# Steady States and Long Time Behavior of Some Convective Reaction-Diffusion Equations 

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## 1. Introduction

We are interested in long time behavior of nonnegative solutions of

$$
\begin{gather*}
u_{t}=u_{x x}+g(u)_{x}+\lambda f(u), \quad 0<x<L, t>0  \tag{1.1}\\
u(0, t)=u(L, t)=0, \quad t>0  \tag{1.2}\\
u(x, 0)=u_{0}(x), \quad 0<x<L, \tag{1.3}
\end{gather*}
$$

where $\lambda$ is a positive parameter and $f, g$ are given functions of $u$. This problem occurs in a number of physical applications, including gas dynamics, see T. F. Chen, H. A. Levine and P. E. Sacks [2] and H. A. Levine, L. E. Payne, P. E. Sacks and B. Stranghan [12] for the references. These two papers include a detailed study of the problem, primarily for the power law cases $f(u)=u^{p}$, $g(u)=u^{m}$.

As usual one starts by considering steady states, given by the equation (2.1) below (without loss of generality we assume $L=1$ ). The first question is whether solutions of steady state equation (2.1) lie on locally simple curves as the parameter $\lambda$ varies (which means that the curve can be parametrized as in the Theorem 1.1 below). In [2] the affirmative answer was established under some conditions on $f$ and $g$. We prove this result under a milder restriction on $g(u)\left(g^{\prime}(u)>0\right.$ for $\left.u>0\right)$ and with no assumptions on $f(u)$ (except for smoothness). To simplify the presentation we then specialize to $g(u)=a u$, with constant $a$, although our results hold for more general $g(u)$. We give conditions on $f(u)$ to ensure a global fold in the equation 2.1 (see Theorem 2.1). Namely, we show existence of $\lambda_{0}$, such that our problem has either two, one or zero solutions, depending on whether $\lambda<\lambda_{0}, \lambda=\lambda_{0}$ or $\lambda>\lambda_{0}$. Moreover all solutions lie on a single solution curve, and we are able to identify which branch is stable and which is unstable (which is important for the long time behavior). We also consider two classes of $f(u)$ modelled on cubic nonlinearities. We show that all solutions again lie on a single solution curve, and provide an exact multiplicity result for large $\lambda$. The techniques we use are based
on bifurcation theory of M. G. Crandall and P. H. Rabinowitz [4], as developed in [8-10]. We also make an extensive use of the results of [2] and [12].

In Section 3 turn to the problem ( $u>0$ )

$$
\begin{equation*}
\Delta u+\lambda f(u)=0 \quad \text { for } \quad|x|<R, \quad u=0 \quad \text { when } \quad|x|=R \tag{1.4}
\end{equation*}
$$

in two dimensions, and prove that positive solutions of (1.4) lie on simple curves. After completing this work we learned of the paper by M. Holzmann and H. Kielhöfer [7], where a similar result is established for a more general class of two-dimensional domains. However, since our proof is completely different and much easier, we feel that this short section is still of interest.

Finally, in Section 5 we apply our results to the time-dependent problem (1.1-1.3). We discuss stability of the steady states, and the behavior of solutions as $t \rightarrow \infty$.

Next we state a bifurcation theorem of Crandall-Rabinowitz [4].
Theorem 1.1 [4]. Let $X$ and $Y$ be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \boldsymbol{R} \times X$ and let $F$ be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into $Y$. Let the null-space $N\left(F_{x}(\bar{\lambda}, \bar{x})\right)=\operatorname{span}\left\{x_{0}\right\}$ be one-dimensional and $\operatorname{codim} R\left(F_{x}(\bar{\lambda}, \bar{x})\right)=1$. Let $F_{\lambda}(\bar{\lambda}, \bar{x}) \notin R\left(F_{x}(\bar{\lambda}, \bar{x})\right)$. If $Z$ is a complement of $\operatorname{span}\left\{x_{0}\right\}$ in $X$, then the solutions of $F(\lambda, x)=F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s))=\left(\bar{\lambda}+\tau(s), \bar{x}+s x_{0}+z(s)\right)$, where $s \rightarrow(\tau(s), z(s)) \in \boldsymbol{R} \times Z$ is a continuously differentiable function near $s=0$ and $\tau(0)=\tau^{\prime}(0)=0, z(0)=$ $z^{\prime}(0)=0$.

A remark on terminology. In applied literature the situation described in the above theorem is referred to as "turning point", and the term "bifurcation" is used when two or more solution curves intersect. We refer to it as bifurcation (or "spontaneous bifurcation") following e.g. [14].

## 2. Exact multiplicity result for a class of nonvariational Dirichlet problems

We consider a class of problems

$$
\begin{equation*}
u^{\prime \prime}+g(u)^{\prime}+\lambda f(u)=0 \quad \text { on }(0,1), \quad u(0)=u(1)=0 \tag{2.1}
\end{equation*}
$$

Here $\lambda$ is a positive parameter, the function $g(u) \in C^{2}\left(\boldsymbol{R}_{+}\right)$satisfies

$$
\begin{equation*}
g^{\prime}(u)>0 \quad \text { for almost all } u>0 \tag{2.2}
\end{equation*}
$$

and $f(u)$ a function of class $C^{2}$ for $u>0$.
Lemma 2.1. Any positive solution of (2.1) has only one point of local maximum (which is of course the point of global maximum).

Proof. Assuming otherwise, the function $u(x)$ would have points of local minimum on $(0,1)$. Let $x_{1}$ be the largest such point, $u^{\prime}\left(x_{1}\right)=0$, and let $x_{2}>x_{1}$ be such that $u\left(x_{2}\right)=u\left(x_{1}\right)$. Let $\bar{x}$ be the point of local maximum on ( $x_{1}, x_{2}$ ), and denote $\bar{u}=u(\bar{x})$ and $u_{1}=u\left(x_{1}\right)$. Multiply (2.1) by $u^{\prime}$ and integrate over ( $x_{1}, \bar{x}$ ),

$$
\begin{equation*}
\int_{x_{1}}^{\bar{x}} g^{\prime}(u) u^{\prime 2} d x+\lambda \int_{u_{1}}^{\bar{u}} f(u) d u=0 . \tag{2.3}
\end{equation*}
$$

Similarly, integrating over ( $\bar{x}, x_{2}$ ),

$$
\begin{equation*}
\frac{1}{2} u^{\prime 2}\left(x_{2}\right)+\int_{\bar{x}}^{x_{2}} g^{\prime}(u) u^{\prime 2} d x-\lambda \int_{u_{1}}^{\bar{u}} f(u) d u=0 \tag{2.4}
\end{equation*}
$$

Adding (2.3) and (2.4), we obtain a contradiction.
Remark. Multiplying (2.1) by $u^{\prime}$ and integrating over ( 0,1 ), one sees that $u^{\prime}(0)>\left|u^{\prime}(1)\right|$, and more generally if $x_{1}<x_{2}$ and $u\left(x_{1}\right)=u\left(x_{2}\right)$ then $u^{\prime}\left(x_{1}\right)>$ $\left|u^{\prime}\left(x_{2}\right)\right|$. I.e. solution is "skewed" to the left.

We shall denote by $x_{0}$ the point of global maximum of $u(x)$. We will need to consider the linearization of (2.1)

$$
\begin{equation*}
w^{\prime \prime}+g^{\prime}(u) w^{\prime}+g^{\prime \prime}(u) u^{\prime} w+\lambda f^{\prime}(u) w=0 \quad \text { on }(0,1), \quad w(0)=w(1)=0 \tag{2.5}
\end{equation*}
$$

and its adjoint equation

$$
\begin{equation*}
\mu^{\prime \prime}-g^{\prime}(u) \mu^{\prime}+\lambda f^{\prime}(u) \mu=0 \quad \text { on }(0,1), \quad \mu(0)=\mu(1)=0 \tag{2.6}
\end{equation*}
$$

Lemma 2.2. If the problem (2.5) admits a nontrivial solution, it can be chosen so that $w(x)>0$ on $(0,1)$.

Proof. Differentiate the equation (2.1)

$$
\begin{equation*}
u_{x}^{\prime \prime}+g^{\prime}(u) u_{x}^{\prime}+g^{\prime \prime}(u) u^{\prime} u^{\prime}+\lambda f^{\prime}(u) u^{\prime}=0 . \tag{2.7}
\end{equation*}
$$

Let $p(x)=u^{\prime \prime}(x) w(x)-u^{\prime}(x) w^{\prime}(x)$. Multiplying the equation (2.7) by $w$, and subtracting from it the equation (2.5) multiplied by $u_{x}$, we obtain

$$
\begin{equation*}
p^{\prime}+g^{\prime}(u) p=0 \quad \text { for } \quad x \in(0,1) \tag{2.8}
\end{equation*}
$$

Assume that $w(x)$ vanishes at some $\xi \in(0,1)$. Assume first that $\xi \in\left(0, x_{0}\right)$ is the smallest root of $w(x)$. We can choose $w(x)>0$ on $(0, \xi)$, and then $p(0)=$ $-u^{\prime}(0) w^{\prime}(0)<0$, while $p(\xi)=-u^{\prime}(\xi) w^{\prime}(\xi)>0$. But this is a contradiction, since solutions of (2.8) cannot change sign. If $\xi \in\left(x_{0}, 1\right)$ is the largest root of $w(x)$, and $w(x)>0$ on $(\xi, 1)$, then $p(\xi)>0$ and $p(1)<0$, which is again a contradiction.

The next lemma is a variation of the Lemma 2.3 in [3].
Lemma 2.3. If the problem (2.5) admits a nontrivial solution, then so does the problem (2.6). Moreover, we can choose $\mu(x)>0$ on $(0,1)$.

Proof. We are given that zero is the principal eigenvalue of the operator $L w=w^{\prime \prime}+g^{\prime}(u) w^{\prime}+g^{\prime \prime}(u) u^{\prime} w+\lambda f^{\prime}(u) w$, and $w(x)>0$ is the corresponding eigenfunction. If zero is not the principal eighenvalue of the transpose of $L$, we can find a constant $\theta \neq 0$ and a function $z(x)>0$, such that ( $L^{\prime} z=\theta z$ )

$$
\begin{equation*}
z^{\prime \prime}-g^{\prime}(u) z^{\prime}+\lambda f^{\prime}(u) z=\theta z \quad \text { on }(0,1), \quad z(0)=z(1)=0 . \tag{2.9}
\end{equation*}
$$

Multiplying (2.9) by $w$ and subtracting (2.5) multiplied by $z$, and then integrating over ( 0,1 ), we obtain, after an integration by parts,

$$
\int_{0}^{1} w(x) z(x) d x=0
$$

a contradiction.
The preceding lemmas and Theorem 1.1 imply that solutions of (2.1) for arbitrarily positive $f \in C^{2}$ lie on smooth solution curves, which admit only simple turns (see proof of the next theorem). This extends the previous result of [2, p. 1356], where a similar result was established for $g(u)$ being a power of $u$, and for increasing $f(u) / u$. By imposing some conditions on $f(u)$ we can get a rather detailed picture of the solution set.

For the rest of the section we will consider the problem

$$
\begin{equation*}
u^{\prime \prime}+a u^{\prime}+\lambda f(u)=0 \quad \text { on }(0,1), \quad u(0)=u(1)=0 \tag{2.10}
\end{equation*}
$$

with a constant $a$, which we assume to be positive without loss of generality (as seen by changing variables $x \rightarrow 1-x$ ). The following lemma is known (even in more generality), see $[3,12]$.

Lemma 2.4. Assume $f(u)$ satisfies the condition (2.16) below. Let $\lambda \geq$ $\bar{\lambda}>0$ and $0<\alpha<\beta<1$. If $u$ is a positive solution of (2.10), then there is a constant $c$, such that

$$
\max _{[\alpha, \beta]} u(x) \leq c .
$$

Lemma 2.5. In the conditions of the previous lemma we have an a priori estimate

$$
|u|_{C^{2}[0,1]} \leq c .
$$

Proof. By Lemma 2.1 any positive solution of (2.10) has only one point of maximum, say $x_{0}$. By the standard elliptic theory suffices to estimate $u\left(x_{0}\right)$.

We claim that $0<x_{0}<1 / 2$. Indeed, let $x_{1}<x_{0}<x_{2}$ be such that $u\left(x_{1}\right)=$ $u\left(x_{2}\right)$. Multiplying (2.10) by $u^{\prime}$, and integrating over ( $x_{1}, x_{2}$ ), we conclude that $u^{\prime}\left(x_{1}\right)>\left|u^{\prime}\left(x_{2}\right)\right|$, and the claim follows.

From the equation (2.10) it follows that the function

$$
g(u)=u^{\prime}+a u
$$

is decreasing on $(0,1)$. By Lemma 2.4 we can find $x_{1}$ in say interval $(2 / 3,3 / 4)$ and a constant $\bar{c}>0$, such that $u^{\prime}\left(x_{1}\right) \geq-\bar{c}$, and $u\left(x_{1}\right) \leq \bar{c}$ (since $u(x)$ is a priori bounded over that interval, $\left|u^{\prime}(x)\right|$ cannot be uniformly large there). It follows that (since $\left.g(u(x))>g\left(u\left(x_{1}\right)\right)\right)$

$$
\begin{equation*}
u^{\prime}+a u>-\bar{c} \quad \text { on } \quad\left(x_{0}, x_{1}\right) . \tag{2.11}
\end{equation*}
$$

Integrating (2.11), we obtain

$$
\begin{equation*}
u\left(x_{0}\right) \leq e^{a\left(x_{1}-x_{0}\right)} u\left(x_{1}\right)+\frac{\bar{c}}{a}\left(e^{a\left(x_{1}-x_{0}\right)}-1\right)<e^{a} \bar{c}\left(1+\frac{1}{a}\right) . \tag{2.12}
\end{equation*}
$$

We will also need to consider the corresponding versions of (2.5) and (2.6).

$$
\begin{array}{cll}
w^{\prime \prime}+a w^{\prime}+\lambda f^{\prime}(u) w=0 & \text { on }(0,1), & w(0)=w(1)=0 \\
\mu^{\prime \prime}-a \mu^{\prime}+\lambda f^{\prime}(u) \mu=0 & \text { on }(0,1), & \mu(0)=\mu(1)=0 \tag{2.14}
\end{array}
$$

Theorem 2.1. Assume that $f(u) \in C^{2}\left(\boldsymbol{R}_{+}\right)$satisfies the following conditions

$$
\begin{gather*}
f_{u u}>0 \quad \text { for almost all } \quad u>0  \tag{2.15}\\
f(u) \geq c_{1} u^{p}+c_{2} \quad \text { for all } \quad u>0 \tag{2.16}
\end{gather*}
$$

with constants $c_{1}, c_{2}>0$ and $p>1$, and finally $f(u) \geq 0$ when $u<0$ (e.g. $f(u)=$ $e^{u}$.) Then all solutions of (2.10) lie on a single smooth solution curve. This curve passes through $\lambda=0, u=0$, then it increases in $\lambda$ until a critical $\lambda=\lambda_{0}>0$, where it bends to the left and tends to infinity as $\lambda \rightarrow 0$. Thus the solution curve has two branches, referred to as lower branch $u^{-}(x, \lambda)$ and the upper one $u^{+}(x, \lambda)$. We have moreover $0<u^{-}(x, \lambda)<u^{+}(x, \lambda)$ for all $0<\lambda<\lambda_{0}$ and $x \in(0,1)$. (Notice that in particular the problem (2.10) has either two, one or zero solutions, depending on whether $0<\lambda<\lambda_{0}, \lambda=\lambda_{0}$ or $\lambda>\lambda_{0}$ ).

Proof. When $\lambda=0$ there is a trivial solution $u=0$. It follows by the implicit function theorem that for $\lambda>0$ small there is a smooth curve of solutions passing through $(0,0)$. By the maximum principle all solutions of (2.10) are positive. This curve cannot be continued indefinitely for increasing $\lambda$, since it is well known that under condition (2.16), the problem (2.10) has no solutions for $\lambda$ sufficiently large, see e.g., Amann [1]. Let $\lambda_{0}$ denote the supremum of $\lambda$ 's
for which the solution curve can be continued to the right. Using Lemma 2.5 it is easy to show that there is a solution $u_{0}$ of (2.10) at $\lambda=\lambda_{0}$ lying on our curve of solutions (see [8] for a similar argument).

Rewrite (2.10) as

$$
F(\lambda, u) \equiv u^{\prime \prime}+a u^{\prime}+\lambda f(u)=0
$$

where $F: \boldsymbol{R}_{+} \times C_{0}^{2}(0,1) \rightarrow \boldsymbol{C}(0,1)$. Notice that $F_{u}(\lambda, u) w$ is given by the left hand side of (2.13). By the definition of $\lambda_{0}, F_{u}\left(\lambda_{0}, u_{0}\right)$ has to be singular, i.e., (2.13) has a nontrivial solution $w(x)$, which is positive by Lemma 2.2. Clearly the null-space $N\left(F_{u}\left(\lambda_{0}, u_{0}\right)\right)=\operatorname{span}\{w(x)\}$ is one dimensional (it can be parametrized by $\left.w^{\prime}(0)\right)$. It follows that $\operatorname{codim} R\left(F_{u}\left(\lambda_{0}, u_{0}\right)\right)=1$, since $F_{u}\left(\lambda_{0}, u_{0}\right)$ is a Fredholm operator of index zero. To apply the Crandall-Rabinowitz Theorem 1.1 it remains to check that $F_{\lambda}\left(\lambda_{0}, u_{0}\right) \notin R\left(F_{u}\left(\lambda_{0}, u_{0}\right)\right)$. Assuming otherwise would imply existence of $v(x) \not \equiv 0$, such that

$$
\begin{equation*}
v^{\prime \prime}+a v^{\prime}+\lambda_{0} f^{\prime}\left(u_{0}\right) v=f\left(u_{0}\right), 0<x<1, \quad v(0)=v(1)=0 . \tag{2.17}
\end{equation*}
$$

Multiplying (2.17) by $\mu$, (2.14) by $v$, subtracting and integrating over ( 0,1 ), we obtain

$$
\int_{0}^{1} f\left(u_{0}\right) \mu d x=0
$$

which is impossible, since both $f\left(u_{0}\right)$ and $\mu$ are positive.
Applying the Crandall-Rabinowitz Theorem 1.1, we conclude that $\left(\lambda_{0}, u_{0}\right)$ is a turning point, near which the solutions of (2.10) form a curve $\left(\lambda_{0}+\tau(s)\right.$, $u_{0}+s w+z(s)$ ) with $s$ near $s=0$, and $\tau(0)=\tau^{\prime}(0)=0, z(0)=z^{\prime}(0)=0$. It follows that for $\lambda$ close to $\lambda_{0}$ and $\lambda<\lambda_{0}$ there are two solutions with $0<$ $u^{-}(x, \lambda)<u^{+}(x, \lambda)$ for all $x \in(0,1)$, and that $u^{-}(x, \lambda)$ is strictly increasing in $\lambda$, while $u^{+}(x, \lambda)$ is strictly decreasing. We show next that if a branch of solutions is increasing for $\lambda$ 's in some interval (i.e., $u_{\lambda}(x, \lambda)>0$ for all $x \in(0,1)$ ), then it keeps increasing until a turn occurs. Indeed, assuming otherwise, we infer existence of $\lambda_{1}$, such that either $u_{\lambda}\left(x, \lambda_{1}\right) \geq 0$ for all $x \in(0,1)$ and $u_{\lambda}\left(x_{1}, \lambda_{1}\right)=0$ for some $x_{1} \in(0,1)$, or $u_{\lambda}^{\prime}\left(0, \lambda_{1}\right)=0$ (or $u_{\lambda}^{\prime}\left(1, \lambda_{1}\right)=0$, which is similar). Differentiate (2.10) in $\lambda$,

$$
\begin{equation*}
u_{\lambda}^{\prime \prime}+a u_{\lambda}^{\prime}+\lambda f^{\prime}(u) u_{\lambda}+f(u)=0 \quad \text { on }(0,1), \quad u_{\lambda}(0)=u_{\lambda}(1)=0 . \tag{2.18}
\end{equation*}
$$

Since $f(u)>0$, we conclude by the sharp form of the strong maximum principle, as in [6], that $u_{\lambda}\left(x, \lambda_{1}\right)>0$ for all $x \in(0,1)$ and $u_{\lambda}^{\prime}\left(0, \lambda_{1}\right)>0\left(u_{\lambda}^{\prime}(1, \lambda)<0\right)$, a contradiction. In particular we conclude that the branch $u^{-}(x, \lambda)$ is increasing in $\lambda$ for all $\lambda \in\left(0, \lambda_{0}\right)$.

Next we compute the direction of bifurcation at $\left(\lambda_{0}, u_{0}\right)$ and any other turning point. We only sketch the derivation, since it is standard (see also
[8]). Near $\left(\lambda_{0}, u_{0}\right)$ we represent $\lambda=\lambda(s), u=u(s)$ with $\lambda_{0}=\lambda(0)$ and $u_{0}=$ $u(0)$. Differentiating (2.10) in $s$ twice, and setting $s=0$, we obtain (using $\left.\lambda^{\prime}(0)=0, u_{s}(0)=w\right)$

$$
\begin{array}{r}
u_{s s}^{\prime \prime}+a u_{s s}^{\prime}+\lambda_{0} f^{\prime}(u) u_{s s}+\lambda_{0} f_{u u}(u) w^{2}+\lambda^{\prime \prime}(0) f(u)=0  \tag{2.19}\\
u_{s s}(0)=u_{s s}(1)=0
\end{array}
$$

Multiplying the equation (2.19) by $\mu$, (2.14) by $u_{s S}$, integrating over $(0,1)$ and subtracting, we have by Lemma 2.3,

$$
\begin{equation*}
\lambda^{\prime \prime}(0)=-\lambda_{0} \frac{\int_{0}^{1} f_{u u} w^{2} \mu d x}{\int_{0}^{1} f(u) \mu d x}<0 \tag{2.20}
\end{equation*}
$$

This means that at $\left(\lambda_{0}, u_{0}\right)$, as well as any other turning point, the curve of solutions will bend to the left in ( $\lambda, u$ ) "plane".

We now continue the branch $u^{+}(x, \lambda)$ for decreasing $\lambda>0$. We claim that all solutions are nondegenerate, i.e. the only solution of the corresponding linearized equation (2.13) is zero. Indeed, at any singular point a turn to the left has to occur, which is impossible. Hence $u^{+}(x, \lambda)$ can be continued for all $\lambda>0$, and since at $\lambda=0$ there is only a trivial solution, it has to tend to infinity as $\lambda \rightarrow 0$ (it cannot tend to infinity at a positive $\lambda$ by Lemma 2.5).

Finally, we rule out any solutions, not lying on the above curve. Assuming such solution exists, we continue it for increasing $\lambda$ until a turning point is reached (it cannot go to infinity by Lemma 2.5). At the turning point we get an increasing in $\lambda$ lower branch which has to tend to zero as $\lambda \rightarrow 0^{+}$(it cannot lose its positivity by the maximum principle). We now get two solutions near $\lambda=0$, $u=0$, which contradicts the implicit function theorem.

## 3. The set of positive solutions in a two-dimensional ball

We apply the techniques of the previous section to study the positive solutions of the Dirichlet problem

$$
\begin{equation*}
\Delta u+\lambda f(u)=0 \quad \text { for } \quad|x|<R, \quad u=0, \quad \text { when } \quad|x|=R \tag{3.1}
\end{equation*}
$$

in two dimensions $\left(x=\left(x_{1}, x_{2}\right)\right)$, with a parameter $\lambda>0$. By stretching the variables we may assume that $R=1$. By the theorem of B. Gidas, W.-M. Ni and L. Nirenberg [6], positive solutions of (3.1) are radial, and so (3.1) becomes (with $r=|x|, u=u(r)$ )

$$
\begin{equation*}
u^{\prime \prime}+\frac{1}{r} u^{\prime}+\lambda f(u)=0 \quad \text { for } \quad r \in(0,1), \quad u^{\prime}(0)=u(1)=0 \tag{3.2}
\end{equation*}
$$

We assume that $f(u) \in C^{2}\left(\boldsymbol{R}_{+}\right)$satisfies $f(0) \geq 0$ (which implies that $u^{\prime}(1)<0$ ). We shall need the linearized equation

$$
\begin{equation*}
w^{\prime \prime}+\frac{1}{r} w^{\prime}+\lambda f^{\prime}(u) w=0 \quad \text { for } \quad r \in(0,1), \quad w^{\prime}(0)=w(1)=0 \tag{3.3}
\end{equation*}
$$

Lemma 3.1. Assume (3.3) admits a nontrivial solution. Assume that $w(r)$ is positive near $r=1$. Then

$$
\begin{equation*}
\int_{0}^{1} f(u) w r d r>0 \tag{3.4}
\end{equation*}
$$

Proof. We have the uniqueness theorem for ODE's $w^{\prime}(1)<0$. Differentiate (3.2)

$$
\begin{equation*}
u^{\prime \prime \prime}+\left(\frac{1}{r} u^{\prime}\right)^{\prime}+\lambda f^{\prime}(u) u^{\prime}=0 \tag{3.5}
\end{equation*}
$$

Multiply (3.3) by $r^{2} u^{\prime}$, subtract (3.5) multiplied by $r^{2} w$, then integrate over $(0,1)$. Obtain

$$
\begin{equation*}
\int_{0}^{1}\left[r^{2} u^{\prime} w^{\prime \prime}-r^{2} u^{\prime \prime \prime} w+r u^{\prime} w^{\prime}-r^{2}\left(\frac{1}{r} u^{\prime}\right)^{\prime} w\right] d x=0 \tag{3.6}
\end{equation*}
$$

Integrate by parts in the first, second and fourth terms:

$$
\left.r^{2} u^{\prime} w^{\prime}\right|_{0} ^{1}+\int_{0}^{1}\left[-2 r u^{\prime} w^{\prime}-r^{2} u^{\prime \prime} w^{\prime}+2 r u^{\prime \prime} w+r^{2} u^{\prime \prime} w^{\prime}+r u^{\prime} w^{\prime}+2 u^{\prime} w+r u^{\prime} w^{\prime}\right] d r=0
$$

After cancellations:

$$
u^{\prime}(1) w^{\prime}(1)+2 \int_{0}^{1} r\left(u^{\prime \prime}+\frac{1}{r} u^{\prime}\right) w d r=0
$$

Finally, using the equation (3.2)

$$
\int_{0}^{1} f(u) w r d r=\frac{1}{2 \lambda} u^{\prime}(1) w^{\prime}(1)>0
$$

Theorem 3.1. All positive solutions of (3.1) lie on simple smooth solution curves, that is, at each $(\lambda, u)$ either implicit function theorem or CrandallRabinowitz Theorem 1.1 applies.

Proof. If (3.3) has only trivial solution, then the implicit function theorem applies. So we assume (3.3) has nontrivial solutions at $(\bar{\lambda}, \bar{u})$. We shall verify the conditions of the Crandall-Rabinowitz Theorem. Define $B$ to be a unit ball in $\boldsymbol{R}^{2}, X=\left\{u \in C^{2, \alpha}(\bar{B}) \mid u=0\right.$ on $\left.\partial B\right\}$ and $Y=C^{\alpha}(\bar{B})$. Let $F: \boldsymbol{R}_{+} \times X \rightarrow Y$ be
given by $F(\lambda, u)=\Delta u+\lambda f(u)$. We also rewrite (3.3) as

$$
\begin{equation*}
\Delta w+\lambda f^{\prime}(u) w=0 \quad \text { in } B, \quad w=0 \quad \text { on } \partial B . \tag{3.7}
\end{equation*}
$$

That the null-space of $F_{u}(\bar{\lambda}, \bar{u})$ is one-dimensional is seen from (3.3) (it can be parametrized by $\left.w^{\prime}(1)\right)$. Since $F_{u}(\bar{\lambda}, \bar{u})$ is a Frecholm operator of index zero it follows that $\operatorname{codim} R\left(F_{u}(\bar{\lambda}, \bar{u})\right)=1$. Finally, if the condition $F_{\lambda}(\bar{\lambda}, \bar{u}) \notin$ $R\left(F_{u}(\bar{\lambda}, \bar{u})\right)$ was violated, one could find $z \in X$ satisfying

$$
\begin{equation*}
\Delta z+\bar{\lambda} f^{\prime}(\bar{u}) z=f(u) \quad \text { in } B, \quad z=0 \quad \text { on } \partial B . \tag{3.8}
\end{equation*}
$$

From the equation (3.7) at $(\bar{\lambda}, \bar{u})$ and (3.8)

$$
0=\int_{B} f(u) w=2 \pi \int_{0}^{1} f(u) w r d r
$$

contradicting Lemma 3.1.
We give a simple application of Theorem 3.1 next. We consider $f(u) \in$ $C^{2}\left(\boldsymbol{R}_{+}\right)$satisfying for some $0<a<b$,

$$
\begin{gather*}
f(0)=f(a)=f(b)=0  \tag{3.9}\\
f(u)<0 \quad \text { on }(0, a), \quad f(u)>0 \quad \text { on }(a, b)  \tag{3.10}\\
\int_{0}^{b} f(u) d u>0 \tag{3.11}
\end{gather*}
$$

(e.g. $f(u)=u(u-a)(b-u)$ with $b>2 a)$.

It is known (see e.g. [5]) that for $\lambda$ large the problem (3.1) has exactly two solutions: a "large" one that is close to $b$, and a "small" one close to zero for $r \neq 0$. By maximum principle any nontrivial solution satisfies $0<u<b$.

Theorem 3.2. The "large" and "small" solutions lie on a single smooth solution curve.

Proof. It is known that for large $\lambda$ the problem (3.1) has exactly two positive solutions, see [5]. Continue either one of these solutions for decreasing $\lambda$. Since (3.1) has no positive solutions for small $\lambda>0$ (multiply it by $u$ and integrate), a critical point must be reached, at which the Crandall-Rabinowitz Theorem 1.1 will apply, and a turn will occur. After finitely many such turns the curve will join up with the other solution for large $\lambda$.

## 4. Equations with nonlinearities generalizing cubic

We will study a class of problems on an interval $(0, \ell)$ :

$$
\begin{equation*}
u^{\prime \prime}+a u^{\prime}+\lambda f(u)=0 \quad \text { on }(0, \ell), \quad u(0)=u(\ell)=0 \tag{4.1}
\end{equation*}
$$

Corresponding to (4.1), we consider the equations:

$$
\begin{array}{rc}
w^{\prime \prime}+a w^{\prime}+\lambda f^{\prime}(u) w=0 & \text { on }(0, \ell), \\
\mu^{\prime \prime}-a \mu^{\prime}+\lambda f^{\prime}(u) \mu=0 & \text { on }(0, \ell), \\
u_{x}^{\prime \prime}+a u_{x}^{\prime}+\lambda f^{\prime}(u) u_{x}=0 &  \tag{4.4}\\
\text { on }(0, \ell)
\end{array}
$$

We recall that Lemmas $2.1-5$ apply to (4.1). We shall now derive some additional properties of the problem (4.1) with general nonlinearities $f(u)$, which we will apply to derive other multiplicity results. We list our assumptions on $f(u)$ next.

We assume that the function $f(u) \in C^{2}[0,1]$ satisfies the following properties:

$$
\begin{equation*}
f(u)>0 \quad \text { on }(0,1), \quad f(0)=f(1)=0 \tag{4.5}
\end{equation*}
$$ there is a number $\alpha \in(0,1)$, such that $f^{\prime \prime}(u)>0$

$$
\begin{equation*}
\text { on }(0, \alpha) \text { and } f^{\prime \prime}(u)<0 \text { on }(\alpha, 1) \tag{4.6}
\end{equation*}
$$

We will be interested in the positive solutions of (4.1) satisfying

$$
\begin{equation*}
0<\max _{(0,1)} u(x)<1 \tag{4.7}
\end{equation*}
$$

We recall that a solution $u(x)$ of (4.1) is called singular (or critical) if (4.2) admits a nontrivial solution. We will then refer to $(\lambda, u(x))$ as a singular (or critical) point.

It is clear from our assumptions that there is exactly one point where a ray out of the origin touches the graph of $f(u)$. We denote this point by $\beta$, i.e. $\beta$ is the unique solution of

$$
\begin{equation*}
f^{\prime}(\beta)=\frac{f(\beta)}{\beta} \tag{4.8}
\end{equation*}
$$

Clearly $\beta>\alpha$.
The following lemma generalizes a similar one in [10].
Lemma 4.1. Let $u(x)$ be any singular solution of (4.1), and $x_{0}$ its point of maximum. Then

$$
\begin{equation*}
u\left(x_{0}\right)>\beta \tag{4.9}
\end{equation*}
$$

Proof. We will show that if $u\left(x_{0}\right) \leq \beta$, then the only solution of (4.2) is $w \equiv 0$. We begin by noticing that

$$
\begin{equation*}
f^{\prime}(u)>\frac{f(u)}{u} \quad \text { for } \quad 0<u<\beta \tag{4.10}
\end{equation*}
$$

This was proved in [10], we repeat the argument for completeness. Denoting $p(u)=u f^{\prime}(u)-f(u)$, we see that $p(0)=p(\beta)=0$, and $p^{\prime}(u)=u f^{\prime \prime}(u)$. It follows that $p^{\prime}(u)>0$ near $u=0$ and $p^{\prime}(u)<0$ near $u=\beta$. Since $p(u)$ has no roots on $(0, \beta)$, it follows that $p(u)>0$ on $(0, \beta)$, establishing (4.10). We now rewrite (4.1) and (4.2) correspondingly as

$$
\begin{array}{lll}
\left(e^{a x} u^{\prime}\right)^{\prime}+\lambda \frac{f(u)}{u} e^{a x} u=0 & \text { on }(0, \ell), & u(0)=u(\ell)=0 \\
\left(e^{a x} w^{\prime}\right)^{\prime}+\lambda f^{\prime}(u) e^{a x} w=0 & \text { on }(0, \ell), & w(0)=w(\ell)=0 \tag{4.12}
\end{array}
$$

Using the Sturm comparison theorem (see e.g. [11, p. 3]) and (4.10), we see that the assumption $u\left(x_{0}\right) \leq \beta$ implies that $w(x)$ must vanish on $(0, \ell)$, contradicting Lemma 2.2.

Lemma 4.2. Assume $f(u)$ satisfies (4.5). When $\lambda>0$ is sufficiently small the problem (4.1) has no solution satisfying (4.7).

Proof. Multiplying (4.1) by $u$ and integrating, we obtain using Poincare's inequality,

$$
c_{1} \lambda \int_{0}^{\ell} u^{2} \geq \lambda \int_{0}^{\ell} f(u) u d x=\int_{0}^{\ell} u^{\prime 2} d x \geq \frac{\pi^{2}}{\ell^{2}} \int_{0}^{\ell} u^{2} d x
$$

with some constant $c_{1}>0$. The proof follows.
We now distinguish between the cases

$$
\begin{equation*}
f^{\prime}(0)=0 \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime}(0)>0 . \tag{4.14}
\end{equation*}
$$

Theorem 4.1. Consider the problem (4.1) with $f(u)$ satisfying the conditions (4.5), (4.6) and (4.13). Then there is a critical $\lambda_{0}>0$ such that for $\lambda<\lambda_{0}$ the problem (4.1) has no positive solutions, it has at least one positive solution for $\lambda=\lambda_{0}$, and at least two positive solutions for $\lambda>\lambda_{0}$. There are exactly two solutions for $\lambda>\lambda_{1} \geq \lambda_{0}$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda>\lambda_{1}$ has two branches, denoted by $u^{-}(x, \lambda)<u^{+}(x, \lambda)$, with $u^{+}(x, \lambda)$ strictly monotone increasing in $\lambda$, and $\lim _{\lambda \rightarrow \infty} u^{+}(x, \lambda)=1$ for all $x \in(0, \ell)$. For the lower branch, $\max _{x \in[0,1]} u^{-}(x, \lambda)$ is monotone decreasing in $\lambda$, and $\lim _{\lambda \rightarrow \infty} \max _{x} u^{-}(x, \lambda)=0$.

Proof. We begin by showing that for sufficiently large $\lambda$ the problem (4.1) has a positive solution. Choose $\alpha>0$ such that

$$
\begin{equation*}
f(u) \geq \alpha u^{2} \quad \text { for } \quad u \in[0,2 / 3] . \tag{4.15}
\end{equation*}
$$

Consider the problem

$$
\begin{equation*}
v^{\prime \prime}+a v^{\prime}+\lambda \alpha v^{2}=0 \quad \text { on }(0, \ell), \quad v(0)=v(\ell)=0 \tag{4.16}
\end{equation*}
$$

Setting $v=(1 / \lambda \alpha) w$, we transform it into

$$
\begin{equation*}
w^{\prime \prime}+a w^{\prime}+w^{2}=0 \quad \text { on }(0, \ell), \quad w(0)=w(\ell)=0 . \tag{4.17}
\end{equation*}
$$

By the Theorem 3.3 in [12, p. 138] for any $a$ the problem (4.17) has exactly one positive solution. Since any solution of (4.16) is a subsolution of (4.1), it follows that for $\lambda$ large, the problem (4.1) has a positive subsolution smaller than $2 / 3$. Since $u=1$ is a supersolution of (4.1), we conclude that for large $\lambda$ the problem (4.1) has positive solutions. (Here $\lambda$ is large, but fixed. By a standard device, involving addition of $M u$ to both sides of the equation, with a large constant $M>0$, we can obtain monotone nonlinearity.)

Next, starting with the maximal solution at some $\lambda$ we continue it for decreasing $\lambda$, using either the implicit function theorem or the Crandall-Rabinowitz Theorem 1.1 (similarly to the Theorem 2.1). By Lemma 4.2, we cannot continue this curve for all $\lambda>0$. Let $\lambda_{0}$ be the infimum of $\lambda$ for which we can continue the curve to the left. As before, we conclude existence of a solution $u_{0}(x)$ of (4.1) at $\lambda=\lambda_{0}$, lying on our curve of solutions. Also ( $\lambda_{0}, u_{0}$ ) is a singular point at which a turn to the right occurs. Hence near $\left(\lambda_{0}, u_{0}\right)$ we have two solution branches, an upper one $u^{+}(x, \lambda)$ and a lower one $u^{-}(x, \lambda)$, with $u^{-}(x, \lambda)<$ $u^{+}(x, \lambda)$ for all $x \in(0,1)$.

We show next that $u^{+}(x, \lambda)$ is strictly increasing in $\lambda$ for all $\lambda>\lambda_{0}$. For $\lambda$ close to $\lambda_{0}$ this follows from Crandall-Rabinowitz Theorem and Lemma 2.2. That it continues to increase for all $\lambda$ is proved in exactly the same way as monotonicity of the lower branch in the Theorem 2.1.

We turn to the monotonicity of the lower branch next. By Lemma 2.1 $u^{-}(x, \lambda)$ has its maximum at a unique point $x_{0}=x_{0}(\lambda)$. Clearly,

$$
\begin{equation*}
u_{x}\left(x_{0}(\lambda), \lambda\right)=0 \tag{4.18}
\end{equation*}
$$

From (4.1) express (using 4.18))

$$
u_{x x}\left(x_{0}(\lambda), \lambda\right)=-\lambda f\left(u\left(x_{0}(\lambda), \lambda\right)\right)<0
$$

which implies by the implicit function theorem that $x_{0}(\lambda)$ is a differentiable function of $\lambda$. We claim that

$$
\begin{equation*}
u_{\lambda}\left(x_{0}(\lambda), \lambda\right)<0 \quad \text { for all } \quad \lambda>\lambda_{0} . \tag{4.19}
\end{equation*}
$$

Assuming the claim for the moment, denote $m(\lambda)=\max _{x \in[0,1]} u^{-}(x, \lambda)=$
$u^{-}\left(x_{0}(\lambda), \lambda\right) . \quad$ Then

$$
\begin{equation*}
m^{\prime}(\lambda)=u_{x}\left(x_{0}(\lambda), \lambda\right) \frac{d x_{0}(\lambda)}{d \lambda}+u_{\lambda}\left(x_{0}(\lambda), \lambda\right)=u_{\lambda}\left(x_{0}(\lambda), \lambda\right)<0 \tag{4.20}
\end{equation*}
$$

proving that $\max _{x} u^{-}(x, \lambda)$ is decreasing in $\lambda$.
To prove (4.19) we differentiate (4.1) in $\lambda$

$$
\begin{equation*}
u_{\lambda}^{\prime \prime}+a u_{\lambda}^{\prime}+\lambda f^{\prime}(u) u_{\lambda}+f(u)=0 \quad \text { on }(0, \ell), \quad u_{\lambda}(0)=u_{\lambda}(\ell)=0 . \tag{4.21}
\end{equation*}
$$

For $\lambda$ close to $\lambda_{0}, u_{\lambda}^{-}(x, \lambda)<0$ for all $x$, as follows by the Crandall-Rabinowitz Theorem 1.1. Assuming (4.19) to be false, let $\lambda_{1}$ be the first value of $\lambda$, where $u_{\lambda}\left(x_{0}\left(\lambda_{1}\right), \lambda_{1}\right)=0$. Notice that $u_{\lambda}\left(x, \lambda_{1}\right)$ cannot have a local minimum at $x_{0}\left(\lambda_{1}\right)$ (as is seen from (4.21)), and hence it has to become negative either to the left of $x_{0}\left(\lambda_{1}\right)$ or to the right. Assume first that it becomes negative to the left of $x_{0}\left(\lambda_{1}\right)$. Let $0 \leq \xi<x_{0}\left(\lambda_{1}\right)$ be the adjacent root of $u_{\lambda}\left(x, \lambda_{1}\right)$. Letting $p(x)=$ $u^{\prime} u_{\lambda}^{\prime}-u_{\lambda} u^{\prime \prime}$, we conclude from (4.4) and (4.21),

$$
\begin{equation*}
p^{\prime}+a p=-f(u) u^{\prime}<0 \quad \text { on } \quad\left(\xi, x_{0}\left(\lambda_{1}\right)\right) . \tag{4.22}
\end{equation*}
$$

Notice, $p(\xi)=u^{\prime}(\xi) u_{\lambda}^{\prime}(\xi)<0$ and $p\left(x_{0}\right)=0$. But integrating (4.22), we obtain

$$
\begin{equation*}
0=e^{a x_{0}} p\left(x_{0}\right)<e^{a \xi} p(\xi)<0, \tag{4.23}
\end{equation*}
$$

a contradiction, proving that $m(\lambda)=\max _{x} u^{-}(x, \lambda)$ is decreasing in $\lambda$. The case when $u_{\lambda}\left(x_{0}\left(\lambda_{1}\right), \lambda_{1}\right)$ becomes negative to the right of $x_{0}\left(\lambda_{1}\right)$ is handled similarly. (We proved that $m(\lambda)$ is decreasing until a possible turning point is reached. But if a turn "to the left" occurs, then we are now on the upper branch relative to the next turning point, and so by above $u(x)$ is uniformly decreasing as we follow the solution curve until the turn "to the right" occurs, and our argument is repeated.)

Define $m_{0}=\lim _{\lambda \rightarrow \infty} m(\lambda) \geq 0$. (The limit is defined as we trace the lower branch through all of the possible turning points.) We exclude the possibility that $m_{0}>0$ next. Since $u(x)$ is a bounded and unimodular function, its first and second derivatives cannot become large over any subinterval of $(0,1)$. It follows that $f(u(x)) \rightarrow 0$ for all $x \in(0,1)$ as $\lambda \rightarrow \infty$. For the increasing upper branch $u^{+}(x, \lambda)$ this implies that $\lim _{\lambda \rightarrow \infty} u^{+}(x, \lambda)=1$. For the lower branch we conclude that $\lim _{\lambda \rightarrow \infty} u^{-}(x, \lambda)=0$, but there is still a possibility of nonuniform convergence with $m_{0}>0$ (and $\left.x_{0}(\lambda) \rightarrow 0\right)$. Arguing as in Lemma 2.5, we see that in formula (2.12) we may assume that both $u\left(x_{1}\right)$ and $\bar{c}$ are arbitrarily small, and hence the maximum of $u^{-}(x, \lambda)$ will get small as $\lambda \rightarrow \infty$.

We show next that for large $\lambda$ no further turns are possible (i.e. (4.2) has only the trivial solution). For the lower branch this follows by Lemma 4.1. We now turn to the upper branch, where we know that $\lim _{\lambda \rightarrow \infty} u^{+}(x, \lambda)=1$ for all $x \in(0, \ell)$. Assume that at $u=u^{+}$the problem (4.2) has a nontrivial solution,
which we normalize: $\int_{0}^{\ell} w^{2} d x=1$. Since $f^{\prime}(u(x))$ is negative for large $\lambda$, except near $x=0$ and $x=\ell$, if we multiply (4.1) by $u$ and integrate, we see that $w(x)$ has to become large near $x=\ell$ (or $x=0$, which is similar). We claim that

$$
\begin{equation*}
\left|u^{\prime}(x)\right| \geq c_{0} \sqrt{\lambda} \tag{4.24}
\end{equation*}
$$

for some $c_{0}>0$ when $\lambda$ is large, for all $x \in(\eta, 1]$, where $u(\eta)=\alpha$ and $\alpha$ is the larger root of $f^{\prime}(u)$. Indeed, multiplying (4.1) by $u$ and integrating, we first conclude that $\int_{0}^{\ell} u^{\prime 2} d x=o(\lambda)$ as $\lambda \rightarrow \infty$, and then multiplying (4.1) by $u^{\prime}$ and integrating over $\left(x_{0}, x\right)$, we conclude (4.24). Since $f^{\prime}(u)<0$ except near $x=0, \ell$, it follows from (4.2) that $w(x)$ can have local maximums only near $x=0$ or $x=\ell$. Since $\int_{0}^{\ell} w^{2} d x=1$, it follows that $w(x)$ is bounded over say the interval $I=((3 / 7) \ell,(5 / 7) \ell)$ as $\lambda \rightarrow \infty$. Similarly one can find a constant $c$ so that the measure of points where the inequality $\left|w^{\prime}(x)\right| \leq c$ is violated is $<|I| / 3$. Since $u(x, \lambda) \rightarrow 1$, we can find a constant $c$, so that measures of each of the subsets of $I$ where the inequalities $\left|u^{\prime}(x)\right| \leq c$ and $\left|u^{\prime \prime}(x)\right| \leq c$ are violated are also $<|I| / 3$. Combining, we can find $\xi=\xi(\lambda) \in I$, such that $|w(\xi)|,\left|w^{\prime}(\xi)\right|,\left|u^{\prime}(\xi)\right|$ and $\left|u^{\prime \prime}(\xi)\right|$ are all bounded as $\lambda \rightarrow \infty$. Denoting $p(x)=u^{\prime \prime} w-u^{\prime} w^{\prime}$, we easily conclude from (4.2) and (4.4) that

$$
p^{\prime}+a p=0 \quad \text { for } \quad x \in(0, \ell)
$$

i.e., $p(\ell)=e^{a(\ell-\xi)} p(\xi)$. Since by above $p(\xi)$ is bounded, the same must be true for $p(\ell)=-u^{\prime}(\ell) w^{\prime}(\ell)$ as $\lambda \rightarrow \infty$. By (4.24) this implies that $\left|w^{\prime}(\ell)\right|=O(1 / \sqrt{\lambda})$. Also by (4.24) we conclude that when moving from $x=\ell$ leftwards, $u(x)$ must reach $u(\eta)=\alpha, f^{\prime}(\alpha)=0$ over the interval $(\eta, \ell)$ of length $\ell-\eta=O(1 / \sqrt{\lambda})$. Since $w(x)$ cannot take local maximums where $f^{\prime}(u)<0$, it follows that $w(x)$ takes its maximum on $(\eta, \ell)$ (if not, argue from $x=0$ end). And the maximum value must be at least $1 / 3$, since $\int_{0}^{\ell} w^{2} d x=1$. On $(\eta, 1)$ we set $t=\ell-x$, and estimate from (4.2), which can be written as $\left(e^{-a t} w^{\prime}\right)^{\prime}=-\lambda f^{\prime}(u) w e^{-a t}$,

$$
\left(e^{-a t} w^{\prime}\right)^{\prime} \leq c \lambda w, \quad w(0)=0, \quad w^{\prime}(0)=O\left(\frac{1}{\sqrt{\lambda}}\right), \quad 0<t<\frac{c_{1}}{\sqrt{\lambda}}
$$

with positive constants $c$ and $c_{1}$. Integrating

$$
w(t) \leq c \lambda \int_{0}^{t}(t-s) w(s) d s+w^{\prime}(0) t \leq c_{2}\left(\sqrt{\lambda} \int_{0}^{t} w(s) d s+\frac{1}{\lambda}\right) .
$$

Applying Gronwall's inequality, we conclude that $w(t)=O(1 / \lambda)$ on $(\eta, 1)$. This is a contradiction, proving that there are no turns on $u^{+}(x, \lambda)$ for $\lambda$ large enough.

Finally, we rule out the possibility of more than one solution curve. If there was another curve, it would have the same structure as the one described previously, in particular it would have a lower curve $v^{-}(x, \lambda)$ tending uniformly to zero as $\lambda \rightarrow 0$. Denoting $w(x)=u^{-}(x, \lambda)-v^{-}(x, \lambda)$, we express

$$
\begin{equation*}
w^{\prime \prime}+a w^{\prime}+\lambda c(x) w=0 \quad \text { on }(0, \ell), \quad w(0)=w(\ell)=0 \tag{4.25}
\end{equation*}
$$

with $c(x)=\int_{0}^{1} f^{\prime}\left(\theta u^{-}+(1-\theta) v^{-}\right) d \theta$. By Lemma 2.1 in [2] we may assume that $u^{-}(x, \lambda)>v^{-}(x, \lambda)$ for all $x \in(0,1)$, i.e. $w>0$. For large $\lambda$ both $u^{-}$and $v^{-}$ are small, and hence

$$
\begin{equation*}
c(x)>\int_{0}^{1} f^{\prime}\left(\theta v^{-}+(1-\theta) v^{-}\right) d \theta=f^{\prime}\left(v^{-}\right) \tag{4.26}
\end{equation*}
$$

Since $v^{-}$solves (4.1), it is the principal eigenfunction of

$$
z^{\prime \prime}+a z^{\prime}+\lambda \frac{f\left(v^{-}\right)}{v^{-}} z=\mu z, \quad z(0)=z(\ell)=0
$$

corresponding to the principal eigenvalue $\mu=0$. Since for large $\lambda, f^{\prime}\left(v^{-}\right)>$ $f\left(v^{-}\right) / v^{-}$, the principal eigenvalue of

$$
w^{\prime \prime}+a w^{\prime}+c(x) w=\mu w, \quad w(0)=w(\ell)=0
$$

must be positive, which implies that (4.25) cannot have positive solutions, a contradiction.

We turn to the case $f^{\prime}(0)>0$ next. We denote by $\lambda_{1}$ the principal eigenvalue of

$$
\begin{equation*}
w^{\prime \prime}+a w^{\prime}+\lambda f^{\prime}(0) w=0 \quad \text { on }(0, \ell), \quad w(0)=w(\ell)=0, \tag{4.27}
\end{equation*}
$$

and $w(x)>0$ the corresponding eigenfunction. We recall that we are looking for solutions of (4.1) satisfying (4.7).

Theorem 4.2. Consider the problem (4.1) with $f(u)$ satisfying the conditions (4.5), (4.6) and (4.14). Then there is a critical value of the parameter $\lambda=\lambda_{0}$, $0<\lambda_{0}<\lambda_{1}$, such that for $\lambda<\lambda_{0}$ the problem (4.1) has no solutions, it has at least two solutions for $\lambda_{0}<\lambda<\lambda_{1}$, and at least one solution for $\lambda \geq \lambda_{1}$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda>\lambda_{0}$ has, after at most finitely many turns, two branches denoted by $u^{-}(x, \lambda)<u^{+}(x, \lambda)$, with $u^{+}(x, \lambda)$ strictly monotone increasing in $\lambda$, and $\lim _{\lambda \rightarrow \infty} u^{+}(x, \lambda)=1$ for all $x \in(0, \ell)$. For the lower branch, $\max _{x} u^{-}(x, \lambda)$ is monotone decreasing in $\lambda$, and $\lim _{\lambda \rightarrow \lambda_{1}} \max _{x} u^{-}(x, \lambda)=0$.

Proof. It is easy to see that at $\lambda=\lambda_{1}$ and $u=0$ a theorem of CrandallRabinowitz on "bifurcation from a simple eigenvalue" applies. We show next
that the bifurcation direction is to the "left" in $(\lambda, u)$ plane. Differentiate (4.1) in $s$ twice

$$
\begin{align*}
u_{s s}^{\prime \prime}+a u_{s s}^{\prime}+\lambda f^{\prime}(u) u_{s s}+\lambda f^{\prime \prime}(u) u_{s}^{2}+2 \lambda^{\prime} f^{\prime}(u) u_{s}+\lambda^{\prime \prime} f(u) & =0  \tag{4.28}\\
u_{s s}(0)=u_{s s}(\ell) & =0
\end{align*}
$$

Setting $s=0$, and using that $\lambda(0)=\lambda_{1},\left.u_{s}\right|_{s=0}=w$, we have

$$
\begin{equation*}
u_{s s}^{\prime \prime}+a u_{s s}^{\prime}+\lambda_{1} f^{\prime}(0) u_{s s}+\lambda_{1} f^{\prime \prime}(0) w^{2}+2 \lambda^{\prime}(0) f^{\prime}(0) w=0 \tag{4.29}
\end{equation*}
$$

Let $\mu>0$ be solution of

$$
\begin{equation*}
\mu^{\prime \prime}-a \mu^{\prime}+\lambda_{1} f^{\prime}(0) \mu=0 \quad \text { on }(0, \ell), \quad \mu(0)=\mu(\ell)=0 \tag{4.30}
\end{equation*}
$$

From (4.29) and (4.30),

$$
\lambda^{\prime}(0)=-\frac{1}{2} \lambda_{1} \frac{f^{\prime \prime}(0) \int_{0}^{\ell} w^{2} \mu d x}{f^{\prime}(0) \int_{0}^{\ell} w \mu d x}<0
$$

We continue the curve of positive solutions bifurcating from $\left(\lambda_{1}, w(x)\right)$ for decreasing $\lambda<\lambda_{1}$. By Lemma 4.2 we cannot continue this process for all $\lambda>0$. If $\lambda_{0}$ is the infimum of $\lambda$ for which (4.1) is solvable, then proceeding as in Theorem 4.1 we conclude that a turn to the right occurs at $\lambda_{0}$, and both solution branches have the properties described in the theorem. Similarly to [9] one sees that solution curve can have only finitely many turns. (If there were infinitely many turns, they would accumulate to a singular point of (4.1) near which the solution set of (4.1) is not a simple curve, contradicting the Crandall-Rabinowitz Theorem 1.1.) Other properties of the solution curve are proved the same way as in the previous theorem.

It remains to exclude the possibility of more than one solution curve. Assuming existence of another solution curve, we continue it for decreasing $\lambda$ until a turning point, at which we will obtain two branches with the properties described above. In particular, we will get another lower branch $v^{-}(x, \lambda)$, which has to tend uniformly to zero as $\lambda \rightarrow \infty$ (it cannot tend to ( $\lambda_{1}, 0$ ) by uniqueness in Crandall-Rabinowitz Theorem.) Multiplying (4.1) by $\mu$ and using (4.30), we have:

$$
\begin{equation*}
-\lambda_{1} f^{\prime}(0) \int_{0}^{\ell} v \mu d x+\lambda \int_{0}^{\ell} f(v) \mu d x=0 \tag{4.31}
\end{equation*}
$$

Since for small $v, f(v) \geq c v$ for some $c>0$, we have a contradiction in (4.31) for large $\lambda$.

Remark. Condition (4.6) in Theorems 4.1 and 4.2 can be considerably relaxed. However its present form allows us to obtain a sharp estimate in Lemma 4.1, and to state the following conjecture: the solution curve has exactly one turn in both Theorems 4.1 and 4.2.

## 5. The time dependent problem

We shall apply our results from Section 2 to discuss the long time behavior for the parabolic problem

$$
\begin{gather*}
u_{t}=u_{x x}+a u_{x}+\lambda f(u), \quad 0<x<1, t>0  \tag{5.1}\\
u(0, t)=u(1, t)=0, \quad t>0 \\
u(x, 0)=u_{0}(x) \geq 0, \quad 0<x<1 .
\end{gather*}
$$

Here $a$ is a real constant, $\lambda \geq 0$ is a parameter. We assume that $f(u)$ satisfies all conditions of the Theorem 2.1, so that the steady-states of (5.1) are described by that theorem. We assume that $u_{0}(x) \in L^{\infty}(0,1)$. This implies existence of solution $u(x, t)$ of (5.1) on some interval $0 \leq t<T_{\max }$ for some $T_{\max } \in(0, \infty]$, and if $T_{\text {max }}<\infty$ then $\lim _{t \rightarrow T_{\text {max }}}\|u(\cdot, t)\|_{L^{\infty}(0,1)}=\infty$ (see [2]).

Steady states of (5.1) are solutions of (2.10). We recall that a steady state is stable, if all eigenvalues of the linearization of (2.10) are negative, i.e. for the problem

$$
\begin{equation*}
w^{\prime \prime}+a w^{\prime}+\lambda f^{\prime}(u) w=v w \quad \text { on }(0,1), \quad w(0)=w(1)=0 \tag{5.2}
\end{equation*}
$$

all eigenvalues $v$ are negative. Solution of (2.10) is unstable if the principal eigenvalue of (5.2) is positive.

Theorem 5.1. In the conditions of the Theorem 2.1, solution $u^{-}(x, \lambda)$ is stable and $u^{+}(x, \lambda)$ is unstable for all $0<\lambda<\lambda_{0}$.

Proof. Let $u=u^{-}(x, \lambda)$, and let $v$ be the principal eigenvalue of (5.2) with $w(x)>0$ the corresponding eigenfunction. We wish to show that

$$
\begin{equation*}
v<0 . \tag{5.3}
\end{equation*}
$$

Notice that for both $u^{-}(x, \lambda)$ and $u^{+}(x, \lambda)$ we have $v \neq 0$ for all $0<\lambda<\lambda_{0}$. Indeed, if $v=0$ at some $\bar{\lambda}$ and corresponding $u=\bar{u}$, it follows that $(\bar{\lambda}, \bar{u})$ is a critical point of the map $F(\lambda, u)$, defined in the Theorem 2.1. According to the proof of that theorem, a turn must occur at $(\bar{\lambda}, \bar{u})$, which is impossible, since there is only one turning point. So assuming (5.3) to be violated, will imply that $v>0$. We claim that then $\varphi_{\varepsilon}=u-\varepsilon w$ will be a supersolution of (2.10) for
sufficiently small $\varepsilon$. Indeed,

$$
\varphi_{\varepsilon}^{\prime \prime}+a \varphi_{\varepsilon}^{\prime}+\lambda f\left(\varphi_{\varepsilon}\right)=-\varepsilon \nu w+o(\varepsilon)<0, \quad \text { for } \quad \varepsilon \text { small. }
$$

Fix a supersolution $\varphi_{\varepsilon_{0}}=u-\varepsilon_{0} w$. Notice that $u^{-}(x, \mu)$ is a subsolution of (2.10) for all $0<\mu<\lambda$, and $u^{-}(x, \mu)<u^{-}(x, \lambda)$ for all $x \in(0,1)$. Increasing $\mu$, we obtain a contradiction at the first $\mu_{1}$ where the graph of $u^{-}\left(x, \mu_{1}\right)$ touches $\varphi_{\varepsilon_{0}}$, by using a maximum principle (the possibility of $u^{-}\left(x, \mu_{1}\right)$ and $\varphi_{\varepsilon_{0}}$ having the same tangents at $x=0$ or $x=1$ is also ruled out by the maximum principle). This proves stability of $u^{-}(x, \lambda)$.

Turning to the upper branch, assume that $u=u^{+}(x, \lambda)$ is stable, i.e. $v<0$. Then as above we infer existence of $\varepsilon_{0}>0$, such that for all $0<\varepsilon<\varepsilon_{0}$, $\varphi_{\varepsilon}=u-\varepsilon w$ is a subsolution of (2.10). Clearly, $\varphi_{\varepsilon}$ is a subsolution of (2.10) for all $\mu>\lambda$. Fix $\mu$ so close to $\lambda$ that $u^{+}(x, \mu)>\varphi_{\varepsilon_{0}}(x)$ for all $x \in(0,1)$. Now decrease $\varepsilon$ from $\varepsilon_{0}$ to 0 . Since stability does not change along a branch, we may assume $\lambda$ to be close to $\lambda_{0}$, and then $u^{+}(x, \lambda)>u^{+}(x, \mu)$ for all $x \in(0,1)$ by Theorem 1.1 and Lemma 2.2. At the first $\varepsilon$ where the subsolution $\varphi_{\varepsilon}(x)$ touches the solution $u^{+}(x, \mu)$ we obtain a contradiction, completing the proof.

Recall that the problem (5.1) admits a Lyapunov function (see [2]), which implies that any bounded solution of (5.1) converges to the set of steady states as $t \rightarrow \infty$. Combining this with the Theorem 5.1, we conclude that for $0<\lambda \leq$ $\lambda_{0}$ almost all bounded solutions of (5.1) converge to $u^{-}(x, \lambda)$.

The following proposition shows that only positive solutions of (2.10) have a chance to be stable. It generalizes a similar result (when $a=0$ ) in R. Schaaf [13].

Proposition 1. Assume $f(u) \in C^{1}(\boldsymbol{R})$ and $u(x)$ is a solution of (2.10) which changes sign on $(0,1)$. Then $u(x)$ is unstable.

Proof. Let $v$ be the principal eigenvalue of (5.2), w(x)>0 the corresponding eigenfunction. We need to show that $v>0$. Since $u(x)$ changes sign, we can find $0<x_{1}<x_{2}<1$, such that $u^{\prime}\left(x_{1}\right)=u^{\prime}\left(x_{2}\right)=0$ and say $u^{\prime}(x)<0$ on ( $x_{1}, x_{2}$ ) (the case $u^{\prime}(x)>0$ on ( $x_{1}, x_{2}$ ) is similar). Differentiate (2.10)

$$
\begin{equation*}
u_{x}^{\prime \prime}+a u_{x}^{\prime}+\lambda f^{\prime}(u) u_{x}=0 \tag{5.4}
\end{equation*}
$$

Notice that $u^{\prime \prime}\left(x_{1}\right)<0$ and $u^{\prime \prime}\left(x_{2}\right)>0$. Denoting $p(x)=u^{\prime \prime} w-u^{\prime} w^{\prime}$, we obtain from (5.2) and (5.4)

$$
\begin{equation*}
p^{\prime}+a p=-v u^{\prime} w . \tag{5.5}
\end{equation*}
$$

If one assumes $v \leq 0$, then the right side of (5.5) is non-positive on ( $x_{1}, x_{2}$ ), while $p\left(x_{1}\right)=u^{\prime \prime}\left(x_{1}\right) w\left(x_{1}\right)<0$ and $p\left(x_{2}\right)=u^{\prime \prime}\left(x_{2}\right) w\left(x_{2}\right)>0$, which is impossible.

A similar result holds for the PDE case. Its proof is a modification of the one above, and is therefore omitted.

Proposition 2. Assume $f(u) \in C^{1}(\boldsymbol{R})$ and $u(r)$ is a radially symmetric solution of (3.1) which changes sign on $(0, R)$. Then $u(r)$ is unstable.

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