AN EXACT MULTIPLICITY RESULT FOR A CLASS OF SEMILINEAR EQUATIONS

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Abstract

For a class of Dirichlet problems in two dimensions, generalizing the model case

$$\Delta u + \lambda u(u-b)(c-u) = 0 \text{ in } |x| < R, u = 0 \text{ on } |x| = R,$$

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we show existence of a critical $\lambda_0 > 0$, so that there are exactly 0, 1 or 2 nontrivial solutions (in fact, positive), depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. We show that all solutions lie on a single smooth solution curve, and study some properties of this curve. We use bifurcation approach. The crucial thing is to show that any nontrivial solution of the corresponding linearized problem is of one sign.

Key words and phrases: exact multiplicity, positivity of solution for the linearized problem.

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1 Introduction

We study a class of semilinear Dirichlet problems

$$\Delta u + \lambda f(u) = 0 \text{ in } |x| < R, u = 0 \text{ on } |x| = R, \tag{1.1}$$

on a ball in two dimensions (i.e. $x = (x_1, x_2)$). Here λ is a positive parameter, the nonlinearity $f(u)$ is assumed to generalize a model case, $f(u) = u(u - b)(c - u)$, with positive constants b and c, and $c > 2b$. (The last condition is necessary for existence of nontrivial solutions.) The one-dimensional version of this problem was studied previously in a number of papers, see [10] and the references given there. We prove existence of a critical $\lambda_0 > 0$, such that the problem (1.1) has exactly 0, 1 or 2 nontrivial solutions, depending whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$ (all nontrivial solutions are, in fact, positive by the maximum principle.) Moreover, we show that all solutions of (1.1) lie on a single smooth parabola-like curve of solutions, and study how both branches of this curve behave as $\lambda \to \infty$. This appears to be the first exact multiplicity result for a polynomial nonlinearity (compare with [1], [2]).

We use techniques from bifurcation theory, particularly a theorem of M.G. Crandall and P.H. Rabinowitz [3], which we recall below. The crucial thing is to prove that any nontrivial solution of the linearized equation

$$\Delta w + \lambda f'(u) w = 0 \ \text{ in } \ |x| < R, w = 0 \ \text{ on } \ |x| = R, \tag{1.2}$$

is of one sign, i.e. we can assume that $w(x) > 0$ for $|x| < R$. This is done in Section 2. This result appears to be also of independent interest. In Section 3 we begin by showing that the Crandall-Rabinowitz theorem applies at any critical point (λ, u) of (1.1) (i.e. when (1.2) admits a nontrivial solution), and that a "turn to the right" occurs in (λ, u) "plane". We then combine this information with a study of stability of both branches to show that there is a single solution curve.

Without loss of generality we shall assume $R = 1$.

Next we state a bifurcation theorem of Crandall-Rabinowitz [3].

Theorem 1.1 *[3] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) =$ span $\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

Our condition (2.2) implies that $f(u) > 0$ for $u < 0$ and hence u would have been superharmonic if u were to be negative and it would have contradicted with the maximum principle. Therefore all solutions of (1.1) are positive, hence by a well-known result of B. Gidas, W.-M. Ni and L. Nirenberg [6] they are radially symmetric. By a theorem of C.S. Lin and W.-M. Ni [13] all solutions of (1.2) are also radially symmetric. Accordingly, we shall study the ODE versions of these equations. For a class of nonlinearities generalizing $f(u) = u(u - b)(c - u)$ (see (2.1)-(2.7) and (2.21)) our results can be roughly summarized as follows.

Theorem 1.2 *Any nontrivial solution of (1.2) can be assumed to be positive.*

Positivity of the solution of linearized equation is crucial to our analysis. It is here that the restriction on space dimension comes in. All other arguments in this paper will apply (with slight modifications) in all dimensions. In this direction Wei [14] proves an exact multiplicity result in balls for $f(u) = u(u-b)(c-u)$ for large λ.

Theorem 1.3 *For the problem (1.1) there is a critical $\lambda_0 > 0$ such that the problem (1.1) has exactly* 0, 1 *or* 2 *nontrivial solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^+(r,\lambda)$ strictly monotone increasing in λ and $\lim_{\lambda\to\infty} u^+(r,\lambda) = c$ for $r \in [0,1)$. For the lower branch $\lim_{\lambda\to\infty} u^-(r,\lambda) = 0$ for $r \neq 0$, while $u^-(0,\lambda) > b$ for all $\lambda > \lambda_0$.*

A word on notation. We shall denote derivatives of $u(r)$ by either $u'(r)$ or u_r, and mix both notations when it helps to make our proofs more transparent. We denote by $B(0,1)$ the unit ball around the origin in R^2.

2 Positivity of Solutions of the Linearized Equation

We begin by listing our assumptions on the nonlinearity $f(u)$. We assume that $f(u) \in C^2(R)$ has the following properties

$$f(0) = f(b) = f(c) = 0 \text{ for some constants } 0 < b < c, \tag{2.1}$$

$$\begin{aligned} f(u) &< 0 \text{ for } u \in (0,b) \cup (c,\infty), \\ f(u) &> 0 \text{ for } u \in (-\infty,0) \cup (b,c), \end{aligned} \tag{2.2}$$

$$f'(0) < 0, \tag{2.3}$$

$$\int_0^c f(u)du > 0, \tag{2.4}$$

(2.5) There exists $\alpha \in (0, c)$, such that

$$f''(u) > 0 \text{ for } u \in (0, \alpha) \text{ and } f''(u) < 0 \text{ for } u \in (\alpha, c).$$

It is clear from the above assumptions that there is exactly one point in (b, c) where a ray out of the origin touches the graph of $f(u)$. We denote this point by β, i.e. β is the unique solution of

$$f'(\beta) = \frac{f(\beta)}{\beta}. \tag{2.6}$$

We shall place the following assumption on $f(u)$.

(2.7) The function $g_\mu(s)$, defined in (2.14) below,
can have at most one sign change when $s \in (0, c)$,
for any value of the parameter $\mu \in (0, \infty)$.

We shall show that this assumption follows for example from the following two more explicit assumptions: $f(u) \in C^4(R)$ satisfies

$$f'''(u) \leq 0 \text{ and } f^{(4)}(u) \leq 0 \text{ with}$$
$$f'''(u) + f^{(4)}(u) < 0, \text{ for } u \in (0, \beta), \tag{2.8}$$

$$2f'(u) > \mu u f''(u) \text{ for any } \mu \in (0, 1] \text{ and } u \in (b, \alpha). \tag{2.9}$$

For example, a function $f(u) = u(u - b)(c - u)$ with $c > 2b$ satisfies all of the above conditions. By the maximum principle all solutions of (1.1) are positive, hence by the well-known result of B. Gidas, W.-M. Ni and L. Nirenberg [6] they are radially symmetric. Therefore in this section we will consider the following ODE version of (1.1) for $R = 1$ in two dimensions

$$u''(r) + \frac{1}{r}u'(r) + \lambda f(u) = 0 \quad r \in (0, 1), \quad u'(0) = u(1) = 0, \tag{2.10}$$

and its linearization

$$w''(r) + \frac{1}{r} w'(r) + \lambda f'(u) w = 0 \ \ r \in (0,1), \ \ w'(0) = w(1) = 0. \tag{2.11}$$

The goal of this section is to show that if (2.11) admits a nontrivial solution, then we have $w(r) > 0$.

We shall use $v(x) = r u_r + \mu u$, with a constant μ to be specified, as a test function. The same test function was used by many authors for very different problems, see e.g., V. Komornik [8, p. 35] for applications in control theory and many references. One easily checks that v satisfies the equation

$$v'' + \frac{1}{r} v' + \lambda f'(u) v = \lambda [\mu f'(u) u - (\mu + 2) f(u)] \equiv \lambda g_\mu(u). \tag{2.12}$$

We recall that (λ, u) is called a critical point of (2.10) if (2.11) has nontrivial solutions. Let us assume $u = u(r)$ is such a point, we are interested in the sign properties of $g_\mu(u(r))$. But first we notice that by our assumptions

$$f'(u) > \frac{f(u)}{u} \ \text{ for } \ 0 < u < \beta. \tag{2.13}$$

Indeed, denote $p(u) = u f'(u) - f(u)$. Then $p(0) = p(\beta) = 0$, and $p'(u) = u f''(u)$. It follows that $p'(u) > 0$ near $u = 0$, and $p'(u) < 0$ near $u = \beta$. Since $p(u)$ has no roots on $(0, \beta)$, it follows that $p(u) > 0$ on $(0, \beta)$, establishing (2.13). The same argument shows that for $u > \beta$ the inequality sign in (2.13) is reversed.

We now show that (2.7) can be derived from (2.8) and (2.9). More precisely

Lemma 2.1 *Conditions (2.2), (2.3), (2.5), (2.8) and (2.9) imply (2.7), that is, for any $\mu \in (0, \infty)$ the function $g_\mu(u(r))$ can have at most one sign change when $r \in (0,1)$.*

Proof. We write (with $p(u)$ as above)

$$g_\mu(u) = \mu (f'(u) u - f(u)) - 2 f(u) = \mu p(u) - 2 f(u). \tag{2.14}$$

When $u \in (0, b)$ both functions $p(u)$ and $-f(u)$ are positive, while if $u \in (\beta, c)$ both of the above functions are negative. It follows that the roots of $g_\mu(u)$ may occur only when $u \in (b, \beta)$. Compute

$$g'_\mu(u) = -2f'(u) + \mu f''(u)u, \tag{2.15}$$

$$g''_\mu(u) = (\mu - 2)f''(u) + \mu f'''(u)u, \tag{2.16}$$

$$g'''_\mu(u) = (2\mu - 2)f'''(u) + \mu f^{(4)}(u)u. \tag{2.17}$$

Case (i) $0 < \mu \leq 1$. Then by (2.9) we see that

$$g'_\mu(u) < 0 \ \text{ for } \ u \in (b, \beta),$$

and the lemma follows. (Notice that (2.9) holds when $u \in [\alpha, \beta)$, as follows from (2.5).)

Case (ii) $\mu > 1$. Then $g_\mu(0) = 0$ and by (2.3)

$$g'_\mu(0) > 0, \tag{2.18}$$

while by (2.8)

$$g'''_\mu(u) < 0 \ \text{ for } \ u \in (0, \beta). \tag{2.19}$$

Assume that $0 < u_1 < u_2$ are the smallest two roots of $g_\mu(u)$, and u_1 is a simple root (the other case is similar). Then by (2.18) $g_\mu(u)$ is negative on (u_1, u_2), and hence g''_μ is positive somewhere on (u_1, u_2). By (2.19), $g''_\mu(u)$ would have to be positive on $(0, u_1)$, which combined with (2.18) makes it impossible for $g_\mu(u)$ to vanish at $u = u_1$. Hence $g_\mu(u)$ cannot have more than one root, concluding the proof of the lemma.

The condition guaranteeing that $g_\mu(u(r))$ vanishes exactly once is

$$u(0) > \beta. \tag{2.20}$$

After we prove positivity of the solution of linearized equation, we will be able to show that the condition (2.20) holds at any turning point. At the moment we consider this condition as one of two possibilities (the other one is $u(0) \leq \beta$).

We now impose the final condition on the function $f(u)$:

$$N \equiv f'^2 u - f'f - ff''u > 0 \;\text{ for }\; b < u < \beta. \tag{2.21}$$

In case $f(u) = u(u-b)(c-u)$ one computes

$$N = u^2[(b+c)u^2 - 4bcu + bc(b+c)] > 0 \;\text{ for all }\; u > 0.$$

Define $0 < \rho_1 < \rho_0$ by $u(\rho_0) = b$ and $u(\rho_1) = \beta$ respectively.

Lemma 2.2 *Under the conditions (2.20) and (2.21) the equation*

$$g_\mu(u(r)) = 0 \tag{2.22}$$

defines for $\rho_1 < r < \rho_0$ a smooth curve $\mu = \mu(r)$, with the properties

$$\mu(\rho_0) = 0 \tag{2.23}$$

$$\lim_{r \to \rho_1 +} \mu(r) = \infty \tag{2.24}$$

$$\mu'(r) < 0 \;\; \text{for} \;\; r \in (\rho_1, \rho_0). \tag{2.25}$$

If instead of (2.20) we have

$$u(0) \leq \beta, \tag{2.26}$$

then the curve $\mu = \mu(r)$ is defined for $0 < r < \rho_0$, and (2.24) is replaced by

$$\lim_{r \to 0^+} \mu(r) = \begin{cases} \infty, & \text{if } \; u(0) = \beta \\ \frac{2f(u(0))}{f'(u(0))u(0) - f(u(0))}, & \text{if } \; u(0) < \beta. \end{cases} \tag{2.27}$$

Proof. Solving (2.22),

$$\mu = 2\frac{f(u)}{f'(u)u - f(u)}.$$

By (2.21)

$$\mu_r = 2\frac{f'^2 u - f'f - ff''u}{[f'u - f]^2} u_r < 0,$$

and the proof follows.

The sign of our test function $v = ru_r + \mu u$ is governed by the function $h(r) = \frac{ru_r}{u}$.

Lemma 2.3 *For all* $r \in (0,1)$*, we have*

$$h'(r) < 0. \tag{2.28}$$

Proof. Assume first that $0 < r < \rho_0$. Then

$$h'(r) = \frac{(ru_{rr} + u_r)u - ru_r^2}{u^2} = \frac{-\lambda r f(u)u - ru_r^2}{u^2} < 0, \tag{2.29}$$

Since $f(u) > 0$ when $b = u(\rho_0) < u < u(0)$.

Turning to the general case, notice that $h(r) \to -\infty$ as $r \to 1$. Hence if (2.28) was violated somewhere on $(\rho_0, 1)$, we would have on this interval at least one point of local maximum, call it $\overline{r}$ (and also at least one point of local minimum). At $\overline{r}$, $h'(\overline{r}) = 0$, which in view of (2.29) implies

$$u_r^2(\overline{r}) + \lambda f(u(\overline{r}))u(\overline{r}) = 0, \tag{2.30}$$

and $h''(\overline{r}) \le 0$. On the other hand, using (2.29), (2.30) and (2.13), we compute

$$\begin{aligned} h''(\overline{r}) &= -\frac{(ru_r^2 + \lambda f(u)u)'(\overline{r})}{u^2(\overline{r})} \\ &= -\frac{ru_r(2u_{rr} + \lambda f'(u)u)}{u^2} = -\frac{ru_r\left(-\frac{2}{r}u_r + \lambda f'(u)u - \lambda f(u)\right)}{u^2} > 0, \end{aligned}$$

a contradiction.

Notice that $-h(0) = 0$ and $-h(\rho_0) > 0$. It follows by Lemmas 2.2 and 2.3 that the curves $\mu = \mu(r)$ and $\mu = -h(r)$ intersect exactly once on (ρ_1, ρ_0). Let $r_0 \in (\rho_1, \rho_0)$ denote the point of intersection, and $\mu_0 = \mu(r_0)$. We now fix our test function $v = ru_r + \mu_0 u$.

Lemma 2.4 *With μ_0 as fixed above, we have*

$$v(r) > 0 \;\; and \;\; g_{\mu_0}(u(r)) < 0 \;\; for \;\; 0 < r < r_0, \qquad (2.31)$$
$$v(r) < 0 \;\; and \;\; g_{\mu_0}(u(r)) > 0 \;\; for \;\; r_0 < r < 1.$$

Proof.

Case (i) $u(0) \geq \beta$. Using (2.3) we see that $g_{\mu_0}(0) = 0$, $g'_{\mu_0}(0) > 0$, which implies that the function $g_\mu(u(r))$ is positive near $r = 1$, and we also have $g_{\mu_0}(u(0)) < 0$. Since by Lemma 2.1 $g_{\mu_0}(u(r))$ can have at most one sign change, it follows that it has exactly one sign change at $r = r_0$. When $r < r_0$ we have

$$-h(r) < \mu_0,$$

i.e. $v = ru_r + \mu_0 u > 0$. Similarly $-h(r) > \mu_0$ for $r_0 < r < 1$, implying that $v < 0$ there.

Case (ii) $u(0) < \beta$. We still have $g_{\mu_0}(0) = 0$ and $g'_{\mu_0}(0) > 0$. Also $g_{\mu_0}(u(r_0)) = 0$. We claim that at $u(r_0)$, $g_{\mu_0}(u)$ must change sign (from positive to negative for increasing u). One observes that

$$g'_{\mu_0}(u(r_0)) = -\frac{2N(u(r_0))}{p(u(r_0))} < 0, \qquad (2.32)$$

using the fact that $g_{\mu_0}(u(r_0)) = 0$ and (2.21). It follows that $g_{\mu_0}(u(r))$ has exactly one sign change at $r = r_0$, and the rest follows as in Case (i).

Theorem 2.1 *Assume $f(u)$ satisfies the assumption (2.7) and the conditions (2.1-2.5) and (2.21). Let $w(r)$ be a nontrivial solution of (2.11). Then we can choose it such that*

$$w(r) > 0 \;\; for \; all \;\; r \in [0, 1). \qquad (2.33)$$

(By a theorem of C.S. Lin and W.-M. Ni [13] this result classifies not only radial, but all solutions of the linearized problem (1.2).)

Proof. By a result of Holzmann and Kielhöfer [7, p. 227], $w(0) \neq 0$, so that we may assume $w(0) > 0$. With μ_0 as fixed above, we multiply the equation (2.12) by rw and subtract from this the equation (2.11) multiplied by rv, obtaining

$$\frac{d}{dr}[r(wv' - vw')] = \lambda g_{\mu_0}(u(r))w(r)r. \tag{2.34}$$

Assuming the contrary to (2.33) let $r_1 \in (0,1)$ be the first (smallest) zero point of $w(r)$. Let r_0 be as defined in Lemma 2.4.

Case (i) $r_1 < r_0$. Integrate (2.34) over $(0, r_1)$,

$$-r_1 v(r_1)w'(r_1) = \lambda \int_0^{r_1} g_{\mu_0}(u(r))w(r)r\,dr. \tag{2.35}$$

Using (2.31) we see that the left hand side of (2.35) is non-negative, while the right hand side is negative, a contradiction.

Case (ii) $r_1 \geq r_0$. The function $w(r)$ has to change sign at $r = r_1$ (since otherwise we would have $w'(r_1) = 0$ and hence $w \equiv 0$, a contradiction). Let $r_2 \leq 1$ be the next root of $w(r)$, with $w(r) < 0$ on (r_1, r_2). Integrate (2.34) over (r_1, r_2)

$$-r_2 v(r_2)w'(r_2) + r_1 v(r_1)w'(r_1) = \lambda \int_{r_1}^{r_2} g_{\mu_0}(u(r))w(r)r\,dr. \tag{2.36}$$

Using (2.31), we see that the quantity on the left in (2.36) is non-negative, while the one on the right is negative, a contradiction, concluding the proof of the theorem.

We recall that (λ, u) is called a critical point of (2.10) if (2.11) has non-trivial solutions.

Lemma 2.5 *Let $(\lambda, u(r))$ be a critical point of (2.10). Then*

$$u(0) > \beta.$$

Proof. We will show that if $u(0) \le \beta$, then the only solution of (2.11) is $w \equiv 0$. We recall that

$$f'(u) > \frac{f(u)}{u} \ \text{ for } \ 0 < u < \beta. \tag{2.37}$$

If $u(r)$ is a nontrivial solution of (2.10), it follows that the principal eigenvalue of

$$\Delta z + \lambda \frac{f(u)}{u} z = \mu z \ \text{ for } \ r \in (0,1), \ \ z|_{r=1} = 0$$

is $\mu = 0$ (and $z = u$ is the corresponding principal eigenfunction). By (2.37), the principal eigenvalue of

$$\Delta z + \lambda f'(u) z = \mu z \ \text{ for } \ r \in (0,1), \ \ z|_{r=1} = 0 \tag{2.38}$$

must be positive. On the other hand, using Theorem 2.1, we see that $w(r) > 0$ is an eigenfunction of (2.38), corresponding to an eigenvalue $\mu = 0$, a contradiction.

3 The global solution curve

In this section we shall prove an exact multiplicity result for our problem

$$u''(r) + \frac{1}{r}u'(r) + \lambda f(u) = 0 \ \ r \in (0,1), \ \ u'(0) = u(1) = 0. \tag{3.1}$$

We recall the linearized equation

$$w'' + \frac{1}{r}w' + \lambda f'(u) w = 0 \ \ r \in (0,1), \ \ w'(0) = w(1) = 0. \tag{3.2}$$

We separate this section into several subsections.

3.1 Some preliminaries

The following lemma was proved in [9]. We repeat its proof for completeness.

Lemma 3.1 *Let $f(u) \in C^2(R_+)$ satisfy $f(0) \geq 0$. Assume (3.2) admits a nontrivial solution. Then*

$$\int_0^1 f(u) w r dr > 0. \tag{3.3}$$

Proof. Notice that condition $f(0) \geq 0$ and the Hopf's boundary Lemma imply that (see e.g. [4, p. 484])

$$u'(1) < 0. \tag{3.4}$$

We may also assume $w'(1) < 0$. Differentiate (3.1)

$$u''' + \left(\frac{1}{r} u'\right)' + \lambda f'(u) u' = 0. \tag{3.5}$$

Multiply (3.2) by $r^2 u'$, subtract (3.5) multiplied by $r^2 w$, then integrate over $(0,1)$. Obtain

$$\int_0^1 \left[r^2 u' w'' - r^2 u''' w + r u' w' - r^2 \left(\frac{1}{r} u'\right)' w \right] dx = 0. \tag{3.6}$$

Integrate by parts in the first, second and fourth terms:

$$r^2 u' w' |_0^1 + \int_0^1 [-2ru'w' - r^2 u'' w' + 2ru'' w + r^2 u'' w'$$
$$+ru'w' + 2u'w + ru'w'] dr = 0.$$

After cancellations:

$$u'(1) w'(1) + 2 \int_0^1 r \left(u'' + \frac{1}{r} u' \right) w \; dr = 0.$$

Finally, using the equation (3.1)

$$\int_0^1 f(u) w r \; dr = \frac{1}{2\lambda} u'(1) w'(1) > 0.$$

Lemma 3.2 *In the conditions of the previous lemma, with $f(0) = 0$, we have*

$$\int_0^1 f''(u) u_r^2 w r \; dr = 0. \tag{3.7}$$

Proof. Rewrite (3.5)

$$u_r'' + \frac{1}{r}u_r' + \lambda f'(u)u_r - \frac{1}{r^2}u_r = 0. \tag{3.8}$$

Differentiate (3.2)

$$w_r'' + \frac{1}{r}w_r' + \lambda f'(u)w_r - \frac{1}{r^2}w_r + \lambda f''(u)u_r w = 0. \tag{3.9}$$

Multiply (3.8) by rw_r, (3.9) by ru_r and subtract. Denoting $p(r) = u''w' - u'w''$, we express

$$\frac{d}{dr}[rp] = \lambda r f''(u)u_r^2 w. \tag{3.10}$$

Integrate (3.10) over $(0,1)$

$$\lambda \int_0^1 f''(u)u_r^2 wr\, dr = p(1) = u''(1)w'(1) - u'(1)w''(1). \tag{3.11}$$

¿From the corresponding equations we express

$$u''(1) = -u'(1),$$

$$w''(1) = -w'(1).$$

Using these expressions in (3.11), we conclude that $p(1) = 0$, and the lemma follows.

We now state our main result whose proof will occupy the rest of the paper.

Theorem 3.1 *Assume that $f(u)$ satisfies assumption (2.7) and the the conditions (2.1-2.5) and (2.21). Then there is a critical $\lambda_0 > 0$, such that for $\lambda < \lambda_0$ the problem (3.1) has no nontrivial solutions, it has exactly one nontrivial solution for $\lambda = \lambda_0$, and exactly two nontrivial solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^+(r,\lambda)$ strictly monotone increasing in λ and $\lim_{\lambda\to\infty} u^+(r,\lambda) = c$ for $r \in [0,1)$. For the* *lower branch,* $\lim_{\lambda \to \infty} u^-(r,\lambda) = 0$ *for* $r \neq 0$*, while* $u^-(0,\lambda) > \gamma$ *for all* $\lambda > \lambda_0$ *, where* γ *is the unique number* $\in (b,c)$ *such that* $\int_0^\gamma f(u)du = 0$*. (Recall that any nontrivial solution is positive by the maximum principle.)*

Proof of the existence part. We begin by noticing that for sufficiently small λ the problem (3.1) has no positive solutions. Indeed, under our assumptions there is a constant $\gamma > 0$, such that $f(u) \leq \gamma u$ for all $u > 0$. Multiplying the PDE version of (3.1) by u, and integrating over the unit ball

$$\lambda\gamma \int_{r<1} u^2 dx \geq \lambda \int_{r<1} f(u)u\, dx = \int_{r<1} |\nabla u|^2 dx \geq \lambda_1 \int_{r<1} u^2\, dx,$$

where λ_1 is the principal eigenvalue of the Laplacian, and the claim follows. Existence of positive solutions for large λ is a known fact, see e.g. [5].

3.2 Bifurcations of solution curves

Notice that when a nontrivial solution of (3.1) exists, there is a maximal solution, call it $u(r,\lambda)$ (which can be obtained by monotone iterations, starting with a supersolution $u = c$). We now continue $u(r,\lambda)$ for decreasing λ. If the corresponding linearized equation (3.2) at some (λ_1, u_1) has only the trivial solution $w = 0$, then by the implicit function theorem we can solve (3.1) for $\lambda < \lambda_1$ and λ close to λ_1, obtaining a smooth in λ curve of solutions. This process of decreasing λ cannot be continued indefinitely, since for sufficiently small $\lambda > 0$ the problem (3.1) has no solutions. Let λ_0 be the infimum of λ's for which we can continue the curve of solutions to the left. It is easy to show (see [11], [12] for a similar argument) that there is a solution on this curve at λ_0, call it $u(r,\lambda_0) \equiv u_0(r)$. Clearly the linearized equation (3.2) at $\lambda = \lambda_0$ and $u = u_0$ must have a nontrivial solution, and by Theorem 2.1, $w(r) > 0$ for all r in $[0,1)$. We shall finish the rest of the proof of Theorem 3.1 in a few lemmas.

Lemma 3.3 *Let* (λ_0, u_0) *be a critical point of (3.1), i.e. (3.2) has a nontrival solution. Then* (λ_0, u_0) *is a bifurcation point, near which the solutions of (3.1)* *form a curve* $(\lambda_0+\tau(s), u_0+sw+z(s))$ *with* s *near* $s=0$*, and* $\tau(0)=\tau'(0)=0$*,* $z(0)=z'(0)=0$.

Furthermore

$$\tau''(0)>0. \tag{3.12}$$

Proof. We shall first show that at the critical point (λ_0, u_0) the Crandall-Rabinowitz theorem applies. Define $B=B(0,1)$ to be a unit ball in $R^2, X = \{u \in C^{2,\alpha}(\overline{B}) | u=0 \text{ on } \partial B\}$ and $Y=C^\alpha(\overline{B})$. Let $F: R_+ \times X \to Y$ be given by $F(\lambda, u) = \Delta u + \lambda f(u)$. We also rewrite (3.2) as

$$\Delta w + \lambda f'(u) w = 0 \ \text{ in } \ B, w=0 \ \text{ on } \ \partial B. \tag{3.13}$$

That the null-space of $F_u(\lambda_0, u_0)$ is one-dimensional is seen from (3.2) (it can be parametrized by $w'(1)$). Since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero, it follows that codim $R(F_u(\lambda_0, u_0))=1$. Finally, if the condition $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$ was violated, one could find $z \in X$ satisfying

$$\Delta z + \lambda_0 f'(u_0) z = f(u) \ \text{ in } \ B, z=0 \ \text{ on } \ \partial B. \tag{3.14}$$

¿From the equation (3.13), written at (λ_0, u_0), and from (3.14) we have

$$0 = \int_B f(u_0) w \, dx = 2\pi \int_0^1 f(u_0) w r \, dr,$$

contradicting Lemma 3.1.

Applying the Crandall-Rabinowitz theorem, we conclude that (λ_0, u_0) is a bifurcation point, near which the solutions of (3.1) form a curve $(\lambda_0 + \tau(s), u_0+sw+z(s))$ with s near $s=0$, and $\tau(0)=\tau'(0)=0$, $z(0)=z'(0)=0$. We claim that (3.12) holds, which implies that only "turns to the right" in (λ, u) "plane" are possible. We use the formula

$$\tau''(0) = -\lambda_0 \frac{\int_0^1 f''(u_0) w^3 r \, dr}{\int_0^1 f(u_0) w r \, dr}. \tag{3.15}$$

To derive (3.15), we differentiate the PDE version of (3.1) twice in s,

$$\Delta u_{ss} + \lambda f'(u)u_{ss} + 2\lambda' f'(u)u_s + \lambda f''(u)u_s^2 + \lambda'' f(u) = 0. \tag{3.16}$$

Setting here $s = 0$, and using that $\tau'(0) = 0$ and $u_s|_{s=0} = w(x)$, we obtain

$$\Delta u_{ss} + \lambda_0 f'(u)u_{ss} + \lambda_0 f''(u)w^2 + \tau''(0)f(u) = 0. \tag{3.17}$$

Multiplying (3.17) by w, and the equation (3.13) by u_{ss}, subtracting and integrating, we obtain (3.15).

By Lemma 3.1 the denominator in (3.15) is positive, so we only need to show that

$$\int_0^1 f''(u_0)w^3 r\,dr < 0. \tag{3.18}$$

We shall establish (3.18) by comparing this integral with the one in Lemma 3.2. We claim that $f''(u_0(r))$ changes sign exactly once on $(0,1)$. Indeed, $f''(u)$ is positive for small u, and hence $f''(u_0(r)) > 0$ for r close to 1. By Lemma 2.5, $u_0(0) > \beta$, and hence $f''(u_0(0)) < 0$. The claim follows by our conditions on $f(u)$. We claim next that the functions $-u_r$ and w intersect exactly once on $(0,1)$. Differentiating (3.1),

$$u_r'' + \frac{1}{r}u_r' + \lambda f'(u)u_r = \frac{1}{r^2}u_r < 0 \;\text{ for }\; r \in (0,1),$$

which implies that u_r is a supersolution of the same equation (3.2) that w satisfies. If $-u_r$ and w intersected more than once, then, considering their values at $r = 0, 1$, we would find an interval $(r_1, r_2) \subset (0,1)$, so that $w(r) < -u_r$ on (r_1, r_2). Then we can find a constant $1 < \mu$, such that $-u_r \le \mu w$ for all $r \in (r_1, r_2)$ and $-u_r(r_0) = \mu w(r_0)$ for some $r_0 \in (r_1, r_2)$. Since $-u_r$ and μw are respectively a subsolution and solution of the same equation (3.2), we obtain a contradiction, proving that $-u_r$ and w intersect exactly once on $(0,1)$.

We will show that

$$\int_0^1 f''(u_0)w^3 r\,dr < \int_0^1 f''(u_0)u_{0_r}^2 w r\,dr. \tag{3.19}$$

Let $\overline{r}$ be the point where $f''(u_0(r))$ changes sign on $(0,1)$. By considering $\mu w(r)$ with a proper constant μ, we may assume that $-u_r$ and $w(r)$ intersect at the same point $\overline{r}$. Returning to (3.19), we see that on the interval $(0,\overline{r})$, where $f''(u_0) < 0$, we have $w^2 > u_{0_r}^2$, while on the interval $(\overline{r},1)$, where $f''(u_0) > 0$, we have $w^2 < u_{0_r}^2$. So that the integrand on the right in (3.19) is larger than the one on the left for all $r \neq \overline{r}$, which proves (3.19). In view of Lemma 3.2, both inequality (3.18) and our claim (3.12) follow.

It follows that at the critical point (λ_0, u_0) the curve of solutions turns to the "right" in (λ, u) plane. We denote by $u^+(x,\lambda)$ and $u^-(x,\lambda)$ its upper and lower branches respectively. It is clear from Lemma 3.3 that

$$u_\lambda^+ = \frac{1+o(1)}{\sqrt{2\tau''(0)(\lambda-\lambda_0)}} w \ \text{ for } \ \lambda > \lambda_0 \ \text{ and close to } \ \lambda_0, \tag{3.20}$$

and

$$u_\lambda^- = -\frac{1+o(1)}{\sqrt{2\tau''(0)(\lambda-\lambda_0)}} w \ \text{ for } \ \lambda > \lambda_0 \ \text{ and close to } \ \lambda_0. \tag{3.21}$$

After the turn we can continue this curve of solutions for increasing λ, using the implicit function theorem, so long as (λ, u) is a nonsingular point of $F(\lambda, u)$. However, there can be no critical points on either upper or lower branches, since we know precisely the structure of solutions at any critical point, namely a turn to the right must always occur, which is impossible. It remains to show that there is only one such curve, and to establish the behavior of its branches as $\lambda \to \infty$.

3.3 Uniqueness of solution curve

Lemma 3.4 *The upper branch u^+ is increasing in λ for all $\lambda > \lambda_0$.*

Proof. Let u be a solution on an upper branch. For λ close to λ_0 the Lemma follows from (3.20) and Theorem 2.1, since u_λ is then positive. Our goal is to show that u_λ remains positive for all $\lambda > \lambda_0$. For that we shall show that u_λ cannot develop a zero in the interior of the interval $(0,1)$, or zero slope at $r = 1$. But first we establish a preliminary inequality. We show next that for any solution of (3.1)

$$\int_0^1 f(u)u'r\,dr < 0. \tag{3.22}$$

Indeed, write (3.1) in the form

$$(ru')' + \lambda r f(u) = 0. \tag{3.23}$$

Multiply (3.23) by ru' and integrate over $(0,1)$

$$\int_0^1 f(u)u'r^2 dr = -\frac{1}{2\lambda}u'^2(1) < 0. \tag{3.24}$$

The function $f(u(r))u'(r)$ is negative near $r = 0$ and positive near $r = 1$, and it changes sign once, say at $r = r_0$. It follows that

$$\begin{aligned}\int_0^1 f(u)u'r^2 dr &> r_0\int_0^{r_0} f(u)u'r\,dr + r_0\int_{r_0}^1 f(u)u'r\,dr\\ &= r_0\int_0^1 f(u)u'r\,dr.\end{aligned}$$

In view of (3.24), the inequality (3.22) follows. If r_1 is any point in $(0,1)$, then we also have

$$\int_0^{r_1} f(u)u'r\,dr < \int_0^1 f(u)u'r\,dr < 0. \tag{3.25}$$

Let now λ_1 be the supremum of $\lambda > \lambda_0$ where the inequality $u_\lambda^+(r,\lambda) > 0$ (for all $r \in (0,1)$) holds. Several cases are possible.

Case (i) $u_\lambda^+(r,\lambda_1) \geq 0$ for all $r \in (0,1)$, and $u_\lambda^+(r_1,\lambda_1) = 0$ for some $r_1 \in (0,1)$. For the rest of the argument we shall write u for u^+ and u_λ for u_λ^+. Notice that u_λ satisfies

$$u_\lambda'' + \frac{1}{r}u_\lambda' + \lambda f'(u)u_\lambda + f(u) = 0 \;\; r \in (0,1), \;\; u_\lambda'(0) = u_\lambda(1) = 0. \tag{3.26}$$

Since r_1 is a point of minimum for $u_\lambda(r, \lambda_1)$, it follows that $u'_\lambda(r_1, \lambda_1) = 0$ and $u''_\lambda(r_1, \lambda_1) \geq 0$. From (3.26) we then see that $f(u(r_1, \lambda_1)) \leq 0$, and hence

$$0 < u(r_1, \lambda_1) \leq b. \tag{3.27}$$

¿From the equations (3.8) and (3.26) we obtain as before

$$(u'_\lambda u' - u_\lambda u'')' + \frac{1}{r}(u'_\lambda u' - u_\lambda u'') + \frac{1}{r^2} u_\lambda u' + f(u)u' = 0. \tag{3.28}$$

Letting $p = u'_\lambda u' - u_\lambda u''$, we express from (3.28)

$$(rp)' = -\frac{1}{r} u_\lambda u' - f(u)u'r. \tag{3.29}$$

Integrate (3.29) over $(0, r_1)$, and use (3.25)

$$r_1 p(r_1) \geq -\int_0^{r_1} f(u)u'r \, dr > 0. \tag{3.30}$$

But $p(r_1) = 0$, a contradiction.

Case (ii) $u'_\lambda(1, \lambda_1) = 0$. Integrating (3.29) over $(0, 1)$ we obtain the same contradiction.

Case (iii) $u_\lambda(0, \lambda_1) = 0$. Again from (3.26) we see that $f(u(0, \lambda_1)) \leq 0$, a contradiction. Hence the upper branch is increasing for all $\lambda > \lambda_0$.

Since the upper branch is increasing and bounded above by c, it tends to a limit at any $r \in (0, 1)$ as $\lambda \to \infty$. This limit cannot be different from either b or c over any subinterval of $(0, 1)$, since otherwise from the equation (3.1), $(ru')'$ would have to be large over that subinterval, which is impossible. (The function ru' would have a large change over that subinterval, which would imply a large change in u, contradicting to the total variation of u being less than c). Since $u''(r) > 0$ below $u = b$, the upper branch cannot tend to b over any subinterval of $(0, 1)$. It follows that the upper branch $u^+(x, \lambda)$ tends to c over $[0, 1)$ as $\lambda \to \infty$.

Before taking a look at the asymptotic behavior of lower branch we will prove the following result about the eigenvalue estimates of the linearized equation of (3.1) at regular points.

$$w'' + \frac{1}{r}w' + \lambda f'(u)w = \mu w \;\; r \in (0,1), \;\; w'(0) = w(1) = 0. \tag{3.31}$$

Lemma 3.5 *Everywhere on lower branch the principal eigenvalue of (3.31) is positive and everywhere on upper branch the principal eigenvalue of (3.31) is negative.*

Proof. Let u be a solution on a lower branch, and w be a solution of the corresponding equation (3.31). ¿From the PDE versions of the equations (3.26) and (3.31) we obtain, multiplying the equations by w and u_λ respectively, subtracting and integrating over the region $r < 1$,

$$\mu = -\frac{\int_0^1 f(u)wr\,dr}{\int_0^1 u_\lambda wr\,dr}. \tag{3.32}$$

For λ near λ_0 (the turning point) we know that $u_\lambda < 0$ by (3.21), and hence the denominator is negative. We show next that the numerator in (3.32) is positive. Differentiate (3.1)

$$u''' + \frac{1}{r}u'' + \lambda f'(u)u' - \frac{1}{r^2}u' = 0. \tag{3.33}$$

¿From (3.33) and (3.31)

$$(u''w - u'w')' + \frac{1}{r}(u''w - u'w') - \frac{1}{r^2}u'w = -\mu u'w.$$

Setting $p(r) = u''w - u'w'$, we express

$$(rp)' = \left(\frac{1}{r} - \mu r\right) u'w. \tag{3.34}$$

The function $q(r) = rp(r)$ satisfies $q(0) = 0$, $q(1) = -u'(1)w'(1) < 0$, and from (3.34) $q' < 0$ near $r = 0$. Also from (3.34) we see that $q(r)$ can have at most one critical point on $(0,1)$, no matter what μ is. It follows that $q(r)<0$ on $(0,1)$. Hence

$$
\begin{aligned}
0 > \int_0^1 rp(r)dr &= \int_0^1 r(u''w - u'w')dr = \\
&= \int_0^1 2ru''wdr + \int_0^1 u'wdr > 2\int_0^1 r(u'' + \frac{1}{r}u')wdr.
\end{aligned}
$$

Using (3.1) we conclude $\int_0^1 f(u)wr\,dr > 0$, i.e. the numerator in (3.32) is positive.

So $\mu > 0$ near $\lambda = \lambda_0$ on the lower branch. However, the same is true for the entire lower branch. Indeed, assuming otherwise, since μ changes continuously, we would have a point on the lower branch where $\mu = 0$. This is a singular point, where a turn to "the right" must occur, impossible. So $\mu > 0$ on the entire lower branch. The same argument shows that $\mu < 0$ on the entire upper branch.

Next, we study the asymptotic behavior of lower branch.

Lemma 3.6 *For any lower branch,* $\lim_{\lambda\to\infty} u^-(r,\lambda) = 0$ *for* $r \neq 0$*, while* $u^-(0,\lambda) > \gamma$ *for all* $\lambda > \lambda_0$*.*

Proof. Let u be a solution on a lower branch. Multiplying (3.1) by u' and integrating over (0,1), we obtain that

$$
0 < \frac{1}{2}u'^2(1) + \int_0^1 \frac{1}{r}u'^2 dr = \lambda \int_0^{u(0)} f(u)du, \tag{3.35}
$$

which shows that $u(0) > \gamma$ for any solution of (3.1). On the other hand no subsequence of the lower branch solutions can approach to any limit bigger than $\frac{b}{2}$ for any $r \in (0,1)$, since by Lemma 9 in [5] a solution with such property (so-called large solutions in [5, p. 55]) has to be linearly stable, namely the principal eigenvalue of (3.31) is negative, contradicting the result of Lemma 3.5. Therefore we have that $\limsup_{\lambda\to\infty} u^-(r,\lambda) \leq \frac{b}{2}$. However since the only zero of f in $[0,\frac{b}{2}]$ is 0, we conclude as in the proof of Lemma

3.4 that the set of points $r \in (0,1)$ where $u^-(r,\lambda)$ does not converge to zero cannot contain any intervals. Combined with the fact that $u'(r) < 0$ for all $r \in (0,1)$, we conclude that this set is empty, and hence we must have that

$$\lim_{\lambda \to \infty} u^-(r,\lambda) = 0, \tag{3.36}$$

which proves this lemma.

Lemma 3.7 *Define $\zeta \in (b,c)$ to be the larger root of $f'(u)$. Then any two solutions $u(r)$ and $v(r)$ of (3.1) cannot intersect in the region where they are both greater than ζ. In particular, if λ is sufficiently large, any solution on the lower branch cannot intersect any solution on the upper branch (because the latter tends to c in $[0,1)$ as $\lambda \to \infty$).*

Proof. Assuming the contrary, we can find $\bar{r} \in (0,1)$ such that $v(r) > u(r) \geq \zeta$ for $r \in (0,\bar{r})$ but $v(\bar{r}) = u(\bar{r})$. Since $f'(u) < 0$ for u in (ζ, c), it follows that $f(u(r)) > f(v(r))$ for $r \in (0,\bar{r})$. Integrating the difference of the corresponding equations

$$(r(u' - v'))' + r\lambda(f(u) - f(v)) = 0$$

over $(0,\bar{r})$, we conclude that $u'(\bar{r}) < v'(\bar{r}) < 0$. This is a contradiction, since clearly $v(r)$ must be "steeper" than $u(r)$ at $\bar{r}$.

Finally, we rule out the possibility of another curve of solutions. By the above analysis any solution curve would have an upper branch tending to c as $\lambda \to \infty$. By Lemma 3.5 the principal eigenvalue of (3.31) for upper branch is negative. Let now $v(x,\lambda)$ be another curve of solutions, with the branches $v^-(r,\lambda) < v^+(r,\lambda)$, and with a turning point $(\overline{\lambda}_0, \overline{u}_0)$, and say $\overline{\lambda}_0 \geq \lambda_0$. We claim that for $\lambda > \overline{\lambda}_0$

$$v^-(r,\lambda) > u^-(r,\lambda) \text{ and } v^+(r,\lambda) < u^+(r,\lambda) \text{ for all } r \in [0,1). \tag{3.37}$$

Assume that the first inequality in (3.37) is violated at some $\lambda_1 > \overline{\lambda}_0$. Follow both curves $v^-(x,\lambda)$ and $u^-(x,\lambda)$ as λ decreases from λ_1 to $\overline{\lambda}_0$. Then at each

$\lambda \in [\overline{\lambda}_0, \lambda_1)$ the inequality (3.37) has to be violated, since otherwise we could find $\hat{\lambda} \in (\overline{\lambda}_0, \lambda_1)$, such that $v^-(r, \hat{\lambda}) \geq u^-(r, \hat{\lambda})$ and $v^-(r, \hat{\lambda}) \not\equiv u^-(r, \hat{\lambda})$, but that is impossible by the maximum principle. At $\overline{\lambda}_0$ we switch to $v^+(r, \lambda)$, i.e. we compare $v^+(r, \lambda)$ with $u^-(x, \lambda)$ as λ increases from $\overline{\lambda}_0$ to infinity. By the same analysis, the inequality $v^+(r, \lambda) > u^-(r, \lambda)$ has to be violated for all λ, but that is impossible from Lemma 3.7. The second inequality in (3.37) is proved similarly.

Next we notice that on any finite in λ interval, one can have only finitely many solution curves. Assuming otherwise, let (λ_n, u_n) be the turning points of the infinite family of curves on a finite interval. Along a subsequence $\lambda_{n_k} \to \overline{\lambda}$ and $u_{n_k} \to \overline{u}$ (in C^2). The point $(\overline{\lambda}, \overline{u})$ is singular, but the solution set near it is not a simple curve, contradicting the Crandall-Rabinowitz theorem.

We are now ready to exclude the possibility of more than one solution curve. We use the argument essentially of [5, p. 68]. Assume that at some $\hat{\lambda}$ there are several solution curves. Let $v^-(r, \lambda)$ be the largest of the lower branches at $\hat{\lambda}$. Let $\hat{\mu} > 0$ and $\hat{w}(r) > 0$ be the corresponding eigenpair, obtained by solving (3.31). It is easy to check that for small enough ε, $v \equiv v^-(r, \hat{\lambda}) + \varepsilon \hat{w}(r)$ is a strict subsolution of (3.1), which lies above all solutions on the lower branches. Define an open set

$$O = \left\{ u \in C^1[0,1], \ \ u'(0) = u(1) = 0, \ \ v(r) < u(r) < c \right\}$$

Define $H = \max_{[0,c]} |f'(s)|$, and the map $T(u) = (\Delta - H)^{-1}(-\lambda f(u) - Hu)$. As in [5] we see that $T(O) \subset O$, and hence $\deg(T, O, 0) = 1$. But the index of each of the solutions on the upper branches is also 1, hence there can be only one upper branch, and hence only one solution curve.

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