

## Exact multiplicity results for boundary value problems with nonlinearities generalising cubic

**Philip Korman\***

Department of Mathematical Sciences, University of Cincinnati,  
Cincinnati, Ohio 45221-0025, U.S.A.

**Yi Li†**

Department of Mathematics, University of Rochester, Rochester,  
NY 14627, U.S.A.

**Tiancheng Ouyang‡**

Department of Mathematics, Brigham Young University, Provo,  
Utah 84602, U.S.A.

(MS received 15 June 1994. Revised MS received 18 January 1995)

Using techniques of bifurcation theory we present two exact multiplicity results for boundary value problems of the type

$$u'' + \lambda f(x, u) = 0 \quad \text{for } -L < x < L, \quad u(-L) = u(L) = 0.$$

The first result concerns the case when the nonlinearity is independent of  $x$  and behaves like a cubic in  $u$ . The second one deals with a class of nonlinearities with explicit  $x$  dependence.

### 1. Introduction

We present exact multiplicity results for boundary value problems of the type

$$u'' + \lambda f(x, u) = 0 \quad \text{for } -L < x < L, \quad u(-L) = u(L) = 0, \quad (1.1)$$

with the nonlinearity  $f$  behaving like a cubic polynomial in  $u$ . Here  $\lambda$  is a positive parameter, and we may assume without loss of generality that  $L = 1$ . Our first result concerns the case when  $f = f(u)$ , i.e.  $f$  does not depend explicitly on  $x$ , and  $f(u)$  has three simple and distinct positive roots  $0 < a < b < c$ , with  $f(u) > 0$  for  $u \in (-\infty, a) \cup (b, c)$ , and  $f(u) < 0$  for  $u \in (a, b) \cup (c, \infty)$ . Our prototype is the problem

$$u'' + \lambda(u - a)(u - b)(c - u) = 0 \quad \text{for } -1 < x < 1, \\ u(-1) = u(1) = 0. \quad (1.2)$$

Assuming that the area under  $f(u)$  from  $b$  to  $c$  (the positive hump) is greater than the area under  $f(u)$  from  $a$  to  $b$  (the negative hump), and a technical assumption,

\* Supported in part by the Taft Faculty Grant at the University of Cincinnati.

† Supported in part by the National Science Foundation.

‡ Supported in part by a College of Science Research Grant, Brigham Young University.

which restricts  $a$  from being large, we show existence of a critical value of the parameter  $\lambda = \lambda_0$ , so that for  $0 < \lambda < \lambda_0$  the problem (1.1) has exactly one solution, for  $\lambda = \lambda_0$  it has exactly two solutions, and exactly three solutions for  $\lambda > \lambda_0$  (all solutions are positive by the maximum principle, and throughout the paper we consider only the classical solutions). For the special case of (1.2) a similar result was proved in the papers of J. Smoller and A. Wasserman [10] and S.-H. Wang [11]. These authors used rather involved phase-plane analysis. We can treat more general nonlinearities by using more flexible techniques of bifurcation theory. Our approach is applicable in many other situations, and in fact it was used by two of the present authors in [4–6] to derive multiplicity results (some of which were exact multiplicity results) for a number of problems of the type (1.1) in case  $f$  is even in  $x$ .

Our second result is on exact multiplicity of solutions for a cubic in  $u$  nonlinearity with explicit dependence on  $x$ . Namely, we consider a model problem

$$u'' + \lambda u^2(b(x) - u) = 0 \text{ on } (-1, 1), \quad u(-1) = u(1) = 0. \quad (1.3)$$

Under certain conditions on  $b(x)$ , we prove existence of a critical  $\lambda_0 > 0$ , so that the problem (1.3) has no nontrivial solutions for  $0 < \lambda < \lambda_0$ , exactly one (positive) solution for  $\lambda = \lambda_0$  and exactly two (positive) solutions for  $\lambda > \lambda_0$ . This appears to be the first such result. The phase-plane analysis is, of course, not applicable here. In [5] it was proved that all solutions of (1.3) lie on a single smooth solution curve, and that for  $\lambda$  large there are exactly two solutions.

A word on notation. We shall denote derivatives of  $u(x)$  by either  $u'(x)$  or  $u_x$  and mix both notations to make our proofs more transparent ( $u_x''$  will denote the second derivative of  $u(x)$ , when convenient).

Next we list some background results. Recall that a function  $\gamma(x) \in C^2(-L, L) \cap C^0[-L, L]$  is called a supersolution of (1.1) if

$$\gamma'' + \lambda f(x, \gamma) \leq 0 \text{ on } (-L, L), \quad \gamma(-L) \geq 0, \quad \gamma(L) \geq 0. \quad (1.4)$$

A subsolution  $\psi(x)$  is defined by reversing the inequalities in (1.4). The following result is standard:

**LEMMA 1.1.** *Let  $\gamma(x)$  and  $\psi(x)$  be respectively super- and subsolutions of (1.1), and  $\gamma(x) \geq \psi(x)$  on  $(-L, L)$  with  $\gamma(x) \not\equiv \psi(x)$ ; then  $\gamma(x) > \psi(x)$  on  $(-L, L)$ .*

We shall often use this lemma with either  $\gamma(x)$  or  $\psi(x)$  or both being solutions of (1.1). The following lemma is a consequence of the first.

**LEMMA 1.2.** *Let  $u(x)$  be a nontrivial solution of (1.1) with  $f(x, 0) \equiv 0$ . If  $u(x) \geq 0$  on  $(-L, L)$  then  $u > 0$  on  $(-L, L)$ .*

Next we state a bifurcation theorem of Crandall and Rabinowitz [1].

**THEOREM 1.3 [1].** *Let  $X$  and  $Y$  be Banach spaces. Let  $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X$  and let  $F$  be a continuously differentiable mapping of an open neighbourhood of  $(\bar{\lambda}, \bar{x})$  into  $Y$ . Let the null-space  $N(F_x(\bar{\lambda}, \bar{x})) = \text{span}\{x_0\}$  be one-dimensional and  $\text{codim } R(F_x(\bar{\lambda}, \bar{x})) = 1$ . Let  $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$ . If  $Z$  is a complement of  $\text{span}\{x_0\}$  in  $X$ , then the solutions of  $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$  near  $(\bar{\lambda}, \bar{x})$  form a curve  $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$ , where  $s \rightarrow (\tau(s), z(s)) \in \mathbb{R} \times Z$  is a continuously differentiable function near  $s = 0$  and  $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$ .*

## 2. Exact multiplicity results for a class of nonlinearities generalising cubic

We study the exact multiplicity of nontrivial solutions of the problem ( $u = u(x)$ )

$$u'' + \lambda f(u) = 0 \quad \text{for } x \in (-1, 1), \quad u(-1) = u(1) = 0. \quad (2.1)$$

A corresponding linearised problem will often be used ( $w = w(x)$ ):

$$w'' + \lambda f'(u)w = 0 \quad \text{for } x \in (-1, 1), \quad w(-1) = w(1) = 0. \quad (2.2)$$

It is known that the positive solutions of (2.1) and (2.2) are even functions, and moreover  $u' < 0$  for  $x \in (0, 1]$ , see e.g. [4]. Since  $u'(x)$  also satisfies the equation in (2.2), by using the Sturm Comparison Theorem we easily conclude that any nontrivial solution of (2.2) can be chosen to be positive. If the equation (2.2) has a nontrivial solution, we shall refer to  $(\lambda, u(x))$  as a critical solution (or critical point) of (2.1).

Before stating our assumptions on  $f(u)$ , we prove an important lemma relating solutions of the problems (2.1) and (2.2), which essentially does not require any assumptions.

**LEMMA 2.1.** *Let  $u(x) \in C^3(-1, 1) \cap C^0[-1, 1]$  and  $w(x) \in C^2(-1, 1) \cap C^0[-1, 1]$  be solutions of (2.1) and (2.2) respectively, and  $f \in C^1(\mathbb{R})$ . Then*

$$\int_0^1 f(u)w \, dx = \frac{1}{2} w(0)f(u(0)). \quad (2.3)$$

*Proof.* Differentiate the equation (2.1)

$$u_x'' + \lambda f_u u_x = 0. \quad (2.4)$$

From (2.4) and (2.2) we get

$$(wu'' - u'w')' = 0 \quad \text{for all } x \in [0, 1].$$

Hence, the quantity  $wu'' - u'w'$  is constant, and so

$$w(x)u''(x) - u'(x)w'(x) = -\lambda w(0)f(u(0)). \quad (2.5)$$

Integrating (2.5),

$$\int_0^1 (wu'' - u'w') \, dx = -\lambda w(0)f(u(0)). \quad (2.6)$$

On the other hand,

$$\begin{aligned} \int_0^1 (wu'' - u'w') \, dx &= \int_0^1 wu'' \, dx - \left[ wu'|_0^1 - \int_0^1 wu'' \, dx \right] \\ &= 2 \int_0^1 wu'' \, dx = -2\lambda \int_0^1 f(u)w \, dx. \end{aligned} \quad (2.7)$$

From (2.6) and (2.7) the lemma follows.  $\square$

We begin with a special case when  $f(0) = 0$ . Namely, we assume that the function

$f(u) \in C^2(R)$  has the following properties:

$$f(0) = f(b) = f(c) = 0 \quad \text{for some constants } 0 < b < c, \quad (2.8)$$

$$f(x) > 0 \quad \text{for } x \in (-\infty, 0) \cup (b, c), \quad (2.9)$$

$$f(x) < 0 \quad \text{for } x \in (0, b) \cup (c, \infty),$$

$$\int_0^c f(u) du > 0, \quad (2.10)$$

$$f''(u) \text{ changes sign exactly once when } u > 0, \text{ and } f''(u) \text{ has} \quad (2.11)$$

exactly one positive root (the root assumption can be relaxed).

Our canonical example is  $f(u) = u(u - b)(c - u)$  with constants  $u < b < c$ , such that  $b < \frac{1}{2}c$ . Using the maximum principle and Lemma 1.1, we see that any nontrivial solution of (2.1) satisfies

$$0 < u(x) < c \quad \text{for all } x \in (-1, 1). \quad (2.12)$$

Similarly,

$$b < u(0) < c. \quad (2.13)$$

By our assumptions the function  $f(u)$  is concave up near  $u = 0$  and concave down for  $u > u_0 > 0$ , where  $f''(u_0) = 0$ . It is clear that there is exactly one point where a ray out of the origin touches the graph of  $f(u)$ . We denote this point by  $\beta$ , i.e.  $\beta$  is the unique solution of

$$f'(\beta) = \frac{f(\beta)}{\beta}. \quad (2.14)$$

We recall from the analysis in [6] that turning (or critical) points of (2.1) can occur only when (2.2) has a nontrivial solution  $w(x)$ , and also that in such a case we can choose  $w(x)$  to be strictly positive on  $(-1, 1)$ .

LEMMA 2.2. *Let  $u(x)$  be any critical point of (2.1). Then*

$$u(0) > \beta. \quad (2.15)$$

*Proof.* We will show that if  $u(0) \leq \beta$ , then the only solution of (2.2) is  $w \equiv 0$ . First, we claim that

$$f'(u) > \frac{f(u)}{u} \quad \text{for } 0 < u < \beta. \quad (2.16)$$

Indeed, denote  $p(u) = uf'(u) - f(u)$ . Then  $p(0) = p(\beta) = 0$ , and  $p'(u) = uf''(u)$ . It follows that  $p'(u) > 0$  near  $u = 0$ , and  $p'(u) < 0$  near  $u = \beta$ . Since  $p(u)$  has no roots in  $(0, \beta)$  (since the solution of (2.14) is unique) it follows that  $p(u) > 0$  on  $(0, \beta)$ , establishing (2.16). We now rewrite (2.1) in the form

$$u'' + \lambda \frac{f(u)}{u} u = 0.$$

Using the Sturm Comparison Theorem and (2.16), we conclude that (2.2) cannot have a positive solution  $w(x)$ . (By (2.16) any solution of (2.2) would have to vanish

on  $(-1, 1)$ .) Since any nontrivial solution of (2.2) has to be positive, we conclude the lemma.  $\square$

**THEOREM 2.3.** *Under the conditions (2.8)–(2.11) there is a critical  $\lambda_0 > 0$  such that for  $\lambda < \lambda_0$  the problem (2.1) has no nontrivial solutions, it has exactly one nontrivial solution for  $\lambda = \lambda_0$ , and exactly two nontrivial solutions for  $\lambda > \lambda_0$ . Moreover, all solutions lie on a single solution curve, which for  $\lambda > \lambda_0$  has two branches denoted by  $u^-(x, \lambda) < u^+(x, \lambda)$ , with  $u^+(x, \lambda)$  strictly monotone increasing in  $\lambda$ ,  $u^-(0, \lambda)$  strictly monotone decreasing in  $\lambda$ , and  $\lim_{\lambda \rightarrow \infty} u^+(x, \lambda) = c$ ,  $\lim_{\lambda \rightarrow \infty} u^-(x, \lambda) = 0$  for  $x \in (-1, 1) \setminus \{0\}$ , while  $u^-(0, \lambda) > b$  for all  $\lambda > \lambda_0$ .*

*Proof.* We begin by noticing that for sufficiently small  $\lambda > 0$  the problem (2.1) has no positive solutions. Indeed under our assumptions there is a constant  $\gamma > 0$ , such that  $f(u) \leq \gamma u$  for all  $u > 0$ . Then

$$\lambda \gamma \int_{-1}^1 u^2 dx \geq \lambda \int_{-1}^1 f(u) u du = \int_{-1}^1 u'^2 dx \geq \frac{\pi^2}{4} \int_{-1}^1 u^2 dx,$$

and the claim follows. Next we show that positive solutions exist for large  $\lambda$ . We outline the argument, which is due to A. Ambrosetti and P. H. Rabinowitz [8, p. 12]. Solutions of (2.1) are critical points on  $H_0^1(-1, 1)$  of the functional

$$J(u) = \int_{-1}^1 \left[ \frac{1}{2} u'^2 - \lambda F(u) \right] dx, \quad F(u) \equiv \int_0^u f(z) dz.$$

Clearly  $J(0) = 0$ , and by Poincaré's inequality  $J(u)$  is positive in a sufficiently small neighbourhood of zero in  $H_0^1(-1, 1)$ . Let  $u_\varepsilon(x) \in H_0^1(-1, 1)$  be such that  $0 < u_\varepsilon(x) \leq c$  for all  $x \in (-1, 1)$  and  $u_\varepsilon(x)$  is different from  $c$  only on a set of measure  $\varepsilon$ . Then  $\int_{-1}^1 F(u_\varepsilon(x)) dx > 0$  for  $\varepsilon$  small enough. Then  $J(u_\varepsilon) < 0$  for sufficiently large  $\lambda$ . By the Mountain Pass Theorem, see [8], it follows that  $J(u)$  has a nontrivial critical point at some  $\lambda = \lambda_1$ , where  $J(u) > 0$ . (Actually, with a little more care one can show existence of a second critical point where  $J(u) < 0$ .) We denote by  $u(x, \lambda_1)$  the maximal solution of (2.1) (which can be obtained by monotone iterations, starting with a supersolution  $u = c$ ).

We now continue  $u(x, \lambda)$  for decreasing  $\lambda$ . If the corresponding linearised equation (2.2) has only the trivial solution  $w = 0$ , then by the Implicit Function Theorem we can solve (2.1) for  $\lambda < \lambda_1$  and  $\lambda$  close to  $\lambda_1$ , obtaining a continuous in  $\lambda$  curve of solutions  $u(x, \lambda)$ . This process of decreasing  $\lambda$  cannot be continued indefinitely, since for sufficiently small  $\lambda > 0$  the problem (2.1) has no solution. Let  $\lambda_0$  be the infimum of  $\lambda$  for which we can continue the curve of solutions to the left. It is easy to show (see [4, 5] for a similar argument) that there is a solution  $u(x, \lambda_0) \equiv u_0(x)$ . Clearly the linearised equation at  $\lambda = \lambda_0$  and  $u = u_0$  must have a nontrivial solution, and by the result of [5] we have  $w(x) > 0$  for all  $x \in (-1, 1)$ .

We rewrite the equation (2.1) in the operator form

$$F(\lambda, u) = u'' + \lambda f(u) = 0, \quad (2.17)$$

where  $F: R \times C_0^2[-1, 1] \rightarrow C[-1, 1]$ . Notice that  $F_u(\lambda, u)w$  is given by the left-hand side of (2.2). We show next that at the critical point  $(\lambda_0, u_0)$  the Crandall–Rabinowitz Theorem applies. Indeed,  $N(F_u(\lambda_0, u_0)) = \text{span}\{w(x)\}$  is one-dimensional, and  $\text{codim } R(F_u(\lambda_0, u_0)) = 1$  by the Fredholm alternative. It remains to check that

$F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$ . Assuming the contrary would imply existence of  $v(x) \not\equiv 0$ , such that

$$v'' + \lambda_0 f'(u_0)v = f(u_0) \quad \text{for } x \in (-1, 1), \quad v(-1) = v(1) = 0. \quad (2.18)$$

Multiply (2.18) by  $w$ , integrate and use Lemma 2.1, and (2.13)

$$0 = \int_{-1}^1 f(u_0)w \, dx = w(0)f(u_0(0)) > 0,$$

a contradiction.

Applying the Crandall–Rabinowitz Theorem, we conclude that  $(\lambda_0, u_0)$  is a bifurcation point, near which the solutions of (2.1) form a curve  $(\lambda_0 + \tau(s), u_0 + sw + z(s))$  with  $s$  near  $s = 0$ , and  $\tau(0) = \tau'(0) = 0$ ,  $z(0) = z'(0) = 0$ . We claim that

$$\tau''(0) > 0, \quad (2.19)$$

i.e. only ‘turns to the right’ in  $(\lambda, u)$  ‘plane’ are possible. We use the formula  $(u_0(x) = u(x, \lambda_0))$

$$\tau''(0) = -\lambda_0 \frac{\int_{-1}^1 f''(u_0)w^3 \, dx}{\int_{-1}^1 f(u_0)w \, dx} = -\lambda_0 \frac{\int_0^1 f''(u_0)w^3 \, dx}{\int_0^1 f(u_0)w \, dx}. \quad (2.20)$$

For completeness, we present next the derivation of (2.20). Differentiate (2.1) in  $s$  twice

$$u''_{ss} + \lambda f''u_s^2 + \lambda f'u_{ss} + 2\tau'f'u_s + \tau''f(u) = 0.$$

Setting here  $s = 0$ , and using that  $\tau'(0) = 0$  and  $u_s|_{s=0} = w(x)$ , we obtain

$$u''_{ss} + \lambda_0 f''w^2 + \lambda_0 f'u_{ss} + \tau''(0)f = 0. \quad (2.21)$$

Multiplying (2.21) by  $w$ , (2.2) by  $u_{ss}$ , integrating and subtracting, we obtain (2.20). By Lemma 2.1 the denominator in (2.20) is positive, so we only need to show that

$$\int_0^1 f''(u_0)w^3 \, dx < 0. \quad (2.22)$$

Differentiating the equations (2.1) and (2.2) respectively, we obtain (at  $u = u_0$ ),

$$u'_x + \lambda f'(u)u_x = 0, \quad (2.23)$$

$$w'_x + \lambda f'(u)w_x + \lambda f''(u)u_x w = 0. \quad (2.24)$$

Multiply (2.24) by  $u_x$ , (2.23) by  $w_x$ , integrate and subtract,

$$(u_x w'_x - w_x u'_x)|_0^1 + \lambda \int_0^1 f''u_x^2 w \, dx = 0. \quad (2.25)$$

Notice that  $w''(1) = -f_u w(1) = 0$ , and  $u''(1) = -\lambda f(u(1)) = 0$ , so that from (2.25) at  $u = u_0$ ,

$$\int_0^1 f''(u)u_x^2 w \, dx = 0. \quad (2.26)$$

We will show that

$$\int_0^1 f''(u_0)w^3 dx < \int_0^1 f''(u_0)u_{0x}^2 w dx, \quad (2.27)$$

from which (2.22) and (2.19) will follow. We claim that  $f''(u_0(x))$  changes sign exactly once on  $(0, 1)$ . Indeed,  $f''(u)$  is positive for small  $u$ , and hence  $f''(u_0(x)) > 0$  for  $x$  close to 1. By Lemma 2.2,  $u_0(0) > \beta$  and it is clear that  $f''(u) < 0$  for  $u > \beta$ . Hence  $f''(u_0(0)) < 0$ . Since  $u_0(x)$  is decreasing on  $(0, 1)$ , the claim follows. (It is interesting to illustrate the last point for the special case of  $f = u(u - b)(c - u)$ . Indeed,  $f''(u_0) = -6u_0 + 2(b + c)$ , so clearly  $f''(u_0(1)) > 0$ . One easily computes the number  $\beta$  defined in (2.14) to be  $\beta = (b + c)/2$ . By Lemma 2.2,  $f''(u_0(0)) \leq -6\beta + 2(b + c) = -(b + c) < 0$ . Since  $u_0(x)$  is decreasing on  $(0, 1)$ , it follows that  $f''(u_0(x))$  changes sign exactly once.) Let  $\bar{x}$  be such that  $f''(u_0(\bar{x})) = 0$ . We claim next that the functions  $u'(x)$  and  $-w(x)$  intersect exactly once on  $(0, 1)$ . Since  $u'(0) = 0$  and  $u'(1) < 0$ , while  $-w(0) < 0$  and  $-w(1) = 0$ , the functions  $u'(x)$  and  $-w(x)$  intersect at least once. To see that  $u'(x)$  and  $-w(x)$  cannot intersect more than once, notice that these functions are solutions of the same linear equation (2.23), and hence cannot intersect twice in the region where they are both negative (or positive). Indeed, if say  $u'(x) > -w(x)$  on some interval  $(x_1, x_2) \subset (-1, 1)$  then we can find a constant  $\lambda$ ,  $0 < \lambda < 1$ , such that  $-\lambda w(x) \geq u'(x)$  on  $(x_1, x_2)$ , and for some  $\bar{x} \in (x_1, x_2)$  we have  $-\lambda w(\bar{x}) = u'(\bar{x})$  and  $-\lambda w'(\bar{x}) = u''(\bar{x})$ . Since  $\lambda w(x)$  is also a solution of (2.23), we have two solutions satisfying the same initial conditions at  $\bar{x}$ , a contradiction. It follows that  $-w(x)$  and  $u'(x)$  intersect exactly once on  $(0, 1)$ . By considering  $\lambda w(x)$  with a proper constant  $\lambda$ , we may assume that  $-w(x)$  and  $u'(x)$  intersect at  $\bar{x}$ , the point where  $f''(u_0(x))$  changes sign. Returning to (2.27), we see that on the interval  $(0, \bar{x})$ , where  $f''(u_0) < 0$ , we have  $w^2 > u_{0x}^2$ , while on the interval  $(\bar{x}, 1)$ , where  $f''(u_0) > 0$ , we have  $w^2 < u_{0x}^2$ . So that the integrand on the right in (2.27) is larger than the one on the left for all  $x \in (-1, 1) \setminus \{\bar{x}\}$ , which proves (2.27) and (2.19).

It follows that at critical point  $(\lambda_0, u_0)$  the curve of solutions turns to the 'right' in  $(\lambda, u)$  plane. After the turn we can continue this curve of solutions for increasing  $\lambda$ , using the Implicit Function Theorem, so long as  $(\lambda, u)$  is a nonsingular point of  $F(\lambda, u)$ . However, there can be no critical points on the lower branch, since we know precisely the structure of solutions at any critical point, namely a turn to the right always occurs, which is impossible at the lower branch. Hence the lower branch can be continued for all  $\lambda > \lambda_0$ . The same is true for the upper branch, and we obtain a parabola-like curve of solutions. It remains to show that there is only one such curve, and to prove the monotonicity properties of its branches.

We claim that the upper branch is increasing for all  $\lambda > \lambda_0$ . For  $\lambda$  close to  $\lambda_0$ , this follows from the Crandall-Rabinowitz Theorem ( $u_\lambda(x, \lambda) \simeq w(x) > 0$  for all  $x$ ). Assuming the claim to be false, denote by  $\lambda_1$  the first  $\lambda$  where the condition  $u_\lambda > 0$  is violated, i.e.  $u_\lambda(x, \lambda_1) \geq 0$  for all  $x \in (-1, 1)$ , and  $u_\lambda(x_1, \lambda_1) = 0$  for some  $x_1 \in (0, 1)$ . (The possibility that  $u'_\lambda(1, \lambda_1) = 0$  is easily excluded multiplying (2.4) by  $u_\lambda$ , subtracting from it the equation (2.29) multiplied by  $u'$ , and then integrating from 0 to 1.) Since  $x_1$  is a point of minimum,  $u'_\lambda(x_1, \lambda_1) = 0$  and  $u''_\lambda(x_1, \lambda_1) \geq 0$ . It follows from the equation (2.29) below that

$$0 < u(x_1) \leq b. \quad (2.28)$$

Differentiate (2.1) in  $\lambda$ ,

$$u''_\lambda + \lambda f'(u)u_\lambda + f(u) = 0. \quad (2.29)$$

Multiply the equation (2.29) by  $u_x$ , the equation (2.23)  $u_\lambda$ , integrate over  $(0, x_1)$  and subtract,

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^{x_1} + \int_{u(0)}^{u(x_1)} f(u) du = 0. \quad (2.30)$$

The first term in (2.30) is equal to  $u''(0)u_\lambda(0) \leq 0$ . We claim that the second term in (2.30) is negative, which will lead to a contradiction. Multiply (2.1) by  $u'$  and integrate from 0 to 1,

$$\lambda \int_0^{u(0)} f(u) du = \frac{1}{2} u'^2(1) > 0.$$

Using (2.28) and our conditions (2.8)–(2.10) on  $f(u)$ ,

$$\int_{u(x_1)}^{u(0)} f(u) du > \int_0^{u(0)} f(u) du > 0.$$

So we have a contradiction in (2.30), and the monotonicity of the upper branch is proved.

Since the upper branch is increasing (and is bounded by  $c$ ) it tends to a limit at any  $x \in (-1, 1)$  as  $\lambda \rightarrow \infty$ . Over any subinterval of  $(-1, 1)$  this limit may only be equal to either  $b$  or  $c$ , since otherwise from the equation (2.1)  $u''(x)$  would have to be large over a subinterval, which is impossible. Since  $u(x)$  is convex below  $u = b$ , the upper branch cannot tend to  $b$  over any subinterval of  $(-1, 1)$ . It follows that the upper branch tends to  $c$  over  $(-1, 1)$ .

We now rule out the possibility of another curve of solution. By the above analysis such a curve would have an upper branch tending to  $c$  as  $\lambda \rightarrow \infty$ . We show next that any solution of (2.1) tending to  $c$  is stable. Then uniqueness of such solution follows by a degree argument exactly as in [2, p. 68].

Indeed if  $u = u(x, \lambda)$  were not stable, we could find a constant  $\mu \geq 0$  and  $w(x) > 0$ , so that

$$w'' + \lambda f'(u)w = \mu w \quad \text{for } x \in (-1, 1), \quad w(-1) = w(1) = 0. \quad (2.31)$$

We may assume that  $\int_0^1 w^2 dx = 1$ . Multiplying (2.1) by  $u'$  and integrating over  $(0, x)$ , we conclude that

$$|u'(x)| \geq c_0 \sqrt{\lambda}, \quad (2.32)$$

for some  $c_0 > 0$  when  $\lambda$  is large, for all  $x \in (\eta, 1]$ , where  $u(\eta) = \alpha$  and  $\alpha$  is the larger root of  $f'(u)$ . Since  $u(x, \lambda) \rightarrow c$ , we can find a constant  $A$  independent of  $\lambda$  and  $\xi = \xi(\lambda)$  near  $x = 0$  (say in  $(0, 1/4)$ ), such that  $|u''(\xi)| \leq A$ . As in Lemma 2.1, we derive

$$-u'(1)w'(1) + u'(\xi)w'(\xi) - w(\xi)u''(\xi) + \mu \int_\xi^1 w(x)u'(x) dx = 0. \quad (2.33)$$

From (2.31)  $w''(x) > 0$  on  $(0, 1)$ , except near  $x = 1$ . It follows that  $w'(\xi) > 0$  and  $w(\xi)$  is bounded (because  $\int_0^1 w^2 dx = 1$ ). It follows that the third term in (2.33) is bounded,

while the second and the fourth are negative. The first term in (2.33) is negative, and we show next that it is large in absolute value (as  $\lambda \rightarrow \infty$ ), which leads to a contradiction. Indeed, assuming otherwise would imply by (2.32) that

$$|w'(1)| = O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Also by (2.32) we conclude that when moving from  $x = 1$  leftwards,  $u(x)$  must reach  $u(\eta) = \alpha$ ,  $f'(\alpha) = 0$  ( $\alpha$  the larger root of  $f'(u)$ ) over the interval  $(\eta, 1)$  of length

$$1 - \eta = O\left(\frac{1}{\sqrt{\lambda}}\right).$$

Since  $w(x)$  is convex on  $(0, \eta)$  ( $f'(u) < 0$  there), it must take its maximum, on  $(\eta, 1)$ . The maximum value must be at least 1, since  $\int_0^1 w^2 dx = 1$ . On  $(\eta, 1)$  we set  $t = 1 - x$ , and estimate from (2.31)

$$w'' \leq c\lambda w, \quad w(0) = 0, \quad w'(0) = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad 0 < t < \frac{c_1}{\sqrt{\lambda}},$$

with positive constants  $c$  and  $c_1$  (it can be easily seen that  $\mu \leq c\lambda$  for some  $c > 0$ ). Integrating

$$w(t) \leq c\lambda \int_0^t (t-s)w(s) ds + w'(0)t \leq c_2 \left( \sqrt{\lambda} \int_0^t w(s) ds + \frac{1}{\lambda} \right).$$

Applying Gronwall's inequality, we conclude that  $w(t) = O(1/\lambda)$  on  $(\eta, 1)$ . This is a contradiction, which in turn implies a contradiction in (2.33), proving uniqueness of the solution curve.

Finally, we prove that  $u(0, \lambda)$  is decreasing on the lower branch. By the Crandall-Rabinowitz Theorem, we know that  $u_\lambda(x, \lambda) < 0$  for  $\lambda$  close to  $\lambda_0$  and all  $x \in (-1, 1)$  on the lower branch. Let  $\lambda_1$  be the first  $\lambda$  where  $u_\lambda(0, \lambda_1) = 0$ . From (2.29) we see that  $u'_\lambda(0, \lambda_1) < 0$ , and so  $x = 0$  is not a point of minimum of  $u_\lambda(x, \lambda_1)$ , and then we conclude that  $u_\lambda(x, \lambda_1)$  is negative for  $x$  positive and close to zero. Multiply (2.29) by  $u_x$ , (2.23) by  $u_\lambda$  and integrate from 0 to 1, then subtract

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^1 + \int_{u(0)}^0 f(u) du = 0. \quad (2.34)$$

The integral term in (2.34) is negative, as was proved earlier. From the first term only  $u'_\lambda(1)u'(1)$  survives. Hence

$$u'_\lambda(1, \lambda_1) < 0,$$

and so  $u_\lambda(x, \lambda_1)$  is positive near  $x = 1$ , and then  $u_\lambda(x, \lambda_1)$  must have at least one zero on  $(0, 1)$ . Let  $x_1$  be the smallest zero, i.e.

$$u'_\lambda(x_1, \lambda_1) \geq 0. \quad (2.35)$$

Multiply (2.29) by  $u_x$ , (2.23) by  $u_\lambda$ , integrate from 0 to  $x_1$ , and subtract

$$(u'_\lambda u_x - u'_x u_\lambda)|_0^{x_1} + \int_{u(0)}^{u(x_1)} f(u) du = 0. \quad (2.36)$$

The integral term in (2.36) is smaller than the one in (2.34), and so is negative. The first term in (2.36) is equal to  $u'(x_1)u'_\lambda(x_1) \leq 0$ . We have a contradiction in (2.36), which shows that  $u_\lambda(0, \lambda) < 0$  for all  $\lambda > \lambda_0$ , and this finishes the proof of the theorem.  $\square$

We now turn to the case of three positive roots. With  $f(u)$  as described by (2.8)–(2.10), and

$$\text{for } u > 0, \quad f''(u - a) \text{ changes sign exactly once and has exactly one root,} \quad (2.11)'$$

we consider

$$u'' + \lambda f(u - a) = 0 \quad \text{for } x \in (-1, 1), \quad u(-1) = u(1) = 0, \quad (2.37)$$

where  $a$  is a positive constant. Our prototype is  $f(u - a) = (u - a)(u - a - b)(c + a - u)$ . The corresponding linearised equation is

$$w'' + \lambda f'(u - a)w = 0 \quad \text{for } x \in (-1, 1), \quad w(-1) = w(1) = 0. \quad (2.38)$$

For the  $a > 0$  case, we need to assume additionally that

$$f(\beta)\beta - 2[F(\beta) - F(-a)] \geq 0, \quad (2.39)$$

with  $\beta$  as defined by (2.14).

The equation satisfied by  $u_x$  is

$$u_x'' + \lambda f'(u - a)u_x = 0. \quad (2.40)$$

It follows by the maximum principle that any solution of (2.37) satisfies

$$0 < u(x) < c + a \quad \text{for all } x \in (-1, 1). \quad (2.41)$$

Since all solutions of (2.37) are positive, it follows by [3] that they are even functions, with  $u_x < 0$  for  $x > 0$ . As  $u_x$  vanishes exactly once in the interval  $(-1, 1)$  (at  $x = 0$ ), and since  $w$  and  $u_x$  satisfy the same equation, it follows by the Sturm Comparison Theorem that any nontrivial solution of (2.38) does not vanish inside  $(-1, 1)$ , and so we can choose  $w(x) > 0$  on  $(-1, 1)$ . The solution of (2.37) depends now on two parameters  $\lambda$  and  $a$ ; however, we will denote it by  $u(x, a)$ , or even  $u(x)$ , when dependence on the other parameters is secondary. Note that the case  $a = 0$  was covered in the previous theorem.

LEMMA 2.4. *Let  $u(x, a)$  be any critical point (i.e. (2.38) has a nontrivial solution). Then*

$$u(0, a) - a > u(0, 0). \quad (2.42)$$

*Proof.* Multiply the equation (2.37) by  $u'$  and integrate from 0 to  $x$ . Denoting  $F_a(u) = \int_0^u f(z - a) dz$ , we express

$$\frac{u'^2(x)}{2} + \lambda(F_a(u(x)) - F_a(u(0))) = 0.$$

Rearranging and integrating from 0 to 1,

$$g_a(u(0)) \equiv \int_0^{u(0)} \frac{dt}{\sqrt{F_a(u(0)) - F_a(t)}} = \sqrt{2\lambda}. \quad (2.43)$$

Denoting  $F(u) = F_0(u) = \int_0^u f(z) dz$ , we express

$$F_a(u) = \int_0^u f(z-a) dz = \int_{-a}^{u-a} f(z) dz = F(u-a) - F(-a),$$

and then we rewrite (2.43) as

$$g_a(s) \equiv \int_0^s \frac{dt}{\sqrt{F(s-a) - F(t-a)}} = \sqrt{2\lambda}, \quad (2.44)$$

where  $s = u(0, a)$ . We may assume that  $s > a$ , since  $f_u < 0$  in the region where  $u < a$ , and then (2.38) could not have a nontrivial solution  $w(x)$ . Express

$$\begin{aligned} g_a(s) &= \int_0^a \frac{dt}{\sqrt{F(s-a) - F(t-a)}} + \int_0^{s-a} \frac{dt}{\sqrt{F(s-a) - F(t)}} \\ &\equiv h_a(s) + g_0(s-a), \end{aligned}$$

where  $h_a(s)$  denotes the first integral and  $g_0(s-a)$  was defined by (2.44). By the Crandall–Rabinowitz Theorem, solutions of (2.37) are monotone in  $\lambda$  near a critical point (if a turn occurs, one branch is monotone increasing, and the other one monotone decreasing in  $\lambda$ ). Hence, we can take  $s = u(0, a) = \max_{x \in (-1, 1)} u(x, a)$  as the parameter used in the Crandall–Rabinowitz Theorem. Hence at a critical point we have  $d\lambda/ds = 0$ , and then from (2.44),

$$g'_a(s) = 0. \quad (2.45)$$

(That the function  $g_a(s)$  is differentiable is not obvious, but it can be seen by a change of variables  $t-a = (s-a)\tau$  in its definition.) We can also interpret Theorem 2.3 in terms of the function  $g_0(s)$ :  $g'_0(s) < 0$  for  $s < u(0, 0)$ ,  $g'_0(s) = 0$  for  $s = u(0, 0)$ , and  $g'_0(s) > 0$  for  $s > u(0, 0)$ . (Indeed since the maximum value is decreasing on the lower branch, we have  $ds/d\lambda < 0$  for  $s < u(0, 0)$ .)

Compute

$$h'_a(s) = -\frac{1}{2} \int_0^a f(s-a) [F(s-a) - F(t-a)]^{-3/2} dt < 0, \quad (2.46)$$

since from the equation (2.37)

$$f(s-a) = f(u(0, a) - a) = -\frac{1}{\lambda} u''(0, a) > 0.$$

We now rewrite (2.45),

$$0 = g'_a(s) = h'_a(s) + g'_0(s-a), \quad (2.47)$$

and conclude the proof of the lemma, since the first term on the right in (2.47) is always negative, and assuming that  $s-a < u(0, 0)$  ( $s = u(0, a)$ ) would imply that the second term is negative as well.  $\square$

**LEMMA 2.5.** *Let  $u(x)$  be any critical point of (2.37), and  $w(x) > 0$  the corresponding solution of (2.38). Then*

$$\int_0^1 f''(u-a) u_x w^2 dx > 0. \quad (2.48)$$

*Proof.* We begin by deriving a convenient expression for the integral in (2.48). Differentiating (2.38) yields

$$w_x'' + \lambda f'(u-a)w_x + \lambda f''(u-a)u_x w = 0. \quad (2.49)$$

Combining (2.49) and (2.38) gives

$$(ww_x' - w_x w')' + \lambda f''(u-a)u_x w^2 = 0.$$

Integrating, we express

$$\begin{aligned} -\lambda \int_0^1 f''(u-a)u_x w^2 dx &= (ww'' - w'^2)|_0^1 = -w'^2(1) - w(0)w''(0) \\ &= -w'^2(1) + \lambda w^2(0)f'(u(0)-a). \end{aligned} \quad (2.50)$$

Proceeding as in the derivation of (2.5), we obtain

$$w(x)u''(x) - u'(x)w'(x) = -\lambda w(0)f(u(0)-a) \quad \text{for all } x \in [-1, 1].$$

Setting  $x = 1$ , we express

$$w'(1) = \frac{\lambda w(0)f(u(0)-a)}{u'(1)}. \quad (2.51)$$

From the proof of Lemma 2.4,

$$u'^2(x) = 2\lambda(F(u(0)-a) - F(u(x)-a)).$$

Setting here  $x = 1$  and using this formula in (2.51), we express

$$w'^2(1) = \frac{\lambda w^2(0)f^2(u(0)-a)}{2(F(u(0)-a) - F(-a))}.$$

Using this in (2.50), we obtain

$$\begin{aligned} \int_0^1 f''(u-a)u_x w^2 dx &= \frac{w^2(0)}{2[F(u(0)-a) - F(-a)]} \{f^2(u(0)-a) \\ &\quad - 2[F(u(0)-a) - F(-a)]f'(u(0)-a)\} \\ &\equiv \frac{w^2(0)}{2[F(u(0)-a) - F(-a)]} I(\alpha), \end{aligned} \quad (2.52)$$

where  $\alpha = u(0, a) - a$ , and

$$I(\alpha) = f^2(\alpha) - 2[F(\alpha) - F(-a)]f'(\alpha). \quad (2.53)$$

By Lemma 2.4 and (2.41),  $u(0, 0) < \alpha < c$ , and to prove the present lemma we need to show that  $I(\alpha) > 0$  for all  $u(0, 0) < \alpha < c$ . If  $f'(\alpha) < 0$  then from (2.53) we see that  $I(\alpha) > 0$ , since  $F(\alpha) - F(-a) > F(\alpha) > F(u(0, 0)) > 0$ , so assume that

$$f'(\alpha) > 0. \quad (2.54)$$

Notice that then (since  $\alpha > \beta$ )

$$f''(\alpha) < 0. \quad (2.55)$$

Compute

$$I'(\alpha) = -2[F(\alpha) - F(-a)]f''(\alpha) > 0. \quad (2.56)$$

Notice that condition (2.39) implies that  $I(\beta) \geq 0$ . Then by (2.54) we have for all  $\alpha$  satisfying  $u(0, 0) < \alpha < c$ ,

$$I(\alpha) > I(\beta) \geq 0,$$

and the lemma follows.  $\square$

REMARK 2.6. For the case  $f(u)$  is a cubic, S.-H. Wang [11] assumes instead of (2.39) that

$$rf(r) - 2F(a) > 0, \quad (2.57)$$

where  $r > b$  is such that  $F(r) = 0$ . It is easy to see that these conditions do not imply each other.

We are now ready to prove the main result of this section. We recall that all nontrivial solutions of (2.37) are positive.

**THEOREM 2.7.** *Consider the problem (2.37) with  $f(u)$  as described by (2.8)–(2.10), (2.39), and (2.11). Then there exists a critical  $\lambda_0$  such that for the problem (2.37) there exist exactly one positive solution for  $0 < \lambda < \lambda_0$ , exactly two positive solutions for  $\lambda = \lambda_0$ , and exactly three positive solutions for  $\lambda > \lambda_0$ . Moreover, all solutions lie on two smooth in  $\lambda$  solution curves, and all different solutions of (2.37) at the same  $\lambda$  are strictly ordered on  $(-1, 1)$ . One of the curves, referred to as the lower curve, starts at  $\lambda = 0$ ,  $u = 0$ , it is strictly increasing in  $\lambda$ , and  $\lim_{\lambda \rightarrow \infty} u(x, \lambda) = a$ . The upper curve is a parabola-like curve, consisting of two branches  $u^-(x, \lambda) < u^+(x, \lambda)$ . The upper branch is monotone increasing in  $\lambda$  and  $\lim_{\lambda \rightarrow \infty} u^+(x, \lambda) = a + c$  for all  $x \in (-1, 1)$ . The lower branch approaches a spike-layer, namely  $\lim_{\lambda \rightarrow \infty} u^-(x, \lambda) = a$  for all  $x \in (-1, 1) \setminus \{0\}$ , while  $u^-(0, \lambda) > a + b$  for all  $\lambda > \lambda_0$ .*

*Proof.* The proof is similar to that of Theorem 2.3 (and also [6, Theorem 3.3]), so that we will just outline it.

Using the Implicit Function Theorem, one sees that for sufficiently small  $\lambda$  there is a curve of solutions emanating from  $u = 0$ ,  $\lambda = 0$ . By Lemma 1.1 the maximum value of these solutions stays below  $a$ . Since  $f'(u - a) < 0$  when  $u < a$ , it follows that (2.38) has only the trivial solution, and hence by the implicit function theorem this curve of solutions can be continued for all  $\lambda > 0$ . By differentiating the equation (2.37) in  $\lambda$ , we conclude that  $u_\lambda > 0$  for  $x \in (-1, 1)$ . By writing (2.37) in the equivalent integral form, we conclude that solutions on this curve tend to  $a$  as  $\lambda \rightarrow \infty$ .

Solutions of (2.37) are critical points in  $H_0^1(-1, 1)$  of the functional

$$J(u) = \int_{-1}^1 \left[ \frac{1}{2} u'^2 - \lambda F_a(u) \right] dx.$$

On the lower curve, which we denote by  $\bar{u} = \bar{u}(x, \lambda)$ ,

$$J(\bar{u}) \geq -\lambda \int_{-1}^1 F_a(x, \bar{u}) dx \simeq -\lambda \int_{-1}^1 F_a(x, a) dx,$$

for  $\lambda$  large. By modifying  $u = a + c$  near  $x = \pm 1$  to obtain a function of class  $H_0^1(-1, 1)$ , we produce a function  $\hat{u} \in H_0^1(-1, 1)$ , such that  $J(\hat{u}) < J(\bar{u})$ . Since the functional  $J(u)$  is bounded from below, it will have a point of minimum different

from  $\bar{u}(x, \lambda)$ . Hence for large  $\lambda$ , say  $\lambda \geq \bar{\lambda}$ , we will have solutions, not on the lower curve. In particular the maximal solution at  $\bar{\lambda}$  is not on the lower curve.

We now continue the maximal solution at  $\bar{\lambda}$  for decreasing  $\lambda$ ,  $\lambda \leq \bar{\lambda}$ . When the solution is noncritical, we can use the Implicit Function Theorem. For the critical  $(\lambda_0, u_0)$ , the Crandall–Rabinowitz Theorem applies, since the crucial condition  $F_\lambda \notin R(F_u)$  is verified in exactly the same way as in Theorem 2.3. We claim that at any critical point a turn ‘to the right’ occurs. Namely (compare with (2.20)) we need to show that

$$\tau''(0) = -\lambda_0 \frac{\int_0^1 f''(u_0(x) - a)w^3 dx}{\int_0^1 f(u_0(x) - a)w dx} > 0. \quad (2.58)$$

From Lemma 2.1 it follows that the denominator in (2.58) is positive. By Lemmas 2.4 and 2.2, it follows that  $f''(u_0(x) - a)$  changes sign exactly once on  $(0, 1)$ , say at  $x_0 > 0$ . By stretching  $w(x)$ , as in Theorem 2.3, we can arrange for  $w(x)$  and  $-u_x$  to have their unique intersection point at  $x_0$ . Then using Lemma 2.5

$$\int_0^1 f''(u_0(x) - a)w^2 w dx < \int_0^1 f''(u_0(x) - a)w^2 (-u_x) dx < 0,$$

which proves (2.58).

We now return to the curve of solutions through  $\bar{u}$ . We cannot continue it for decreasing  $\lambda$  indefinitely, since for  $\lambda > 0$  small the problem (2.37) has only one solution (lying on the lower curve, described earlier). Indeed, assuming two solutions, denoting by  $z$  their difference, writing the equation for  $z$ , and observing boundedness of  $f'(u)$  for  $u \in [0, a + c]$ , we obtain a contradiction. Let  $\lambda_0$  be the infimum of  $\lambda$ 's for which the upper curve can be continued in  $\lambda$ . One easily sees the existence of a solution of (2.37), corresponding to  $\lambda_0$ . This solution has to be singular, and by the previous analysis a turn to the right occurs. We then continue both branches of this curve for increasing  $\lambda$ , where we cannot encounter any more singular solutions, since at such solutions a turn ‘to the right’ would have to occur, which is impossible. The rest of the proof, including monotonicity of the upper branch of the upper curve, limiting behaviour of the branches, and the uniqueness of the upper curve, are all similar to the corresponding parts of Theorem 2.1.  $\square$

### 3. An exact multiplicity result for a cubic nonlinearity with $x$ dependence

To simplify the presentation, we consider a model equation

$$u'' + \lambda u^2(b(x) - u) = 0 \text{ on } (-1, 1), \quad u(-1) = u(1) = 0, \quad (3.1)$$

although our results can be easily generalised in various directions. (We could consider  $f(x, u)$  which is even in  $x$  with negative  $f_x, f_{xx}, f_{xxx}$  and such that for each  $x$  there is a  $\beta$ , such that  $f_{uu} > 0$  when  $u \in (0, \beta)$  and  $f_{uu} < 0$  when  $u \in (\beta, 1)$ . Also  $f(x, u)$  cannot change too much in  $x$ . We then extend the results of [12].) We assume that

the positive function  $b(x) \in C^3[-1, 1]$  satisfies the following conditions

$$b(-x) = b(x) \quad \text{for all } x \in [-1, 1], \quad (3.2)$$

$$b'(x) < 0 \quad \text{for } x \in (0, 1], \quad (3.3)$$

$$b''(x) < 0 \quad \text{for } x \in (0, 1], \quad (3.4)$$

$$b'''(x) \leq 0 \quad \text{for } x \in (0, 1], \quad (3.5)$$

$$b(1) \geq \frac{2}{3}b(0) > 0. \quad (3.6)$$

For example,  $b(x) = a - x^2$  with constant  $a \geq 3$ , satisfies all of the above conditions.

Notice that condition (3.4) implies that  $b(x)$  is a supersolution of (3.1). Combining this with the maximum principle and Lemma 1.1, we conclude that any nontrivial solution of (3.1) satisfies

$$0 < u(x) < b(x) \quad \text{for all } x \in (-1, 1). \quad (3.7)$$

Also, any solution of (3.1) is even, with  $u' < 0$  for  $x \in (0, 1]$ , see [3] or [4].

**THEOREM 3.1.** *Under the conditions (3.2)–(3.6) there is a critical  $\lambda_0 > 0$  such that for  $\lambda < \lambda_0$  the problem (3.1) has no nontrivial solutions, it has exactly one nontrivial solution for  $\lambda = \lambda_0$ , and exactly two solutions for  $\lambda > \lambda_0$ . Moreover, all solutions lie on a single solution curve, which for  $\lambda > \lambda_0$  has two branches denoted by  $u^-(x, \lambda) < u^+(x, \lambda)$ , with  $u^+(x, \lambda)$  strictly monotone increasing in  $\lambda$ ,  $u^-(x, \lambda)$  strictly monotone decreasing in  $\lambda$ , and  $\lim_{\lambda \rightarrow \infty} u^+(x, \lambda) = b(x)$ ,  $\lim_{\lambda \rightarrow \infty} u^-(x, \lambda) = 0$  for all  $x \in (-1, 1)$ .*

*Proof.* All of the assertions except for the exact multiplicity have already been proved in [5]. We briefly recall the steps in [5]. Multiplying the equation (3.1) by  $u$  and integrating over  $(-1, 1)$ , we easily conclude that (3.1) has no solutions for  $\lambda > 0$  small. On the other hand, by a variational approach there exist solutions for  $\lambda$  large. We follow the curve of maximal solutions for decreasing  $\lambda$  until a turning point. Once we show that only turns to 'the right' are possible, the theorem will follow. At a turning point  $\lambda'(0) = 0$  and

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^1 f_{uu} w^3 dx}{\int_{-1}^1 f w dx}, \quad (3.8)$$

where  $f = u^2(b(x) - u)$ , and  $w = w(x)$  is the solution of the linearised problem

$$w'' + \lambda(2b(x)u - 3u^2)w = 0 \quad \text{on } (-1, 1), \quad w(-1) = w(1) = 0. \quad (3.9)$$

It was shown in [5] that at a turning point we can take  $w(x) > 0$  on  $(-1, 1)$ . It follows that the denominator in (3.8) is positive. We need to show that the numerator in (3.8) is negative. Since both  $u(x)$  and  $w(x)$  are even functions, it suffices to prove that

$$\int_0^1 f_{uu} w^3 dx < 0. \quad (3.10)$$

The proof of (3.10) will be accomplished in four steps.

Step 1. We claim that

$$\int_0^1 f_{uu} w u_x^2 dx < 0. \quad (3.11)$$

Differentiating the equation (3.1) twice (we have denoted  $f(x, u) = u^2(b(x) - u)$ )

$$u_x'' + \lambda f_u u_x + \lambda f_x = 0, \quad (3.12)$$

$$u_{xx}'' + \lambda f_u u_{xx} + \lambda f_{uu} u_x^2 + 2\lambda f_{ux} u_x + \lambda f_{xx} = 0. \quad (3.13)$$

We now multiply the equation (3.13) by  $w$ , and subtract from it the equation (3.9) multiplied by  $u_{xx}$ . The result is then integrated over  $(0, 1)$ ,

$$w u_{xx}'|_0^1 - u_{xx} w'|_0^1 + \lambda \int_0^1 f_{uu} u_x^2 w dx + \lambda \int_0^1 (2f_{ux} u_x + f_{xx}) w dx = 0. \quad (3.14)$$

Notice that  $u'''(0) = 0$ , since  $u'''(x)$  is odd, and  $u''(1) = -\lambda f(1, u(1)) = -\lambda f(1, 0) = 0$ . Hence all the boundary terms in (3.15) vanish, and then

$$\int_0^1 f_{uu} u_x^2 w dx + \int_0^1 (2f_{ux} u_x + f_{xx}) w dx = 0. \quad (3.15)$$

Denote by  $I$  the second integral in (3.15),  $I = \int_0^1 (2f_{ux} u_x + f_{xx}) w dx$ . To prove (3.11) we need to show that  $I > 0$ . Computing  $f_{ux} = 2b'u$ , and  $f_{xx} = b''u^2$ , we rewrite

$$I = \int_0^1 (4b'uu_x + b''u^2) w dx. \quad (3.16)$$

Using that  $b'(0) = 0$ , since  $b'(x)$  is odd, we express

$$\int_0^1 b''u^2 w dx = - \int_0^1 2b'uu_x w dx - \int_0^1 b'u^2 w' dx.$$

Using this in (3.16),

$$I = \int_0^1 b'u(2u'w - uw') dx \equiv \int_0^1 b'uJ dx, \quad (3.17)$$

where

$$J = 2u'w - uw'.$$

It suffices to show that  $J < 0$  for all  $x \in (0, 1)$ . Differentiate  $J$ , and use (3.1) and (3.9) to express the second derivatives,

$$\begin{aligned} J' &= 2u''w - uw'' + u'w' \\ &= \lambda(-2bu^2w + 2u^3w + 2bu^2w - 3u^3w) + u'w' \\ &= u'w' - \lambda u^3w. \end{aligned}$$

Similarly,

$$\begin{aligned} J'' &= u''w' + u'w'' - 3\lambda u^2u'w - \lambda u^3w' \\ &= \lambda(-bu^2w' + u^3w' - 2buu'w + 3u^2u'w - 3u^2u'w - u^3w') \end{aligned}$$

$$\begin{aligned}
&= \lambda bu(-uw' - 2u'w) = \lambda bu(2u'w - uw' - 4u'w) \\
&= \lambda buJ - 4\lambda buu'w > \lambda buJ.
\end{aligned}$$

Hence

$$J'' - \lambda buJ > 0, \quad (3.18)$$

and also

$$J(0) = J(1) = 0. \quad (3.19)$$

Using the maximum principle, we see from (3.18) and (3.19) that  $J < 0$  over  $(0, 1)$ , i.e.  $I > 0$ , and the claim (3.11) follows.

Step 2. We show that  $f_{uu}(x, u)$  changes sign exactly once on  $(0, 1)$ . We denote

$$\frac{1}{6}f_{uu} = \frac{1}{3}b(x) - u(x) \equiv p(x).$$

Compute

$$p''' = \frac{1}{3}b''' - (u'')' = \frac{1}{3}b''' + \lambda(2bu - 3u^2)u' + \lambda b'u^2. \quad (3.20)$$

Clearly  $p(1) = \frac{1}{3}b(1) > 0$ . We claim next that the inequality  $b(x) > 2u(x)$  cannot hold for all  $x \in [0, 1]$ , which will imply that  $p(x)$  cannot be non-negative on  $[0, 1]$ . Indeed, assuming otherwise, we would have  $2b(x)u - 3u^2 > b(x)u - u^2$  for all  $x \in (-1, 1)$ . Then writing our equation (3.1) in the form ( $u > 0$ )

$$u'' + \lambda(b(x)u - u^2)u = 0 \text{ on } (-1, 1), \quad u(-1) = u(1) = 0,$$

and comparing it to (3.9), we obtain a contradiction. Hence  $p(x)$  is negative somewhere in  $(0, 1)$ . We claim that

$$p(0) < 0. \quad (3.21)$$

Indeed, assuming otherwise we would have for all  $x \in (0, 1)$ , in view of (3.6),

$$u(x) < u(0) \leq \frac{1}{3}b(0) \leq \frac{1}{2}b(1) < \frac{1}{2}b(x),$$

which is impossible by the preceding argument.

Since  $p(x)$  changes sign on  $(0, 1)$ , it has at least one zero. Assume, contrary to what we want to prove, that  $p(x)$  changes sign more than once on  $(0, 1)$ . Then it has at least three zeros on  $(0, 1)$ , and we denote by  $x_0$  the smallest one. We have

$$0 = 3p(x_0) = b(x_0) - 3u(x_0) < 2b(x) - 3u(x) \quad (3.22)$$

for all  $x \in (x_0, 1)$  by the assumption (3.6) (the maximum of  $b(x)$  is smaller than the minimum of  $2b(x)$ , so that once  $3u(x)$  gets below  $b(x)$ , it will stay below  $2b(x)$ ). Using (3.20) and (3.22), we conclude that

$$p'''(x) < 0 \quad \text{for all } x \in (x_0, 1), \quad (3.23)$$

or  $(p'')' < 0$  so that  $p''(x)$  is decreasing and has at most one zero on  $(x_0, 1)$ , i.e.  $p(x)$  has at most one inflexion point on  $(x_0, 1)$ . Also,  $p''(1) = \frac{1}{3}b''(1) < 0$ . If  $p''(x) < 0$  on  $(x_0, 1)$ , then it is clearly impossible for the concave  $p(x)$  to have three roots on  $[x_0, 1)$ , a contradiction. If  $p''(x_0) > 0$ , then on  $(x_0, 1]$  the function  $p(x)$  starts out to be convex, has two roots inside  $(x_0, 1)$ , and ends up concave at  $x = 1$ , and yet it has only one inflexion point, which is again a contradiction.

*Step 3.* We show that  $-w(x)$  and  $u_x$  have exactly one intersection point on  $(0, 1)$ . We know that  $u_x(0) = 0$  and  $(u_x)_x < 0$ , i.e.  $u_x$  is decreasing on  $(0, 1)$ . Assume there is more than one intersection point. Then  $-w(x)$  cannot be increasing between two intersection points. Let  $0 < x_1 < x_2 < 1$  be the largest two points of intersection. Then we can assume that  $-w(x) < u_x$  on  $(x_1, x_2)$ , because the other case, when  $-w(x)$  and  $u_x$  touch at  $x_2$  ( $-w(x_2) = u_x(x_2)$  and  $-w'(x_2) = u'_x(x_2)$ ) and  $-w(x) > u_x$  on  $(x_1, x_2)$ , will be excluded in the subsequent discussion. We can find a constant  $0 < \gamma < 1$  such that  $-\gamma w(x) \geq u_x$  on  $(x_1, x_2)$  and  $-\gamma w$  touches  $u_x$  at some  $\bar{x} \in (x_1, x_2)$  (i.e.  $-\gamma w(\bar{x}) = u_x(\bar{x})$  and  $-\gamma w'(\bar{x}) = u'_x(\bar{x})$ ). The function  $-\gamma w$  is a solution of (3.9). From (3.12) with  $f_x < 0$ , we see that  $u_x$  is a subsolution of the same equation. We have a subsolution touching a solution from below, which is impossible by Lemma 1.1.

*Step 4.* Let  $x_0$  be the unique point in  $(0, 1)$  where  $f_{xx}(x_0, u) = 0$ . By scaling  $w(x)$  we can arrange for  $-w$  and  $u_x$  to have their unique intersection point also at  $x_0$ . We have  $f_{uu} > 0$  and  $u_x^2 > w^2$  on  $(x_0, 1)$  and the reverse inequalities on  $(0, x_0)$ . Using (3.11), we then have

$$\int_0^1 f_{uu} w^3 dx < \int_0^1 f_{uu} u_x^2 w dx < 0.$$

This concludes the proof of (3.10), and of the theorem.  $\square$

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(Issued 17 June 1996)