Differential and Integral Equations, Volume 6, Number 6, November 1993, pp. 1507–1517.

EXACT MULTIPLICITY RESULTS FOR TWO CLASSES OF BOUNDARY VALUE PROBLEMS

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(Submitted by: A.R. Aftabizadeh)

Abstract. We give a complete description of the set of positive solutions for two classes of boundary value problems, involving both convex and non-convex nonlinearities.

1. Introduction. We consider Dirichlet problem of the type

$$u'' + \lambda f(x,u) = 0 \quad \text{on } (a,b), \quad u(a) = u(b) = 0 \tag{1.1}$$

for two classes of nonlinearities depending on a parameter λ, and prove existence and exact multiplicity results for positive solutions. Our main tool is a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [3]. In both cases we obtain a complete description of the set of positive solutions for all values of λ.

We begin by considering convex in u nonlinearities. We assume that $f(x,u)$ is even in x relative to the midpoint $\frac{a+b}{2}$ of the interval (a,b). This allows us to get rather detailed information about the solution $u(x)$ of (1.1), as well as on the solution of the corresponding linearized problem, which makes it possible to apply the Crandall-Rabinowitz bifurcation theorem. We outline our arguments next. The problem is easily solvable for small λ. Since we assume $f(x,u)$ to be positive on $[-1,1]$ and grow superlinearly in u, there are no positive solutions for large λ. We show that the curve of solutions "bends back" at a critical λ_0, and then study the curve after the bend, obtaining rather detailed description of the solution curve. We show then that there are no other positive solutions by excluding all other possible situations.

Our second class involves non-convex nonlinearities of the type studied by T. Ouyang [7]. Using the results of those papers, we can apply a similar analysis to obtain a complete description of the set of positive solutions.

Our approach appears to be quite general. We have some multiplicity results for polynomial in u nonlinearities, which we defer to future publications. Our results can be used to describe the structure of global attractors for the corresponding parabolic problems, see e.g., J. Hale [5].

Next, we list some background results. Recall that a function $\varphi(x) \in C^2(a,b) \cap C^0[a,b]$ is called a supersolution of (1.1) if

$$\varphi'' + \lambda f(x,\varphi) \le 0 \quad \text{on } (a,b), \quad \varphi(a) \ge 0,\ \varphi(b) \ge 0. \tag{1.2}$$

A subsolution $\psi(x)$ is defined by reversing the inequalities in (1.2). The following result is standard.

Received April 1992.
AMS Subject Classification: 34B15.

Lemma 0. *Let $\varphi(x)$ and $\psi(x)$ be, respectively, super- and subsolutions of* (1.1), *and $\varphi(x) \geq \psi(x)$ on (a,b) with $\varphi(x) \not\equiv \psi(x)$, then $\varphi(x) > \psi(x)$ on (a,b).*

We shall often use this lemma with either $\varphi(x)$ or $\psi(x)$ or both being a solution of (1.1).

Next we state a bifurcation theorem of Crandall-Rabinowitz [3].

Theorem 0. [3] *Let X and Y be Banach spaces. Let $(\bar\lambda, \bar x) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar\lambda, \bar x)$ into Y. Let the null-space $N(F_x(\bar\lambda, \bar x)) = \operatorname{span}\{x_0\}$ be one dimensional and codim $R(F_x(\bar\lambda, \bar x)) = 1$. Let $F_\lambda(\bar\lambda, \bar x) \notin R(F_x(\bar\lambda, \bar x))$. If Z is a complement of $\operatorname{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar\lambda, \bar x)$ near $(\bar\lambda, \bar x)$ form a curve $(\lambda(s), x(s)) = (\bar\lambda + \tau(s), \bar x + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

Throughout the paper we consider only the classical solutions. Also, we assume without loss of generality, that $(a,b) = (-1,1)$.

2. A class of convex nonlinearities.

On the interval $[-1,1]$ we consider the following boundary-value problem

$$u'' + \lambda f(x,u) = 0, \ -1 < x < 1, \ u(-1) = u(1) = 0. \tag{2.1}$$

We assume that $f(x,u) \in C^2([-1,1] \times R_+)$ and satisfies the following conditions:

$$f(-x,u) = f(x,u) \quad \text{for } x \in (-1,1) \ \text{ and } u > 0; \tag{2.2}$$

$$x f_x(x,u) < 0 \quad \text{for } x \in (-1,1) \setminus \{0\} \ \text{ and } u > 0; \tag{2.3}$$

$$f_{uu}(x,u) > 0 \quad \text{for } x \in (-1,1) \quad \text{ and } u > 0; \tag{2.4}$$

$$f(x,u) \geq c_1 u^p + c_2 \quad \text{for } x \in (-1,1), \quad u > 0, \tag{2.5}$$

with constants c_1, $c_2 > 0$ and $p > 1$. A good example is $f(x,u) = h(x)e^u$ with even and positive $h(x)$, such that $h'(x) < 0$ for $x > 0$, a nonlinearity of a type that arises in combustion theory [2]. We shall denote the solution of (2.1) by $u(x,\lambda)$, and by $F(\lambda, u)$ we denote the left hand side of (2.1).

Lemma 1. *Assume that $f(x,u)$ satisfies* (2.2) *and* (2.3). *Then any positive solution of* (2.1) *is an even function, such that $u'(x) < 0$ for $x \in (0,1]$.*

Proof. We show first that $u(x)$ has only one (global) maximum on $(-1,1)$. Assuming the contrary, the function $u(x)$ would have points of local minimum, and assuming some of those are nonnegative (negative are treated similarly) let $x_0 \geq 0$ be the largest point of local minimum. Let $x_0 < x_1 < 1$ be such that $u(x_1) = u(x_0)$, and let $\bar x$ be the point of local maximum on (x_0, x_1). On $(x_0, \bar x)$ we can represent the curve $u = u(x)$ by $x = x_1(u)$, and on $(\bar x, x_1)$ by $x = x_2(u)$, with $x_1(u) < x_2(u)$ for all $u \in (u(x_0), u(\bar x))$. Multiply the equation (2.1) by u' and integrate from x_0, to x_1,

$$\frac{1}{2}u'^2(x_1) + \lambda \int_{x_0}^{x_1} f(x,u)u' \, dx = 0. \tag{2.6}$$

But

$$\int_{x_0}^{x_1} f(x,u)u'\,dx = \int_{x_0}^{\bar{x}} f(x,u)u'\,dx + \int_{\bar{x}}^{x_1} f(x,u)u'\,dx$$
$$= \int_{u(x_0)}^{u(\bar{x})} f(x_1(u),u)\,du - \int_{u(x_0)}^{u(\bar{x})} f(x_2(u),u)\,du > 0.$$

This leads to a contradiction by (2.6).

We show next that any positive solution of (2.1) is even. Assume on the contrary some solution $u(x)$ has its maximum at $\bar{x} > 0$, and let $\bar{u} = u(\bar{x})$. Then by (2.2), $v(x) = u(-x)$ is also a solution of (2.1), with the same maximal value $\bar{u}$, and $v(0) = u(0) = u_0$, $v'(0) = -u'(0)$. We claim that

$$|u'(1)| > |v'(1)|. \tag{2.7}$$

Indeed, if (2.7) is violated then $|u'(1)| < |v'(1)|$ (if $u'(1) = v'(1)$ the solutions coincide). We can find ξ, $\bar{x} < \xi < 1$, such that $u(\xi) = v(\xi) = u_1$, and $v'(\xi)| < |u'(\xi)|$ (if there was no such ξ, $v(x)$ would have a larger maximum value than $u(x)$). Multiply the equation (2.1) by u', integrate from ξ to1, and denote by $x = x_1(u)$ the inverse function of $u(x)$. Obtain

$$\frac{1}{2}u'^2(1) - \frac{1}{2}u'^2(\xi) + \lambda \int_{u_1}^{0} f(x_1(u),u)\,du = 0.$$

Similarly,

$$\frac{1}{2}v'^2(1) - \frac{1}{2}v'^2(\xi) + \lambda \int_{u_1}^{0} f(x_2(v),v)\,dv = 0,$$

with $x_2(u) > x_1(u)$ for all $u \in (0,u_1)$. Subtracting

$$\frac{1}{2}(u'^2(1) - v'^2(1)) + \frac{1}{2}(v'^2(\xi) - u'^2(\xi)) + \lambda \int_0^{u_1} [f(x_2(u),u) - f(x_1(u),u)]du = 0.$$

Since the first term on the left is negative, and the others are non-positive, we obtain a contradiction, which establishes (2.7).

Let us now denot by $\eta > \bar{x}$ the point where $u(\eta) = u(0) = u_0$. Let us denote by $x_1(u)$ and $x_2(u)$ the inverse functions of $u(x)$ on $(0,\bar{x})$ and $(\bar{x},\eta)$ respectively. Multiply (2.1) by u' and integrate from 0 to η. As above,

$$\frac{1}{2}(u'^2(\eta) - u'^2(0)) + \lambda \int_{u_0}^{\bar{u}} [f(x_1(u),u) - f(x_2(u),u)]du = 0,$$

from which it follows that

$$|v'(0)| = |u'(0)| > |u'(\eta)|. \tag{2.8}$$

Let now $x_3(v)$ denote the inverse of $v9x)$ on $(0,1)$, and $x_4(u)$ the inverse of $u(x)$ on $(\eta,1)$. Multiply (2.1) by u' and integrate from η to 1,

$$\frac{1}{2}(u'^2(1) - u'^2(\eta)) + \lambda \int_{u_0}^{0} f(x_4(u),u)du = 0.$$

Multiply the equation (2.1) for $v(x)$ by v', and integrate from 0 to 1,

$$\frac{1}{2}(v'^2(1) - v'^2(0)) + \lambda \int_{u_0}^{0} f(x_3(v), v)dv = 0.$$

Subtracting,

$$\frac{1}{2}(u'^2(1) - v'^2(1)) + \frac{1}{2}(v'^2(0) - u'^2(\eta)) + \lambda \int_0^{u_0} [f(x_3(u), u) - f(x_4(u), u)]du = 0.$$

By (2.7) and (2.8) all three terms on the left are positive. The resulting contradiction proves that any solution is even.

Since $u(x)$ is even, positive and unimodular, it follows that $u'(x) \leq 0$ for $x \in (0,1)$, $u'(0) = 0$, $u'(1) < 0$. To show that $u'(x) < 0$ for $x \in (0,1)$, we differentiate the equation (2.1) on $(0,1)$

$$u_x'' + \lambda f_u u_x = -\lambda f_x \geq 0, \tag{2.9}$$

and apply the Lemma 0 to u_x on $(0,1)$.

If $u(x)$ is a solution of (2.1), then the corresponding linearized problem is

$$w'' + \lambda f_u(x,u)w = 0, \quad w(-1) = w(1) = 0. \tag{2.10}$$

Lemma 2. *If (2.10) has a nontrivial solution, then $w(x)$ does not change sign on $(-1,1)$; i.e., we can choose it so that $w(x) > 0$.*

Proof. Assume that $w(x)$ changes sign on $(-1,1)$. Assume that $w(x)$ has a zero on $[0,1)$, the other case being similar. Without loss of generality (taking $-w$ if necessary), we may assume that $w(x) < 0$ on (x_1, x_2), $0 \leq x_1 < x_2 \leq 1$, $w(x_1) = w(x_2) = 0$, and $w(x) > 0$ for $x < x_1$ and close to x_1, and for $x > x_2$ and close to x_2 (unless $x_2 = 1$). Multiply the equation (2.10) by u', the equation (2.9) by w, subtract and integrate on both sides. Obtain

$$[w'u' - w(u')'] \,|_{x_1}^{x_2} = \lambda \int_{x_1}^{x_2} f_x w \, dx. \tag{2.11}$$

The quantity on the right side of (2.11) is positive by our assumptions. The one on the left is equal to

$$w'(x_2)u'(x_2) - w'(x_1)u'(x_1),$$

which is negative by Lemma 1. This contradiction proves the lemma.

Theorem 1. *Consider the problem (2.1) under the assumptions (2.2–2.5). There exists a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$, the problem has no solution, for $\lambda = \lambda_0$ it has exactly one solution, and exactly two strictly ordered solutions for $0 < \lambda < \lambda_0$. Moreover, for $0 < \lambda < \lambda_0$, the solutions lie on two continuous λ curves $u^-(x,\lambda)$ and $u^+(x,\lambda)$, with $u^-(x,\lambda) < u^+(x,\lambda)$ for all $x \in (-1,1)$. The lower branch $u^-(x,\lambda)$ is strictly monotone increasing in λ and $\lim_{\lambda\to 0^+} u^-(x,\lambda) = 0$ for* *all* $x \in (-1,1)$. *For the upper branch* $\lim_{\lambda \to 0^+} \max_x u^+(x,\lambda) = \infty$. *(See Figure 1 for a diagram of the solution set.)*

Proof. It is well known that under condition (2.5), the problem (2.1) has no positive solutions for λ sufficiently large, see e.g., Amann [1]. When $\lambda = 0$ there is a trivial solution $u = 0$. It follows by the implicit function theorem that for $\lambda > 0$ small there is a continuous λ curve of solutions passing through $(0,0)$.

We show next that for any $c > 0$, one can find $\bar{\lambda} > 0$, such that for $0 < \lambda < \bar{\lambda}$ the problem (2.1) has at most one positive solution satisfying $\max_x u(x) < c$. This will imply both local uniqueness of the branch passing through $\lambda = 0$, $u = 0$, and that solutions on any other branch tend to infinity as $\lambda \to 0^+$. Indeed, if u and v are two solutions of (2.1), and $w = u - v$, then for some $0 < \theta < 1$,

$$w'' + \lambda f_u(x, \theta u + (1-\theta)v)w = 0, \quad w(-1) = w(1) = 0,$$

so that $w = 0$ for λ sufficiently small.

Let us return now to the branch of solutions passing through $\lambda = 0$, $u = 0$. If the corresponding linearized equation (2.10) has only a trivial solution $w = 0$, at $\lambda = \lambda_1$ and $u = u_1(x)$ on that branch, then by the implicit function theorem we can solve (2.1) for $\lambda > \lambda_1$ and λ close to λ_1. We cannot continue this process of increasing λ indefinitely, since we know that for $\lambda > 0$ sufficiently large (2.1) has no solution. Let λ_0 be the supremum of λ for which we can continue the branch to the right. We claim that as $\lambda \to \lambda_0$ the solution $u(x,\lambda)$ of (2.1) remains uniformly bounded. We sketch the argument, which is almost identical to that Lemma 4.5 in [3]. Define for $c > 0$

$$I(c) = \{x \in (-1,1) : u(x) > c\} = (-a(c), a(c)).$$

Assume $I(c)$ is non-empty for λ close to λ_0 (otherwise there is nothing to prove). Rewrite the equation (2.1) in the form $u'' + \lambda \rho_c(x) u = 0$ on I_c, where $\rho_c(x) = \frac{f(x,u)}{u}$. By (2.5), $\rho_c(x) \ge c_1 c^{p-1}$ on I_c. Since $u(x)$ has no zeros on I_c, it follows by the Sturm comparison theorem that (for λ close to λ_0)

$$2a(c) < \frac{\pi}{\sqrt{\lambda c_1 c^{p-1}}} < \frac{\pi}{\sqrt{\frac{\lambda_0}{2} c_1 c^{p-1}}}.$$

Choose c_0 so large that $a(c) < \frac{1}{2}$ for all $c \ge c_0$. This means that as $\lambda \to \lambda_0$, the function $u(x)$ is bounded by c_0 on $(a(c_0), 1)$. Since $u(x)$ is also a concave function, this means that it cannot become unbounded.

It follows that $u(x)$ is bounded in $C_0^2(-1,1)$ as $\lambda \to \lambda_0$. Passing to the limit in the integral form of (2.1) we see that at $\lambda = \lambda_0$ the problem (2.1) has a solution $u_{\lambda_0}(x) \in C_0^2(-1,1)$.

By the definition of λ_0 it follows that $F_u(\lambda_0, u_{\lambda_0})$ is singular, i.e., the problem (2.10) has a nontrivial solution, which is positive by Lemma 2. Using Lemma 2 it follows that $N(F_u(\lambda_0, u_{\lambda_0})) = \text{span}\,\{w(x)\}$ is one dimensional (by simplicity of the principal eigenvalue), and then codim $R(F_u(\lambda_0, u_{\lambda_0})$ is a Fredholm operator of index zero. To apply the Crandall-Rabinowitz Theorem 0, it remains to check that $F_\lambda(\lambda_0, u_{\lambda_0}) \notin R(F_u(\lambda_0, u_{\lambda_0}))$. Assuming the contrary would imply existence of $v(x) \not\equiv 0$, such that

$$v'' + \lambda_0 f_u(x, u_{\lambda_0})v = f(x, u_{\lambda_0}) > 0, \; v(-1) = v(1) = 0.$$

By the Fredholm alternative

$$\int_{-1}^{1} f(x, u_{\lambda_0}) w(x)\, dx = 0,$$

which is a contradiction. It follows that the Crandall-Rabinowitz Theorem 0 applies at $(\lambda_0, u_{\lambda_0})$.

Next we compute the direction of bifurcation. Near $(\lambda_0, u_{\lambda_0})$, represent $\lambda = \lambda(s)$, $u = u(s)$, with $\lambda_0 = \lambda(0)$, $u_{\lambda_0} = u(0)$. Notice that $\dot{\lambda}(0) = 0$, since the branch of solutions does not extend beyond λ_0. Also $u_s(0) = w$, a solution of (2.1). Differentiating the equation (2.1) twice in s, we obtain

$$u_{ss}'' + \lambda f_u u_{ss} + \lambda f_{uu} u_s^2 + 2\lambda' f_u u_s + \lambda'' f = 0, \; u_{ss}(-1) = u_{ss}(1) = 0.$$

Letting $s = 0$,

$$u_{ss}'' + \lambda_0 f_u u_{ss} + \lambda_0 f_{uu} w^2 + \lambda''(0) f = 0, \; u_{ss}(-1) = u_{ss}(1) = 0. \tag{2.12}$$

Multiplying the equation (2.12) by w, (2.10) by u_{ss} integrating and subtracting, we express

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^{1} f_{uu} w^3\, dx}{\int_{-1}^{1} f w\, dx} < 0.$$

This means that at $(\lambda_0, u_{\lambda_0})$, as well as at any other bifurcation point, the curve of solutions will bend leftwards in (λ, u) "plane".

Applying the Crandall-Rabinowitz theorem, we conclude that near the bifurcation point $(\lambda_0, u_{\lambda_0})$ the solutions of (2.1) form a curve $(\lambda_0 + \tau(s), u_{\lambda_0} + sw + z(s))$ for s near $s = 0$, with $\tau(s) < 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. It follows that for λ close to λ_0 and $\lambda < \lambda_0$ we have two solutions $u^-(x, \lambda)$ and $u^+(x, \lambda)$, with $u^-(x, \lambda)$ strictly increasing in λ, while $u^+(x, \lambda)$ is strictly decreasing. We show next that the lower branch $u^-(x, \lambda)$ is strictly increasing for all $0 < \lambda < \lambda_0$ (and the same is true for the lower branch at any other bifurcation point).

We know by above that $u_\lambda^-(x, \lambda) > 0$ for λ close to λ_0 and all $x \in (-1, 1)$. Let λ_1 be the largest λ where this inequality is violated, i.e., $u_\lambda^-(x, \lambda_1) \geq 0$ and $u_\lambda^-(x_0, \lambda_1) = 0$ for some $x_0 \in (-1, 1)$. Differentiating (2.1) in λ

$$u_\lambda'' + \lambda_1 f_u u_\lambda = -f < 0, \quad u_\lambda(-1) = u_\lambda(-1) = 0.$$

By the strong maximum principle $u_\lambda^-(x, \lambda_1) > 0$, a contradiction.

From the above discussion we know that the curve of solutions passing through $\lambda = 0$, $u = 0$ increases monotonically in λ until it reaches a critical λ_0, where it bends to the left. After the turn $F_u(\lambda, u)$ is never singular, for otherwise at a point $(\bar{\lambda}, \bar{u})$ where F_u is singular we must have a bifurcation with a turn to the left, which is clearly impossible. So that after the turn the curve can be continued for all $0 < \lambda < \lambda_0$, is single-valued in λ and tends to infinity as $\lambda \to 0$.

Finally, we claim that there are no other positive solutions, not lying on the curve just described. Indeed, let $(\bar{\lambda}, \bar{u})$ be another solution. If $F_u(\bar{\lambda}, \bar{u})$ is nonsingular, we can continue it for increasing λ until we reach a bifurcation point where $F_u(\bar{\lambda}, \bar{u})$ is singular. According to the Crandall-Rabinowitz theorem, at the bifurcation point we have a lower branch, which is strictly increasing in λ. This lower branch has no place to go as $\lambda \to 0^+$. Indeed, it cannot go to $\lambda = 0$, $u = 0$ by local uniqueness there, and it cannot go to either λ or u axes, as can be seen by writing (2.1) in an equivalent integral form. Hence, all positive solutions lie on a single curve, and the theorem is proved.

3. A class of non-convex nonlinearities.

Consider the problem

$$u'' + \lambda u + h(x)u^p = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0 \tag{3.1}$$

with $p > 1$ and λ real parameter. Using the techniques of the previous section, we shall obtain an exact multiplicity result for (3.1). We assume that

$$h(x) \in C^1(-1,1) \cap C^0[-1,1] \text{ and } h(-x) = h(x) \text{ for all } x; \tag{3.2}$$

$$h(0) > 0 \text{ and } h'(x) < 0 \text{ for all } x \in (0,1); \tag{3.3}$$

$$\int_{-1}^{1} h(x)\varphi_1^{p+1}(x)\,dx < 0, \tag{3.4}$$

where $\varphi_1 = \sin\frac{\pi}{2}(x+1)$ is the principal eigenfunction of $-D^2$ on $(-1,1)$ corresponding to the principal eigenvalue $\lambda_1 = \frac{\pi^2}{4}$.

Lemma 3. *Under the above assumptions, any positive solution of* (3.1) *is an even function, such that* $u'(x) < 0$ *for* $x \in (0,1]$.

Proof. Follows from Lemma 1.

Lemma 4. *Consider the linearized problem for* (3.1),

$$w'' + \lambda w + ph(x)u^{p-1}w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0. \tag{3.5}$$

If the problem (3.5) *has a nontrivial solution, then* $w(x)$ *does not change signs on* $(-1,1)$, *i.e., we can choose it so that* $w(x) > 0$.

Proof. The proof is similar to that of Lemma 2. As there, we assume that $w(x) < 0$ on (x_1, x_2), $0 \le x_1 < x_2 \le 1$, $w(x_1) = w(x_2) = 0$, etc. Differentiate (3.1),

$$u_x'' + \lambda u_x + ph(x)u^{p-1}u_x + h'(x)u^p = 0. \tag{3.6}$$

Multiply the equation (3.6) by w, (3.5) by u_x, subtract and integrate. Obtain

$$(wu_x' - u_x w')\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} h'u^p w\,dx = 0. \tag{3.7}$$

The integral term in (3.7) is positive. The other term is equal to

$$-u'(x_2)w'(x_2) + u'(x_1)w'(x_1),$$

which is also positive, a contradiction.

Theorem 2. *Consider the problem (3.1) under the assumptions (3.2), (3.3) and (3.4). Then there exists a critical $\lambda_0 > \lambda_1$, such that for $-\infty < \lambda < \lambda_1$, the problem (3.1) has a unique positive solution, for $\lambda_1 < \lambda < \lambda_0$ there are exactly two positive solutions, exactly one for $\lambda = \lambda_0$, and no positive solutions for $\lambda > \lambda_0$. Moreover, solutions lie on a continuous in λ curve, which bifurcates from $\lambda = \lambda_1$, and $\max_x u(x) \to \infty$ as $\lambda \to -\infty$. (See Figure 2).*

Proof. It follows from the results of T. Ouyang [7] that there is a curve of solutions starting at $\lambda = \lambda_1$, $u = 0$ which is increasing in λ, and it continues until $(\lambda_0, u_{\lambda_0})$, $\lambda_0 > \lambda_1$, where $F(u, \lambda) = u'' + \lambda u + hu^p$ (with $u(-1) = u(1) = 0$) is singular, i.e., (3.5) has a nontrivial solution. Further, it was proved in [7] that at any bifurcation point the curve of solutions bends leftward, and that (3.1) has no positive solutions for $\lambda > \lambda_0$.

We claim next that after the turn, the curve of solutions can be continued for all $-\infty < \lambda < \lambda_0$, and it is single-valued in λ. Indeed, if $F(\lambda, u)$ was singular at $(\bar{\lambda}, \bar{u})$, then as in the proof of the Theorem 1, the Crandall-Rabinowitz theorem would apply, giving us bifurcation at $(\bar{\lambda}, \bar{u})$ with the curve bending leftwards, which is impossible.

We claim that the solution curve cannot become unbounded at a finite λ after the turn. For $\lambda_1 < \lambda < \lambda_0$, we know from Ouyang [7] that there are at least two positive solutions of (3.1). We show next that our curve of solutions, call it Γ, continues smoothly for all $\lambda_1 < \lambda < \lambda_0$ after the turn. Since bifurcation is impossible, it remains to rule out the possibility of

$$\max_x u(x, \lambda) \to \infty, \quad \text{as } \lambda \downarrow \bar{\lambda}, \quad \text{for some } \lambda_1 < \bar{\lambda} < \lambda_0. \tag{3.8}$$

Assume (3.8) holds, and let $\bar{u}(x)$ be a positive solution at $\bar{\lambda}$, which is not on the lower branch of Γ. We claim that for $\lambda > \bar{\lambda}$ sufficiently close to $\bar{\lambda}$

$$u(x, \lambda) > \bar{u}(x) \quad \text{for all} \quad x \in (-1, 1). \tag{3.9}$$

Indeed, it follows from the proof of Lemma 1 that two different solutions of (3.1) cannot intersect. Since $u(0, \lambda) > \bar{u}(0)$ for λ close to $\bar{\lambda}$, (3.9) follows.

If $F(x, u)$ is invertible at $\bar{u}(x)$, we can apply the implicit function theorem; if it is not, then the Crandall-Rabinowitz Theorem 0. In either case, we would have another curve of solutions $v(x, \lambda)$, call it Γ_1, passing through $\bar{u}(x)$. In case $F_u(x, \bar{u})$ is invertible by continuing Γ_1 for increasing λ we would have to reach a bifurcation point, since there are no solutions for $\lambda > \lambda_0$, and solutions on Γ_1 cannot become unbounded, since then we would be able to find a λ_2, $\bar{\lambda} < \lambda_2 < \lambda_0$ such that $u(x, \lambda_2) \geq v(x, \lambda_2)$, $u(x, \lambda_2) \not\equiv v(x, \lambda_2)$ and $u(x_1, \lambda_2) = v(x_1, \lambda_2)$ at some $x_1 \in (-1, 1)$, which leads to a contradiction by the maximum principle. (The possibility that $u(x, \lambda_2) \equiv v(x, \lambda_2)$ is ruled out by the Crandall-Rabinowitz Theorem 0, since otherwise we would have an inadmissible bifurcation at λ_2). Once Γ_1 reaches a bifurcation point, we would have, by the Crandall-Rabinowitz Theorem 0, a lower branch, which is monotone increasing in λ, and this branch would have nowhere to go for decreasing λ, as we show later on in the proof.

Next we show that the problem (3.1) has a positive solution at $\lambda = \lambda_1$. To this end, we minimize the functional

$$F(u) = \int_{-1}^{1} (u'^2 - \lambda_1 u^2)\, dx$$

on the set $B = \{u \in H_0^1(-1,1) : \int_{-1}^1 h(x)|u|^{p+1}\,dx = 1\}$. Notice that by (3.4), constant multiples of $\varphi_1(x)$ do not belong to B. This, together with the variational characterization of λ_1, implies that $\mu = \min_B F(u) > 0$. If $u(x)$ is a point of minimum we can assume that $u(x) \geq 0$ on $(-1,1)$ (replacing it by $|u|$ if necessary). At the point of minimum we have for any $\varphi \in C_0^1(-1,1)$

$$\int_{-1}^1 u'\varphi'\,dx - \lambda_1 \int_{-1}^1 u\varphi\,dx = \mu \int_{-1}^1 h(x)u^p\varphi\,dx.$$

Since for (3.1) any solution of class $H_0^1(-1,1)$ also belongs to $C_0^2(-1,1)$, after integration by parts and rescaling, we obtain a solution of (3.1) which we denote $\bar{u}(x)$. By the strong maximum principle $\bar{u}(x) > 0$ on $(-1,1)$.

It follows that our solution curve remains bounded as $\lambda \downarrow \lambda_1$. For otherwise, we would have another curve through $\lambda = \lambda_1$, $u = \bar{u}(x)$ (using the implicit function theorem if $F_u(\lambda_1, \bar{u})$ is invertible, and the Crandall-Rabinowitz Theorem, otherwise). But this leads to a contradiction the same way as above.

Next, we show that the solution curve cannot become unbounded at any $\lambda < \lambda_1$. Differentiate (3.1) in λ,

$$\begin{aligned} &u_\lambda'' + \lambda u_\lambda + ph(x)u^{p-1}u_\lambda + u = 0, \\ &-1 < x < 1,\ u_\lambda(-1) = u_\lambda(1) = 0. \end{aligned} \tag{3.10}$$

Multiplying (3.1) by u_λ, (3.10) by u, then integrating and subtracting, we express

$$\frac{d}{d\lambda}\int_{-1}^1 h(x)u^{p+1}\,dx = -\frac{p+1}{p-1}\int_{-1}^1 u^2\,dx. \tag{3.11}$$

Multiply (3.1) by u, integrate, and use the Poincaré's inequality

$$(\lambda_1 - \lambda)\int_{-1}^1 u^2\,dx \leq \int_{-1}^1 u'^2\,dx - \lambda\int_{-1}^1 u^2\,dx = \int_{-1}^1 h(x)u^{p+1}\,dx. \tag{3.12}$$

Denote $s = \lambda_1 - \lambda$. Using (3.11) in (3.12),

$$s\frac{p-1}{p+1}\frac{d}{ds}\int_{-1}^1 hu^{p+1}\,dx \leq \int_{-1}^1 hu^{p+1}\,dx.$$

This implies that the quantity $\int_{-1}^1 hu^{p+1}\,dx$ cannot become unbounded at a finite $\lambda < \lambda_1$. Returning to (3.12), we conclude that first $\int_{-1}^1 u^2\,dx$ and then $\int_{-1}^1 u'^2\,dx$ cannot become unbounded. By the Sobolev imbedding theorem the same is true for $\max_x u(x)$.

Next, we claim that $\max_x u(x) \to \infty$ as $\lambda \to -\infty$. Indeed, assuming otherwise, we would have a sequence $\{\lambda_n\}$, $\lambda_n \to -\infty$, such that $\max_x u(x,\lambda_n) \leq c$, with some positive c. Rewrite (3.1) in the form (for $\lambda > 0$)

$$u(x) = -\int_{-1}^1 G(x,\xi)h(\xi)u^p(\xi)\,d\xi,$$

where $G(x,\xi)$ is the corresponding Green's function

$$G(x,\xi) = \frac{\sinh \mu(x+1) \sinh \mu(\xi - 1)}{\mu \sinh 2\mu}$$

for $x \le \xi$ with $\mu = \sqrt{|\lambda|}$, and $G(\xi, x) = G(x,\xi)$. It follows that $\max_x u(x,\lambda_n) \to 0$ as $n \to \infty$, Choose n so large that $u(x,\lambda_n) < \varphi_1(x)$ for all $x \in (-1,1)$, and $-\lambda_n \ge \max_x h(x) = h(0)$. Then, $\varepsilon\varphi_1(x)$ is a supersolution (but not a solution) of (3.1) at λ_n for any $0 \le \varepsilon \le 1$. At the first $0 < \varepsilon_0 < 1$, where $u(x,\lambda_n)$ and $\varepsilon_0\varphi_1(x)$ touch, we get a contradiction by the maximum principle.

It follows that the problem (3.1) has a curve of solutions as described in the theorem. We claim that there are no other solutions. Indeed, any other solution can be continued for increasing λ until a bifurcation point, or until it becomes unbounded (since it cannot be continued indefinitely). At the bifurcation point, the Crandall-Rabinowitz Theorem 0 provides us with a lower branch which is locally increasing in λ, (i.e., solution decreases when λ decreases). As in the Theorem 1, we show that the lower branch is increasing for all λ. But then it has nowhere to go. Indeed, by above, no solution can tend to 0 as $\lambda \to -\infty$. If $u(x,\lambda) \equiv 0$ at some λ, then λ must be an eigenvalue of $-D^2$. But in view of another bifurcation theorem of Crandall-Rabinowitz [4], at $\lambda = \lambda_1$, we have local uniqueness of bifurcating solution, while at other eigenvalues no bifurcation of positive solutions is possible.

Finally, we rule out the possibility of another curve Γ_1 of solutions $v(x,\lambda)$ which becomes unbounded for increasing λ. By above, we can find a $v(x,\lambda_2)$ on Γ_1 so that $v(x,\lambda_2) > u(x,\lambda_2)$ for all $x \in (-1,1)$. However, $v(x,\lambda_2)$ is a strict supersolution, (i.e., never a solution) for (3.1) for all $-\infty < \lambda < \lambda_2$. This means that as solutions on Γ go to infinity as $\lambda \to -\infty$, they are unable by the maximum principle to cross above $v(x,\lambda_2)$, which is a contradiction.

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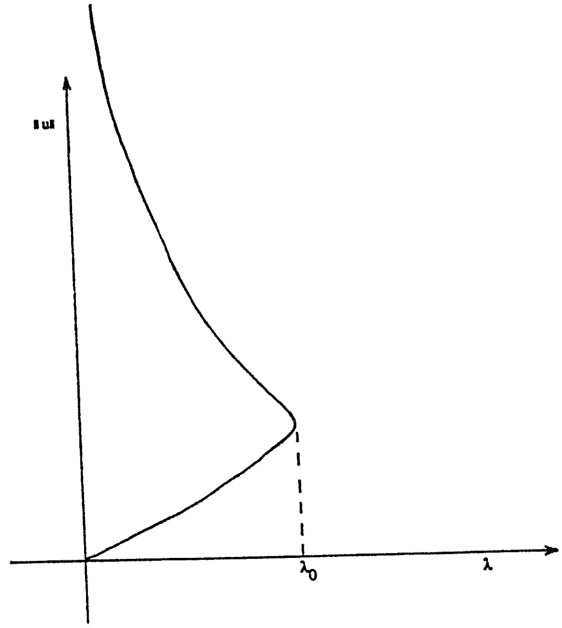

Figure 1

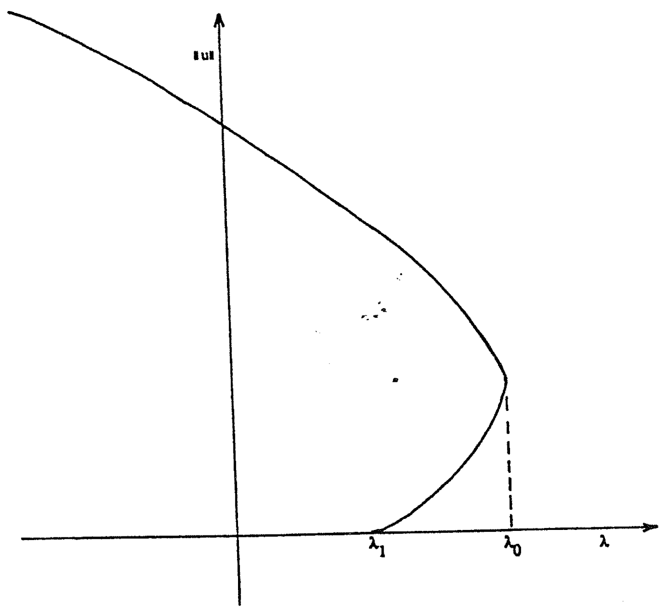

Figure 2