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MULTIPLICITY RESULTS FOR TWO CLASSES OF BOUNDARY-VALUE PROBLEMS*

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Abstract. Multiplicity results are provided for two classes of boundary-value problems with cubic nonlinearities, depending on a parameter λ. In particular, it is proved that for sufficiently large λ, there are exactly two solutions, and that all solutions lie on a single smooth solution curve. The last fact allows one to use continuation techniques to compute all solutions.

Key words. multiplicity results, bifurcation of solutions

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1. Introduction.

We consider a Dirichlet problem of the type

$$u'' + \lambda f(x,u) = 0 \quad \text{on } (a,b), \qquad u(a) = u(b) = 0 \tag{1}$$

for two classes of cubic nonlinearities depending on a parameter λ, and we prove existence and multiplicity results. We also study in detail the solution branches as $\lambda \to \infty$. For both types of nonlinearities we show existence of a critical λ_1, such that for $0 < \lambda < \lambda_1$, (1) has no nontrivial solution; it has at least one solution at $\lambda = \lambda_1$; and it has at least two solutions for $\lambda > \lambda_1$, with precisely two solutions for λ sufficiently large (nontrivial solutions that we find are positive by the maximum principle). Moreover, all solutions lie on a single curve of solutions. The last assertion is important for computational purposes, since it allows one to use efficient continuation techniques to compute all solutions of (1).

Exact multiplicity results are usually difficult to establish; see, e.g., Lions [5]. Our main tools are a bifurcation theorem of Crandall and Rabinowitz [2], and a variational argument due to Ambrosetti and Rabinowitz; see [7]. For both problems it is relatively easy to show that there are no solutions for sufficiently small $\lambda > 0$. We then show that for sufficiently large λ the functional corresponding to (1) has at least two critical points: a minimum point (corresponding to the stable maximal solution of (1)), and a saddle point (corresponding to the unstable minimum solution). To show that there are exactly two solutions for sufficiently large λ, we show that all solutions must lie on certain curves in the (λ, u) "plane." We then study the properties of these curves and exclude the possibility of more than two solutions.

The equations that we study have attracted considerable attention. For constant $a(x)$ and $b(x)$, problems (3) and (21) were studied by Smoller and Wasserman [10] (see also [11] and [12]), who obtained exact multiplicity results by a very nontrivial phase plane analysis. The Neumann problem for (3) was studied in detail by Angenent, Mallet-Paret, and Peletier [1] and Rocha [8]; see also Hale [3]. For f independent of x, both Neumann and Dirichlet problems were studied extensively by Schaaf [9].

Our approach appears to be quite general. We intend to consider other equations where exact multiplicity might be three or more for some values of λ. We are also working to extend our results to partial differential equations.

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Next we list some background results. Recall that a function $\varphi(x) \in C^2(a,b) \cap C^0[a,b]$ is called a supersolution of (1) if

$$\varphi'' + \lambda f(x,\varphi) \leq 0 \quad \text{on } (a,b), \quad \varphi(a) \geq 0, \quad \varphi(b) \geq 0. \tag{2}$$

A subsolution $\psi(x)$ is defined by reversing the inequalities in (2). The following result is standard.

LEMMA 1. *Let $\varphi(x)$ and $\psi(x)$ be, respectively, super- and subsolutions of* (1), *and $\varphi(x) \geq \psi(x)$ on (a,b) with $\varphi(x) \not\equiv \psi(x)$; then $\varphi(x) > \psi(x)$ on (a,b).*

We shall often use this lemma with either $\varphi(x)$ or $\psi(x)$ or both being the solution of (1). The following lemma is a consequence of the first.

LEMMA 2. *Let $u(x)$ be a nontrivial solution of* (1) *with $f(x,0) \equiv 0$. If $u(x) \geq 0$ on (a,b) then $u > 0$ on (a,b).*

We proved the following proposition in [4].

PROPOSITION 1. *Consider the problem* (1) *and assume that $f(x,u) \in C^1([-1,1] \times R_+)$ satisfies*

(i) *$f(-x,u) = f(x,u)$ for all $x \in (-1,1)$ and $u > 0$;*

(ii) *$xf_x(x,u) < 0$ for all $x \in (-1,1)\backslash\{0\}$ and $u > 0$.*

Then any positive solution of (1) *is an even function with $u'(x) < 0$ on* (0, 1]. *Moreover, any two positive solutions of* (1) *do not intersect.*

Remark. Except for the last statement, this proposition is included in the Gidas–Ni–Nirenberg theorem.

Next we state a bifurcation theorem of Crandall and Rabinowitz [2].

THEOREM 1 [2]. *Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and $\operatorname{codim} R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \rightarrow (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

Throughout this paper we consider only the classical solutions (which is not a serious restriction in the one-dimensional case). We also assume, without loss of generality, that $(a,b) = (-1,1)$.

2. A class of cubic nonlinearities with double root.

On the interval $[-1,1]$ we consider the following boundary-value problem:

$$u'' + \lambda a(x) u^2 (1 - b(x)u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{3}$$

We assume throughout this section that $a(x)$ and $b(x)$ are even functions $a(x) \in C^1(-1,1) \cap C^0[-1,1], b(x) \in C^2(-1,1) \cap C^0[-1,1]$, satisfying the following conditions:

$$a(x), b(x) > 0 \quad \text{for } -1 \leq x \leq 1; \tag{4}$$

$$xb'(x) > 0 \quad \text{and} \quad xa'(x) < 0 \quad \text{for } x \in (-1,1)\backslash\{0\}; \tag{5}$$

$$b''(x)b(x) - 2b'^2(x) > 0 \quad \text{for } -1 < x < 1. \tag{6}$$

For example, $b(x) = x^2 + \alpha$ with $\alpha > 3$ satisfies the above conditions. Notice that condition (6) implies that $1/b(x)$ is a supersolution of (3). To prove our multiplicity result we need the following lemmas. Recall that by maximum principle any solution of (3) is positive on $(-1,1)$.

LEMMA 3. *Every solution of* (3) *is strictly concave, i.e.,* $u'' < 0$ (*or* $1 - b(x)u > 0$) *for all* $x \in (-1, 1)$.

Proof. Denote $w(x) = b(x)u(x)$. Then one computes

$$w'' + \frac{\lambda a(x)}{b(x)} w^2(1 - w) = 2b'u' + b''u. \tag{7}$$

If x_0 is a maximum point of $w(x)$, then

$$0 = w'(x_0) = b'(x_0)u(x_0) + b(x_0)u'(x_0),$$

i.e.,

$$u'(x_0) = -\frac{b'(x_0)}{b(x_0)} u(x_0).$$

Using this in (7), we obtain

$$w''(x_0) + \frac{\lambda a(x_0)}{b(x_0)} w^2(x_0)(1 - w(x_0)) = \frac{u(x_0)}{b(x_0)}(b''(x_0)b(x_0) - 2b'^2(x_0)). \tag{8}$$

By our assumptions, the right-hand side of (8) is positive, while $w''(x_0) \leq 0$. Hence $w(x_0) < 1$, i.e., $1 - b(x)u(x) > 0$ for all $x \in (-1, 1)$, and the proof follows.

LEMMA 4. *Every solution of* (3) *is an even function with* $u'(x) < 0$ *for* $x \in (0, 1]$.

Proof. Using Lemma 3 one sees that Proposition 1 applies, giving the conclusions of the lemma.

LEMMA 5. *Let* $u_\lambda(x)$ *be a continuous-in-*λ *branch of solutions of* (3). *Then either* $\lim_{\lambda \to \infty} u_\lambda(x) = 0$ *or* $\lim_{\lambda \to \infty} u_\lambda(x) = 1/b(x)$ *for all* $x \in (-1, 1)$.

Proof. Rewrite (3) in the form

$$u_\lambda(x) = \lambda \int_{-1}^{1} G(x, \xi) a(\xi) u_\lambda^2(\xi)(1 - b(\xi)u_\lambda(\xi))\, d\xi, \tag{9}$$

where $G(x, \xi)$ is the corresponding Green's function, which is easily seen to be strictly positive and bounded on $(-1, 1) \times (-1, 1)$. By Lemma 3, $u_\lambda(x)$ is bounded as $\lambda \to \infty$ (by $1/b(x)$), and the integral on the right in (9) is positive. It follows that for each $\xi \in (-1, 1)$ either $\lim_{\lambda \to \infty} u_\lambda(\xi) = 0$ or $\lim_{\lambda \to \infty} u_\lambda(\xi) = 1/b(\xi)$. Finally, since by Lemma 4 $u'_\lambda(\xi) < 0$ for $\xi \in (0, 1)$, it follows that only one of the above possibilities holds for all ξ.

If $u(x)$ is a solution of (3), then the corresponding linearized problem will be used in the sequel

$$w'' + \lambda a(x)(2u - 3b(x)u^2)w = 0, \qquad w(-1) = w(1) = 0. \tag{10}$$

LEMMA 6. *If* (11) *has a nontrivial solution, then* $w(x)$ *does not change sign on* $(-1, 1)$, *i.e., we can choose it so that* $w(x) > 0$ *on* $(-1, 1)$.

Proof. Assume that $w(x)$ changes sign in $(-1, 1)$. Assume that $w(x)$ has a zero on $[0, 1)$, and the other case is similar. Without loss of generality (taking $-w$ if necessary), we may assume that $w(x) < 0$ on $(x_1, x_2), 0 \leq x_1 < x_2 \leq 1, w(x_1) = w(x_2) = 0$, and $w(x) > 0$ for $x < x_1$ and close to x_1, and for $x > x_2$ and close to x_2 (unless $x_2 = 1$). Differentiating (3), we obtain

$$(u')'' + \lambda a(x)(2u - 3b(x)u^2)u' = -\lambda a'u^2(1 - bu) + \lambda ab'u^3. \tag{11}$$

Multiply (10) by u', (11) by w, and subtract and integrate both sides. Obtain

$$(12) \qquad [w'u' - w(u')']|_{x_1}^{x_2} = \lambda \int_{x_1}^{x_2} [a'u^2(1-bu) - ab'u^3]w\,dx.$$

The quantity on the right side in (12) is positive by our assumptions. The one on the left is equal to

$$w'(x_2)u'(x_2) - w'(x_1)u'(x_1),$$

which is negative by Lemma 4. The contradiction proves the lemma.

LEMMA 7. *Let* $u(x)$, *the solution of* (3), *be such that* $\max_{[-1,1]} b(x)u(x) \le \frac{1}{2}$. *Then the only solution of* (10) *is* $w \equiv 0$.

Proof. Since $u(x) > 0$ solves (3), it is the principal eigenfunction of

$$z'' + \lambda a(x)(u - b(x)u^2)z = \mu z, \qquad z(-1) = z(1) = 0,$$

corresponding to the principal eigenvalue $\mu = 0$. The principal eigenvalue of

$$(13) \qquad w'' + \lambda a(x)(2u - 3b(x)u^2)w = \mu w, \qquad w(-1) = w(1) = 0$$

must be positive, since $2u - 3bu^2 \ge u - bu^2$ for all $x \in (-1,1)$, with inequality being strict near $x = \pm 1$, by our assumption. If $w(x)$ is a nontrivial solution of (10), it is a nonprincipal eigenfunction of (13) (corresponding to $\mu = 0$), and so it must change sign on $[-1,1]$. But this contradicts the previous lemma.

THEOREM 2. *There exists a critical* λ_1, *such that for* $0 < \lambda < \lambda_1$ *the problem* (3) *has no solution*; *it has at least one solution at* $\lambda = \lambda_1$; *and it has at least two solutions for* $\lambda > \lambda_1$. *All solutions lie on a single curve of solutions, which is smooth in* λ. *For each* $\lambda > \lambda_1$ *there are finitely many solutions, and different solutions are strictly ordered on* $(-1,1)$. *Moreover, there exists* $\lambda_2 \ge \lambda_1$, *so that for* $\lambda > \lambda_2$ *the problem* (3) *has exactly two solutions denoted by* $u^-(x,\lambda) < u^+(x,\lambda)$, *with* $u^+(x,\lambda)$ *strictly monotone increasing in* $\lambda, u^-(0,\lambda)$ *strictly monotone decreasing in* λ, *and* $\lim_{\lambda\to\infty} u^+(x,\lambda) = 1/b(x), \lim_{\lambda\to\infty} u^-(x,\lambda) = 0$ *for all* $x \in (-1,1)$. (*Recall that all solutions of* (3) *are positive by maximum principle.*)

Proof. Multiply (3) by u and integrate

$$(14) \qquad \int_{-1}^{1} u'^2 dx = \lambda \int_{-1}^{1} a(x)u^2u(1-b(x)u)\,dx.$$

By the Poincaré inequality,

$$\int_{-1}^{1} u'^2 dx \ge \frac{\pi^2}{4} \int_{-1}^{1} u^2 dx.$$

On the other hand,

$$\int_{-1}^{1} a(x)u^2u(1-b(x)u)\,dx \le \frac{a(0)}{4b(0)} \int_{-1}^{1} u^2\,dx.$$

Thus (3) has no solution for $\lambda < \pi^2 b(0)/a(0)$.

Existence of at least two solutions for sufficiently large λ follows similarly to the proof of a theorem of Ambrosetti and Rabinowitz; see [7, p. 12]. We outline the argument. Solutions of (3) are critical points on $H_0^1(-1,1)$ of the functional

$$J(u) = \int_{-1}^{1} \left(\frac{1}{2}u'^2 - \lambda a(x)\frac{u^3}{3} + \lambda a(x)b(x)\frac{u^4}{4} \right) dx.$$

It is easy to show that $J(u)$ is bounded from below, so that it must have a global minimum. By the Poincaré inequality, $J(u)$ is positive in a small neighborhood of zero in $H_0^1(-1,1)$. If we now can exhibit a function for which $J(u) < 0$, then in addition to a global minimum, where $J(u) < 0$, the functional $J(u)$ will have another critical point, where $J(u) > 0$, in view of the well-known mountain pass theorem; see [7]. It is easy to check that

$$J\left(\frac{1}{b(1)} \cos \frac{\pi}{2} x \right) < 0$$

for sufficiently large λ. (Alternatively, we could consider the evolution equation corresponding to (3) with the initial data

$$u(x,0) = \frac{1}{b(1)} \cos \frac{\pi}{2} x.$$

It is easy to show that $0 < u(x,t) \le c$ for some $c > 0$, and so by well-known results, $u(x,t)$ would have to converge as $t \to \infty$ to the set of solutions of (3). Since $J(u(x,0)) < 0$ for sufficiently large λ, and $J(u(x,t))$ is nonincreasing in t, it follows that $u(x,t)$ cannot converge to zero. This would provide us with at least one positive solution of (3), which is sufficient for the arguments that follow.)

It is clear that the problem (3) has a maximal solution for λ large. We now study the curve of maximal solutions for decreasing λ. Rewrite (3) as

$$F(\lambda, u) = u'' + \lambda a(x) u^2 (1 - b(x)u) = 0, \tag{15}$$

where $F : R \times C_0^2[-1,1] \to C[-1,1]$. Notice that $F_u(\lambda, u)w$ is given by the left-hand side of (10).

Now let $(\lambda_1, u(x))$ be a solution of (15). If the corresponding linearized equation (10) has only a trivial solution $w = 0$, then by the implicit function theorem we can solve (15) for $\lambda < \lambda_1$ and λ close to λ_1, obtaining a continuous-in-λ branch of solutions. We cannot continue this process of decreasing λ indefinitely, since we know that for $\lambda > 0$ sufficiently small, (15) has no solution. Let λ_0 be the infimum of λ for which we can continue the branch to the left. We claim there is a sequence $\{\lambda_n\}$ and $u_{\lambda_0} \in C_0^2(-1,1)$, a solution of (15) at $\lambda = \lambda_0$, so that as $\lambda_n \downarrow \lambda_0, u_{\lambda_n} \to u_{\lambda_0}$. Indeed, using Lemma 3, we conclude that there is a number $M > 0$, such that for any solution of (15),

$$\|u_\lambda\|_{C_0^2[-1,1]} \le M.$$

It follows that a subsequence of $\{u_{\lambda_n}\}$ converges uniformly on $[-1,1]$. Passing to the limit in the integral version of (15) (see (9)), we establish the claim.

By the definition of λ_0 it follows that $F_u(\lambda_0, u_{\lambda_0})$ is singular, i.e., (10) has a nontrivial solution, which is positive by Lemma 6. By Lemma 6 one sees that $N(F_u(\lambda_0, u_{\lambda_0})) = \text{span}\{w(x)\}$ is one-dimensional, and then $\text{codim} R(F_u(\lambda_0, u_{\lambda_0})) = 1$, since $F_u(\lambda_0, u_{\lambda_0})$ is a Fredholm operator of index zero. To apply the Crandall–Rabinowitz theorem (Theorem 1) it remains to check that $F_\lambda(\lambda_0, u_{\lambda_0}) \notin R(F_u(\lambda_0, u_{\lambda_0}))$. Assuming the contrary would imply the existence of $v(x) \not\equiv 0$, such that

$$(16) \qquad v'' + \lambda_0(2au_0 - 3abu_0^2)v = au_0^2(1 - bu_0), \quad -1 < x < 1, \quad v(-1) = v(1) = 0.$$

Multiplying (16) by w, (10) by v, and integrating and subtracting, we obtain

$$\int_{-1}^{1} a(x)u_0^2(x)(1 - b(x)u_0(x))w(x)\,dx = 0,$$

which is a contradiction in view of Lemmas 3 and 6.

Applying Theorem 1, we conclude that $(\lambda_0, u_{\lambda_0})$ is a bifurcation point, near which the solutions of (3) form a curve $(\lambda_0 + \tau(s), u_{\lambda_0} + sw + z(s))$ with s near $s = 0$, and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. It follows that for λ close to λ_0 and $\lambda > \lambda_0$ we have two solutions $u^-(x,\lambda)$ and $u^+(x,\lambda)$ with $u^-(x,\lambda) < u^+(x,\lambda)$ for all $x \in (-1,1)$, and that $u^+(x,\lambda)$ is strictly increasing in λ while $u^-(x,\lambda)$ is strictly decreasing. We show next that the upper branch $u^+(x,\lambda)$ is increasing in λ for all $\lambda > \lambda_0$. Differentiate (3) in λ:

$$(17) \qquad u_\lambda'' + \lambda a(2u - 3bu^2)u_\lambda = -au^2(1 - bu), \qquad u_\lambda(-1) = u_\lambda(1) = 0.$$

We know by the above that $u_\lambda(x,\lambda) > 0$ for λ close to λ_0 and all $x \in (-1,1)$. Let λ_1 be the first λ where this inequality is violated, i.e., $u_\lambda(x,\lambda_1) \geq 0$ and $u_\lambda(x_0,\lambda_1) = 0$ for some $x_0 \in (-1,1)$. Applying the strong maximum principle to (17), we conclude that $u_\lambda(x,\lambda_1) > 0$ for all $x \in (-1,1)$. Thus $u^+(x,\lambda)$ is strictly increasing in λ for all $\lambda > \lambda_0$.

After turning right the curve of solutions will decrease in λ, until a possible turn to the left occurs. If that happens, Theorem 1 applies exactly as above, and monotonicity of the branches follows similarly, so that after the turn the curve of solutions is increasing in λ (i.e., as we follow the curve for decreasing λ, the solution is decreasing). By the same reasoning as used previously, the curve will eventually have to turn to the right and decrease in λ, and so on. Denote by $(\lambda_i, u_i(x))$ the turning points (i.e., $F(\lambda_i, u_i)$ is singular).

We claim that the set of turning points is finite. Indeed, assuming the contrary, we first rule out a finite accumulation point $\bar{\lambda}$, i.e., $\lambda_{i_k} \to \bar{\lambda}$ along a subsequence. As previously, we show that a subsequence of u_{i_k} converges uniformly on $[-1,1]$ to a solution $\bar{u}(x)$ of (3). Clearly $F_u(\bar{\lambda}, \bar{u})$ is singular (since otherwise the implicit function theorem would imply local uniqueness of the solution near $(\bar{\lambda}, \bar{u}(x))$). But then we have a contradiction with Theorem 1, which tells us that there can be no more than two solutions near $(\bar{\lambda}, \bar{u}(x))$. Next we rule out an infinite sequence of $\lambda_i \to \infty$. Notice that $u_{i+1}(x) < u_i(x)$ for all $i \geq 1$ and all $x \in (-1,1)$. By Lemma 5, $u_i(x) \to 0$ as $i \to \infty$, but then we get a contradiction with Lemma 7, which tells us that there can be no bifurcations for sufficiently small u.

We now return to the curve of maximal solutions and follow it for increasing λ. If it turns to the left then Theorem 1 applies, and the curve is decreasing in λ after the turn (i.e., $u(x)$ is increasing when λ is decreasing). Since solutions of (3) are bounded, it follows as above that over any finite interval of λ's there is only a finite number of turns, and the final turn is to the right. Since all the while the solution is increasing, it follows by Lemma 5 that it approaches $1/b(x)$ as $\lambda \to \infty$. We show next that for sufficiently large λ bifurcation is impossible, so that the curve of solutions keeps moving to the right in the (λ, u) "plane." Indeed, let $w(x)$ be a nontrivial solution of the linearized equation (10), normalized so that $\int_{-1}^{1} w^2\,dx = 1$. Multiply (10) by w, integrate by parts, and use the Poincaré inequality, obtaining

$$\int_{-1}^{1} a(x)(2u - 3b(x)u^2)w^2\,dx \geq \frac{\pi^2}{4\lambda}. \tag{18}$$

Since the quantity on the left is negative for u close to $1/b(x)$, we have a contradiction, which shows that (10) can have only trivial solution for λ large. (That $w(x)$ cannot concentrate near $x = +1$ follows similarly to Lemma 6.)

To recapitulate, we have a smooth curve of solutions which after a possible finite number of turns has a decreasing and single-valued-in-λ lower branch tending to zero, and a monotone increasing and single-valued-in-λ upper branch tending to $1/b(x)$. We show next that there is only one such curve. Indeed, assuming two such curves we would have for sufficiently large λ two upper branches, $v = v(x, \lambda)$ and $u = u(x, \lambda)$, both tending to $1/b(x)$. Denoting $w = u - v$, we express

$$w'' + p(x)w = 0 \quad -1 < x < 1, \quad w(-1) = w(1) = 0,$$

where $p(x) = a(x)[u + v - b(x)(u^2 + uv + v^2)]$ is negative for $u(x)$ and $v(x)$ close to $1/b(x)$. This leads to the same contradiction as previously, unless $w \equiv 0$.

Remark 2.1. Consider an interesting class of problems with the nonlinearity resembling the logistic one,

$$u'' + \lambda u^2(b(x) - u) = 0, \qquad u(-1) = u(1) = 0. \tag{19}$$

If $b(x)$ is an even function satisfying $b(x) > 0$ on $[-1, 1], b'(x) < 0$ for $x > 0$, and $b''(x) < 0$ for all $x \in (-1, 1)$, then it is easy to check that Theorem 2 applies.

Remark 2.2. Lemma 7 provides a lower estimate for the maximum value of any solution where bifurcation occurs, $u_m > 1/2b(0)$.

Remark 2.3. If u_m is the maximum value of the solution on the lower branch then

$$\frac{c_1}{\lambda} \leq u_m \leq \frac{c_2}{\lambda}, \quad \lambda > \lambda_2, \quad c_1, c_2 > 0. \tag{20}$$

Indeed, multiplying (3) by u and integrating,

$$\frac{\pi^2}{4}\int_{-1}^{1} u^2\,dx \leq \int_{-1}^{1} u'^2\,dx < \lambda a(0)u_m \int_{-1}^{1} u^2\,dx.$$

On the other hand, since all solutions are concave down, we have $u(x) \geq u_m|x - 1|$. Using this in (9), we easily obtain the second inequality in (20).

Remark 2.4. Based on the numerical evidence we believe that at $\lambda = \lambda_1$ the solution is unique, while for $\lambda > \lambda_1$ there are exactly two solutions.

3. Cubic nonlinearities with distinct roots. In this section we consider the problem

$$u'' + \lambda u(u - a(x))(b - u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{21}$$

Here b is a positive constant, and the function $a(x) \in C^1[-1,1]$ satisfies the following conditions:

$$(22)\quad a(x) \geq a_0 > 0, \quad a'(x) > 0 \quad \text{for } x \in (0,1), \quad a(-x) = a(x) \quad \text{for all } x \in (-1,1);$$

$$(23)\qquad a(x) < \frac{1}{2}b \quad \text{for all } x \in (-1,1).$$

From the maximum principle every solution of (21) satisfies $0 < u < b$ in $(-1,1)$. Notice that, unlike (3), solutions of (21) are concave up near $x = \pm 1$.

LEMMA 8. *The solution of* (21) *is an even function. Moreover, $u_x < 0$ for $x > 0$.*

Proof. Since $0 < u(x) < b$ for all $x \in (-1,1)$, one easily sees that Proposition 1 applies.

LEMMA 9. *Let $u(x,\lambda)$ be a nontrivial solution of* (21) *for $\lambda > \lambda_0$. Then there are only three possibilities for $\lim_{\lambda\to\infty} u(x,\lambda)$: $0, a(x)$, and b. If the solution is increasing in λ then $\lim_{\lambda\to\infty} u(x,\lambda) = b$ for all $x \in (-1,1)$.*

Proof. The first part follows from the integral representation of the solution as before. From the previous lemma we know that for any $\lambda > \lambda_0, u(0,\lambda) > a(0)$. If the solution is increasing in λ this leaves us with $\lim_{\lambda\to\infty} u(0,\lambda) = b$. Indeed, the solution cannot tend to $a(x)$ over a subinterval, since $u_x < 0$ while $a'(x) > 0$, and it cannot tend to $a(x)$ at a point for the same reason.

As previously, we need to consider the linearization of (21),

$$(24)\qquad w'' + \lambda[-3u^2 + 2(a+b)u - ab]w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0.$$

LEMMA 10. *If* (24) *has a nontrivial solution, we can choose it so that $w(x) > 0$ in $(-1,1)$.*

Proof. Assume on the contrary that $w(x)$ changes sign on $(-1,1)$. Assume $w(x)$ has a zero on $(-1,0]$ (the proof is similar if it has a root on $(0,1]$). We may then assume that $w(x) < 0$ on (x_1,x_2) with $-1 \leq x_1 < x_2 \leq 0$, and $w(x_1) = w(x_2) = 0, w'(x_1) < 0, w'(x_2) > 0$ (by changing if necessary to $-w$). Differentiate (21):

$$(25)\qquad u_x'' + \lambda[-3u^2 + 2(a+b)u - ab]u_x = \lambda a' u(b-u).$$

Multiply (25) by w, (24) by u_x, and integrate and subtract:

$$(26)\qquad (u_x' w - u_x w')|_{x_1}^{x_2} = \lambda \int_{x_1}^{x_2} a'(x)u(b-u)w\,dx.$$

The quantity on the right in (26) is positive by our assumptions, while the one on the left is

$$(27)\qquad -u'(x_2)w'(x_2) + u'(x_1)w'(x_1) < 0$$

by Lemma 8.

THEOREM 3. *There exists a critical λ_1, such that for $0 < \lambda < \lambda_1$ the problem* (21) *has no solution; it has at least one solution at $\lambda = \lambda_1$; and it has at least two solutions for $\lambda > \lambda_1$. All solutions lie on a single smooth curve of solutions. For each $\lambda > \lambda_1$ there are finitely many solutions, and different solutions are strictly ordered. Moreover, there exists $\lambda_2 \geq \lambda_1$ so that for $\lambda > \lambda_2$ the problem* (21) *has exactly two solutions denoted by $u^-(x,\lambda) < u^+(x,\lambda)$, and $\lim_{\lambda\to\infty} u^+(x,\lambda) = b$ for all $x \in (-1,1)$. Solution $u^-(x,\lambda)$ develops a spike layer at $x = 0$ as $\lambda \to \infty$.*

Proof. The proof is similar to that of Theorem 2, so we shall not repeat all the details but concentrate on the points that are different. As before we show that (21) has no solutions for sufficiently small $\lambda > 0$. To show existence of at least two solutions for sufficiently large λ, we need to consider the functional

$$J(u) = \int_{-1}^{1} \left(\frac{1}{2}u'^2 + \lambda ab\frac{u^2}{2} - \lambda(a+b)\frac{u^3}{3} + \lambda\frac{u^4}{4} \right) dx$$

on $H_0^1(-1,1)$, and produce a function for which $J(u) < 0$. Consider the functional

$$\bar{J}(u) = \int_{-1}^{1} \left(ab\frac{u^2}{2} - (a+b)\frac{u^3}{3} + \frac{u^4}{4} \right) dx.$$

Using the condition (23) one computes $\bar{J}(b) < 0$. The function $u \equiv b$ does not satisfy the boundary conditions; however, it is clear that one can now construct $u_0(x) \in H_0^1(-1,1)$ with $\bar{J}(u_0(x))$ arbitrarily close to $\bar{J}(b)$, i.e., $\bar{J}(u_0) < 0$. Then for sufficiently large λ we have $J(u_0) < 0$, as desired.

To apply Theorem 1 it remains to verify that $F_\lambda(\lambda_0, u_{\lambda_0}) \notin R(F_u(\lambda_0, u_{\lambda_0}))$, where the map F and $(\lambda_0, u_{\lambda_0})$ are defined the same way as in the proof of Theorem 2. Assuming the contrary, we have $\int_{-1}^{1} u''w\,dx = 0$ (u is solution of (21), w of (24)). Notice that $w(x)$ is an even function (for otherwise the linear problem (24) would have another positive solution $w(-x)$, which is impossible). We then conclude that

$$\int_0^1 u''w\,dx = \int_0^1 u'w'\,dx = \int_0^1 uw''\,dx = 0.$$

Next we multiply (24) by xu_x, (25) by xw, and integrate and subtract. Using the above formula,

$$u'(1)w'(1) + \int_0^1 xa'(x)wu(b-u)\,dx = 0,$$

which is a contradiction, since both terms on the left are positive.

Proceeding as in the proof of Theorem 2, we follow the curve of maximal solutions left until a turning point $\lambda = \lambda_0$. Near that point, Theorem 1 implies existence of two solutions with $u^-(x,\lambda) < u^+(x,\lambda)$ for all $x \in (-1,1)$, and that u^- is decreasing in λ while u^+ is increasing in λ (for λ close to λ_0).

By Lemma 9, as $\lambda \to \infty$, any solution $u(x,\lambda)$ of (21) can only approach $0, b$, or $a(x)$. By Lemma 8, $u(x,\lambda)$ cannot approach $a(x)$ over any interval, since u_x and a' have opposite signs over $(-1,1)\backslash\{0\}$. On the other hand, $u(0,\lambda) > a(0)$, since $x = 0$ is the maximum point of $u (u_{xx}(0,\lambda) < 0)$. It follows that there are just two possibilities as $\lambda \to \infty$: either the solution approaches b for all $x \in (-1,1)$, or the solution approaches zero for $x \in (-1,1)\backslash\{0\}$, while $u(0,\lambda) > a(0)$, i.e., a spike-layer shape. (The possibility that $u^-(x,\lambda)$ approaches b on some proper subinterval of $(-1,1)$, and zero on its complement, is easily ruled out by the argument used in the proof of Proposition 1.)

As in Theorem 2 we show the existence of a smooth curve of solutions, which after possibly finitely many turns, has an upper branch $u^+(x,\lambda)$ single-valued in λ, and tending to b as $\lambda \to \infty$ (notice that for u close to b, (24) has only the trivial solution). The lower branch can also have only (possibly) finitely many turns, and it cannot tend to zero at a finite λ (as can be seen by converting (21) into an equivalent integral equation). It is easy to see that the lower branch cannot approach b as $\lambda \to \infty$ (setting $w = u^+(x, \lambda) - u^-(x, \lambda)$, we obtain an equation similar to (24)). Hence the lower branch has to approach a spike layer shape described above. We next show that as this happens, further bifurcations (turns) are impossible. From (24) we obtain, as previously (normalizing w),

$$\int_{-1}^{1} [-3u^2 + 2(a+b)u - ab]w^2\, dx \geq \frac{\pi^2}{4\lambda}.$$

Since the quantity on the left is negative for u close to the spike layer, it follows that (24) has only the trivial solution.

We now have a smooth curve of solutions, which after a finite number of turns has an upper branch strictly monotone increasing and single-valued in λ and tending to b as $\lambda \to \infty$, and a lower branch single-valued in λ and tending to the spike-layer shape. We next show that there are no other solutions. Indeed, any other solution would have to lie on another curve of solutions, having the same properties. In particular, we would have another upper branch, tending to b, which was already ruled out previously.

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