < \ell$$
$$u'(0) \geq 0,\ u'(\ell) \leq 0.$$

Then u attains its minimum at either $x = 0$ or $x = \ell$.

This corollary generalizes the maximum principle in S.N. Chow, D.R. Dunninger and A. Lasota [1], who proved it for $a(x) \equiv 1$. Later, J.R. Kuttler [4] noticed that this result follows from the Theorem 1, which he proved for $a(x) \equiv 1$. His proof used the following representation

$$\begin{aligned} u(x) = \int_0^{\ell} k(x,\xi)Lu(\xi)d\xi + p_0(x)u(0) + q_0(x)u'(0) \\ + p_1(x)u(\ell) - q_1(x)u'(\ell), \end{aligned} \tag{2}$$

where $k(x,\xi)$ is the Green's function of the operator L with homogeneous boundary conditions, and the functions p_0, q_0, p_1, q_1 are solutions of the following problems

$$\begin{aligned} &Lp_0 = 0, \ p_0(0) = 1, \ \ p_0'(0) = p_0(\ell) = p_0'(\ell) = 0, \\ &Lq_0 = 0, \ q_0'(0) = 1, \ \ q_0(0) = q_0(\ell) = q_0'(\ell) = 0, \\ &Lp_1 = 0, \ p_1(\ell) = 1, \ \ p_1(0) = p_1'(0) = p_1'(\ell) = 0, \\ &Lq_1 = 0, \ q_1'(\ell) = -1, \ q_1(0) = q_1'(0) = q_1(\ell) = 0. \end{aligned} \tag{3}$$

In case $a(x) \equiv 1$, the functions k, p_0, q_0, p_1, q_1 are relatively simple, so that by inspection one sees that they are all positive in $(0,\ell)$, which proves the Theorem 1.

For a general $a(x)$ the representation (2) is still valid, but is far more complicated, so that a direct proof as above does not appear to be feasible. Indeed, to write down $k(x,\xi)$ we need first to define the following functions

$$y(x) = \int_0^x \frac{dt}{a(t)}, \ v(x) = \int_0^x y(t)dt, \ w(x) = \int_0^x \frac{tdt}{a(t)}, \ z(x) = \int_0^x w(t)dt,$$
$$\Delta = y(\ell)z(\ell) - v(\ell)w(\ell).$$

Then for $x \le \xi$ we have (with $G(\xi,x) = G(x,\xi)$; see [2, p. 84])

$$\begin{aligned} k(x,\xi) = \frac{1}{\Delta}\{[-z(\xi)w(\ell) + v(\xi)(\ell w(\ell) - z(\ell)) + \xi\Delta]v(x) \\ + [z(\xi)y(\ell) - v(\xi)w(\ell) - \Delta]z(x)\}. \end{aligned} \tag{4}$$

We establish positivity of k, p_0, q_0, p_1, q_1 by an indirect argument, which involves counting of inflection points of $u(x)$. It appears that this idea is new.

The proof of the Theorem 1 follows from the following lemmas, which appear to be of independent interest.

Lemma 1. Let $u(x)$ be a real-valued function of class $C^{[illegible]}[0,\ell]$, which satisfies

$$\begin{aligned} &Lu \equiv (a(x)u'')'' > 0 \qquad \text{for } 0 \le x \le \ell \\ &u(0) = u'(0) = u(\ell) = u'(\ell) = 0, \end{aligned} \tag{5}$$

where $a(x) \in C^2[0,\ell]$, $a(x) > 0$ for $0 \le x \le \ell$. Then $u(x) > 0$ in $(0,\ell)$.

Proof. Denote $U = a(x)u''$. Since $U'' > 0$ on $[0,\ell]$ it follows that U and hence u'' can have at most two zeroes on $[0,\ell]$, so that $u(x)$ can have at most two inflection points. It follows that $u(x)$ is of one sign in a sufficiently small neighborhood of both $x = 0$ and $x = \ell$. Also, since $\int_0^\ell a(x)u''^2 dx = \int_0^\ell uLudx$, it follows that $u(x)$ cannot be negative everywhere on $[0,\ell]$.

Assume first that u is positive near $x = 0$. We claim that $u''(0) > 0$ (obviously $u''(0) \ge 0$). Assume on the contrary that $u''(0) = 0$. Then $u'''(0) \ge 0$. Integrate (5) twice

$$\begin{aligned} a(x)u''(x) &= a(0)u''(0) + x(au'')'(0) + \int_0^x (x-t)Lu(t)dt \\ &= xa(0)u'''(0) + \int_0^x (x-t)Lu(t)dt. \end{aligned}$$

The function on the right is positive for all x in $(0,\ell]$. Then $u''(x) > 0$ on $(0,\ell]$, contradicting the boundary conditions at $x = \ell$. A similar argument shows that $u''(\ell) > 0$ in case u is positive near $x = \ell$.

Suppose next that u is negative near $x = 0$. Then $u''(0) \le 0$. We claim that $u''(x) < 0$ for $x \in (0,z)$ for some sufficiently small z. Suppose $u''(0) = 0$, since the claim is obvious for $u''(0) < 0$. Then $u'''(0) < 0$, for otherwise we would have $u(0) = u'(0) = u''(0) = u'''(0)$

$= 0$, and then from the equation $u''''(0) > 0$, which would imply that $u(x)$ is positive near $x = 0$, a contradiction, and the claim follows.

We are now ready to prove the lemma. Assume that $u(x)$ is negative near $x = 0$. Then u'' is negative near $x = 0$. Since $u(x)$ cannot remain non-positive for the entire interval, it has to turn back, become positive, turn down again, and then (assuming no more zeroes) turn concave up again when entering $x = \ell$, i.e., it would have to possess at least three inflection points, a contradiction. Similar contradictions arise if one assumes $u(x)$ to be negative near $x = \ell$, or positive near the end-points but negative inside of the interval.

Lemma 2. Assume that $a(x) > 0$ on $[0,\ell]$, $a(x) \in C^2[0,\ell]$. Then the functions $p_0(x), q_0(x), p_1(x), q_1(x)$ (defined in (3)) are positive on $(0,\ell)$.

Proof. Since $a(x)u'' = c_1 + c_2 x$ with some constants c_1 and c_2, it follows that $u(x)$ has no more than one inflection point in $(0,\ell)$, and that $u''(x)$ is of one sign near $x = 0$ and $x = \ell$ (possibly in a deleted neighborhood of 0 or ℓ). From the Taylor's formula it follows that $u''u > 0$ near both $x = 0$ and $x = \ell$, provided $u' = 0$ at that point.

(i) Consider $p_0(x)$ ($p_0(0) = 1$). Assume that it becomes negative somewhere in $(0,\ell)$, and that $p_0'' < 0$, $p_0 < 1$ near $x = 0$ (the other case when $p_0'' > 0$, $p_0 > 1$ near $x = 0$ is easier). Then $p_0(x)$ has to become concave up under the x axis, and then (assuming there is no more zeroes) concave down entering $x = \ell$. So that $p_0(x)$ has to possess at least two inflection points, a contradiction.

(ii) Consider $q_1(x)$. It is positive near $x = \ell$. Assume that $q_1(x)$ vanishes somewhere on $(0,\ell)$, and that q_1'' is negative near $x = \ell$ (the other case is easier). Then q_1 has to be concave up under the x-axis, and (assuming no more zeroes) concave down near $x = 0$, which leads to the same contradiction as before.

Positivity of $p_1(x)$ and $q_0(x)$ is established similarly.

Lemma 3. $\Delta < 0$ (Δ appears in (4)).

Proof. One easily sees that

$$\Delta = \left(\int_0^{\ell} \frac{t}{a(t)}\, dt \right)^2 - \int_0^{\ell} \frac{dt}{a(t)} \int_0^{\ell} \frac{t^2}{a(t)}\, dt,$$

so that the proof follows by Schwarz's inequality, applied to the product of the linearly independent functions $\frac{t}{\sqrt{a(t)}}$ and $\frac{1}{\sqrt{a(t)}}$.

Proof of Theorem 1. Define u_ε to be the solution of $Lu_\varepsilon = \varepsilon$ with constant $\varepsilon > 0$ and homogeneous boundary conditions. Then $L(u+u_\varepsilon) > 0$, and by lemmas 1 and 2 it follows that $u+u_\varepsilon > 0$ at each $x \in (0,\ell)$. From the representation (2) it follows that $u_\varepsilon \to 0$ as $\varepsilon \to 0$ (uniformly in x), and hence $u \geq 0$ on $[0,\ell]$. (We could have used a priori estimates instead of the representation (2) to show that $u_\varepsilon \to 0$ as $\varepsilon \to 0$.)

We apply our maximum principle to develop a monotone iteration scheme for the problem

$$(a(x)u'')'' = f(x,u) \qquad 0 < x < \ell$$
$$u(0) = \alpha,\ u'(0) = \beta,\ u(\ell) = \gamma,\ -u'(\ell) = \delta, \tag{6}$$

which describes the displacement curve $u(x)$ of a uniform elastic beam of length ℓ, supporting a distributed load of intensity $f(x,u(x))$. Here $a(x) = EI$, where E is the Young's modulus, I - the moment of inertia, see e.g., [6].

Theorem 2. For the problem (6) assume that $a(x) \in C^2[0,\ell]$ and the following.

(i) There exists a supersolution $\phi(x) \in C^4[0,\ell]$, i.e., $(a(x)\phi'')'' \geq f(x,\phi)$ for $0 < x < \ell$, $\phi(0) \geq \alpha$, $\phi'(0) \geq \beta$, $\phi(\ell) \geq \gamma$, $-\phi'(\ell) \geq \delta$, and a subsolution $\psi(x) \in C^4[0,\ell]$ defined by reversing the above inequalities. Moreover $\psi \leq \phi$ for $0 \leq x \leq \ell$.

(ii) f is continuous and increasing in u for $0 \leq x \leq \ell$ and $\psi \leq u \leq \phi$. Then the problem (6) has a C^4 solution $u(x)$, and $\psi \leq u \leq \phi$.

Proof. Set $u_0 = \psi$ and define inductively a sequence $\{u_n(x)\}$ by solving ($n = 0, 1, \ldots$)

$$(au''_{n+1})'' = f(x,u_n) \quad \text{for } 0 < x < \ell, \; u_{n+1}(0) = \alpha, u'_{n+1}(0) = \beta,$$
$$u_{n+1}(\ell) = \gamma, \; -u'_{n+1}(\ell) = \delta. \tag{7}$$

Similarly, starting with $v_0 = \phi$ one defines a sequence $\{v_n(x)\}$. Using the Theorem 1 one sees that

$$\psi \le u_1 \le u_2 \le \ldots \le u \le \ldots \le v_2 \le v_1 \le \phi.$$

Call $u(x) = \lim_{n\to\infty} u_n(x)$, $v(x) = \lim_{n\to\infty} v_n(x)$. To see that $u(x)$ is a (minimal) solution of (6) we rewrite (7) as

$$u_{n+1}(x) = \int_0^\ell k(x,\xi)f(\xi,u_n(\xi))d\xi + \alpha p_0(x)$$
$$+ \beta q_0(x) + \gamma p_1(x) + \delta q_1(x), \tag{8}$$

and pass to the limit as $n \to \infty$, using the monotone convergence theorem.

Theorem 2 provides a computationally feasible scheme requiring solution of the linear problem (7) at each step. If one uses finite differences to solve (7), one has to deal with very ill-conditioned matrices (for $a(x) \equiv 1$ the condition number $\sim n^4$, where n is the number of subdivision points). This causes round-off errors to accumulate in Gaussian elimination (we have found there was no gain in accuracy when increasing n past some moderate number, $n \simeq 20$ for $\ell = 2$ in our experiments), while if one uses iterations it may take hundreds of thousands of them to solve (7) with moderate accuracy.

We suggest one uses the formula (8) and numerical integration to solve (7) whenever the Green's function $k(x,\xi)$ is not too involved to program. We had tried this approach for $a(x) \equiv 1$, and had no difficulty computing solution with $n = 200$ mesh points for $\ell = 2$ with good accuracy. It turns out that the Green's function $k(x,\xi)$ is "small" for moderate values of ℓ, which makes the convergence fast. Indeed, for $a(x) \equiv 1$ elementary computations show that

$$\max_{0\le x\le \ell} \int_0^\ell k(x,\xi)d\xi = \int_0^\ell k(\tfrac{\ell}{2},\xi)d\xi = \frac{\ell^4}{384} ;$$

$$\max_{0\le x,\xi\le \ell} k(x,\xi) = k(\tfrac{\ell}{2},\tfrac{\ell}{2}) = \frac{\ell^3}{192} .$$

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