New exact multiplicity results with an application to a population model

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Abstract

We obtain some new exact multiplicity results for the Dirichlet boundary value problem

$$\Delta u + \lambda f(u) = 0 \quad \text{for } x \in B^n, \quad u = 0 \ \text{ for } x \in \partial B^n,$$

on a unit ball B^n in R^n. We consider several classes of nonlinearities $f(u)$, including both positive and sign-changing cases. Crucial part of the proof is to establish positivity of solutions for the corresponding linearized problem. As an application we obtain exact multiplicity results for the Holling-Tanner population model.

Key words: Exact multiplicity of solutions, the global solution curve.

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1 Introduction

We study positive solutions of the Dirichlet boundary value problem depending on a positive parameter λ:

$$\Delta u + \lambda f(u) = 0 \quad \text{for } x \in B^n,\ u = 0 \quad \text{for } x \in \partial B^n, \tag{1.1}$$

on the unit ball B^n in $\mathbf{R}^n$, $n \geq 2$. From the well-known result of B. Gidas, W.-M. Ni and L. Nirenberg [6] any positive solution of (1.1) is radially symmetric, i.e. $u = u(|x|)$, and hence (1.1) reduces to an ordinary differential equation of the form (2.2) below.

The set of positive solutions of (1.1) (or equivalently of (2.2)) consists of simple curves in the function space $\mathbf{R}^+ \times X = \{(\lambda, u)\}$, where $X = C^{2,\alpha}(\overline{B^n})$, and each solution curve can be parameterized by the maximal value of the solution, see [12] or [18]. To determine the exact shape of the solution curve, and to count the number of solutions for each given $\lambda > 0$, it is important to study the singular points on the solution curve. The singular points are where the curve makes a turn (so we called them the *turning points*), and at a turning point (λ, u), the linearized equation

$$\Delta z + \lambda f'(u) z = 0 \quad \text{for } x \in B^n,\ u = 0 \quad \text{for } x \in \partial B^n, \tag{1.2}$$

has a nontrivial solution z. A bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [2] can be applied to (1.1) near a turning point, providing a bifurcation theory approach to the exact multiplicity results, that was developed and refined in [11], [12], [9], [18], and [19]. The key to the bifurcation theory approach is

1. Prove for any turning point (λ, u), z can be chosen to be positive;

2. Prove that if z is positive, then there is at most one turning point.

The second part involves an estimate of the integral $\int_{B^n} f''(u) z^3 dx$, so both the positivity of z and the convexity properties of f are relevant. (See Lemma 3.) The first part usually involves comparison arguments of Sturm's type, and the choice of suitable comparison functions is critical and delicate.

In this paper, we mainly use a comparison function $v(r) = r u_r(r) + (n-2)u(r) + \mu$ for some unspecified constant μ, and it results in a condition on the nonlinearity $f(u)$:

$$2[f'(u)]^2 - n f(u) f''(u) \geq 0 \quad \text{for all } u > 0 \ \ (\text{or } c > u > 0). \tag{1.3}$$

Under (1.3) and some other conditions on f, we are able to show the positivity of z, and consequently we obtain some new exact multiplicity results for (1.1). The test function v and the condition (1.3) also appeared in [10], where the problem (1.1) with a nonlinearity f satisfying $f(0) < 0$ was studied. We will consider the nonlinearities f satisfying A) $f(0) \geq 0$, and f is asymptotically superlinear; B) there is a constant $c > 0$, such that $f(u) > 0$ for $u \in (0, c)$ and $f(c) = 0$; and C) there exists $c > b > 0$, such that $f(u) < 0$ for $u \in (0, b)$, $f(u) > 0$ for $u \in (b, c)$ and $f(0) = f(b) = f(c)$. In all of these cases we assume $f(u)$ to satisfy (1.3).

In Section 3, we apply the exact multiplicity results to Holling-Tanner population model:

$$\Delta u + \lambda \left(mu - u^2 - \frac{ku}{1+u} \right) = 0 \quad \text{for } x \in B^n, \; u = 0 \quad \text{for } x \in B^n, \tag{1.4}$$

with constants $k, m > 0$. The solutions of (1.4) are the steady state solutions of the corresponding reaction-diffusion equation:

$$\begin{aligned} u_t = \Delta u + \lambda \left(mu - u^2 - \frac{ku}{1+u} \right) & \quad \text{for } x \in B^n, \\ u(x,t) = 0 & \quad \text{for } x \in \partial B^n, \\ u(x,0) = u_0(x) & \quad \text{for } x \in B^n. \end{aligned} \tag{1.5}$$

When $k = 0$, $f_0(u) = mu - u^2$, and (1.4) is the well-known diffusive logistic equation, which has wide applications in population biology modeling. When $k > 0$, the term $-ku/(1+u)$ is one example of a predation term. Here u is considered to be a population of prey, whose growth rate is decreased because of the existence of some predators. The predation term $-ku/(1+u)$ was introduced by Holling and Tanner, see e.g. [7]. We prove that condition (1.3) is satisfied for nonlinearity here when $1 \leq n \leq 4$, and we completely classify the bifurcation diagrams for all parameters $k > 0$, $m > 0$ and $1 \leq n \leq 4$. Our results allow us to characterize the attractor of (1.5) for any possible combination of parameters.

We mention that Y. Du and Y. Lou [3] have studied the bifurcation curve of solutions of an elliptic system with Holling-Tanner type interaction of two species, and, for some parameters, they obtained an exact S-shaped solution curve. The test function $v(r) = ru_r(r) + (n-2)u(r) + \mu$ was also used in Korman [10]. But the special case of $v(r) = ru_r(r) + (n-2)u(r)$ was well-known, see [17], [13], [19], and on the other hand, when $n = 2$, Y. Du and Y. Lou [4] used a test function $v(r) = ru_r(r) + \mu$ in studying a problem from combustion theory. When $n = 1$, R. Schaaf [20] also obtained some exact multiplicity results with f satisfying similar properties as (1.3). More previous results on using test functions to obtain the exact multiplicity results for (1.1) can be found in [14], [11], [12], [9], [18], [19].

Throughout the paper λ_1 is the principal eigenvalue of the Laplacian on B^n, and $\phi(x) > 0$ the corresponding eigenfunction. And throughout the paper, a solution of (1.1) is always meant a positive solution of (1.1).

2 Exact Multiplicity Results

We consider the positive solutions of the Dirichlet problem

$$\Delta u + \lambda f(u) = 0 \quad \text{for } x \in B^n,\ u = 0 \quad \text{for } x \in \partial B^n, \tag{2.1}$$

on the unit ball B^n in $\mathbf{R}^n$, $n \geq 2$. By the well-known result of B. Gidas, W.-M. Ni and L. Nirenberg [6] any positive solution of (2.1) is radially symmetric, and hence it satisfies

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \quad \text{for } 0 < r < 1,\ u'(0) = u(1) = 0, \tag{2.2}$$

where $r = |x|$. The corresponding linearized problem is

$$\Delta z + \lambda f'(u)z = 0 \quad \text{for } x \in B^n,\ z = 0 \quad \text{for } x \in \partial B^n, \tag{2.3}$$

and if u is a positive radially symmetric solution of (2.1) , then it was shown by C.S. Lin and W.-M. Ni [15] that any solution of (2.3) is also radially symmetric, and thus it satisfies

$$L[z] \equiv z'' + \frac{n-1}{r}z' + \lambda f'(u)z = 0 \quad \text{for } r < 1,\ z'(0) = z(1) = 0. \tag{2.4}$$

Recall that a solution u of (2.2) is called *unstable* if the principle eigenvalue μ_1 of $L[z]+\mu z = 0$, $z'(0) = z(1) = 0$ is negative, otherwise we say it is *stable*. When $\mu_1 < 0$, the number of negative eigenvalues is called the *Morse index* of u. When $\mu_1 = 0$, the solution is *degenerate*, that is (2.4) has a nontrivial solution, and the corresponding eigenfunction z is of one sign, and it can be chosen to be positive. For determining the precise bifurcation diagram of (2.2), it has been shown (see [9], [11], [12], [18], [19]) the importance of proving $\mu_1 = 0$ at a degenerate solution (instead of $\mu_k = 0$ for $k > 1$). That is equivalent to excluding the possibility of higher Morse index solution of (2.2).

The main tool of proving $z > 0$ is a comparison lemma:

Lemma 1 *Suppose $L[z](t) = 0$, $z \not\equiv 0$. If there exists $v \in C^2[a,b]$ such that $L[v](t) \cdot v(t) \leq (\not\equiv) 0$, then z has at most one zero in $[a,b]$. In addition if $z'(a) = 0$, then z does not have any zero in $[a,b]$.*

For the proof, we refer to [18], [10]. Our first result in proving $\mu_1 = 0$ (or $z > 0$) is

Lemma 2 *Assume that the function $f(u) \in C^2[0,\infty)$ satisfies $f(0) \geq 0$, the condition (1.3) for all $u > 0$, and*

$$f'(u) > 0 \quad \text{for all } u > 0. \tag{2.5}$$

Then any non-trivial solution z of the linearized problem (2.4) is of one sign, i.e. we may assume that $z(r) > 0$ for all $r \in [0,1)$.

Proof: We consider a test function $v = ru'(r) + (n-2)u(r) + \alpha$ with a constant α to be specified. Recall that we denote the left side of the linearized equation for (2.4) by $L[z]$. Compute

$$L[v] = \lambda[(n-2)uf'(u) - nf(u) + \alpha f'(u)] \equiv \lambda g_\alpha(u(r)). \tag{2.6}$$

The sign of the test function $v(r)$ is governed by the function $\alpha = h(r) \equiv -ru'(r) - (n-2)u(r)$. Indeed, $v > 0$ (< 0) when $h(r) < \alpha$ ($> \alpha$). Similarly, the sign of $g_\alpha(u)$ is governed by $\alpha = j(r) \equiv \dfrac{nf(u(r)) - (n-2)uf'(u(r))}{f'(u(r))}$. This time, in view of (2.5), $g_\alpha(u) > 0$ (< 0) provided $\alpha > j(r)$ ($\alpha < j(r)$). Notice that $h(0) = -(n-2)u(0) \leq 0$, $h(1) = -u'(1) > 0$ and

$$h'(r) = -ru'' - (n-1)u' = \lambda r f(u) > 0, \tag{2.7}$$

since by our conditions $f(u)$ is positive. Also $j(1) \geq 0$ and

$$j'(r) \leq 0, \tag{2.8}$$

in view of the condition (1.3). It follows that only two cases are possible.

Case 1: The functions $h(r)$ and $j(r)$ intersect exactly once on $[0,1)$, say at $r = r_0$. We select $\alpha = h(r_0) = j(r_0)$. Then on the interval $[0, r_0)$ we have $v > 0$ and $L[v] < 0$, while on the interval $(r_0, 1)$ the opposite inequalities hold. Lemma 1 implies that $z(r)$ cannot have any roots, and hence is of one sign on $[0,1)$.

Case 2: $j(r) \geq h(r)$ for all $r \in [0,1)$. This time we select $\alpha = h(1) = \max_{r \in [0,1]} h(r)$ to obtain $v > 0$ and $L[v] < 0$ on the entire interval $[0,1)$. We again conclude that $z(r)$ is of one sign. $\square$

Remarks.

1. In several previous works on the exact multiplicity results for (2.2), the function $K_f(u) = \dfrac{uf'(u)}{f(u)}$ also played an important role when proving results of Lemma 2 type. $K_f(u)$ can be thought of as an indicator of the growth rate of $f(u)$, for example, $K_f(u) = p$ if $f(u) = u^p$. It was shown in [19] that if $f(u) > 0$ for $u > 0$, $K_f(u)$ is decreasing or $K_f(u) \le \dfrac{n}{n-2}$, then any nontrivial solution $z(r)$ of (2.4) is of one sign. We point out that the condition (1.3) is also related to K_f. In fact, (1.3) is equivalent to that the function $A(u) = \dfrac{nf(u) - (n-2)uf'(u)}{f'(u)}$ is increasing, and since $A(u) = u[nK_f^{-1}(u) - (n-2)]$, we have
$$A'(u) = \frac{n-2}{K_f(u)}\left[\frac{n}{n-2} - K_f(u)\right] - \frac{nuK_f'(u)}{K_f^2(u)}.$$
So (1.3) holds if $K_f(u)$ is decreasing *and* $K_f(u) \le \dfrac{n}{n-2}$. Also if (1.3) holds, then for any $u > 0$, either $K_f(u)$ is decreasing or $K_f(u) \le \dfrac{n}{n-2}$. This observation will be useful in the proof of Lemma 5 later.

2. Condition (1.3) also implies some asymptotic growth restriction on f. In fact, (1.3) is equivalent to
$$\left(\frac{f}{f'}\right)' - \frac{n-2}{n} \ge 0, \tag{2.9}$$
and by integration, we obtain
$$f(u) \le ae^{bu}, \;\; (n=2); \quad \text{and } f(u) \le a(u+b)^{n/(n-2)}, \;\; (n \ge 3), \tag{2.10}$$
for all $u \ge 0$ and some $a, b > 0$.

If the solution z of the linearized equation (2.4) is shown to be of one sign, there is a well established theory on the set of positive solutions of (2.2), which we briefly review here. The details (and also the proof of all quoted facts) can be found in [9], [11], [12], [18], [19].

From the uniqueness of ordinary differential equation, for any $s > 0$, there is at most one $\lambda(s) > 0$ such that (2.2) has a positive solution $u(\cdot, s)$ with $\lambda = \lambda(s)$ and $u(0) = s$. So the set of positive solutions of (2.2) can be globally parameterized by $s = u(0)$, thus the solution set is a curve of the form $\{(\lambda(s), s)\}$, where $s > 0$ belongs to a certain admissible set. If

$\lambda'(s) \neq 0$, then the corresponding solution $u(\cdot, s)$ is nondegenerate, while if $\lambda'(s) = 0$, then the solution is degenerate. At a degenerate solution, we can show that

$$\lambda''(s) = \frac{-\lambda(s) \int_0^1 r^{n-1} f''(u(r)) z^3(r) dr}{\int_0^1 r^{n-1} f(u(r)) z(r) dr}, \tag{2.11}$$

where z is a nontrivial solution of (2.4). Here we assume that $z(r) > 0$ for $r \in [0, 1)$. For the denominator in (2.11), we can show (see [18] or [9])

$$\int_0^1 r^{n-1} f(u(r)) z(r) dr = \frac{1}{2\lambda(s)} u'(1) z'(1) > 0, \tag{2.12}$$

if $u'(1) < 0$ and $z'(1) < 0$, which are both true if $f(0) \geq 0$. So, the direction of the turn of the bifurcation curve is mainly determined by the integral $\int_0^1 r^{n-1} f''(u(r)) z^3(r) dr$. Here we recall the following results from [19]:

Lemma 3 *Suppose that $(\lambda(s), u(s))$ is a degenerate solution of (2.2), and z is the corresponding solution of linearized equation (2.4), and $z(r) > 0$ for $r \in [0, 1)$.*

1. *If $f''(u) > 0$ for $u > 0$, then $\lambda''(s) < 0$;*
2. *If $f''(u) < 0$ for $u > 0$, then $\lambda''(s) > 0$;*
3. *If $f(0) \geq 0$, and there exists $\beta > 0$ such that $f''(u) < 0$ for u in $(0, \beta)$, and $f''(u) > 0$ for u in (β, ∞), and $u(0) > \beta$, then $\lambda''(s) > 0$;*
4. *If $f(0) \leq 0$, there exists $\beta > 0$ such that $f''(u) > 0$ for u in $(0, \beta)$, and $f''(u) < 0$ for u in (β, ∞), and $u(0) > \beta$, then $\lambda''(s) < 0$.*

Combining Lemmas 2 and 3, we have the following exact multiplicity result:

Theorem 1 *Assume $f(u) > 0$ for $u \geq 0$, $f''(u) \geq 0$, $\lim_{u\to\infty} f(u)/u = \infty$ and f satisfies (1.3) and (2.5). Then there exists a constant $\lambda_0 > 0$, so that the problem (2.2) has no solution for $\lambda > \lambda_0$, exactly two solutions for $\lambda < \lambda_0$, and exactly one solution for $\lambda = \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 1.)*

Proof: Since $f(0) > 0$, then $u = 0$ is a nondegenerate solution of (2.2) when $\lambda = 0$, and by implicit function theorem, for any small $\lambda > 0$, there is a unique solution $u(\lambda, \cdot)$ of (2.2) near $u = 0$. And $u(\lambda, \cdot)$ is positive by the maximum principle. From the remarks before Lemma 3, $u(\lambda, \cdot)$ can also be parameterized by $u(\lambda, 0)$. So a solution curve $\Sigma = (\lambda(s), u(s))$ emerges from

$(\lambda, u) = (0, 0)$ and moves to the right as s increases. On the other hand, (1.1) has no positive solution for $\lambda > 0$ large. In fact, if we multiply (1.1) by ϕ, the principal eigenfunction of $\Delta\phi + \lambda\phi = 0$, $\phi(x) = 0$ for $x \in \partial B^n$, then we have

$$\lambda_1 \int_{B^n} u\phi dx = \lambda \int_{B^n} f(u)\phi dx \geq \lambda a \int_{B^n} u\phi dx, \tag{2.13}$$

where $a = \min_{u \geq 0} f(u)/u > 0$. Thus (1.1) has a positive solution only if $\lambda \leq \lambda_1 a^{-1}$. Thus the solution curve cannot continue to $\lambda = \infty$. Let $\lambda_* = \sup\{\lambda > 0 : (2.2) \text{ has a positive solution with this } \lambda\}$. Then $0 < \lambda_* \leq \lambda_1 a^{-1}$.

We claim that there is a turning point (degenerate solution) on the solution curve Σ. Suppose there is no turning points, then $\lambda'(s) > 0$ for all $s > 0$ and $\lim_{s\to\infty} \lambda(s) = \lambda_*$. Since $s = u(0)$ is a global parameter for all positive solutions, it follows that there is no positive solutions other than those on Σ. In particular (2.2) has a unique positive solution for $\lambda \in (0, \lambda_*)$ and no positive solution for $\lambda \geq \lambda_*$. However, since f satisfies the growth condition (2.10) and $f(0) > 0$, by a result of Lions([16] Theorem 2.1 and Remark 1.1), in case $n > 2$ the problem (1.1) has at least two positive solutions for $0 < \lambda < \lambda^*$, which is a contradiction. In case $n = 2$ we use the Theorem 2.3 in [5] to conclude the existence of at least two positive solutions for $0 < \lambda < \lambda^*$. (Since in [5] it was assumed that $f(0) = 0$, the following modification is needed: to prove existence of the second solution, we apply the mountain pass lemma, using the solution bifurcating from zero (rather than the trivial solution) as a minimizer.) Thus there is a turning point on Σ.

At any turning point, by Lemma 2, z can be chosen as positive, and hence by Lemma 3, $\lambda''(s) < 0$, so the curve turns to the left, and after passing the turning point, the curve travels to the left. There is no any other turning points, since at any turning point, the solution curve turns to the left, but when Σ moves to the left, it has to approach a possible turning point from the right. Therefore, Σ is monotone ($\lambda'(s) < 0$) above the turning point. Let $\overline{\lambda} = \lim_{s\to\infty} \lambda(s)$. Then $\overline{\lambda} \geq 0$. Since from the result of [16], (2.2) has at least two solutions for all $0 < \lambda < \lambda_*$, then $\overline{\lambda} = 0$. □

Examples. As a very particular case we recover the well-known result of D. Joseph and T. Lundgren [8] in case $f(u) = e^u$ and $n = 2$ (actually, we get some extra information even in this case).

We have another application of Lemma 2. In [1] Adimurthi gives an ingenious proof that the problem

$$\Delta u + \lambda u e^u = 0 \quad \text{for } B^2,\ u = 0 \quad \text{for } x \in \partial B^2 \tag{2.14}$$

on a ball B^2 in $\mathbf{R}^2$ has at most one positive solution. We have a more detailed result. Let λ_1 denote the principal eigenvalue of the Laplacian on the unit ball in $\mathbf{R}^2$.

Theorem 2 *The set of positive solutions of (2.14) consists of a smooth solution curve, which bifurcates (to the left) at $\lambda = \lambda_1$ from the trivial solution $u = 0$, and continues for all $0 < \lambda < \lambda_1$, tending to infinity as $\lambda \to 0$. (See Figure 2.)*

Proof: It is well-known that the bifurcation at $\lambda = \lambda_1$ occurs. First we claim that the bifurcation is subcritical. In fact, we can compute $\lambda'(0)$ and

$$\lambda'(0) = \frac{-\lambda_0 f''(0) \int_0^1 r^{n-1} \phi^3(r)\, dr}{2f'(0) \int_0^1 \phi^2(r)\, dr}, \tag{2.15}$$

so $\lambda'(0) < 0$ as $f''(0) > 0$, $f'(0) > 0$ and $\phi > 0$. Thus Σ travels to the left initially near $\lambda = \lambda_1$. Since the nonlinearity is convex, while by Lemma 2 any non-trivial solution of the linearized problem is of one sign, it follows by Lemma 3 that the solution curve continues to the left without any turning point (degenerate solution). It follows from the Theorem 2.3 of D.G. De Figueiredo, P.L. Lions and R.D. Nussbaum [5] that the solution curve cannot go to infinity at a positive λ, while for $\lambda = 0$ infinity is the only place the curve can go. (The result in [5] implies that for any sufficiently small λ the problem (2.14) has a positive solution. If the solution curve were to go to infinity at some positive λ_0, then all possible values of $u(0, \lambda)$ would have been taken, and hence no positive solutions could exist for $0 < \lambda < \lambda_0$, which is a contradiction.) $\square$

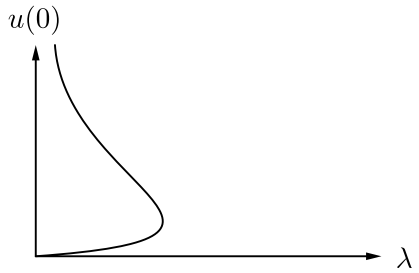

Figure 1: Theorem 1

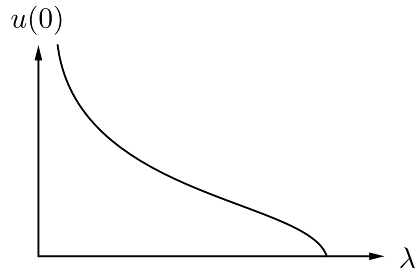

Figure 2: Theorem 2

As our last example for Lemma 2, we point out that for $f(u) = e^{-1/(u+\varepsilon)}$ and $\varepsilon > 0$, (1.3) is satisfied for $n = 2$. In fact, Du and Lou [4] used the test function $v(r) = ru_r(r) + \alpha$ to prove $z > 0$ for that special example.

Next, we turn to the results for f not always increasing. Lemma 2 can be modified as follows:

Lemma 4 *Assume that the function $f(u) \in C^2[0, \infty)$ satisfies $f(0) \geq 0$, $f(u) > 0$ for $0 < u < c$ and there exists $b \in (0, c)$ such that*

$$(2.16) \qquad f'(u) > 0 \quad \text{for } 0 < u < b, \quad f'(u) < 0 \quad \text{for } b < u < c,$$

and f satisfies (1.3). Then any non-trivial solution z of the linearized problem (2.4) is of one sign, i.e. we may assume that $z(r) > 0$ for all $r \in [0, 1)$.

Proof: Let u be a degenerate solution of (2.2). If $0 < u(0) < b$, then $f'(u(r)) > 0$ for $r \in (0, 1)$, and we can just use the proof of Lemma 2. So we assume that $u(0) > b$, and so there exists $r_0 \in (0, 1)$ such that $u(r_0) = b$. Then as in Lemma 2, $h(0) \leq 0$, $h(1) > 0$ and $h'(r) > 0$. On the other hand, for $r \in [0, r_0)$, $L[v(r)] = \lambda g_\alpha(u(r)) < 0$ for any $\alpha > 0$ since $f'(u) < 0$ for $u \in (b, c)$. For $r \in (r_0, 1)$, from (1.3), we still have $j'(r) \leq 0$ and $j(1) \geq 0$. Observe also that $j(r) \to +\infty$ as $r \downarrow r_0$. So only the following two cases are possible:

Case 1: The functions $h(r)$ and $j(r)$ intersect exactly once on $(r_0, 1)$, say at $r = r_1$. We select $\alpha = h(r_1) = j(r_1) > 0$. Then on the interval $[0, r_1)$ we have $v > 0$ and $L[v] < 0$, while on the interval $(r_0, 1)$ the opposite inequalities hold. So we can proceed as in Lemma 2.

Case 2: $j(r) \geq h(r)$ for all $r \in (r_0, 1)$. Again we select $\alpha = h(1) = \max_{r \in [0,1]} h(r) > 0$ to obtain $v > 0$ and $L[v] < 0$ on the entire interval $[0, 1)$. Thus $z \neq 0$ for $r \in [0, 1)$. $\square$

Applying Lemma 4, we obtain two exact multiplicity results as follows:

Theorem 3 *Assume $f(0) = 0$, $f'(0) > 0$ and $f(u) > 0$ for $u \in (0, c)$, where $0 \leq c \leq \infty$. Assume f satisfies (1.3), and for some $c > \beta > 0$ we have*

$$(2.17) \qquad f''(u) > 0 \quad \text{for } 0 \leq u < \beta, \quad f''(u) < 0 \quad \text{for } \beta < u < c.$$

If $c = \infty$, we also assume $\lim_{u \to \infty} f(u)/u = 0$. Define $\lambda_0 = \lambda_1 / f'(0)$. Then there exists a constant $\lambda_* < \lambda_0$, so that the problem (2.2) has no solution for $\lambda < \lambda_*$, exactly two solutions for $\lambda_0 > \lambda > \lambda_*$, and exactly one solution for $\lambda = \lambda_*$ and $\lambda \geq \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 3.)

Proof: It is well-known that if $f(0) = 0$ and $f'(0) > 0$, then $\lambda_0 = \lambda_1/f'(0)$ is a bifurcation point for (1.1), and there is a solution curve bifurcating from $(\lambda, u) = (\lambda_0, 0)$. Similar to Theorem 1, the solution curve $\Sigma = (\lambda(s), u(s))$ has a global parameter $s = u(0)$. And similar to Theorem 2, $\lambda'(0) < 0$ and Σ travels to the left initially near $\lambda = \lambda_0$.

Next we prove that (1.1) has no positive solution for small $\lambda > 0$. In fact, from the conditions on f, there exists $a > 0$ such that $f(u) \leq au$ for all $u > 0$. We multiply (1.1) by u and integrate, then

$$\lambda_1 \int_{B^n} u^2\,dx \leq \int_{B^n} |\nabla u|^2\,dx = \lambda \int_{B^n} f(u)u\,dx \leq \lambda a \int_{B^n} u^2\,dx, \tag{2.18}$$

so $\lambda \geq \lambda_1 a^{-1}$. Hence Σ cannot continue left to $\lambda = 0$, it cannot blow up to $s = \infty$ either, since $s = u(0) < c$, as follows from the maximum principle in case $c < \infty$, and by an easy a priori estimate in case $c = \infty$. So it has to bend back at some turning point.

By the assumptions, f satisfies either (2.5) or (2.16), so $z > 0$ by Lemma 2 or Lemma 4. Similar to Theorem 1, there is a unique turning point on Σ from Lemma 3 (the fourth case), and at the turning point, the curve bends to the right. Above the turning point, $\lambda'(s) > 0$ and so Σ moves to the right monotonously. By an argument in [11] (see also [19]), we can show that there is no other branches, and for the solutions on Σ, when $\lambda \to \infty$, the solution uniformly converges to c for any compact subset of $(0, 1)$. $\square$

Example An example for Theorem 3 is $f(u) = -u(u-2)(u+1)$ and $n = 2, 3$. It is easy to check that f satisfies (2.16) and (2.17) for $u > 0$ with $b = (1+\sqrt{7})/3$, $c = 2$ and $\beta = 1/3$. For (1.3), we obtain

$$\begin{aligned} & [f']^2 - nff'' \equiv B(u) \\ = \;& 2[(9-3n)u^4 + (4n-12)u^3 + (10n-8)u^2 + (8-2n)u + 4]. \end{aligned} \tag{2.19}$$

When $n = 3$, we have $B(u) = 22u^2 + 2u + 4 > 0$ for all $u > 0$, and when $n = 2$, we have $B(u) = 6u^4 - 8u^3 + 24u^2 + 8u + 4 > 0$ for all $u > 0$.

Theorem 4 *Assume $f(0) = 0$, $f'(0) = 0$ and $f(u) > 0$ for $u \in (0, c)$, where $c \leq \infty$. Assume f satisfies (1.3) and (2.17). If $c = \infty$, we also assume $\lim_{u\to\infty} f(u)/u = 0$. Then there exists a constant $\lambda_* > 0$, so that the problem (2.2) has no solution for $\lambda < \lambda_*$, exactly two solutions for $\lambda > \lambda_*$, and exactly one solution for $\lambda = \lambda_*$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 4.)*

The proof of this result is same as that of Theorem 1.3 of [18], except here we use a different way to prove the positivity of z as in Lemma 4. So we refer the reader to [18] and omit the proof here.

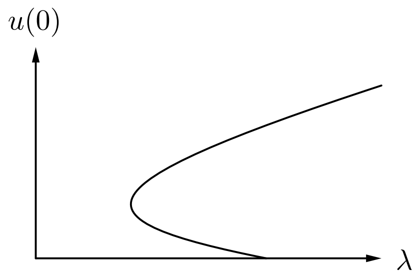

Figure 3: Theorem 3

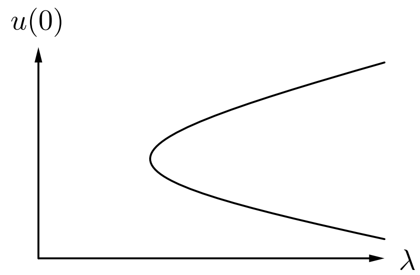

Figure 4: Theorem 4

Finally we consider a problem with sign-changing f.

Lemma 5 *Assume that the function $f(u) \in C^2[0,\infty)$ satisfies $f(0) = 0$, there exists $b \in (0,c)$ such that*

$$f(u) < 0 \quad \text{for } 0 < u < b, \quad f(u) > 0 \quad \text{for } b < u < c, \tag{2.20}$$

$\int_0^c f(u)du > 0$, and f satisfies (1.3) and (2.17). In addition we assume that $n \geq 3$. Then any non-trivial solution z of the linearized problem (2.4) is of one sign, i.e. we may assume that $z(r) > 0$ for all $r \in [0,1)$.

Proof: Since $\int_0^c f(u)du > 0$, there exists a unique $\theta \in (b,c)$ such that $F(\theta) = \int_0^\theta f(u)du = 0$. For any solution u of (2.2), we have $u(0) > \theta$. In fact, multiplying (2.2) by u' and integrating over $(0,1)$, we obtain

$$\frac{1}{2}[u'(1)]^2 + (n-1)\int_0^1 \frac{[u'(r)]^2}{r}dr - \lambda F(u(0)) = 0, \tag{2.21}$$

where $F(u) = \int_0^u f(t)dt$. Thus $F(u(0)) > 0$ and so there exists $r_1 \in (0,1)$ such that $u(r_1) = \theta$. From the result of [14] (see also Lemma 4.9 of [19]), $z(r) \neq 0$ for $r \in [r_1, 1)$. Next we show that z has at most one zero on $(0, r_1]$. For $r \in (0, r_1)$, $f(u(r)) > 0$, so the proof of Lemma 4 can be carried over to here without changes. But we can only conclude that z has at most one zero in $(0, r_1]$, since in the case of Lemma 4 we obtain z has at most one zero in $(0,1]$, while $z(1) = 0$.

Suppose that z has exactly one zero at some $r = r_2$. We exclude this possibility by several steps. Let $u(r_2) = u_2$. First we prove $u_2 < \beta$. If this is not true, then $u(r_2) \geq \beta$, and there exists $r_3 \geq r_2$ such that $u(r_3) = \beta$. Consider the comparison function $v_1(r) = f(u(r))$. It is easy to verify that $L[v_1] = f''(u(r))u_r^2(r)$. So in $[0, r_3]$, $v_1(r) > 0$ and $L[v_1](r) \leq 0$, and by Lemma 1 z has no zero in $[0, r_3]$, which contradicts with $r_2 \in [0, r_3]$.

Next we prove that $K_f(u_2) > n/(n-2)$. Suppose this is not true, then $K_f(u_2) \leq n/(n-2)$. We claim that $K_f(u) \leq n/(n-2)$ for all $u \in [u_2, u(0)]$. In fact, from the remark after the proof of Lemma 2, since f satisfies (1.3), then for any $u \in (0, c)$, either $K_f(u) \leq n/(n-2)$ or $K_f'(u) \leq 0$. So if there exists $u_3 \in [u_2, u(0)]$ such that $K_f(u_3) > n/(n-2)$, then for some $u_4 \in (u_2, u_3)$, $K_f(u_4) > n/(n-2)$ and $K_f'(u_4) > 0$, which contradicts with (1.3). Thus $K_f(u) \leq n/(n-2)$ for all $u \in [u_2, u(0)]$. Define $v_2(r) = ru_r(r) + (n-2)u(r)$, and $v_3(r) = r^{-1}v_2(r)$. We calculate that

$$L[v_2] = \lambda[(n-2)uf'(u) - nf(u)] = \lambda f(u)(n-2)\left[K(u) - \frac{n}{n-2}\right],$$

$$L[v_3] = \lambda\left[\frac{n-2}{r}(uf'(u) - f(u)) - (n-3)\frac{v_2}{r^3}\right].$$

Since $v_2'(r) = ru''(r) + (n-1)u'(r) = -\lambda r f(u) < 0$, and $v_2(0) > 0$, $v_2(1) < 0$, there exists $r_4 \in (0,1)$ such that $v_2(r) > 0$ in $(0, r_4)$ and $v_2(r) < 0$ in $(r_4, 1)$. There are two cases to consider:

Case 1: $r_4 \geq r_2$. On $[0, r_2]$, $v_2(r) \geq 0$, and $L[v_2](r) \leq 0$ since $K_f(u) \leq n/(n-2)$ for all $u \in [u_2, u(0)]$. But that implies z has no zero in $[0, r_2]$, which contradicts with $z(r_2) = 0$.

Case 2: $r_4 < r_2$. On $[r_2, 1)$, $v_3(r) < 0$, and $L[v_3](r) \geq 0$, since $u_2 = u(r_2) < \beta$, then $f''(u(r)) \geq 0$ which implies $uf'(u) - f(u) \geq 0$ for $r \in [r_2, 1)$. That implies z has at most one zero in $[r_2, 1]$, which contradicts with $z(r_2) = z(1) = 0$.

Therefore $K_f(u_2) > n/(n-2)$, and then $K_f'(u_2) \leq 0$ by (1.3). Let $\gamma = K_f(u_2)$. Then for any $u > u_2$, $K_f(u) < \gamma$, otherwise $K_f(u)$ would have to be increasing somewhere above the γ level, and hence there is $u_4 > u_2$ such that $K_f(u_4) > n/(n-2)$ and $K_f'(u_4) > 0$, which contradicts with (1.3). That implies $uf'(u) - \gamma f(u) \leq 0$ for $u > u_2$. Similarly, $uf'(u) - \gamma f(u) \geq 0$ for $u \in [b, u_2]$. And for $u \in [0, b]$, we also have $uf'(u) - \gamma f(u) \geq 0$ since $\gamma > 1$ and $uf'(u) - \gamma f(u) \geq uf'(u) - f(u) \geq 0$. (The latter inequality holds since $f''(u) > 0$, in case $\beta > b$. In case $\beta \leq b$ the lemma follows easily by the remarks in the second paragraph of the present proof.) Hence we obtain that

$$u(r)f'(u(r)) - \gamma f(u(r)) \leq 0, \quad \text{for } r \in [0, r_2], \tag{2.22}$$
$$u(r)f'(u(r)) - \gamma f(u(r)) \geq 0, \quad \text{for } r \in [r_2, 1], \tag{2.23}$$

and $z(r) > 0$ in $[0, r_2)$, $z(r) < 0$ in $(r_2, 1)$. Combining, we obtain

$$\int_0^1 r^{n-1}[u(r)f'(u(r)) - \gamma f(u(r))]z(r)dr < 0. \tag{2.24}$$

However, on the other hand, from a calculation in [18] Lemma 2.3, we have

$$\int_0^1 r^{n-1}[u(r)f'(u(r)) - \gamma f(u(r))]z(r)dr = \frac{1}{2\lambda}(1-\gamma)u'(1)z'(1) > 0, \tag{2.25}$$

since $u'(1) < 0$, $z'(1) > 0$ and $1 - \gamma < 0$. That is a contradiction, and z cannot have exactly one zero in $[0, 1)$. Therefore z must be of one sign in $[0, 1)$. □

Theorem 5 *Suppose that f satisfies the conditions of Lemma 5. If $c = \infty$, we also assume $\lim_{u\to\infty} f(u)/u = 0$. Then there exists a constant $\lambda_* > 0$, so that the problem (2.2) has no solution for $\lambda < \lambda_*$, exactly two solutions for $\lambda > \lambda_*$, and exactly one solution for $\lambda = \lambda_*$. Moreover, all solutions lie on a unique smooth solution curve. (See Figure 5.)*

The proof of Theorem 5 is same as the results in [11], [12] and [18] except that now we use Lemma 5 to prove $z > 0$. So we omit the proof. Some arguments in the proof of Lemma 5 are similar to the proof in [14] and [18].

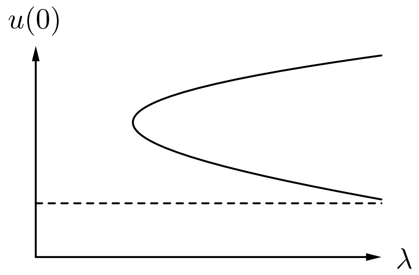

Figure 5: Theorem 5

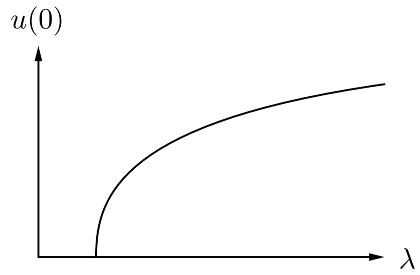

Figure 6: Theorem 6 (1)

3 Applications to A Population Growth Model

We now apply our exact multiplicity results to the Holling-Tanner population model (1.4). The lines $m = k$ and $k = 1$ divide the parameter plane $(m > 0, k > 0)$ into four regions. For three of these regions and a part of the fourth we are able to give exact multiplicity results for $1 \leq n \leq 4$. Notice that the roots of $f(u)$ are 0 and $\frac{(m-1)\pm\sqrt{(m-1)^2+4(m-k)}}{2}$, so that the line $m = k$ separates the regions where $f(u)$ has one or two positive roots. As an application of Theorems 3, 4 and 5, our result for the population model (1.4) is

Theorem 6 *Suppose that $m > 0$, $k > 0$ and $1 \leq n \leq 4$, and define $\lambda_1^0 = \lambda_1/(m-k)$ in case $m - k > 0$.*

1. *If $m > k > 0$ and $1 \geq k$, then (1.4) has a unique solution for $\lambda > \lambda_1^0$ and has no solution for $\lambda_1^0 \geq \lambda > 0$; (See Figure 6)*

2. *If $m > k > 1$, then there exists $\lambda_* \in (0, \lambda_1^0)$ such that (1.4) has exactly two solutions for $\lambda_1^0 > \lambda > \lambda_*$, has exactly one solution for $\lambda = \lambda_*$ and $\lambda \geq \lambda_1^0$, and has no solution for $\lambda_* > \lambda > 0$; (See Figure 3)*

3. *If $n \neq 2$, $k \geq m > 0$, $(1/4)(m+1)^2 > k > 1$ and*

$$\text{for } u_+ = \frac{(m-1)+\sqrt{(m+1)^2-4k}}{2}, \quad F(u_+) > 0, \tag{3.1}$$

where $F(u) = \int_0^u f(t)dt = (m/2)u^2 - (1/3)u^3 - ku + k\ln(1+u)$, then (1.4) has exactly two solutions for $\lambda > \lambda_*$, has exactly one solution for $\lambda = \lambda_*$, and has no solution for $0 < \lambda < \lambda_*$; (See Figures 4 and 5)

4. *If $(k, m) \in \mathbf{R}^+ \times \mathbf{R}^+$ but not in the regions described above, then (1.4) has no solution for $\lambda > 0$.*

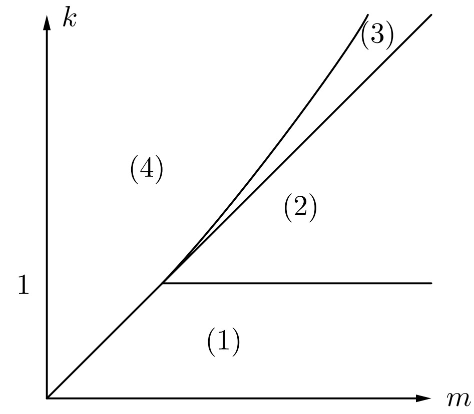

Figure 7: The parameters in Theorem 6

We begin with a lemma.

Lemma 6 *Assume that* $1 \le n \le 4$, $(1/4)(m+1)^2 > k > 1$, $f(u) = mu - u^2 - \dfrac{ku}{1+u}$. *Then* $2[f'(u)]^2 - nf(u)f''(u) \ge 0$ *for* $0 \le u \le u_+$ *if* $u_+ > 0$, *where* u_+ *is defined in (3.1).*

Proof: Note that u_+ is the largest zero of $f(u)$ if $(1/4)(m+1)^2 > k > 1$, so $f(u) < 0$ for $u > u_+$. First we compute

$$f'(u) = m - 2u - \frac{k}{(1+u)^2}, \quad f''(u) = -2 + \frac{2k}{(1+u)^3},$$

$$f'''(u) = -\frac{6k}{(1+u)^4}.$$

If $m \ge k$, then $f'(0) = m - k \ge 0$ and so $f(u) > 0$ for $0 < u < u_+$. If $k \le 1$, then $f''(u) \le 0$ for all $0 < u < u_+$, and $M(u) \equiv 2[f'(u)]^2 - nf(u)f''(u) \ge 0$ for all $0 < u < u_+$. If $k > 1$, $f''(u) > 0$ for $0 < u < \sqrt[3]{k} - 1$ and $f''(u) < 0$ for $\sqrt[3]{k} - 1 < u < u_+$, then again $M(u) \ge 0$ for $\sqrt[3]{k} - 1 < u < u_+$. For $0 < u \le \sqrt[3]{k} - 1$, $M'(u) = (4-n)f'f'' - nff''' > 0$ and $M(0) = 2[f'(0)]^2 \ge 0$, then $M(u) > 0$ for $0 < u \le \sqrt[3]{k} - 1$.

If $m < k$, then $f'(0) < 0$, and we have $u_+ > 0$ only if $(1/4)(m+1)^2 > k > 1$. In such a case, there exists $u_1 \in (0, u_+)$, such that $f(u) < 0$ for $0 < u < u_1$, and $f(u) > 0$ for $u_1 < u < u_+$. For $u_1 < u < u_+$, we can proceed the same way as in the last paragraph to show that $M(u) > 0$. For $0 < u \le u_1$, if $f''(u) > 0$ for all $u \in (0, u_1)$, then $M(u) > 0$. So we assume that there exists $\beta \in (0, u_1)$ such that $f''(u) > 0$ for $0 < u < \beta$ and

$f''(u) < 0$ for $\beta < u < u_1$. For $0 < u < \beta$, $M(u) > 0$ since $f < 0$ and $f'' > 0$. For $\beta < u < u_1$, $M'(u) = (4-n)f'f'' - nff''' < 0$ since $f'(u) > 0$, and $M(u_1) = 2[f'(u_1)]^2 > 0$, so $M(u) > 0$ for $\beta \le u < u_1$. $\square$

Proof of Theorem 6: We first mention that the solution z of (2.4) can always be chosen as positive if $n = 1$, so the corresponding results in theorems 3, 4 and 5 (where $n \ge 2$) are all true for $n = 1$.

We start by dividing the first quadrant of (m,k)-plane into four parts using the lines $m = k$ and $k = 1$. Define

$$
\begin{aligned}
(3.2) \quad I = &\ \{(m,k) : m > k, k > 1\}, \quad II = \ \{(m,k) : m > k, k < 1\}, \\
III = &\ \{(m,k) : m < k, k > 1\}, \quad IV = \ \{(m,k) : m < k, k < 1\}.
\end{aligned}
$$

We will first discuss the exact multiplicity for (m,k) in these regions, and then discuss the cases for the border between them.

(1) From the proof of Lemma 6, we know that $f'(0) = m - k$ and $f''(0) = 2(k-1)$. First, if $(m,k) \in IV$, then $f'(0) < 0$ and $f''(u) < 0$ for all $u > 0$, and hence $f(u) < 0$ for all $u > 0$, so that (1.4) has no positive solution. For $(m,k) \in III$, $f(u) < 0$ near $u = 0$, and it is possible that f has one or two zero(s) in $(0,\infty)$. In fact, the zeros of $f(u)$ are 0 and

$$u_{\pm} = \frac{(m-1) \pm \sqrt{(m+1)^2 - 4k}}{2},$$

so that in III $u_{\pm} > 0$ if and only if $m > 1$, $k > m$ and $(m+1)^2 - 4k > 0$. So if $(m,k) \in III$ and $(m+1)^2 - 4k \le 0$, then $f(u) \le 0$ for all $u > 0$, and (1.4) has no positive solution. Even when $(m+1)^2 - 4k > 0$ and $(m,k) \in III$, (3.1) is a necessary condition for the existence of positive solution, as follows from (2.21). Indeed, $F(u_+) > F(u(0)) > 0$. When $(m+1)^2 - 4k > 0$ and $(m,k) \in III$, $f(u) < 0$ for $u \in (0, u_-)$ and $f(u) > 0$ for $u \in (u_-, u_+)$. So if there is a solution, then $u(0) \in (u_-, u_+)$, and $F(u_+) > 0$.

(2) If $(m,k) \in III$, $(m+1)^2 - 4k > 0$ and (3.1) is satisfied, then all conditions in Lemma 5 and Theorem 5 are satisfied, so we can apply Theorem 5 to prove part 3 of the theorem.

(3) For $(m,k) \in I$, $f'(0) > 0$, $f''(u) > 0$ for $u \in (0, \sqrt[3]{k} - 1)$ and $f''(u) < 0$ for $u \in (\sqrt[3]{k} - 1, \infty)$. From Lemma 6, f also satisfies (1.3). So we can apply Theorem 3.

(4) For $(m,k) \in II$, $f'(0) > 0$, $f''(u) < 0$ for all $u > 0$. Then the result in Theorem 6 is well-known in this case, see for example, [19] Theorem 6.2.

Also in this case, the bifurcation diagram is same as the one with $k = 0$, the classical logistic equation.

(5) Finally we handle the border line cases. For $k = 0$, it is the classical logistic equation, and the bifurcation curve is same as those for $(m, k) \in II$. For $m = 0$, it is obvious that $f(u) < 0$ for all $u > 0$ and thus there is no solution. For the border of II and IV, $m = k$ and $k < 1$, then $f'(0) = 0$ and $f''(0) < 0$, so $f(u) < 0$ for all $u > 0$ and no solution. For the border of I and II, the bifurcation curve is same as those for $(m, k) \in II$ except that at $(\lambda, u) = (\lambda_1^0, 0)$, $\lambda'(0) = 0$ but $\lambda''(0) > 0$. (For a formula for $\lambda''(0)$, see Lemma 3 or [21].) For the border of III and IV, it is obvious that $f(u) < 0$ for all $u > 0$ and hence no solution. For the border of I and III, $f'(0) = 0$, $f''(0) > 0$, and the conditions of Theorem 4 are all satisfied, thus the solution curve is exactly $\subset$-shaped. $\square$

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