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ON NONLINEAR SINGULAR PERTURBATION PROBLEMS

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1. INTRODUCTION

IN THIS paper we consider two types of singular perturbation problems. In the first part we consider boundary value problems of the type

$$\Delta u = \varepsilon f(x, u, \mathrm{D}u, \mathrm{D}^2u, \mathrm{D}^3u) \qquad 0 < x_n < 1, \qquad u(x', 0) = u(x', 1) = 0. \tag{1.1}$$

Here $u = u(x)$, $x = (x', x_n)$, $x' \in T^{n-1}$, $0 \le x_n \le 1$, where T^{n-1} is the $n - 1$ dimensional torus (say $T^{n-1} = [0, 2\pi]^{n-1}$), $\mathrm{D}u$, D^2u, D^3u denote all derivatives of u of orders one, two and three. Our work was motivated by the papers of Rabinowitz [5, 6] and Kato [1], who considered the equation (1.1) on the torus T^n (no boundary conditions), and established existence of solutions for sufficiently small ε. Rabinowitz used the Nash–Moser technique, while Kato was using his abstract results on coercive mappings. Both approaches made use of *a priori* estimates in high order Sobolev spaces for either (1.1) or its linearized version. Such estimates involve differentiation of the equation, after which the boundary conditions are in general lost, and hence the results of [1, 5, 6] were restricted to the torus. We show that for the strip-like domains some *a priori* estimates and existence results are possible. Naturally, our conditions on the nonlinear term f are far more restrictive than those of Rabinowitz [5], in particular we allow only those third order terms which are either of the type $u_{x_ix_jx_n}$ or $u_{x_ix_jx_k}$ with $1 \le i, j, k \le n - 1$.

In the second part we consider equations on the torus. In [1], to prove existence, Kato was deriving *a priori* estimates for fully nonlinear equations, which are rather involved. In Section 5 we show that in order to apply the abstract result of Kato it essentially suffices to derive *a priori* estimates for the linearized equation, which is easier. As an application we extend a well-known result of Moser [4].

Next we discuss the notation and state some preliminary results.

We consider functions on the domain $V = T^{n-1} \times [0, 1]$. We shall abbreviate $\int f \equiv \int_V f$, and in Section 5, $\int f = \int_{T^n} f$. We assume that the Roman letters $i, j, k, \ldots$, run from 1 to $n - 1$, while the Greek $\alpha, \beta, \gamma, \ldots$, from 1 to n and denote $u_i = (\partial u/\partial x_i)$, $u_{ij\alpha} = (\partial^3 u/\partial x_i \partial x_j \partial x_\alpha)$, etc. Summation on repeated indices (as in (2.1)) is implied. By $\|\cdot\|_m$ we denote the mth Sobolev norm in V. We shall also need the norms (in V or T^n)

$$|f|_m = \sum_{|\alpha| \le m} |\mathrm{D}^\alpha f|_{L^\infty}, \qquad m = \text{integer} \ge 0.$$

We shall denote

$$b_m = \max_{i,j,\alpha} |b_{ij\alpha}|_m .$$

We denote $u^\beta \equiv D^\beta u$; $a_{i,j} = (\partial a_i/\partial x_j)$, etc. We adopt the convention

$$D^\alpha(uv) = D^\alpha uv + D^{\alpha-1}uv^1 + D^{\alpha-2}uv^2 + \cdots + u\,D^\alpha v, \tag{1.2}$$

where we denote $u^{\alpha-k}v^k = \sum_{|\gamma|=k} c_\gamma D^{\alpha-\gamma}u\,D^\gamma v$ with $\gamma \le \alpha$ and c_γ the coefficients from the Leibnitz rule. By c we denote various positive constants independent of the unknown functions. Sometimes we wish to distinguish positive constants by denoting them c_0, c_1, γ, etc. Define $G^m_{ij\alpha} = \{u \in H^m(V) | u_{ij\alpha} \in H^{m-2}(V)\}$ with the norm $\|u\|_{G^m_{ij\alpha}} = \|u\|_m + \|u_{ij\alpha}\|_{m-2}$.

LEMMA 1.1. For any i, j, α and any $m \ge 2$, the space $G^m_{ij\alpha}$ is a Banach space.

Proof. We need only to check the completeness. Assume that $\|u^p - u^q\|_{G^m_{ij\alpha}} \to 0$ as $p, q \to \infty$. Then $u^p \to u$ in H^m and $u^p_{ij\alpha} \to v$ in H^{m-2}. We claim that $u_{ij\alpha} = v$, so that $u \in G^m_{ij\alpha}$ and $\|u^p - u\|_{G^m_{ij\alpha}} \to 0$. Indeed, for any $\phi \in C_0^\infty(V)$,

$$\int v\phi = \lim_{p\to\infty} \int u^p_{ij\alpha}\phi = -\lim_{p\to\infty}\int u^p\phi_{ij\alpha} = -\int u\phi_{ij\alpha},$$

i.e. $u_{ij\alpha} = v$ in the weak sense.

The following lemma will be used repeatedly.

LEMMA 1.2. Let $b(x) \in C^2(V)$, then

$$\left|\int b(x)w_{ij\alpha}w\right| \le c|b|_2\|w\|_1^2.$$

Proof. Denoting $I = \int b(x)w_{ij\alpha}w$, and integrating by parts, $I = -I + \ldots$, where the terms not shown have $b(x)$ differentiated once. This allows us to estimate I.

2. *A PRIORI* ESTIMATES FOR THE LINEAR PROBLEM

LEMMA 2.1. For the problem

$$\begin{gathered} \Delta u - \varepsilon b_{ij\alpha}(x)u_{ij\alpha} = f(x) \qquad 0 < x_n < 1,\ x' \in T^{n-1}, \\ u(x', 0) = u(x', 1) = 0, \end{gathered} \tag{2.1}$$

assume that $b_2, b_{2m-2} \le c$ (m-nonnegative integer, ε is real). Then for ε sufficiently small the following estimates hold

$$\begin{gathered} \|u\|_m \le c\|f\|_{2m-2} \qquad m \ge 2, \\ \|u\|_1 \le c\|f\|_0. \end{gathered} \tag{2.2}$$

Proof. Step (1). Multiply (2.1) by u and integrate. Using lemma 1.2 we easily obtain

$$\int |\nabla u|^2 \le c\int f^2,$$

provided $b_2 \leq c$, which proves (2.2) for $m = 1$. Next, we multiply (2.1) by u_{kk} and integrate. Similarly,

$$\sum_{k=1}^{n-1} \int |\nabla u_k|^2 \leq c \int f^2. \tag{2.3}$$

To finish the estimation of $\|u\|_2^2$ we need a bound on $\int u_{nn}^2$. When expressed from (2.1), u_{nn} depends on the third order terms, not yet estimated. This leads us to differentiate (2.1) in the tangential directions.

$$\Delta u_{kl} - \varepsilon b_{ij\alpha} u_{ij\alpha kl} - \varepsilon b_{ij\alpha,k} u_{ij\alpha l} - \varepsilon b_{ij\alpha,l} u_{ij\alpha k} - \varepsilon b_{ij\alpha,kl} u_{ij\alpha} = f_{kl}. \tag{2.4}$$

Multiply (2.4) by u_{kl} and integrate. Since by lemma 1.2 (integrating by parts in x_i, x_j and x_α)

$$\left| \int b_{ij\alpha} u_{ij\alpha kl} u_{kl} \right| \leq c b_2 \sum_{k,l=1}^{n-1} \int |\nabla u_{kl}|^2, \tag{2.5}$$

we obtain (for sufficiently small ε)

$$\sum_{k,l=1}^{n-1} \int |\nabla u_{kl}|^2 \leq c \int f_{kl}^2. \tag{2.6}$$

Expressing now u_{nn} from the equation (2.1), and using (2.6) we estimate

$$\int u_{nn}^2 \leq c\left(\sum_{i=1}^{n-1} u_{ii}^2 + \varepsilon^2 \int b_{ij\alpha}^2 u_{ij\alpha}^2 + f^2 \right) \leq c\|f\|_2^2,$$

which together with (2.6) gives us the estimate (2.2) for $m = 2$.

Step (2). To get higher order estimates we differentiate the equation and proceed similarly. Let multi-index $\beta = (\beta', 0)$ with $|\beta| = m - 1$, and denote $u^\beta = D^\beta u$. Differentiate the equation (2.1) (and use the notation described previously), and then multiply by u^β and integrate,

$$-\int |\nabla u^\beta|^2 - \varepsilon \int b_{ij\alpha} u_{ij\alpha}^\beta u^\beta - \varepsilon \int b_{ij\alpha}^1 u_{ij\alpha}^{\beta-1} u^\beta - \cdots - \int b_{ij\alpha}^\beta u_{ij\alpha} u^\beta = \int f^\beta u^\beta.$$

Using lemma 1.2 on the second term on the left, and summing on all β with $\beta_n = 0$, we easily obtain

$$\sum_\beta \int |\nabla u^\beta|^2 \leq c\|f\|_{m-1}^2, \qquad \text{assuming } b_{m-1} \leq c. \tag{2.7}$$

Next we need to estimate the derivatives of order m, where more than one derivative in x_n is allowed. Let now $\beta = (\beta', 0)$ with $|\beta| = m - k$. We shall prove that

$$\int (D_y^k u^\beta)^2 \leq c\|f\|_{m+k-2}^2, \qquad \text{assuming } b_{m+k-2} \leq c, \tag{2.8}$$

using induction on $k \geq 2$. Let $k = 2$, $|\beta| = m - 2$. Express u_{nn} from the equation (2.1) and differentiate,

$$u_{nn}^\beta = -\sum_{i=1}^{n-1} u_{ii}^\beta + \varepsilon b_{ij\alpha} u_{ij\alpha}^\beta + \varepsilon b_{ij\alpha}^1 u_{ij\alpha}^{\beta-1} + \cdots + \varepsilon b_{ij\alpha}^\beta u_{ij\alpha} + f^\beta.$$

Applying (2.7) (with $|\beta| + 2 = m$, $\gamma_n = 0$),

$$\int (u^\beta_{nn})^2 \le c\left(\|f\|^2_{m-1} + \sum_{|\gamma|=m} \int |\nabla u^\gamma|^2 \right) \le c\|f\|^2_m, \qquad \text{provided } b_m \le c.$$

Assuming (2.8) to be true for k we now prove it for $k+1$ (with $|\beta| = m-k-1$). Differentiating the equation (2.1),

$$\mathrm{D}^{k+1}_n u^\beta = -\sum_{i=1}^{n-1} \mathrm{D}^{k-1}_n u^\beta_{ii} + \varepsilon b_{ij\alpha} \mathrm{D}^{k-1}_n u^\beta_{ij\alpha} + \cdots + D^{k-1}_n f^\beta,$$

where $\cdots$ denotes the lower order terms. The second term on the right involves a derivative of u of order $m+1$, which includes up to k differentiations in x_n. Applying (2.8), we obtain

$$\int (\mathrm{D}^{k+1}_n u^\beta)^2 \le c\|f\|^2_{m+k-1}, \qquad \text{assuming } b_{m+k-1} \le c.$$

(By (2.8) when estimating $\int (u^\gamma)^2$ all that matters is $|\gamma|$ and γ_n.) From the estimates (2.8) the lemma follows.

This lemma can be used to give existence and uniqueness results for the problem (2.1). We start with the simplest one.

THEOREM 2.1. Assume that all $b_{ij\alpha}$ are constants, and $\|f\|_{2m-2} \le c$ for $m \ge 1$. Then for ε sufficiently small the problem (2.1) has a unique solution of class H^m.

Proof. Look for the solution in the form $u(x) = \sum_l u_l(x_n) \mathrm{e}^{il\cdot x'}$, with $l = (l_1, \ldots, l_{n-1})$. Then from (2.1), writing, $f = \sum_l f_l(x_n) \mathrm{e}^{il\cdot x'}$,

$$\begin{aligned} u_l'' - |l|^2 u_l - \varepsilon i b_{ijk} l_i l_j l_k u_l - \varepsilon b_{ijn} l_i l_j u_l' &= f_l(x_n) \\ u_l(0) = u_l(1) &= 0. \end{aligned} \tag{2.9}$$

We claim that (2.9) is uniquely solvable for all multi-indices l. Then the proof will follow, since lemma 1.2 will imply convergence of the series $\sum u_l(x_n)\mathrm{e}^{il\cdot x'}$ and the regularity. To prove the claim, write $u_l(x) = v(x) + iw(x)$ with real v and w. Then for the adjoint equation to (2.9)

$$\begin{aligned} v'' - |l|^2 v + \varepsilon b_{ijn} l_i l_j v' + \varepsilon b_{ijk} l_i l_j l_k w &= \operatorname{Re} f_l(x_n) \\ w'' - |l|^2 w + \varepsilon b_{ijn} l_i l_j w' - \varepsilon b_{ijk} l_i l_j l_k v &= \operatorname{Im} f_l(x_n) \\ v(0) = v(1) = w(0) = w(1) &= 0. \end{aligned} \tag{2.10}$$

Assume for the moment that $f_l(x_n) \equiv 0$. Then multiplying the first equation in (2.10) by v, the second one by w, adding and integrating by parts,

$$\int_0^1 (v'^2 + |l|^2 v^2 + w'^2 + |l|^2 w^2)\, \mathrm{d}x_n = 0,$$

i.e. $v(x_n) = w(x_n) = 0$. Applying the Fredholm alternative we get the solvability of (2.9).

3. FURTHER LINEAR THEORY

The estimate (2.2) has the advantage of being rather general, in the sense that any combination (and any number) of the derivatives of the type $u_{ij\alpha}$ are allowed. However, stronger estimates are needed to prove solvability of (2.1) with variable coefficients and of nonlinear problems. This is done in the present section under additional conditions on the structure of the terms present in (2.1).

In Sections 3 and 4 we assume both Roman and Greek indices to run from 1 to $n-1$, i.e. admit only tangential derivatives in the singular perturbation terms.

Definition. Two derivatives of the third order $u_{ij\alpha}$ and $u_{kl\beta}$ are called conjugate if among the indices $i, j, \alpha, k, l, \beta$ there are three pairs of equal ones (different pairs may contain equal elements). For example, u_{113} and u_{355}, or u_{113} and u_{333} are conjugate. It is clear that conjugacy is an equivalence relation, and that after an even number of integrations by parts

$$\int u_{ij\alpha}u_{kl\beta} = \int (u_{pq\gamma})^2 \qquad (p, q, \gamma \text{ are among } i, j, \alpha, k, l, \beta). \tag{3.1}$$

LEMMA 3.1. Consider the problem $(0 < x_n < 1)$

$$\Delta u - \varepsilon \sum b_{ij\alpha}(x)u_{ij\alpha} = f(x), \qquad u(x', 0) = u(x', 1) = 0. \tag{3.2}$$

Assume that any two derivatives of the third order present in (3.2) (i.e. $b_{ij\alpha} \not\equiv 0$) are mutually conjugate; $b_2 \le c$, and that one of the two sets of inequalities hold: either $b_{ij\alpha} \ge c_0 > 0$ or $b_{ij\alpha} \le -c_0 < 0$ for all i, j, α present in (3.2) and all x.

Then for ε sufficiently small

$$\|u\|_2 + \varepsilon \sum \|u_{ij\alpha}\|_0 \le c\|f\|_0 \qquad (c = c(\varepsilon)). \tag{3.3}$$

Here and later on $\sum$ denotes summation on all i, j, α that are present in (3.2).

Proof. Multiply (3.2) by a derivative $u_{kl\beta}$, integrate and use lemma 1.2,

$$-\varepsilon \int b_{kl\beta}(u_{kl\beta})^2 - \varepsilon \sum{}' \int b_{ij\alpha}u_{ij\alpha}u_{kl\beta} = \int fu_{kl\beta}, \tag{3.4}$$

where $\sum'$ is summation on (i, j, α) present in (3.2), which are different from (k, l, β). Integrating by parts (see 3.1),

$$\int b_{ij\alpha}u_{ij\alpha}u_{kl\beta} = \int b_{ij\alpha}(u_{pq\gamma})^2 + \cdots$$

where all the terms not shown on the right have $b_{ij\alpha}$ differentiated exactly once, so that they can be estimated by $b_2\|u\|_2^2$, using lemma 1.2. Then from (3.4)

$$\begin{aligned}\varepsilon^2 c_0 \int (u_{kl\beta})^2 &\le \varepsilon c b_2\|u\|_2^2 + \varepsilon\left|\int fu_{kl\beta}\right| \\ &\le \varepsilon c b_2\|u\|_2^2 + \varepsilon\varepsilon_1 \int u_{kl\beta}^2 + c(\varepsilon_1)\int f^2,\end{aligned}$$

so that by choosing ε_1 small, we estimate

$$\varepsilon^2 \sum \int u_{ij\alpha}^2 \le c\varepsilon\|u\|_2^2 + c(\varepsilon)\int f^2. \tag{3.5}$$

Next, we multiply (3.2) by u_{kk}, integrate by parts using lemma 1.2, and sum

$$\sum_{k=1}^{n-1} \int |\nabla u_k|^2 \le c\varepsilon\|u\|_2^2 + c(\varepsilon)\int f^2. \tag{3.6}$$

Expressing u_{nn} from the equation (3.2), and using (3.5)-(3.6),

$$\int u_{nn}^2 \le c\left(\sum_{i=1}^{n-1} \int u_{ii}^2 + \varepsilon^2 \sum \int u_{ij\alpha}^2 + c(\varepsilon)\int f^2\right)$$

$$\le c\varepsilon\|u\|_2^2 + c(\varepsilon)\int f^2. \tag{3.7}$$

Adding (3.5–3.7) we conclude the lemma.

THEOREM 3.1. In the conditions of lemma 3.1, with $b_{ij\alpha} \in L^\infty(V)$ for all i, j, α, the problem (3.2) is solvable for sufficiently small ε, i.e. for every $f \in L^2(V)$ there is a unique $u \in \bigcap_{i,j,\alpha} G_{ij\alpha}^2(V) \equiv G$ solving (3.2).

Proof. Assume for definiteness that $b_{ij\alpha} \ge c_0$ for all i, j, α and all x. Consider an auxiliary problem

$$\Delta u - \varepsilon \sum c_0 u_{ij\alpha} = f(x), \qquad u(x', 0) = u(x', 1) = 0. \tag{3.8}$$

As in the theorem 2.1, we can write a Fourier series solution for (3.8), which by the estimate (3.3) converges and belongs to G. Next, for $0 \le t \le 1$ we consider a family of problems

$$\Delta u - \varepsilon \sum [c_0(1-t) + tb_{ij\alpha}]u_{ij\alpha} = f(x), \qquad u(x', 0) = u(x', 1) = 0. \tag{3.9}$$

Denote $S = \{t \in [0, 1] \mid (3.9) \text{ has a solution of class } G\}$. Clearly, $0 \in S$. We shall show that S is both open and closed in $[0, 1]$.

(i) Openness. Assume $t_0 \in S$. Define a map T, $u = T\bar{u}$, by solving

$$\begin{aligned} \Delta u - \varepsilon \sum [c_0(1-t_0) + t_0 b_{ij\alpha}(x)]u_{ij\alpha} &= f \\ - \varepsilon \sum [-c_0(t_0 - t) + (t_0 - t)b_{ij\alpha}(x)]\bar{u}_{ij\alpha}&, \end{aligned}$$

with $u(x', 0) = u(x', 1) = 0$. The estimate 3.3 implies that T takes G into itself, and is a contraction for $|t - t_0|$ small.

(ii) Closedness. Assume there is a sequence $\{t_n\}$, such that $t_n \in S$ and $t_n \to \bar{t}$ as $n \to \infty$. Let u^n be solution of (3.9) corresponding to t_n. By (3.3), it follows that $\|u^n\|_2$, $\|u_{ij\alpha}^n\|_0 \le c$ for all i, j, α, so that without loss of generality we may assume that $u^n \rightharpoonup \bar{u}$ in $H^2(v)$, $u_{ij\alpha}^n \rightharpoonup v$ in $L^2(V)$. We claim that $v = \bar{u}_{ij\alpha}$, so that $u \in G_{ij\alpha}^2$. Indeed, letting $\phi \in C_0^\infty(V)$,

$$\int v\phi = \lim_{n\to\infty} \int u_{ij\alpha}^n \phi = -\lim_{n\to\infty} \int u^n \phi_{ij\alpha} = -\int \bar{u}\phi_{ij\alpha} = \int \bar{u}_{ij\alpha}\phi.$$

Passing to the limit in (3.9), we obtain that $\bar{t} \in S$. (Multiplying (3.9) by a test function ϕ, we can first pass to the limit in the integral form, and then conclude (3.9), since ϕ is arbitrary.)

In the next result the singular perturbation terms need not be small.

LEMMA 3.2. Consider the problem $(0 < x_n < 1)$

$$\Delta u + \sum_{l,m=1}^{n-1} b_{lm} u_{lmn} = f(x), \qquad u(x', 0) = u(x', 1) = 0, \tag{3.10}$$

with constant coefficients β_{lm}. Then

$$\|u\|_2 + \sum_{i,j=1}^{n-1} \|u_{ijn}\|_0 \le c\|f\|_1. \tag{3.11}$$

Moreover, for $f \in H^1(V)$ the problem (3.10) has a solution of class $\bigcap_{l,m=1}^{n-1} G^2_{lmn}$.

Proof. Look for the solution in the form $u(x) = \sum_j u_j(x_n) e^{ij\cdot x'}$, where $j = (j_1, \ldots, j_{n-1}) \in Z^{n-1}$. From (3.10) we obtain (writing $f = \sum f_j e^{ij\cdot x'}$)

$$u_j'' - |j|^2 u_j - \sum_{l,m=1}^{n-1} b_{lm} j_l j_m u_j' = f_j, \; u_j(0) = u_j(1) = 0. \tag{3.12}$$

We shall obtain the estimates (3.11) by deriving corresponding estimates for the problem (3.12) without solving it explicitly. Assume first $j \neq 0$. Conjugating (3.12),

$$\bar{u}_j'' - |j|^2 \bar{u}_j - \sum_{l,m=1}^{n-1} b_{lm} j_l j_m \bar{u}_j' = \bar{f}_j, \; \bar{u}_j(0) = \bar{u}_j(1) = 0. \tag{3.13}$$

Multiply (3.12) by $\bar{u}_j$, (3.13) by u_j, integrate both equations and add,

$$-2\int_0^1 |u_j'|^2 \, dx_n - 2|j|^2 \int_0^1 |u_j|^2 \, dx_n = \int_0^1 (\bar{u}_j f_j + u_j \bar{f}_j) \, dx_n.$$

Since $\int_0^1 |u_j|^2 \, dx_n \le c\int_0^1 |u_j'|^2 \, dx_n$, we estimate

$$\int_0^1 (|u_j'|^2 + |j|^2 |u_j|^2) \, dx_n \le c\int_0^1 |f_j|^2 \, dx_n. \tag{3.14}$$

Multiply (3.12) by $j^4 \bar{u}_j$, (3.13) by $j^4 u_j$, integrate and add

$$-2|j|^4 \int_0^1 |u_j'|^2 \, dx_n - 2|j|^6 \int_0^1 |u_j|^2 \, dx_n = |j|^4 \int_0^1 (f_j \bar{u}_j + \bar{f}_j u_j) \, dx_n,$$

from which it easily follows

$$|j|^4 \int_0^1 |u_j'|^2 \, dx_n + |j|^6 \int_0^1 |u_j|^2 \, dx_n \le c|j|^2 \int_0^1 |f_j|^2 \, dx_n. \tag{3.15}$$

Expressing u_j'' from the equation (3.12) and using (3.15),

$$\int_0^1 |u_j''|^2 \, dx_n \le c|j|^2 \int_0^1 |f_j|^2 \, dx_n. \tag{3.16}$$

In case $j = 0$, $\int_0^1 |u_0''|^2 \, dx_n \le c\int_0^1 |f_0|^2 \, dx_n$, which together with the estimates (3.14)–(3.16) establishes the lemma.

Remark 3.1. We discuss here the third order terms, which were not present in the lemmas 2.1 and 3.1. It appears that the term u_{nnn} cannot be allowed as the following simple example shows. The problem

$$\Delta u - \varepsilon u_{nnn} = 0 \qquad 0 < x_n < 1,\ x' \in T^{n-1}$$
$$u(x', 0) = u(x', 1) = 0,$$

has a nontrivial solution $u = e^{(1/\varepsilon)x_n} - (e^{1/\varepsilon} - 1)x_n - 1$, so that no *a priori* estimate like (2.2) or (3.3) is possible. However, the terms of the type u_{nnk} can be included under some conditions. For example, let $u(x, y)$ be solution of (a = const)

$$u_{xx} + u_{yy} + au_{xyy} = f(x, y) \qquad 0 < y < 1,\ x \in T^1,$$
$$u(x, 0) = u(x, 1) = 0.$$

Using the Fourier series analysis one derives an estimate

$$\|u\|_2 + \|u_{xyy}\|_0 \leq c\|f\|_0. \tag{3.17}$$

which can be used to prove existence of solution.

4. NONLINEAR BOUNDARY VALUE PROBLEMS

THEOREM 4.1. Consider the problem

$$\begin{aligned} \Delta u - \varepsilon \sum b_{ij\alpha}(x)u_{ij\alpha} &= \delta f(x, u, \mathrm{D}u, \mathrm{D}^2u, u_{ij\alpha}) \qquad 0 < x_n < 1, \\ u(x', 0) = u(x', 1) &= 0, \qquad x' \in T^{n-1}. \end{aligned} \tag{4.1}$$

Assume that the coefficients $b_{ij\alpha}$ satisfy all conditions of the theorem 3.1, while the function f depends only on those third order derivatives that are present on the left in (4.1), f is of class C^1 and satisfies

$$|f| \leq c\left(1 + |u| + \sum_i |u_i| + \sum_{i,j} |u_{ij}| + \sum_{i,j,\alpha} |u_{ij\alpha}|\right), \tag{4.1}$$

$$|f_u|,\ |f_{u_i}|,\ |f_{u_{ij}}|,\ |f_{u_{ij\alpha}}| \leq c \qquad \text{for all } i, j, \alpha, \tag{4.2}$$

for all values of its arguments. Then for ε and δ sufficiently small ($\delta = \delta(\varepsilon)$) the problem (4.1) has a solution of class G (G was defined in the theorem 3.1).

Proof. Define a map T, $u = Tv$ by solving

$$\Delta u - \varepsilon \sum b_{ij\alpha}u_{ij\alpha} = \delta f(x, v, \mathrm{D}v, \mathrm{D}^2v, v_{ij\alpha}) \qquad 0 < x_n < 1,\ u(x', 0) = u(x', 1) = 0.$$

By the theorem 3.1, for $v \in G$ the map T is well-defined, and takes G into itself. Using the mean value theorem, one easily shows that T is a contraction.

Further nonlinear existence results can be stated, based on the estimates (3.11) and (3.17).

5. QUASILINEAR SECOND ORDER EQUATIONS ON A TORUS

We start by recalling the set-up in Kato [1], in slightly less generality.

(i) Let $\{Y, Y^*\}$ be a pair of real Banach spaces in metric duality, i.e. there is a nondegenerate continuous bilinear form $\langle , \rangle$ on $Y \times Y^*$, with $|\langle y, f\rangle| \leq \|y\|_Y \|f\|_{Y^*}$. (Nondegeneracy means that condition $\langle y, f\rangle = 0$ for all $f \in Y^*$ implies $y = 0$.) Moreover Y is reflexive and separable.

(ii) There is another pair $\{V, V^*\}$ of Banach spaces in duality, with V separable, such that $V \subset Y$ and $V^* \supset Y^*$ with the injections continuous and dense. Moreover, if $v \in V$ and $f \in Y^*$, $\langle v, f\rangle$ has the same value when taken in $Y \times Y^*$ and $V \times V^*$.

(iii) There is a bounded, closed and convex subset K of Y, containing the origin as an internal point, and a weakly sequentially continuous map A of K into V^* (i.e. it takes weakly convergent sequences into weakly convergent ones), such that

$$\langle v, A(v)\rangle \geq 0 \qquad \text{for all } v \in V \cap \partial K. \tag{5.1}$$

THEOREM I (Kato [1]). Under the assumptions (i), (ii), (iii), the equation

$$A(u) = f \tag{5.2}$$

has a solution $u \in K$, provided $\|f\|_{Y^*}$ is sufficiently small.

Remark. We have relaxed the continuity assumption on A. Examining Kato's proof, one verifies that the assumption in (iii) is sufficient.

LEMMA 5.1. In the above notation, let A be a map from $B_r = \{x \in Y \mid \|x\|_Y \leq r\}$ to V^* of class $C^1(B_r, V^*)$, such that for $r \leq r_0$,

$$\langle A'(0)x, x\rangle \geq c_1\|x\|_Y^2 \qquad \text{for all } x \in B_r \cap V. \tag{5.3}$$

In addition assume that

$$\int_0^1 \langle (A'(tx) - A'(0))x, x\rangle \,\mathrm{d}t \geq I \qquad \text{for } 0 \leq t \leq 1, \tag{5.4}$$

where

$$|I| \leq c_2\|x\|_Y^3.$$

Assume finally that

$$\|A(0)\|_{Y^*} < \varepsilon.$$

Then for ε and $\|f\|_{Y^*}$ sufficiently small the problem (5.2) has a solution $u \in B_r$.

Proof. In view of Kato's theorem I we only need to verify (5.1) (with $K = B_r$). Using the Taylor series expansion

$$\begin{aligned}\langle A(x), x\rangle &= \langle A(0), x\rangle + \langle A'(0)x, x\rangle + \left\langle \int_0^1 (A'(tx) - A'(0))x \,\mathrm{d}t, x\right\rangle \\ &\geq c_1\|x\|_Y^2 - \varepsilon\|x\|_Y - c_2\|x\|_Y^3 \geq c_3 r^2 > 0,\end{aligned}$$

for all $x \in \partial B_r \cap V$, provided r and then ε are chosen small enough.

The following result extends a well-known theorem of Moser.

THEOREM 5.1. Consider the equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + a(x, u, \mathrm{D}u) = f(x), \qquad x \in T^n, \tag{5.5}$$

where T^n is the n-dimensional torus. The unknown function $u(x)$ and the given functions $a_{ij}(x)$ and $f(x)$ are real-valued on T^n, $a_{ij} \in C^s(T^n)$, and $a = a(x, u, p_1, \ldots, p_n)$ is a given function on $T^n \times R^{n+1}$. Denote $a_i(x) = \partial a/\partial p_i(x, 0, 0)$, $a_0(x) = \partial a/\partial u(x, 0, 0)$.

Let s be an integer $\geq \left[\frac{n}{2}\right] + 5$, $a \in C^{s+2}(T^n \times R^{n+1})$. Assume that

$$a(x, 0, 0) \equiv 0, \tag{5.6}$$

$$|\xi|^2\left(a_0 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij,ij}\right) + s \sum_{i,k=1}^{n} a_{i,k}\xi_i\xi_k \geq \gamma|\xi|^2 \tag{5.7}$$

for some $\gamma > 0$ and all $\xi \in R^n$;

$$|a_{ij}|_2 < \delta; \ -\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \qquad \text{for all } \xi \in R^n,\ x \in T^n. \tag{5.8}$$

Then for δ and $\|f\|_s$ sufficiently small the problem (5.5) has a solution $u(x) \in H^s(T^n)$.

Proof. Let $\Lambda = (-\Delta)^{1/2}$. As in Kato [1] we will use the following inner product in the Sobolev space H^s,

$$(u, v)_s = (\Lambda^s u, \Lambda^s v)_0 + \lambda^2(u, v)_0,$$

and the associated norm $\sqrt{(u, u)_s}$, where $(,)_0$ denotes the inner product in L^2. Notice that this norm is equivalent to the usual one in H^s, and that formally

$$(u, v)_s = (-1)^s(u, \Delta^s v)_0 + \lambda^2(u, v)_0,$$

$\sqrt{(v, v)_s}$ is then a new norm on $Y = H^s(T^n)$. However by $\|\cdot\|_s$ we shall denote the usual norm on $H^s(T^n)$.

We shall verify that the assumptions of the Kato's theorem I are satisfied for the operator A defined by

$$A(u(x)) = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + a(x, u, \mathrm{D}u).$$

We introduce the Banach spaces

$$Y = Y^* = H^s(T^n), \qquad V = H^{s+2}(T^n),\ V^* = H^{s-2}(T^n),$$

with the dualities $\langle , \rangle$ given by

$$\langle v, g\rangle = (v, g)_s \qquad \text{for } v, g \in Y,$$
$$\langle v, g\rangle = (\Delta^2 v, g)_{s-2} \qquad \text{for } v \in V,\ g \in V^*.$$

Compute

$$A'(0)v = a_{ij}v_{ij} + a_i v_i + a_0 v \ (a_i, a_0 \text{ were defined above}).$$

According to lemma 5.1 we need to verify

$$\langle A'(0)v, v\rangle = (A'(0)v, v)_s \geq c\|v\|_s^2 \qquad \text{for } v \in B_r \cap V. \tag{5.9}$$

We assume $s = 2l$ with the other case being similar. Then denoting $w = (-\Delta)^{l-1}v$ (and using the notation defined in (1.2)),

$$\begin{aligned}(A'(0)v, v)_s &= ((-\Delta)^l A'(0)v, (-\Delta)^l v)_0 + \lambda^2(v, v)_0\\ &= (a_{ij}\Delta w_{ij} + a_i\Delta w_i + a_0\Delta w, \Delta w)_0\\ &\quad + s\left(\sum_{k=1}^{n} a_{ij,k}w_{ijk} + \sum_{k=1}^{n} a_{i,k}w_{ik} + \sum_{k=1}^{n} a_{0,k}w_k, \Delta w\right)_0\\ &\quad + (a_{ij}^2 v_{ij}^{s-2} + a_i^2 v_i^{s-2} + a_0^2 v^{s-2}, (-\Delta)^l v)_0\\ &\quad + \cdots + (a_{ij}^s v_{ij} + a_i^s v_i + a_0^s v, (-\Delta)^l v)_0 + \lambda^2(v, v)_0.\end{aligned} \tag{5.10}$$

Since for $|\alpha| = s$ and $|\beta| = p < s$,

$$(D^{\alpha}v, D^{\beta}v)_0 \leq \delta\|v\|_s^2 + c(\delta)\|v\|_p^2 \leq 2\delta\|v\|_s^2 + c_1(\delta)\|v\|_0^2, \tag{5.11}$$

we only need to estimate the terms in (5.10) with $|\alpha| + |\beta| \geq 2s$. Integrating by parts

$$\int a_{ij}\Delta w_{ij}\Delta w = -\int a_{ij}\Delta w_i \Delta w_j + \tfrac{1}{2}\int a_{ij,ij}(\Delta w)^2;$$

$$\int a_{i,k}\, w_{ik}\Delta w = \sum_{j=1}^{n}\int a_{i,k}\, w_{ij} w_{kj} + \cdots$$

where the terms not shown can be estimated as in (5.11). The term $(a_{ij,k}\, w_{ijk}, \Delta w)_0$ is estimated by $\delta\|v\|_s^2$ using lemma 1.2. So that by choosing λ sufficiently large,

$$\begin{aligned}(A'(0)v, v)_s \geq{}& \int\left(a_0 - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_i + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij,ij}\right)(\Delta w)^2 \\ &+ s\sum_{i,j,k=1}^{n} a_{i,k}(x) w_{ij} w_{kj} - c\delta\|v\|_s^2 + \tfrac{1}{2}\lambda^2\|v\|_0^2 \\ \geq{}& \gamma_0\int(\Delta w)^2 - c\|w\|_0^2 - c\delta\|v\|_s^2 + \tfrac{1}{2}\lambda^2\|v\|_0^2,\end{aligned}$$

where the last step follows by the assumption (5.7). (Indeed, if a_0, $a_{i,k}$ and $a_{ij,ij}$ were constants, then this would follow by taking the Fourier transform. For the general case proceed as in the Garding's lemma, see [4, p. 311].) So that if δ is sufficiently small, then by choosing λ sufficiently large

$$(A'(0)v, v)_s \geq \frac{\gamma_0}{2}\|v\|_s^2 + \frac{1}{4}\lambda^2\|v\|_0^2 \geq c(v, v)_s.$$

Next we verify the condition (5.4) of lemma 5.1. To simplify the presentation we assume that $a = a(x, Du)$. Then

$$\begin{aligned}(A'(tu) - A'(0))u &= \sum_{i=1}^{n}(a_{p_i}(x, t\,Du) - a_{p_i}(x, 0))u_i \\ &= t\sum_{i,j=1}^{n} a_{p_ip_j}(x, \theta t\,Du)u_i u_j.\end{aligned}$$

Since $u \in H^s$,

$$|u|_2 \leq c\|u\|_s \leq cr < 1 \text{ for sufficiently small } r. \tag{5.12}$$

Setting $u^s = (-\Delta)^{s/2}u$, compute

$$((A'(tu) - A'(0))u, u)_s = \left(\sum_{i,j=1}^{n} a_{p_ip_j}u_i u_j^s, u^s\right)_0 + \left(\sum_{i,j=1}^{n} a_{p_ip_jp_k}u_k^s u_i u_j, u^s\right)_0 + \cdots \tag{5.13}$$

In view of (5.12) both terms shown on the right after one integration by parts are estimated by $c\|u\|_s^3$. Among the terms not shown in (5.13) some have similar structure, others are of the type

$$(\textstyle\sum D_p^{\sigma} D_x^{\sigma_1}a_{p_ip_j}(D^{\alpha_1+\varepsilon_1}u)\cdots(D^{\alpha_k+\varepsilon_k}u)\,D^{\delta}u_i\,D^{\gamma}u_j, u^s)_0, \tag{5.14}$$

with $|\varepsilon_1| = \cdots = |\varepsilon_k| = 1$, $|\alpha_1| + \cdots + |\alpha_k| + |\delta| + |\gamma| \le s$, $k \ge 0$. At this point we recall a well-known inequality (see [1, 4]): if $|\beta_1| + \cdots + |\beta_p| \le s$, then

$$\|D^{\beta_1}f_1 \cdots D^{\beta_p}f_p\|_0 \le \|f_1\|_s|f_2|_0 \cdots |f_p|_0 + \cdots + |f_1|_0 \cdots |f_{p-1}|_0\|f_p\|_s. \tag{5.15}$$

Assume that in (5.14) k, $|\delta|$, $|\gamma| \ge 1$. Write $D^{\gamma_i+\varepsilon_i}u = D^{\beta_i}u^{(i)}$, $D^{\delta}u_i = D^{\beta_{k+1}}u^{(k+1)}$, $D^{\gamma}u_j = D^{\beta_{k+2}}u^{(k+2)}$, where $|\beta_i| = |\gamma_i| - 1$ for $i = 1, \ldots, k$; $|\beta_{k+1}| = |\delta| - 1$, $|\beta_{k+2}| \le |\gamma| - 1$. Then the term in (5.14) is estimated by

$$\begin{aligned} c\|u\|_s\|D^{\beta_1}u^{(1)} \cdots D^{\beta_{k+2}}u^{(k+2)}\|_0 &\le c\|u\|_s(\|u^{(1)}\|_{s-k-2}|u^{(2)}|_0 \cdots |u^{(k+2)}|_0 + \cdots) \\ &\le c\|u\|_s(\|u\|_s|u|_2 \cdots |u|_2 + \cdots) \le c\|u\|_s^3. \end{aligned}$$

If in (5.14) k, $|\sigma| \ge 1$ but $\gamma = 0$, then we estimate $|u_j|$ by $c\|u\|_s$ and proceed as before. Finally if $k = 0$, the term in (5.14) is estimated using (5.15) by

$$c\|u\|_s\|D^{\delta}u_iD^{\gamma}u_j\| \le c\|u\|_s\|u\|_s|u|_2 \le c\|u\|_s^3.$$

Using the mean value theorem one easily verifies that the map $A: B_r \to H^{s-2}$ has two bounded and continuous Fréchet derivatives (see [2, 3] for similar arguments).

Finally to verify the weak sequential continuity of A, assume that $u^j \rightharpoonup u$ in H^s. It follows that $\|u^j\|_s \le c$, and then by the Sobolev's imbedding $|u^j|_2 \le c$. By the well-known Moser's lemma [4], it follows that $\|A(u^j)\|_{s-2} \le c$. The sequence $\{A(u^j)\}$ then has a weakly convergent subsequence in H^{s-2}. Since u^j and u are all $C^2(T^n)$ functions (by the choice of s), it follows that $A(u^{j_k}) \rightharpoonup A(u)$ in H^{s-2}. Repeating this argument for any subsequence of $\{u^j\}$, we see that $A(u^j) \rightharpoonup A(u)$ in H^{s-2}.

Remark. It is natural to try to extend the theorem 5.1 to cover equations of the type

$$a(x, u, Du, D^2u) = f(x), \qquad x \in T^n. \tag{5.16}$$

Under similar assumptions on the linearized operator one verifies condition (5.3) the same way as above. However, we see no way to justify (5.4). The difficulty is that it is not clear how to estimate the terms of the type $(a_{u_{ij}u_{kl}}u_{kl}^s u_{ij}, u^s)_0$ through $c\|u\|_s^3$. In [3] we were able to handle problems of the type (5.16), using the Nash–Moser method and assuming existence of considerably greater number of derivatives than in the theorem 5.1.

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Note added in proof—The estimate mentioned in the above remark can be obtained, provided we additionally assume that

$$\sum_{i,j,k,l} a_{u_{ij}u_{kl}}u_{kl}u_{ij}\xi_k\xi_l \le 0 \qquad \text{for and } (\xi_1, \ldots, \xi_n) \in R^n.$$