$$D_1 = I_{0,0,0} + 3 \, I_{2,2,0} - 3 \, I_{2,0,0} - I_{2;2,2} \ .$$

Démonstration : Analogue à celle de la proposition 3 compte tenu de la remarque 3 .

Exemple 5 : Si $K = [-1,1]^3$ $w \equiv 1$, on obtient :

$$\Pi_w(f) = \frac{1}{27} \, f(\pm 1, \pm 1, \pm 1) + \frac{4}{27} \, [f(\pm 1; \pm 1, 0) + f(\pm 1, 0, \pm 1)]$$
$$+ \frac{64}{27} \, f(0,0,0) \ ;$$

on retrouve la formule produit de SIMPSON cf.[11] .

Remarque 4 : Les formules de quadrature de la proposition 4 et 5 sont à nombre minimal de noeuds, donc à coefficients stricte-ment positifs cf.[6] .

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On the Existence and Uniqueness of Positive Steady States in the Volterra-Lotka Ecological Models with Diffusion

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Abstract

Using the theory of quasimonotone increasing systems developed by P. J. McKenna and W. Walter, we give a rather detailed ana-lysis of the steady state solutions for the Volterra-Lotka model of two cooperating species, and prove some new non-existence results for the competing species case. We indicate generalizations to the case of n > 2 species.

KEY WORDS: Volterra-Lotka ecological model, existence, uni-queness and stability of solutions.

1. INTRODUCTION.

We study interaction of two or more species in the Volterra-Lotka ecological model with diffusion. For two species the model is described as follows. Let $u(x,t)$, $v(x,t)$ be positive solutions of the system

$$u_t = \Delta u + u(a - bu + cv)$$

(1.1)

$$v_t = \Delta v + v(d + eu - fv) \qquad \text{in } \Omega,$$

with $u(x,t) = v(x,t) = 0$ on $\partial\Omega$.

Here $x \in \Omega$, Ω a smooth domain in R^n with boundary $\partial\Omega$. The functions $u(x,t)$, $v(x,t)$ describe populations of two species at position x and time t. Constant coefficients a and d represent

the (intrinsic) growth rates. Positive constants b and f describe self-limitation of each species. Constants c and e describe interactions between the species and can be of either sign. We will be interested in the case $c < 0$, $e < 0$, which describes competing populations, and $c > 0$, $e > 0$, which describes cooperating species (or symbiosis).

The first step in analysing (1.1) is usually to look for the steady state solutions, i.e., the positive solutions of

$$\Delta u + u(a-u+cv) = 0$$

(1.2)

$$\Delta v + v(d+eu-v) = 0 \quad \text{in } \Omega, \ u = v = 0 \text{ on } \partial\Omega .$$

Notice that we can always assume $b = f = 1$ (by stretching the variables u and v). Once existence of a positive steady state is established, one studies its region of attraction.

We study existence, uniqueness and stability of positive steady states, and derive various bounds for them. The case of competing species has attracted considerable recent attention. Existence of positive solutions was proved by A. Leung [6], assuming $c < \dfrac{a-\lambda_1}{d}$, $e < \dfrac{d-\lambda_1}{a}$, where λ_1 is the principal eigenvalue of $-\Delta$ on Ω (see also P. Korman-A. Leung [5]). Stability question for positive solution was studied in [8]. Uniqueness proved to be a hard problem, and only partial results were known, see A. Leung [6], C. Cosner-A. Lazer [3]. A major advance was made recently by P. J. McKenna-W. Walter [9], who transformed the problem to a quasimonotone increasing form (see Section 2), and then applied the general theory they developed for such systems. They show that quasimonotone increasing systems have properties similar to those of a single elliptic equation, in particular the Serrin's sweeping principle holds, which they use to prove uni-

queness under rather general conditions. They also improve the existence result (see Section 4).

We use the results and techniques of [9] to study the case of cooperating species. It turns out that a simple condition $ce < 1$ is necessary and sufficient for existence. We then derive rather tight two-sided bounds for the solution, which are used to give conditions for uniqueness and stability.

For competing species we concentrate on non-existence results. As in [9] we transform the problem to a quasimonotone increasing form, and then use the Serrin's sweeping principle to "sweep" one of the components of solution to zero. This gives us useful non-existence results, different from those obtained by integration by parts (see Section 4 and [2]).

In the last section we consider $n > 2$ species, and describe all systems which can be converted to a quasimonotone increasing form by involution of variables. Then, using results in Korman-Leung [5], we indicate how one can derive existence results for such systems. Such results should be useful in view of recent interest in large systems of biological interactions [4,10].

2. PRELIMINARY RESULTS.

Let Ω be a smooth domain in R^n. By λ_1 we denote the principal eigenvalue of $-\Delta$ on Ω, and by $\phi_1 > 0$ the corresponding eigenfunction, i.e.,

(2.1)
$$\Delta\phi_1 + \lambda_1\phi_1 = 0 \quad \text{in } \Omega, \ \phi_1 = 0 \text{ on } \partial\Omega .$$

Define u_a to be the positive solution of $(a(x)\epsilon C^\alpha)$

(2.2)
$$\Delta u + u(a(x)-u) = 0 \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega .$$

A. Leung proved in [6] that for $a(x) > \lambda_1$ the positive solution exists, is unique and increasing in a. For constant $a > d > \lambda_1$ we

shall denote $u^* = u_a$, $v^* = u_d$. In [9] McKenna and Walter proved that

(2.3)
$$u^* > \frac{a}{d} v^* .$$

They obtained the estimate from the following useful lemma (which is a generalization of the comparison lemma in [6]).

Lemma 2.1 [9].   If $\psi > 0$ and $\phi > 0$ are correspondingly sub- and supersolution of (2.2) then $\psi \leq u \leq \phi$.

It would be very interesting to have some pointwise estimate of u^* in terms of v^*, in particular several results of the present work could be improved (see the theorems 3.3 and 4.2). We conjecture that

(2.4)
$$u^* < \frac{a-\lambda_1}{d-\lambda_1} v^* .$$

We present here a weak version of (2.4).

Lemma 2.2.   Normalize ϕ_1 by $\int_\Omega \phi_1^2 = 1$. Then

(2.5)
$$\int_\Omega u^* \phi_1 < \max_\Omega \phi_1 \int_\Omega \phi_1 \frac{a-\lambda_1}{d-\lambda_1} \int_\Omega v^* \phi_1 .$$

Proof.   Multiply (2.1) by u, (2.2) by ϕ_1, integrate both equations over Ω, and subtract the results. After integration by parts we get

$$(a-\lambda_1) \int_\Omega u^* \phi_1 = \int_\Omega u^{*2} \phi_1 .$$

Since

$$\int_\Omega u^{*2} \phi_1 > (\int_\Omega u^* \phi_1)^2 (\int_\Omega \phi_1)^{-1} ,$$

we get (an interesting estimate by itself)

(2.6)
$$\int_\Omega u^* \phi_1 < (a-\lambda_1) \int_\Omega \phi_1 .$$

One easily checks that $\frac{d-\lambda_1}{\max_\Omega \phi_1} \phi_1$ is a subsolution of (2.2) with d in place of a.

By lemma 2.1,

(2.7)
$$v^* > \frac{d-\lambda_1}{\max_\Omega \phi_1} \phi_1 ,$$

and the proof follows.

In the one dimensional case one can give a pointwise bound of u^* in terms of v^*. Let $\Omega = (0,\pi)$, $\phi_1 = \sin x$. By [9], $u^* < a/a \sin x$. Then in view of (2.7),

(2.8)
$$u^* < \frac{a/a}{d-\lambda_1} v^* .$$

Recall that the system

(2.9)
$$\Delta u + f(x,u) = 0 \text{ in } \Omega , u = 0 \text{ on } \partial\Omega ,$$

where $u = (u_1,\ldots,u_m)$ and $f = (f_1,\ldots,f_m)$, is called quasimonotone increasing if $f_i(x,u)$ is increasing in u_j for all $j \neq i$. The following lemma is a special case of the theorem 4 in McKenna-Walter [9].

Lemma 2.3.   Let $w_\lambda (\alpha < \lambda \leq \beta)$ be a family of supersolutions of (2.9) such that

(2.10)
$$\Delta w_\lambda + f(x,w_\lambda) < 0 \text{ in } \Omega, w_\lambda = 0 \text{ on } \partial\Omega .$$

Assume that w_λ is continuous and increasing in λ, and $u \leq w_\beta$. Also assume that $\frac{\partial w_\lambda}{\partial n}$ changes continuously in λ on $\partial\Omega$, and w_λ does not satisfy (2.9) for any i. Then $u < \inf w_\lambda$.

Let v_λ be a family of subsolutions $(\alpha < \lambda < \beta)$, increasing in λ and satisfying (2.10) with the inequality reversed. Assume also

$u \geq v_\alpha$, v_λ does not satisfy (2.9) for any 1, and that $\frac{\partial v_\lambda}{\partial n}$ changes continuously in λ on $\partial\Omega$. Then $u \geq \sup v_\lambda$.

3. EXISTENCE, UNIQUENESS AND STABILITY FOR COOPERATING SPECIES.

After stretching the variables u and v the system can be put into the form (with $c > 0$, $e > 0$)

$$\Delta u + u(a-u+cv) = 0$$

(3.1)

$$\Delta v + v(d+eu-v) = 0 \quad \text{in } \Omega, \ u = v = 0 \text{ on } \partial\Omega .$$

Throughout this section we assume $a > d > \lambda_1$, and look for a positive solution $u > 0$, $v > 0$. Notice that $u \geq u^*$, $v \geq v^*$, see e.g. [6].

Theorem 3.1. For existence of a positive solution of (3.1) it is necessary and sufficient that $ce < 1$.

Proof. Sufficiency was proved in [5]. To prove necessity we assume that $ce \geq 1$ and consider a family of subsolutions $v_\lambda = (\lambda\phi_1, \lambda\gamma\phi_1)$, with any $\lambda > 0$ and $\gamma > 0$ to be specified. For v_λ to be a subsolution, we need

$$a-\lambda_1-\lambda\phi_1 + c\lambda\gamma\phi_1 > 0, \quad d-\lambda_1+e\lambda\phi_1 - \lambda\gamma\phi_1 > 0 ,$$

which can be achieved by taking e.g., $\gamma = e$. Also notice that $u \geq u^* \geq \lambda_0\phi_1$ for λ_0 sufficiently small, and the same is true for v. By Lemma 2.3, $u \geq \lambda\phi_1$, $v \geq \lambda e\phi_1$ for any $\lambda \geq \lambda_0$, which proves non-existence.

Theorem 3.2. Assume that $ec < 1$. Let $\alpha = \frac{1+c}{1-ec}$, $\beta = \frac{1+e}{1-ec}$. The problem (3.1) has a positive solution, and

(3.2) $\alpha v^* < u < \alpha u^*$, $\beta v^* < v < \beta u^*$.

Proof. (i) Estimates from below. In the spirit of [9] we derive the estimates

(3.3) $u \geq \alpha_n v^*$, $v \geq \beta_n v^*$,

with new α_n and β_n on each step. Clearly, relations (3.3) hold with $\alpha_0 = \frac{d}{a}$, $\beta_0 = 1$. Assuming (3.3) holds, we derive

$$0 = \Delta u + u(a-u+cv) \geq \Delta u + u(d-u+c\beta_n v^*) ,$$

i.e. u is supersolution of $\Delta z + z(d-z+c\beta_n v^*) = 0$ in Ω, $z = 0$ on $\partial\Omega$, while $(1+c\beta_n)v^*$ is a solution of the same problem, and hence by Lemma 2.1 it follows that

$$u \geq (1+c\beta_n)v^* .$$

Similarly,

$$0 = \Delta v + v(d+eu-v) \geq \Delta v + v(d+e\alpha_n v^*-v) ;$$

$$v \geq (1+e\alpha_n)v^* .$$

Hence we can take $\alpha_{n+1} = 1+c\beta_n$, $\beta_{n+1} = 1 + e\alpha_n$. It is not hard to check that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ have finite limits, call them α and β respectively. Then $\alpha = 1+c\beta$, $\beta = 1+e\alpha$ and the proof follows.

(ii) Estimates from above. We introduce a family of supersolutions $(u_\lambda, v_\lambda) = (\alpha\lambda u^*, \beta\lambda u^*)$, $\lambda > 1$, none of which satisfies either equation in (3.1). Indeed, the inequalities

(3.3) $\Delta u_\lambda + u_\lambda(a-u_\lambda+cv_\lambda) < 0$, $\Delta v_\lambda + v_\lambda(d+eu_\lambda-v_\lambda) < 0$,

are equivalent to checking that

$$1 - \alpha\lambda + c\beta\lambda < 0 , \quad d-a + (e\alpha\lambda-\beta\lambda+1)u^* < 0 .$$

But $-\alpha+c\beta = -1$, $e\alpha-\beta = -1$, and (3.3) follows. Clearly $u < \alpha\lambda u^*$, $v < \beta\lambda u^*$ for sufficiently large λ. Letting $\lambda \to 1^+$, and applying Lemma 2.3, we establish the estimates.

Remark 1.   Notice that α_n and β_n in general are not improved at every step. However, the estimates (3.2) are rather tight, as can be seen by letting a = d. Then we conclude that $(\alpha u^*, \beta u^*)$ is the unique solutions of (3.1).

Remark 2.   It is easy to see that $(\alpha u^*, \beta u^*)$ and $(\alpha v^*, \beta v^*)$ are respectively super- and subsolutions of (3.1).

Next we define $\delta = \inf\limits_{\Omega} \frac{v^*}{u^*}$. Notice that $\delta < d/a < 1$. (Our conjecture (2.4) would imply $\delta > \frac{d-\lambda_1}{a-\lambda_1}$.)

Theorem 3.3.   Assume that $a > d > \lambda_1$; $ec < \delta^2$. Then (3.1) has a unique positive solution.

Proof.   Assume there is more than one solution. Then the theorem 2 in [9] guarantees existence of the maximal solution $(\bar{u}, \bar{v})$, i.e. $u < \bar{u}, v < \bar{v}$ for any other solution (u,v) of (3.1). We proceed to prove the opposite inequalities. Consider a family of super-solutions $w_\lambda = (u+\lambda u, v+\lambda\gamma v)$ with any $\lambda > 0$ and $\gamma > 0$ to be specified. In view of (3.2) it is clear that $\bar{u} < u+\lambda u, \bar{v} < v+\lambda v$ for λ sufficiently large. In order for w_λ to be a family of super-solutions it suffices that

$$-\lambda u + c\lambda\gamma v < 0$$

(3.4)

$$e\lambda u - \lambda\gamma v < 0 ,$$

or $c < \frac{u}{\gamma v}, e < \frac{\gamma v}{u}$. Next, by (3.2) we have

$$\frac{u}{v} > \frac{\alpha v^*}{\beta u^*} > \frac{\alpha}{\beta}\delta \; ; \; \frac{v}{u} > \frac{\beta v^*}{\alpha u^*} > \frac{\beta}{\alpha}\delta$$

So that (3.4) will be satisfied if

$$c < \frac{\alpha}{\beta\gamma}\delta \;,\; e < \frac{\gamma\beta}{\alpha}\delta \;.$$

This can clearly be accomplished by choosing γ, since $ec < \delta^2$. Letting $\lambda \to 0$, we conclude the proof, in view of Lemma 2.3.

Remark.   In one dimension taking $\Omega = (0,\pi)$, we have by (2.8),

$$\delta > \frac{d-\lambda_1}{a\sqrt{a}}, \quad \text{and hence the condition } ec < \frac{(d-\lambda_1)^2}{a^3} \text{ will be suf-}$$

ficient for uniqueness.

Theorem 3.4.   In the conditions of the theorem 3.3, the solution of 3.1 is stable in the following sense: it is the limit of a sequence of supersolutions from above and a sequence of sub-solutions from below. (This notion of stability was used by D. Sattinger [11].)

Proof.   We showed that $((1+\lambda)u,(1+\lambda\gamma)v)$ is a supersolution for any $\lambda > 0$ and some fixed γ. Similarly, one checks that $((1-\lambda)u,(1-\lambda\gamma)v)$ is a subsolution for $0 < \lambda < 1$ and the same γ as above.

Next for the positive solution (u,v) of (3.1) we estimate v in terms of u.

Theorem 3.5.   Define $\lambda_0 = \min(\frac{c}{1+e} , \frac{a}{d}\frac{1-ec}{1+e})$. Then $u > \lambda_0 v$.

Proof.   Using (3.2), compute

$$-\Delta(u-\lambda v) + u(u-\lambda v) = au + (c-\lambda-\lambda e)uv - \lambda dv + \lambda v^2$$

$$> (a-\lambda d\beta)u^* + (c-\lambda-\lambda e)uv + \lambda v^2 > 0,$$

provided that $a-\lambda d\beta > 0$, $c-\lambda-\lambda e > 0$. By the maximum principle, $u-\lambda_0 v > 0$.

4. NON-EXISTENCE OF POSITIVE SOLUTIONS FOR COMPETING SPECIES.

We study for which parameters a,d,e,c the problem (with c > 0, e > 0)

$$\Delta u + u(a-u-cv) = 0$$

$$\text{in } \Omega, \; u = v = 0 \text{ on } \partial\Omega$$

(4.1)

$$\Delta v + v(d-eu-v) = 0$$

has no positive solutions, and compare the results with the known existence results of A. Leung [6] and P. J. McKenna-W. Walter [9]. Notice that $u < u^* < a$, $v < v^* < d$. For the most of this section we shall assume a,d to be fixed with $a > d > \lambda_1$. This allows us to concentrate on parameters e and c, in particular to draw pictures of existence and non-existence in e,c plane. If a is sufficiently large, there are no positive solutions (for fixed d,e,c). To see this, multiply the first equation in (4.1) by v, the second one by u and subtract. Integrating over Ω and using the Green's formula, we get

$$\int [a-d+(e-1)u+(1-c)v]uv = 0 .$$

No positive solutions can exist if

(4.2) $A \equiv a-d+(e-1)u+(1-c)v > 0.$

Since $A > \mu a - \nu d$, with $\mu = \min(e,1)$, $\nu = \max(c,1)$, we get non-existence of $a > \frac{\nu}{\mu}d$.

Define $a^* \equiv \max_{\Omega} u^* < d$, $d^* \equiv \max_{\Omega} v^* < d$.

Theorem 4.1. For fixed a and d, in the following four regions in e,c plane there are no positive solutions to (4.1).

    (i) $e > 1$, $c < 1$;

    (ii) $e > 1$, $1 < c < 1 + \frac{a-d}{d^*}$;

    (iii) $1 > e > 1 - \frac{a-d}{a^*}$, $c < 1$;

    (iv) $a-d+(e-1)a^*+(1-c)d^* > 0$ and $e < 1$, $c > 1$.

(The last condition includes $e > \frac{d}{a}c$ and $e < 1$, $c > 1$).

Proof. In each case one shows that A as defined by (4.2) is non-negative (using $u < u^* < a^*$, $v < v^* < d^*$).

The following result provides additional information on the region of non-existence in e,c plane. As in [9] we transform (4.1) to a quasimonotone increasing form by letting $\bar{u} = -u$. Then (with $\bar{u} < 0$, $v > 0$)

$$\Delta\bar{u} + \bar{u}(a+\bar{u}-cv) = 0$$

(4.1)'

$$\Delta v + v(d+e\bar{u}-v) = 0 \quad \text{in } \Omega, \; \bar{u} = v = 0 \text{ on } \partial\Omega.$$

Define $\gamma = \sup_{\Omega} \frac{u^*}{\phi_1}$, with ϕ_1 normalized by $\max_{\Omega} \phi_1 = 1$.

Theorem 4.2. Assume that $a > d > \lambda_1$ and $e < \frac{d-\lambda_1}{\gamma}$, $c > \frac{\gamma}{d-\lambda_1-e\gamma}$. Then the problem (4.1) has no positive solution.

Proof. We consider a family of subsolutions for (4.1)', defined by $w_\mu = (-\mu u^*, \lambda\phi_1)$ with μ decreasing from 1 to 0, $\lambda = \mu\lambda_0 + (1-\mu)(d-\lambda_1-e\gamma)$, and $\lambda_0 > 0$ such that $v > \lambda_0\phi_1$ and $\lambda_0 < d-\lambda_1-e\gamma$. Also we take $\sup_{\Omega} \phi_1 = 1$. In order that w_μ be subsolutions of (4.1)' we need (with $\mu > 0$)

$$\Delta u^* + u^*(a-\mu u^*-c\lambda\phi_1) < 0 , \; \Delta\phi_1 + \phi_1(d-e\mu u^*-\lambda\phi_1) > 0 ,$$

or

$$(1-\mu)u^*-c\lambda\phi_1 < 0 , \; d-\lambda_1-e\mu u^*-\lambda\phi_1 > 0 .$$

It suffices that

$$(1-\mu)\gamma-c\mu\lambda_0-c(1-\mu)(d-\lambda_1-e\gamma) < 0, \; d-\lambda_1 > e\gamma + \lambda .$$

The second inequality holds by the definition of λ_0 and λ, and the first one by our condition on c. The proof follows by Lemma 2.3 (the $\bar{u}$ component of solution is swept to zero).

The following result is useful if a/d is large. By letting $\bar{v} = -v$, we get another quasimonotone increasing form for (4.1) (with $u > 0$, $\bar{v} < 0$)

$$\Delta u + u(a-u+c\bar{v}) = 0$$

(4.1)"

$$\Delta \bar{v} + \bar{v}(d-eu+\bar{v}) = 0 \quad \text{in } \Omega, \quad u = \bar{v} = 0 \text{ on } \partial\Omega.$$

Theorem 4.3. Assume that $a > d > \lambda_1$. Then if $c < a/d$, $e > \frac{1}{a/d-c}$, the problem (4.1) has no positive solution.

Proof. Define $w_\mu = (\lambda u^*, -\mu v^*)$ with μ decreasing from 1 to 0, $\lambda = \mu\lambda_0 + (1-\mu)(1-cd/a)$, where $\lambda_0 > 0$ is such that $u > \lambda_0 u^*$ and $\lambda_0 < 1-cd/a$. It is a family of subsolutions of (4.1)" if

$$\Delta u^* + u^*(a-\lambda u^*-c\mu v^*) > 0 \ , \ \Delta v^* + v^*(d-e\lambda u^*-\mu v^*) < 0.$$

These will hold if

$$\lambda u^* + c\mu v^* < u^* \ , \ -e\lambda u^* - (\mu-1)v^* < 0 \ .$$

We have by (2.3) and our conditions ($\mu > 0$)

$$\lambda u^*+c\mu v^* < (\lambda+cd/a)u^* < u^* \ ;$$

$$-e\lambda u^*-(\mu-1)v^* < [-e\lambda+(1-\mu)\frac{d}{a}]u^*$$

$$< [-e(1-\mu)(1-cd/a)+(1-\mu)d/a]u^* < 0 \ ,$$

and the conclusion follows as before.

We now compare our results with the known existence results for (4.1) if $a > d > \lambda_1$. A. Leung [6] proved existence of a posi-

tive solution for $c < \frac{a-\lambda_1}{d}$, $e < \frac{d-\lambda_1}{a}$ (see also [5]).

Subsequently McKenna-Walter [9] showed that it suffices that $c < a/d$, $e < \min(\frac{d-\lambda_1}{a^*}, \frac{d}{a})$. On the other hand, from the results of Blat-Brown [1] it appears to follow that increasing either e or c will eventually lead us into the non-existence region. We see that most of the first quadrant in e,c plane is the region of non-existence. Also we have more detailed information on where the non-existence region starts if we increase either c or e and keep the other variable fixed and small, rather than when we increase c and e simultaneously. It will be interesting to study how are the existence and non-existence regions separated in e,c plane (by a curve?), and how the picture changes with a and d.

5. ON ESSENTIALLY QUASIMONOTONE INCREASING SYSTEMS.

Our proofs of the theorems 4.2 and 4.3 depended on the trick of converting the problem to a quasimonotone increasing form by changing the sign of some variables. In this section we describe all Volterra-Lotka systems where a similar approach works. Let $u = (u_1,\ldots,u_m)'$, $A = (a_{ij})$ an $m\times m$ matrix with $a_{ii} = -1$ for all i, and $a_{ij} \neq 0$. We consider systems of the form

$$(5.1) \qquad \Delta u_i+u_i(a_i+\sum_{j=1}^{n} a_{ij}u_j) = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad i=1,\ldots,m.$$

Definition. We say that the species i and j cooperate if $a_{ij} > 0$, $a_{ji} > 0$; compete if $a_{ij} < 0$, $a_{ji} < 0$; form a predator-prey pair if $a_{ij}a_{ji} < 0$.

Definition. A system (5.1) is called essentially quasimonotone increasing (EQI) if by changing signs of some variables it can be transformed into quasimonotone increasing form (in the region $u_i > 0$, $\bar{u}_k < 0$).

The following result describes all such systems. We omit its straightforward proof.

Proposition. A system is EQI iff all species can be divided into two groups, such that within each group all species cooperate, and each species competes with all species of the other group. (In particular no predator-prey pairs are allowed).

Existence question for EQI system can be decomposed into studying two smaller problems for cooperating species. As an example, we present an existence result for two species competing against two others, based on [5].

Theorem 5.1. Consider the following problem in the domain Ω

$$\Delta y + y(\alpha-y+b_1 z-c_1 u-d_1 v) = 0$$

$$\Delta z + z(\beta+a_2 y-z-c_2 u-d_2 v) = 0$$

(5.2)

$$\Delta u + u(\gamma-a_3 y-b_3 z-u+d_3 v) = 0$$

$$\Delta v + v(\delta-a_4 y-b_4 z+c_4 u-v) = 0 ,$$

with $y = z = u = v = 0$ for $x \in \partial\Omega$. Assume that

(i) $b_1 a_2 < 1, d_3 c_4 < 1$

(ii) $\alpha > c_1 \alpha_2 u_\gamma + d_1 \beta_2 u_\gamma , \beta > c_2 \alpha_2 u_\gamma + d_2 \beta_2 u_\gamma ,$

$\gamma > a_3 \alpha_1 u_\alpha + b_3 \beta_1 u_\alpha , \delta > a_4 \alpha_1 u_\alpha + b_4 \beta_1 u_\alpha ,$

with $\alpha_1 = \dfrac{1+b_1}{1-b_1 a_2} , \beta_1 = \dfrac{1+a_2}{1-b_1 a_2} , \alpha_2 = \dfrac{1+d_3}{1-d_3 c_4} , \beta_2 = \dfrac{1+c_4}{1-d_3 c_4}$. Also we assume for definiteness $\alpha > \beta, \gamma > \delta$. Then (5.2) has a positive solution.

Proof.   Let $(\bar{y},\bar{z})$ and $(\bar{u},\bar{v})$ be positive solutions of (5.2) with $u = v = 0$ and $y = z = 0$ respectively, whose existence is guaranteed by the theorem 3.1. By the theorem 3.2 we have the estimates

$$\alpha_1 u_\beta < \bar{y} < \alpha_1 u_\alpha , \beta_1 u_\beta < \bar{z} < \beta_1 u_\alpha ;$$

(5.3)

$$\alpha_2 u_\delta < \bar{u} < \alpha_2 u_\gamma , \beta_2 u_\delta < \bar{v} < \beta_2 u_\gamma .$$

We now apply the theorem 2 in [5] (see also [12]) taking $(\bar{y},\bar{z},\bar{u},\bar{v})$ and $(\epsilon\phi_1,\epsilon\phi_1,\epsilon\phi_1,\epsilon\phi_1)$, with ϵ sufficiently small, as a pair of super- and sub-solutions. Conditions on supersolutions follow automatically, the ones on subsolutions from (ii). (Recall that the theorem 2 in [5] provides a monotone scheme to approximate the solution).

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A Note on the Proper Choice of Scales in Two Variable Expansions

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Abstract By way of a class of weakly nonlinear differential equations it is shown that the simplified two scale procedure advocated by Greenlee and Snow in general leads to expansions of which already the second term is secular on the slow scale, while such a secularity can be avoided if the more elaborate choice of scales proposed by Kevorkian and Cole is used. The relevance of this fact for applications is pointed out by demonstrating by means of a numerical example that two terms of Kevorkian and Cole's expansion describe the long time behavior of the solution much more accurately than the same number of terms of Greenlee and Snow's expansion, while being at least as simple to derive.

1. Introduction

Multiple scale methods, as introduced by Cole and Kevorkian [1-3], have been established as powerful and versatile tools for obtaining uniformly valid approximate solutions to weakly nonlinear differential equations of many different types (cf. the discussion in [4] and the references given therein). One such type that frequently occurs in physical applications is the initial value problem

$$y'' + y + \epsilon f(y, y'; t; \epsilon) = 0, \tag{1.1}$$

$$y(0) = a, \quad y'(0) = b. \tag{1.2}$$

For that subclass where f is independent of t, Morrison [5] and