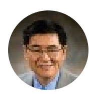
On Homoclinic and Heteroclinic Orbits for Hamiltonian Systems

ARTICLE *in* DIFFERENTIAL AND INTEGRAL EQUATIONS · MAY 2001

Impact Factor: 0.54 · Source: CiteSeer

3 AUTHORS, INCLUDING:

Philip Korman
University of Cincinnati



Yi Li
Wright State University

On Homoclinic and Heteroclinic Orbits for Hamiltonian Systems

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Alan C. Lazer
Department of Mathematics and Computer Science
University of Miami
Coral Gables, FL 33124
and

Yi Li*
Department of Mathematics
University of Rochester
Rochester, NY 14627

Key words: homoclinic solutions, mountain-pass lemma.

AMS subject classification: 34B15, 34A34.

Abstract

We extend some earlier results on existence of homoclinic solutions for a class of Hamiltonian systems. We also study heteroclinic solutions. We use variational approach.

1 Introduction

Recently variational techniques have been used in a number of papers to obtain existence of homoclinic and heteroclinic orbits of the Hamiltonian

*Supported in part by the National Science Foundation.

systems

$$u'' - L(t)u + V_u(t, u) = 0, \tag{1.1}$$

see e.g. A. Ambrosetti and M.L. Bertotti [1], P.H. Rabinowitz [7], W. Omana and M. Willem [5], and P. Korman and A.C. Lazer [3]. Here $L(t)$ is a given positive definite $n \times n$ matrix, the potential $V(t, u)$ is assumed to be superquadratic in u, and the solution is sought in the class $H^1(R, R^n)$, which implies that it is homoclinic at zero, i.e. $\lim_{t\to\pm\infty} u(t) = 0$. The approach used in [1], [5] and [3], was to restrict the problem (1.1) to a bounded interval $(-T, T)$ with Dirichlet boundary conditions $u(-T) = u(T) = 0$, show existence of solutions using the mountain-pass lemma, and then let $T \to \infty$. The crucial observation made in [1], and independently in [3], is that in addition to existence of solutions, the mountain-pass lemma allows one to obtain uniform in T estimate of H^1 norm of the solution. It is then straightforward, via the usual diagonal process, to show existence of a homoclinic solution of (1.1). The problem is to show that this solution is nontrivial. P.H. Rabinowitz and K. Tanaka proved existence of solution under condition that the smallest eigenvalue of $L(t)$ tends to ∞ as $|t| \to \infty$, see [8], and also [5], where an alternative proof is given. The above condition does not seem to be natural, and in fact in [3], P. Korman and A.C. Lazer showed that it can be dropped if $L(t)$ and $V(t, u)$ are even functions in t. In the present paper we prove a similar result for a broad class of problems without assuming evenness. In case of one equation, we prove sharper results, and moreover obtain positive homoclinics.

In Section 3 we use a similar approach and elementary techniques to show existence and uniqueness of an odd heteroclinic solution for a class of equations.

2 Positive homoclinics for a class of equations

In this section we shall prove existence of positive homoclinics for a model equation with a polynomial nonlinearity. Namely, we are looking for a positive solution of

$$u'' - a(x)u + b(x)u^p = 0, \quad -\infty < x < \infty, \quad 1 < p < \infty, \tag{2.1}$$

$$u(-\infty) = u'(-\infty) = u(\infty) = u'(\infty) = 0. \tag{2.2}$$

We assume that the functions $a(x), b(x) \in C^1(-\infty, \infty)$ are strictly positive on $(-\infty, \infty)$, i.e. $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$.

As in [3], we shall obtain solution (2.1-2.2) as the limit when $T \to \infty$ of the solutions of

$$(2.3) \quad u'' - a(x)u + b(x)u^p = 0 \ \text{ for } \ x \in (-T,T), \quad u(-T) = u(T) = 0.$$

Let u_T denote solution of (2.3). To show that a subsequence of $\{u_T\}$ converges to a positive solution of (2.1-2.2) as $T \to \infty$, we need to exclude the possibility of this subsequence converging to zero. Let x_0 be the point of global maximum of $u(x)$. From (2.3), since $u''(x_0) \le 0$, it follows that

$$u(x_0) \ge \left(\frac{a(x_0)}{b(x_0)}\right)^{\frac{1}{p-1}}, \tag{2.4}$$

and hence if we can show that x_0 stays in a bounded interval as $T \to \infty$, it will exclude the possibility of $\{u_{T_k}\} \to 0$. We shall give two sets of conditions, which constrain x_0 to a bounded interval. But first we recall the existence result from [3]. Since we intend to send T to infinity, we shall restrict to $T \ge 1$ in (2.3).

Lemma 2.1 *[3] The problem (2.3) has under our conditions a positive solution for any $T \ge 1$, which is obtained by a variational technique. Moreover, for this (variational) solution we have an estimate*

$$\int_{-T}^{T} (u'^2(x) + a(x)u^2)dx \le c \ \textit{ uniformly in } \ T \ge 1. \tag{2.5}$$

We recall that in the process of proving this lemma it was shown that

$$c_T = \int_{-T}^{T} \left[\frac{u_T'^2}{2} + a(x)\frac{u_T^2}{2} - b(x)\frac{u_T^{p+1}}{p+1} \right] dx$$

is non-increasing in T, which implies that $c_T \le c_1$ for all $T > 1$. Multiplying the equation (2.3) by u and integrating, we easily express

$$\int_{-T}^{T} \left(\frac{u_T'^2}{2} + a(x)\frac{u_T^2}{2} \right) dx = \frac{(p+1)}{p-1} c_T. \tag{2.6}$$

Lemma 2.2 *Assume that*

$$xa'(x) \ge 0 \ \textit{ and } \ xb'(x) \le 0 \ \textit{ for all } x. \tag{2.7}$$

Let $u(x)$ be a positive solution of (2.1-2.2), x_0 its point of maximum. Assume that the following two conditions hold

$$\lim_{x\to\pm\infty} \frac{\left(\sqrt{a(0)}+\sqrt{a(x)}\right)}{2}\left[\frac{(p+1)a(x)}{b(x)}\right]^{\frac{2}{p-1}} > \lim_{T\to\infty}\frac{(p+1)}{p-1}c_T. \tag{2.8}$$

Then x_0 belongs to a bounded interval uniformly in $T>1$.

Proof. We recall that it was proved in Korman-Ouyang [4] that $u(x)$ has only one point of local maximum, which is the point of global maximum, which we denote by x_0, and we assume without loss of generality that $x_0 \geq 0$. Multiplying the equation (2.1) by u' and integrating over (x_0, T) gives (using that $a(x)$ and $-b(x)$ take their minimum at x_0)

$$u(x_0) \geq \left[\frac{(p+1)a(x_0)}{2b(x_0)}\right]^{\frac{1}{p-1}} \tag{2.9}$$

(which is stronger than the estimate (2.4) obtained by maximum principle). For any $T>1$ we have by (2.6)

$$\int_{-T}^{T}\sqrt{a(x)}|uu'|dx \leq \int_{-T}^{T}\left(\frac{1}{2}au^2+\frac{1}{2}u'^2\right)dx = \frac{(p+1)}{p-1}c_T. \tag{2.10}$$

On the other hand, using (2.9),

$$\begin{aligned}
\frac{(p+1)}{p-1}c_T &> \int_{-T}^{T}\sqrt{a(x)}|uu'|\,dx \\
&= \int_{-T}^{x_0}\sqrt{a(x)}\left(\frac{u^2}{2}\right)'dx - \int_{x_0}^{T}\sqrt{a(x)}\left(\frac{u^2}{2}\right)'dx \\
&\geq \sqrt{a(0)}\frac{u^2(x_0)}{2} + \sqrt{a(x_0)}\frac{u^2(x_0)}{2} \\
&\geq \frac{(\sqrt{a(0)}+\sqrt{a(x_0)}))}{2}\left[\frac{(p+1)a(x_0)}{2b(x_0)}\right]^{\frac{2}{p-1}}.
\end{aligned}$$

By (2.8) it then follows that x_0 belongs to a bounded interval.

Remark 1 Condition (2.8) is satisfied if, for example, $\lim_{|x|\to\infty} a(x) = \infty$ and $b(x)$ is bounded.

Remark 2 Instead of (2.7) we could allow a more general condition: $(x-c)a'(x) \geq 0$ and $(x-c)b'(x) \leq 0$ for some $c \in R$ and all x.

A similar result can be given without any symmetry assumptions on $a(x)$ and $b(x)$. Recall that the total variation of the function $f(x)$ on $[a,b]$ is $\int_a^b |f'(x)|\,dx$.

Lemma 2.3 *Assume that*

$$\liminf_{|x|\to\infty} \sqrt{a_0}\left(\frac{a(x)}{b(x)}\right)^{\frac{1}{p-1}} > \lim_{T\to\infty}\frac{(p+1)}{p-1}c_T. \tag{2.11}$$

Then x_0 belongs to a bounded interval.

Proof. Proceeding as in the proof of the previous lemma, we have, using (2.4),

$$\begin{aligned}
\frac{(p+1)}{p-1}c_T &> \int_{-T}^{T}\sqrt{a(x)}\left|\left(\frac{u^2}{2}\right)'\right|\,dx \\
&\geq \min_{[-T,x_0]}\sqrt{a(x)}\int_{-T}^{x_0}\left|\left(\frac{u^2}{2}\right)'\right|\,dx \\
&\quad + \min_{[x_0,T]}\sqrt{a(x)}\int_{x_0}^{T}\left|\left(\frac{u^2}{2}\right)'\right|\,dx \\
&\geq \sqrt{a_0}u^2(x_0) \geq \sqrt{a_0}\left(\frac{a(x_0)}{b(x_0)}\right)^{\frac{1}{p-1}}.
\end{aligned}$$

In view of (2.11) the lemma follows.

Theorem 2.1 *Assume that $a(x)$ and $b(x)$ satisfy either conditions of lemma 2.2 or of lemma 2.3. Then the problem (2.1-2.2) has a positive solution.*

Proof. Take a sequence $\{T_n\}\to\infty$, and denote by u_n the corresponding positive variational solution of the problem (2.3), which exists by lemma 2.1. Using the estimate (2.5), which implies a uniform bound in H^1, one shows exactly in the same way as in [3] that a subsequence of $\{u_n(x)\}$ converges uniformly on bounded intervals to a function $u(x)\in C^2(-\infty,\infty)$, which is a solution of the equation (2.1) for all $x\in(-\infty,\infty)$. Clearly, $u(x)\geq 0$ for all x.

We claim that

$$u(x) > 0 \text{ for all } x\in(-\infty,\infty). \tag{2.12}$$

Indeed, denoting x_{0n} the point of maximum of $u_n(x)$ we have by lemmas 2.2 and 2.3 that $\{x_{0n}\}$ belong to a bounded interval, call it I. Along a subsequence $x_{on_k} \to y \in I$ and by (2.4) $u(y) \geq \min_I \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{p-1}} > 0$. Since $u(x)$ is nonnegative and nontrivial, it is positive by the maximum principle.

The rest of the proof is exactly the same as in [3].

Example. Consider (a is a constant)

$$u'' - a^2 u + 2u^3 = 0, \quad -\infty < x < \infty, \quad u(\pm\infty) = u'(\pm\infty) = 0. \tag{2.13}$$

Multiplying (2.13) by u' and integrating, we obtain a homoclinic solution $u(x) = \frac{a}{\cosh ax}$. In fact, there is an infinite family of homoclinics $u(x) = \frac{a}{\cosh a(x-\gamma)}$ for any constant γ.

3 Odd heteroclinic solutions

We begin with a simple problem

$$u'' + u - u^3 = 0 \ \text{ for } \ x \in (-\infty, \infty), \quad u(\pm\infty) = \pm 1, u'(\pm\infty) = 0. \tag{3.1}$$

Multiplying (3.1) by u' and integrating, we easily compute an odd heteroclinic solution $u = \tanh \frac{x}{\sqrt{2}}$.

Our goal is to obtain a similar result for the problem

$$\begin{gathered} u'' + a(x)(u - |u|^{p-1}u) = 0 \ \text{ for } \ x \in (-\infty, \infty), \\ u(\pm\infty) = \pm 1, u'(\pm\infty) = 0. \end{gathered} \tag{3.2}$$

We assume that $p > 1$ is a real number and the function $a(x)$ is even of class $C^1(-\infty, \infty)$, with

$$a'(x) < 0 \ \text{ for almost all } \ x > 0, \tag{3.3}$$

$$a(\infty) > 0. \tag{3.4}$$

We shall obtain the solution of (3.1) as a limit when $T \to \infty$ of solutions of

$$u'' + a(x)(u - |u|^{p-1}u) = 0 \ \text{ for } \ x \in (-T, T), \quad u(\pm T) = \pm 1. \tag{3.5}$$

Solution of (3.5) will in turn depend on the problem

$$u'' + a(x)(u - |u|^{p-1}u) = 0 \ \text{ on } \ (0, T), \quad u(0) = 0, u(T) = 1. \tag{3.6}$$

Lemma 3.1 *The problem (3.6) has for each $T > 0$ a unique positive solution, which is an increasing function.*

Proof. The function $u \equiv 1$ is a supersolution of (3.6), while $u = \alpha x$ is a subsolution, when the constant α is sufficiently small. It follows that (3.6) has a positive solution ($0 < u < 1$ on $(0, T)$). By the maximum principle any solution of (3.6) satisfies $0 < u < 1$ on $(0, T)$. Turning to the uniqueness, recall that the method of super-subsolutions implies existence of a maximal solution $u(x)$, i.e. $u(x) \geq v(x)$ for all $x \in (0, T)$, if $v(x)$ is any other solution of (3.6). Multiplying (3.6) by v, and the same equation for v by u, subtracting and integrating,

$$\int_0^T a(x)uv(v^{p-1} - u^{p-1})\,dx + u'(T) - v'(T) = 0,$$

which implies that $v \equiv u$.

Finally, assume that $u(x)$ is not monotone. Then it has a point $\overline{x}$ of local minimum on $(0, T)$, at which $u''(\overline{x}) \geq 0$ and $u(\overline{x}) - u^p(\overline{x}) > 0$, which implies a contradiction in (3.6).

Lemma 3.2 *The problem (3.5) has under our conditions a unique solution, which is an odd and increasing function.*

Proof. Let $u(x)$ be the solution of (3.6) for $x \in [0, T]$, obtained in the previous lemma. We extend it to $[-T, 0]$ as $-u(-x)$. The resulting function is an odd and increasing solution of (3.5). Uniqueness follows as above (-1 and $+1$ are respectively sub- and supersolution).

Theorem 3.1 *The problem (3.2) has, under the conditions (3.3) and (3.4), a unique solution, which is an odd and strictly increasing function.*

Proof. Take a sequence $T_n \to \infty$, and consider the problem (3.5) on the interval $(-T_n, T_n)$, i.e. consider

$$\begin{aligned} u'' + a(x)(u - |u|^{p-1}u) = 0 \ \text{ on } \ (-T_n, T_n), \\ u(-T_n) = -1, u(T_n) = 1. \end{aligned} \tag{3.7}$$

By lemma 3.2 the problem (3.11) has a unique solution $u_n(x)$. Since $|u_n(x)| < 1$, we conclude that

$$|u_n''(x)| \leq c \ \text{ for all } \ x \in (-T_n, T_n) \ \text{ uniformly in } \ n. \tag{3.8}$$

Since $u_n(x)$ is monotone the estimate (3.12) implies

$$|u_n'(x)| \le c \text{ for all } x \in (-T_n, T_n) \text{ uniformly in } n. \tag{3.9}$$

(If $u_n'(x)$ were to become large at some x, then by (3.8) $u_n'(x)$ would stay large over a long interval, which would contradict the total variation of $u_n(x)$ being equal to 2).

Arguing as in [3], we see via the usual diagonal process that there is a function $u(x) \in C^2(-\infty, \infty)$ such that along a subsequence we have for all $x \in (-\infty, \infty)$

$$u_{n_k}(x) \to u(x) \text{ and } u_{n_k}'(x) \to u'(x) \quad \text{uniformly on bounded intervals,} \tag{3.10}$$

and that $u(x)$ is a solution of (3.2).

We claim that there is a constant $c_0 > 0$ such that

$$u_n'(0) \ge c_0 \text{ uniformly in } n. \tag{3.11}$$

Indeed, introducing the "energy" function for $x \ge 0$ (where $u_n(x) \ge 0$)

$$E(x) = \frac{1}{2}u_n'^2 + a(x)\left(\frac{u_n^2}{2} - \frac{u_n^{p+1}}{p+1}\right),$$

we compute using (3.5)

$$E'(x) = a'(x)\left(\frac{u_n^2}{2} - \frac{u_n^{p+1}}{p+1}\right) < 0.$$

Therefore

$$E(0) = \frac{1}{2}u_n'^2(0) > E(T_n) > a(\infty)\frac{p-1}{2(p+1)},$$

and (3.11) follows. It follows that $u(x) \not\equiv 0$.

By (3.10) $u'(x) \ge 0$. Since also $-1 \le u(x) \le 1$, it follows that $\lim_{x\to\pm\infty} u(x)$ exist, and the only possibility in view of (3.4) is that $\lim_{x\to\pm\infty} u(x) = \pm 1$ (since $u''(x)$ must be small for x large). Since $u(x)$ is nondecreasing it follows that $\lim_{x\to\pm\infty} u'(x) = 0$. Notice that $u(x)$ is, in fact, strictly increasing, since otherwise we would have $u'(x_0) = 0$ at some $x_0 > 0$, and then integrating the equation (3.2) over (x_0, ∞), we would get a contradiction.

Turning to the uniqueness, let $v(x)$ be another solution of (3.2). We consider four possible cases.

(i) $u(x)$ and $v(x)$ intersect at least twice on $[0,\infty)$. I.e. we can find $0 \le x_1 < x_2 < \infty$, such that $u(x_1) = v(x_1) \equiv u_1$, $u(x_2) = v(x_2) \equiv u_2$ and say $u(x) > v(x)$ on (x_1, x_2). As in lemma 3.1 we obtain

$$u_2(u'(x_2) - v'(x_2)) - u_1(u'(x_1) - v'(x_1)) + \int_{x_1}^{x_2} auv(v^{p-1} - u^{p-1})\, dx = 0,$$

which is impossible, since $u'(x_1) > v'(x_1)$ and $u'(x_2) < v'(x_2)$.

(ii) $u(x)$ and $v(x)$ intersect exactly once on $[0,\infty)$, say at $x_1 \ge 0$. Integrating over (x_1, R) and letting $R \to \infty$, we obtain the same contradiction.

(iii) $u(x)$ and $v(x)$ have only negative points of intersection. By considering $-u(-x)$ and $-v(-x)$ (which are also solutions of (3.2)) we reduce this case to one of the previous cases.

(iv) $u(x)$ and $v(x)$ never intersect. Integrating over $(-R, R)$ and letting $R \to \infty$, we again obtain a contradiction.

Clearly, we have also proved the following theorem.

Theorem 3.2 *Consider the problem*

$$\begin{aligned} u'' + a(x)(u - u^p) = 0 \quad &\text{for } x \in (0,\infty) \\ u(0) = 0, \quad u(\infty) = 1, \quad &u'(\infty) = 0, \end{aligned} \tag{3.12}$$

with $a(x) \in C^1[0,\infty)$ satisfying the conditions (3.3) and (3.4), and p is a real number with $p \ge 1$. Then the problem (3.12) has a unique positive solution, which is a strictly increasing function.

4 Homoclinic solutions for a class of Hamiltonian systems

We are looking for nontrivial solutions $u(t) \in H^1(R, R^n)$ of the system

$$u'' - L(t)u + V_u(t,u) = 0 \qquad -\infty < t < \infty, \tag{4.1}$$

$$u(\pm\infty) = u'(\pm\infty) = 0. \tag{4.2}$$

Here V_u is the gradient of V with respect to u variables. We assume that

$$\begin{aligned} L(t) \quad = \quad &[\ell_{ij}(t)] \;\text{ is a positive definite matrix of class} \\ &C^1(R), \;\text{ and there is } \alpha(t) \in C(R,R) \text{ such that} \\ &\alpha(t) \ge \alpha_0 > 0 \;\text{ for all } t \in R \text{ and } (L(t)u,u) \ge \alpha(t)|u|^2; \end{aligned} \tag{4.3}$$

(4.4) $\quad V(t,u) \in C^1(R \times R^n, R)$, and for some constant $\gamma > 2$

$$0 < \gamma V(t,\xi) \le (V_\xi(t,\xi),\xi) \ \text{ for all } \xi \in R^n \backslash \{0\} \text{ and } t \in R.$$

As in Section 2 we approximate (4.1-4.2) by the problem (with say $T > 1$)

(4.5) $\; u'' - L(t)u + V_u(t,u) = 0 \;$ for $\; t \in (-T,T), \;\; u(-T) = u(T) = 0.$

We recall that under our conditions the problem (4.5) has a nontrivial solution $u = u_T$, which is a critical point of the functional

$$J(u) = \int_{-T}^{T} \left[\frac{1}{2}|u'|^2 + \frac{1}{2}(L(t)u,u) - V(t,u) \right] dt,$$

and that $c_T \equiv J(u_T)$ is non-increasing in T, see [3]. Let t_0 denote (any) point of global maximum of $|u_T|$. Similarly to the scalar case, we wish to constrain t_0 to a bounded region. To this end we assume existence of a function $\beta : R \to R$ and a constant $t_1 > 0$, such that for $|t| > t_1$,

(4.6) $\quad (L(t)u,u) > (V_u(t,u),u)$ provided that $|u|^2 \le \beta(t)$.

Remark 3 It was shown in [3] that under the condition (4.4) the function $V(t,u)$ is superquadratic in u near the origin. While the condition (4.6) does not seem to follow from (4.4), it is clear that it is not a very restrictive condition.

Theorem 4.1 *For the problem (4.1-4.2) assume that conditions (4.3), (4.4) and (4.6) hold, and in addition assume that*

(4.7) $$\liminf_{t\to\infty} \alpha_0 \beta(t) > \lim_{T\to\infty} \frac{2\gamma}{\gamma-2} c_T.$$

Then the problem (4.1-4.2) has a nontrivial solution.

(Keep in mind that c_T is decreasing in T. So that (4.7) will follow, if for example, $\liminf_{t\to\infty} \alpha_0\beta(t) > \frac{2\gamma}{\gamma-2}c_1$*.)*

Proof. As in the previous section (and as in [3]) we approximate our problem by (4.5) and let $T_k \to \infty$. In [3] it was shown that H^1 norm of solutions u_{T_k} is bounded uniformly in k. As before this allows us to conclude that a subsequence of $\{u_{T_k}\}$ converges uniformly on bounded intervals to a function $u(x) \in C^2(R,R^n)$, which is a solution of (4.1). It remains to show that $u(t)$ is nontrivial (that $u(t)$ satisfies (4.2) follows exactly as in [3]).

Define $q(t) = |u(t)|^2$. Compute

$$q''(t) = 2|u'|^2 + 2u \cdot u''. \tag{4.8}$$

It t_0 is the point of maximum of $q(t)$, then $q''(t_0) \leq 0$, and it follows from (4.8) that

$$u(t_0) \cdot u''(t_0) \leq 0. \tag{4.9}$$

We may assume that $|t_0| > t_1$, since otherwise t_0 already belongs to a bounded interval.

Multiplying the $i-$th equation in (4.1) by u_i and summing, we obtain in view of (4.9)

$$-(L(t_0)u(t_0), u(t_0)) + (V_u(t_0, u(t_0)), u(t_0)) \geq 0.$$

Comparing this with (4.6) we conclude

$$|u(t_0)|^2 > \beta(t_0). \tag{4.10}$$

We recall that it was shown in [3] that

$$\int_{-T}^{T} \left[\frac{1}{2}|u_T'|^2 + \frac{1}{2}(L(t)u_T, u_T) \right] dt \leq \frac{2\gamma}{\gamma - 2} c_T \leq \frac{2\gamma}{\gamma - 2} c_1, \tag{4.11}$$

where as before $c_T = J(u_T)$.

On the other hand, proceeding as in lemma 2.3, and using (4.10) and (4.11),

$$\begin{aligned} \frac{2\gamma}{\gamma - 2} c_T \geq \int_{-T}^{T} \sum_{i=1}^{n} \sqrt{\alpha(t)} |u_i u_i'| \, dt \quad &\geq \quad \alpha_0 \int_{-T}^{T} \left| \frac{d}{dt} \frac{1}{2} |u|^2 \right| \, dt \\ &\geq \quad \alpha_0 |u(t_0)|^2 > \alpha_0 \beta(t_0). \end{aligned} \tag{4.12}$$

Condition (4.7) then implies that t_0 stays in a bounded interval as $T_k \to \infty$. As in theorem 2.1 we show existence of $\bar{t}$ such that

$$|u(\bar{t})|^2 > \liminf_{t \to \infty} \beta(t) > 0.$$

(For the second inequality use (4.7) and that $c_T > 0$, since c_T is the value of $J(u)$ at the mountain pass).

Hence $u(t)$ is a nontrivial solution of (4.1). As in [3] one sees that it also satisfies (4.2), completing the proof.

Remark 4 Condition (4.7) can be generalized to read

$$\liminf_{t\to\infty} \beta(t) \min_{(-t,t)} \alpha(s) > \lim_{T\to\infty} \frac{2\gamma}{\gamma-2} c_T.$$

Remark 5 If $|V_u(t,u)| < c_0 u^{1+\delta}$ for some constants $c_0, \delta > 0$ uniformly in $t \in R$, then

$$(L(t)u,u) \geq \alpha(t)|u|^2 \geq c_0|u|^{2+\delta} > (V_u(t,u),u),$$

provided $\alpha(t) \geq c_0|u|^\delta$. Hence we can take $\beta(t) = \left(\frac{\alpha(t)}{c_0}\right)^{2/\delta}$, and if we are given that $\lim_{|t|\to\infty} \alpha(t) = \infty$, then condition (4.7) holds and our theorem applies. This corollary appears to be roughly equivalent to the theorem of P.H. Rabinowitz and K. Tanaka (see [5, p. 1116]). Our result is considerably more general than this corollary.

Remark 6 Our numerical calculations for the problem

$$u'' - 2u + u^3 = 0 \text{ on } (-T,T), \quad u(-T) = u(T) = 0$$

suggest that $\lim_{T\to 0} c_T = \infty$, while $\lim_{T\to\infty} c_T > 0$.

5 A curious maximum principle for elliptic systems

Our argument in section 4 suggests a maximum principle for elliptic systems, which is quite unlike the classical one in [6] or its recent generalizations, see e.g. [2]. In particular we do not require the system to be of cooperative type.

Let Ω be a bounded domain in R^d. We consider the system of m weakly coupled equations with m unknown functions $u^k(x)$, $k = 1, \ldots, m$,

$$\sum_{i,j=1}^{d} a_{ij}(x) u^k_{ij} + \sum_{\ell=1}^{m} b_{k\ell}(x) u^\ell = f_k(x,u), \quad x \in \Omega, \quad k = 1, \ldots, m. \tag{5.1}$$

Here $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, and we assume that for some constant $\theta > 0$

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for all } x \in \Omega \text{ and } \xi \in R^d. \tag{5.2}$$

We denote $u = (u^1, \dots, u^m)$.

We do not impose any smoothness assumptions on $a_{ij}(x), b_{k\ell}(x)$ and $f_k(x,u)$, however we do assume that we have a classical solution of (5.1), i.e., $u^k \in C^2(\Omega)$. Let B be the $m \times m$ matrix, $B = [b_{k\ell}(x)]$.

Theorem 5.1 *Assume that $\frac{1}{2}\left(B + B^T\right)$ is negative semidefinite, i.e.,*

$$\sum_{k,\ell=1}^{m} b_{k\ell}(x) u^k u^\ell \le 0 \ \textit{ for all } \ u \in R^m \ \textit{ and } \ x \in \Omega. \tag{5.3}$$

We assume also

$$\sum_{k=1}^{m} f_k(x,u) u^k \ge 0 \ \textit{ for all } \ u \in R^m \ \textit{ and } \ x \in \Omega. \tag{5.4}$$

Assume finally that at each $x \in \Omega$ at least one of the above two inequalities is strict. Then $|u(x)|^2 = \sum_{k=1}^m u^{k^2}(x)$ has no points of maximum inside Ω.

Proof. Denote $q(x) = |u(x)|^2$ and let $x_0 \in \Omega$ be its point of maximum. Compute

$$q_{ij}(x) = 2\sum_{k=1}^{m} u_i^k u_j^k + 2\sum_{k=1}^{m} u^k u_{ij}^k. \tag{5.5}$$

Since $\displaystyle\sum_{i,j=1}^{d} a_{ij}(x_0) q_{ij}(x_0) \le 0$, and

$$\sum_{i,j=1}^{d} a_{ij}(x_0) u_i^k u_j^k \ge \theta |\nabla u^k|^2 \ge 0,$$

we conclude using (5.5)

$$\sum_{k=1}^{m} \sum_{i,j=1}^{d} a_{ij}(x_0) u^k(x_0) u_{ij}^k(x_0) \le 0. \tag{5.6}$$

We now multiply the $k-$th equation in (5.1) by u^k and sum. In view of (5.3), (5.4) and (5.6) we have a contradiction at $x = x_0$.

Corollary 1 *Assume that homogeneous Dirichlet conditions are imposed*

$$u^k(x) = 0 \ \textit{ for } \ x \in \partial\Omega, \ \ k = 1, \dots, m. \tag{5.7}$$

Then the trivial solution (if it exists) is the only possible solution of (5.1), (5.7).

Remark 7 If non-negative solutions of (5.1) are considered, i.e. $u^k(x) \geq 0$ for all $x \in \Omega$ and $k = 1, \dots, m$ then (5.4) will follow from the condition

$$f_k(x,u) \geq 0 \text{ for all } u \in R_+^m, \;\; k = 1, \dots, m, \text{ and } x \in \Omega.$$

Remark 8 For the corresponding parabolic system one can prove the same way that $|u(x,t)|^2$ can have points of maximum only on the parabolic boundary. In [6, p.194] there are references to some related results.

References

[1] A. Ambrosetti and M.L. Bertotti, Homoclinics for second order conservative systems, In Partial differential equations and related subjects, M. Miranda (Ed.), *Longman Scientific & Technical* (1992).

[2] D.G. de Figueiredo and E. Mitidieri, A maximum principle for an elliptic system and applications to semilinear problems, *SIAM J. Math. Anal.* **17**, 836-849 (1986).

[3] P. Korman and A.C. Lazer, Homoclinic orbits for a class of symmetric Hamiltonian systems, *EJDE* Volumes/1994/01-Korman.

[4] P. Korman and T. Ouyang, Solution curves for two classes of boundary-value problems, To appear in *Nonlinear Analysis TMA*.

[5] W. Omana and M. Willem, Homoclinic orbits for a class of Hamiltonian systems, *Differential and Integral Equations* **5**, 1115-1120 (1992).

[6] M.H. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Englewood Cliffs (1967).

[7] P.H. Rabinowitz, Some recent results on heteroclinic and other connecting orbits of Hamiltonian systems, In Progress in variational methods in Hamiltonian systems and elliptic equations, M. Girardi, M. Matzeu and F. Pacella (Eds.), *Longman Scientific & Technical* (1992).

[8] P.H. Rabinowitz and K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, *Math Z.* **206**, 473-499 (1991).