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ON LANE-EMDEN TYPE SYSTEMS

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Abstract. We consider a class of singular systems of Lane-Emden type

$$\begin{cases} \Delta u + \lambda u^{p_1} v^{q_1} = 0, & x \in D, \\ \Delta v + \lambda u^{p_2} v^{q_2} = 0, & x \in D, \\ u = v = 0, & x \in \partial D, \end{cases}$$

with $p_1 \le 0$, $p_2 > 0$, $q_1 > 0$, $q_2 \le 0$, and D a smooth domain in $\mathcal{R}^n$. In case the system is sublinear we prove existence of a positive solution. If D is a ball in $\mathcal{R}^n$, we prove both existence and uniqueness of positive radially symmetric solution.

1. Introduction. In this paper we study the existence and uniqueness of the positive radially symmetric solutions of the semilinear elliptic system:

$$\begin{cases} \Delta u + \lambda u^{p_1} v^{q_1} = 0, & x \in B_1, \\ \Delta v + \lambda u^{p_2} v^{q_2} = 0, & x \in B_1, \\ u = v = 0, & x \in \partial B_1, \end{cases} \tag{1}$$

where B_1 is the unit ball in $\mathcal{R}^n$, $n \ge 1$, and $p_i, q_i \in \mathcal{R}$, $i = 1, 2$. (1) is called a generalized Lane-Emden system.

We notice that a transformation $U(y) = u(\lambda^{-1/2} y)$, and $V(y) = v(\lambda^{-1/2} y)$ will convert (1) into

$$\begin{cases} \Delta U + U^{p_1} V^{q_1} = 0, & y \in B_R, \\ \Delta V + U^{p_2} V^{q_2} = 0, & y \in B_R, \\ U = V = 0, & y \in \partial B_R, \end{cases} \tag{2}$$

where B_R is the ball in $\mathcal{R}^n$ with center at $y = 0$ and radius R, and $R = \sqrt{\lambda}$. Thus the structure of the solution set of (2) is same as that of (1). We shall study (1) instead of (2) in this paper since the domain is a ball with fixed radius.

Our approach to the uniqueness is based on two ingredients: (a) the parameterization of the set of all solutions; and (b) the scaling of the homogeneous equation

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(1). To illustrate the ideas, we consider the positive radial solutions of scalar equation:

$$\Delta u + \lambda u^p = 0, \quad x \in B_1, \qquad u = 0, \quad x \in \partial B_1. \tag{3}$$

Similar to (2), the solutions of (3) are equivalent to those of

$$\Delta U + U^p = 0, \quad y \in B_R, \qquad U = 0, \quad y \in \partial B_R, \tag{4}$$

via the same change of variables above. Then from the uniqueness of the initial value problem of ordinary differential equation, the radius R in (4) is uniquely determined by $U(0) = \max_{y \in \overline{B_R}} U(y)$, and so is $\lambda = R^2$. Thus the solution set of (4) is parameterized by a single parameter $U(0)$. On the other hand, if $u_1(x)$ is a solution of (3) with $\lambda = 1$, then $u_\lambda(x) = \lambda^{1/(1-p)} u_1(x)$ is a solution of (3) for general $\lambda > 0$, and the range of $\{u_\lambda(0)\}$ is $\mathcal{R}^+$. The curve $\Sigma = \{(\lambda, u_\lambda) : \lambda > 0\}$ is monotone, hence we obtain the uniqueness of the solution for each $\lambda > 0$. (Here we assume the existence of u_1, which in fact can also be proved.)

We follow a similar approach for the uniqueness of solutions to the system (1). While the scaling of homogeneous nonlinearities can still be done, the parametrization of the solutions for system is not simple, and it may not be always possible that the solution set is parameterized by one parameter. We generalize an idea of Dalmasso[3] to prove that the solution set $\{(\lambda, u, v)\}$ of (1) (or equivalently $\{(R, U, V)\}$ of (2)) can be parameterized by a single variable $u(0)$ (or $U(0)$ respectively) under certain conditions on the nonlinearities. (For a particular system this was also observed by Korman [5].) In particular we prove the uniqueness of the solution of (1) for any fixed λ when $p_1, q_2 \le 0$ and $p_2, q_1 > 0$, which generalizes results of Dalmasso[2, 3] and Korman [4]. We also prove a new existence result for (1) with general bounded smooth domain when $p_1, q_2 < 0$ and $p_2, q_1 > 0$ and also satisfying some extra conditions by using sub-supersolution method, thus obtaining the existence and uniqueness for that case.

We will prove a general parametrization result in Section 2, and apply it to the uniqueness problem in Section 3. In Section 4, we prove the existence result for the singular case of $p_1, q_2 > 0$ and $p_2, q_1 < 0$. After submitting this paper, we learned that Maniwa [7] proved the uniqueness of positive solution to a higher dimensional version of (1) for general bounded domain when the exponents are sublinear (see below and [7].) Our uniqueness is for all exponents, and our approach is quite different.

2. Parametrization of the solution set. We consider a system of semilinear equations:

$$\begin{cases} \Delta u + \lambda f(u, v) = 0, & x \in B_1, \\ \Delta v + \lambda g(u, v) = 0, & x \in B_1, \\ u = v = 0, & x \in \partial B_1, \end{cases} \tag{5}$$

where $f, g : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}$ are C^1 functions. We assume that f and g satisfy

$$\frac{\partial f(u,v)}{\partial u} \le 0, \quad \frac{\partial f(u,v)}{\partial v} > 0, \quad \frac{\partial g(u,v)}{\partial u} > 0, \quad \frac{\partial g(u,v)}{\partial v} \le 0, \qquad \text{for any } u \ge 0, v \ge 0. \tag{6}$$

Lemma 1. *Assume that $f(u, v)$ and $g(u, v)$ satisfy (6), and (u_1, v_1) and (u_2, v_2) are two radially symmetric solutions of (5) with the same parameter λ. If $u_1(0) = u_2(0)$, then $v_1(0) = v_2(0)$.*

Proof. Suppose not, we can then assume $v_1(0) > v_2(0)$ without loss of generality. Then $v_1(x) - v_2(x) > 0$ in a neighborhood of 0. Since $\Delta(u_1 - u_2)(0) = \lambda[f(u_2(0), v_2(0)) - f(u_1(0), v_1(0))] < 0$ and $u_1(0) - u_2(0) = 0$ and $\nabla u_1(0) = \nabla u_2(0) = 0$, then $u_1(x) - u_2(x) < 0$ for $x \in B_\delta \backslash \{0\}$ for some $\delta > 0$. (Keep in mind, both solutions are radially symmetric.) We define $r_0 = \sup\{r > 0 : v_1(x) - v_2(x) > 0, u_1(x) - u_2(x) < 0, 0 < |x| < r_0\}$. From the above, $r_0 > 0$. Suppose that $r_0 < 1$, then either $v_1(x) - v_2(x) = 0$ or $u_1(x) - u_2(x) = 0$ when $|x| = r_0$. We assume that $u_1(x) - u_2(x) = 0$ when $|x| = r_0$. Then for $x \in B_{r_0}$, $\Delta(u_1 - u_2) = \lambda[f(u_2, v_2) - f(u_1, v_1)] = \lambda[f(u_2, v_2) - f(u_2, v_1)] + \lambda[f(u_2, v_1) - f(u_1, v_1)] < 0$ using that $v_1 - v_2 > 0$ and $u_1 - u_2 < 0$ in B_{r_0}. Since we also have $u_1 - u_2 = 0$ on ∂B_{r_0}, then $u_1 - u_2 \geq 0$ in B_{r_0} by the maximum principle, which contradicts with $u_1(x) - u_2(x) < 0$ for $x \in B_\delta \backslash \{0\}$. We can reach a similar contraction if we assume that $v_1(x) - v_2(x) = 0$ when $|x| = r_0$. Hence $r_0 = 1$. When $r_0 = 1$, we have $v_1 - v_2 = 0$ and $u_1 - u_2 = 0$ on ∂B_1 because of boundary conditions. But then we reach the same contradiction by the above arguments. $\square$

Proposition 1. *Assume that $f(u, v)$ and $g(u, v)$ satisfy (6). Then the set of positive radial solutions of (5) can be parameterized by $d = u(0)$, i.e. for each $d > 0$, there exists at most one solution (λ, u, v) such that $u(0) = d$; The set of solution is a differentiable curve*

$$\Sigma = \{(\lambda(d), u(\cdot, d), v(\cdot, d)) : d \in T\}, \tag{7}$$

where T is an open subset of $\mathcal{R}^+$, and $u(0, d) = d$.

Proof. Let $d > 0$. From Lemma 1, there is at most one $e = v(0)$ such that (5) has a solution with $(u(0), v(0)) = (d, e)$. If such a solution exists, λ can also be uniquely determined since via a change of variables, we have

$$\begin{cases} \Delta U + f(U, V) = 0, & y \in B_R, \\ \Delta V + g(U, V) = 0, & y \in B_R, \\ U = V = 0, & y \in \partial B_R, \end{cases} \tag{8}$$

where $R = \sqrt{\lambda}$. R can be determined from the shooting problem of the initial value problem:

$$\begin{cases} U'' + \dfrac{n-1}{t} U' + f(U, V) = 0, & t > 0, \\ V'' + \dfrac{n-1}{t} V' + g(U, V) = 0, & t > 0, \\ U'(0) = V'(0) = 0, \quad U(0) = d, \quad V(0) = e. \end{cases} \tag{9}$$

Note here Lemma 1 also implies that there is at most one $e = v(0)$ such that U and V in (9) will hit 0 at simultaneous R. Since f and g are C^1, then we can differentiate (9) with respect to d, thus $(\lambda(d), U(d), V(d), R(d))$ is differentiable, so is $(\lambda(d), u(d), v(d))$. $\square$

Remark. The results above are motivated by Dalmasso[3], but our proof is simpler and it could be generalized to some other problems like p-Laplacian equations. We also notice that it is well-known that λ in (5) can be uniquely determined by the pair $(u(0), v(0))$ for any f and g (see e.g. Korman[4], Lemma 4.1), which follows from the uniqueness of solution to initial value problem of ordinary differential equations. It would be interesting to know if there are examples of f and g so that the solution set of (5) cannot be determined by $u(0)$ alone, but must depend on both $u(0)$ and $v(0)$.

3. Uniqueness. In this section we consider the uniqueness of radially symmetric solution of (1), assuming that

$$p_1 \leq 0,\ p_2 > 0,\ q_1 > 0,\ q_2 \leq 0. \tag{10}$$

Here we assume the existence of a radially symmetric solution of (1) for some $\lambda_0 > 0$, and study the uniqueness based on this assumption. We will prove one existence result in Section 4.

Theorem 1. *We assume that p_i, q_i satisfy* (10), *and there exists $\lambda_0 > 0$ such that* (1) *has a positive radially symmetric solution $(u_{\lambda_0}, v_{\lambda_0})$. Then*

1. *If $(1-q_2)(1-p_1) - q_1p_2 \neq 0$, then for each $\lambda > 0$, there exists exactly one positive radially symmetric solution (u_λ, v_λ);*
2. *If $(1-q_2)(1-p_1) - q_1p_2 = 0$, then* (1) *has no positive radially symmetric solution for any $\lambda > 0$ and $\lambda \neq \lambda_0$, and* (1) *has infinitely many positive radially symmetric solutions at $\lambda = \lambda_0$, which can be represented as $\{(ku_{\lambda_0}, k^\theta v_{\lambda_0}) : k > 0\}$, with $\theta = (1-p_1)/q_1 = p_2/(1-q_2)$.*

Proof. First we assume that $(1-q_2)(1-p_1) - q_1p_2 \neq 0$. Suppose that there exists a $\lambda_0 > 0$ such that (1) has a solution $(u_{\lambda_0}, v_{\lambda_0})$, then it is easy to verify that $(u_1, v_1) = (\lambda_0^{(1-q_2+q_1)/[(1-q_2)(1-p_1)-q_1p_2]} u_{\lambda_0}, \lambda_0^{(1-p_1+p_2)/[(1-q_2)(1-p_1)-q_1p_2]} v_{\lambda_0})$ is a solution of (1) with $\lambda = 1$.

For each $\lambda > 0$, we define

$$(u_\lambda, v_\lambda) = (\lambda^{-(1-q_2+q_1)/[(1-q_2)(1-p_1)-q_1p_2]} u_1, \lambda^{-(1-p_1+p_2)/[(1-q_2)(1-p_1)-q_1p_2]} v_1), \tag{11}$$

then it is a solution of (1) with the given λ. Clearly $\{(u_\lambda, v_\lambda) : \lambda > 0\}$ is a smooth curve, and the map $P : \lambda \mapsto u_\lambda(0)$ is also smooth and monotone. Indeed $P(\lambda) = \lambda^{(1-q_2+q_1)/[(1-q_2)(1-p_1)-q_1p_2]} P(1)$, thus it is strictly increasing when $(1 - q_2)(1-p_1) - q_1p_2 > 0$, and it is strictly decreasing when $(1-q_2)(1-p_1) - q_1p_2 < 0$. Moreover the range of the map P is $(0, \infty)$. From Proposition 1, for each $d > 0$, there is at most one solution of (1), with $u(0) = d$. Therefore (1) has no other solutions besides the ones on the curve $\{(u_\lambda, v_\lambda) : \lambda > 0\}$, which proves the uniqueness of the solution for each $\lambda > 0$.

If $(1-q_2)(1-p_1) - q_1p_2 = 0$, then we can see that $(ku_{\lambda_0}, k^\theta v_{\lambda_0})$ is a solution of (1), where $k > 0$, and $\theta = (1-p_1)/q_1 = p_2/(1-q_2)$. We define the map $Q : k \mapsto ku_{\lambda_0}(0)$ for $k > 0$, then the range of Q is also $(0, \infty)$. From Proposition 1, for each $d > 0$, there is at most one solution of (1), with $u(0) = d$. Therefore (1) has no other solutions besides the ones on $\{(ku_{\lambda_0}, k^\theta v_{\lambda_0}) : k > 0\}$. □

Remark. The special case of $p_1 = q_2 = 0$ is proved in Dalmasso[3]. Similar to the scalar equation case, when $(1-q_2)(1-p_1) - q_1p_2 > 0$, we call the system sublinear, and when $(1-q_2)(1-p_1) - q_1p_2 < 0$, we call it superlinear. The bifurcation diagrams for three cases are as in Figure 1. (The middle one is the case $(1-q_2)(1-p_1) - q_1p_2 = 0$, which we refer as "linear" due to the linear structure of the solution set.)

4. Sublinear singular systems. In this section we prove an existence result for the system in (1) on general domains, for the exponents satisfying (10) and being sublinear (see the definition in Section 3.) We begin by recalling the following well-known result, see [1], [6], [8].

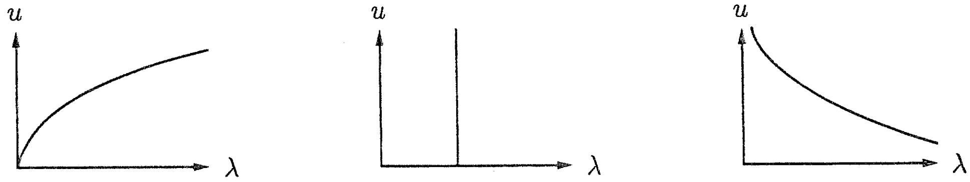

FIGURE 1. Bifurcation Diagrams: (a) sublinear; (b)"linear"; (c) superlinear

Theorem 2. *Consider the problem*

$$\Delta u + g(x)u^p = 0, \quad x \in D, \quad u = 0, \quad x \in \partial D, \tag{12}$$

with $q(x) > 0$ *on* D, *and* $q(x) \in C^{\alpha}(\bar{D})$, *and* $p < 0$. *Assume that* ∂D *is of class* $C^{2+\alpha}$, $0 < \alpha < 1$. *Then the problem (12) has a unique positive solution* $u \in C^{2+\alpha}(D) \cap C(\bar{D})$. *This solution is monotone increasing in* $q(x)$. *Moreover, if* D *is a ball and* $q(x)$ *is radially symmetric, then so is the solution* $u = u(|x|)$.

Consider the system

$$\begin{cases} \Delta u + u^{p_1} v^{q_1} = 0, & x \in D, \\ \Delta v + u^{p_2} v^{q_2} = 0, & x \in D, \\ u = v = 0, & x \in \partial D, \end{cases} \tag{13}$$

where D is a smooth bounded domain in $\mathcal{R}^n$. We assume that p_i and q_i satisfy (10).

We make a change of variables $u = \alpha U$, $v = \beta V$, with constant α and β, and choose β, which will equalize the coefficients in front of the nonlinear terms of the resulting system, *i.e.* $\lambda^{p_1-1}\beta^{q_1} = \alpha^{p_2}\beta^{q_2-1}$, which implies that $\beta = \alpha^{\frac{p_2-p_1+1}{q_1-q_2+1}}$. The system (13) transforms to

$$\begin{cases} \Delta U + \alpha^{\theta} U^{p_1} V^{q_1} = 0, & x \in D, \\ \Delta V + \alpha^{\theta} U^{p_2} V^{q_2} = 0, & x \in D, \\ u = v = 0, & x \in \partial D, \end{cases} \tag{14}$$

with

$$\theta = -\frac{(1-q_2)(1-p_1) - q_1 p_2}{q_1 - q_2 + 1}. \tag{15}$$

We call the original system (13) *sublinear* if $\theta < 0$ *i.e.*

$$(1-q_2)(1-p_1) - q_1 p_2 > 0. \tag{16}$$

Theorem 3. *Any sublinear system (13), satisfying* (10), *admits a positive (componentwise) classical solution. Moreover, if* D *is a ball, there exists a positive radially symmetric solution.*

We will prove this theorem by using monotone iterations. However, because of the singularity, one cannot convert (13) to an system increasing in both u and v, by the usual trick of adding a constant times u (or v). We introduce a modification, where on each step we solve two singular semilinear equations. We recall the following definition.

Definition. We say that a pair of functions $(u_0, v_0) \in C^2(D) \times C^2(D)$ is a *subsolution* of (13) if

$$\begin{cases} \Delta u_0 + u_0^{p_1} v_0^{q_1} \geq 0, & x \in D, \\ \Delta v_0 + u_0^{p_2} v_0^{q_2} \geq 0, & x \in D, \\ u_0,\ v_0 \leq 0, & x \in \partial D. \end{cases} \tag{17}$$

A *supersolution* is defined by reversing the inequalities in (15).

Proof of Theorem 3. We present a pair of ordered super- and subsolutions of (13), and then set up monotone iterations. Let $\bar{D}$ by any smooth domain, which contains D as a proper subset, and let $\bar{\phi}(x)$ denote the principal eigenfunction of the minus Laplacian on $\bar{D}$, corresponding to the eigenvalue $\bar{\lambda}_1 > 0$. Then the pair $(U_0, V_0) = (\alpha\bar{\phi}(x), \alpha^{\frac{p_2-p_1+1}{q_1-q_2+1}}\bar{\phi}(x))$ is a supersolution of (13), if α is sufficiently large. Indeed, $U_0, V_0 > 0$ on ∂D, and

$$\begin{cases} \Delta U_0 + U_0^{p_1} V_0^{q_1} = \alpha(-\bar{\lambda}_1\bar{\phi} + \alpha^\theta \bar{\phi}^{p_1+q_1}), \\ \Delta V_0 + U_0^{p_2} V_0^{q_2} = \alpha^{\frac{p_2-p_1+1}{q_1-q_2+1}}(-\bar{\lambda}_1\bar{\phi} + \alpha^\theta \bar{\phi}^{p_2+q_2}), \end{cases}$$

where $\theta < 0$ is defined by (15). Clearly the right hand sides are negative for large enough α. Turning to the subsolution, let $\phi(x)$ denote the principal eigenfunction of the minus Laplacian on D, corresponding to the eigenvalue $\lambda_1 > 0$. We construct a subsolution in the form $(u_0, v_0) = (\epsilon\phi(x)^\delta, \epsilon^{\frac{p_2-p_1+1}{q_1-q_2+1}}\phi(x)^\gamma)$, with constants $\delta > 1$ and $\gamma > 1$ to be selected, and $\epsilon > 0$ sufficiently small. We compute

$$\begin{cases} \Delta u_0 + u_0^{p_1} v_0^{q_1} = \epsilon(-\lambda_1\delta\phi^\delta + \delta(\delta-1)\phi^{\delta-2}|\nabla\phi|^2 + \epsilon^\theta\phi^{\delta p_1+\gamma q_1}), \\ \Delta v_0 + u_0^{p_2} v_0^{q_2} = \epsilon^{\frac{p_2-p_1+1}{q_1-q_2+1}}(-\lambda_1\gamma\phi^\gamma + \gamma(\gamma-1)\phi^{\gamma-2}|\nabla\phi|^2 + \epsilon^\theta\phi^{\delta p_2+\gamma q_2}). \end{cases} \tag{18}$$

Let us assume that

$$\delta p_1 + \gamma q_1 \leq \delta. \tag{19}$$

This implies that near the boundary ∂D the term $\phi^{\delta p_1+\gamma q_1}$ cannot vanish faster than ϕ^δ. Hence for ϵ small (*i.e.* ϵ^θ large) and $\delta > 1$, the right hand side of the first equation in (18) is positive. Similarly if we assume that

$$\delta p_2 + \gamma q_2 \leq \gamma, \tag{20}$$

then for ϵ small and $\gamma > 1$, the right hand side of the second equation in (18) is positive. It remains to show that we can select $\delta > 1$ and $\gamma > 1$, satisfying (19) and (20). The inequality (19) is satisfied by (δ, γ), which lie below the line L_1 : $\gamma = \frac{-p_1+1}{q_1}\delta$. Similarly, (20) describes the region above the line $L_2 : \gamma = \frac{p_2}{-q_2+1}\delta$. We need that the slope of L_1 is greater than the slope of L_2, which is exactly equivalent to the sublinearity condition (16). Then there are infinitely many solutions of (19) and (20). This concludes the construction of a subsolution. By decreasing ϵ, if necessary, we may assume that

$$u_0 < U_0, \text{ and } v_0 < V_0 \text{ for all } x \in D. \tag{21}$$

Starting with a subsolution (u_0, v_0) we now construct a sequence of iterates (u_n, v_n) by solving two scalar semilinear singular boundary values problems

$$\begin{cases} \Delta u_n + u_n^{p_1} v_{n-1}^{q_1} = 0, & x \in D, \quad u_n = 0, \quad x \in \partial D, \\ \Delta v_n + u_{n-1}^{p_2} v_n^{q_2} = 0, & x \in D, \quad v_n = 0, \quad x \in \partial D. \end{cases} \tag{22}$$

[9] Troy, William C., *Symmetry properties in systems of semilinear elliptic equations.* J. Differential Equations, **42** (1981), no. 3, 400–413.

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