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A general monotone scheme for elliptic systems with applications to ecological models

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Synopsis

We consider weakly-coupled elliptic systems of the type

$$\Delta u_i + f_i(u_1, \ldots, u_m) = 0$$

with each f_i being either an increasing or a decreasing function of each u_j. Assuming the existence of coupled super- and subsolutions, we prove the existence of solutions, and provide a constructive iteration scheme to approximate the solutions. We then apply our results to study the steady-states of two-species interaction in the Volterra–Lotka model with diffusion.

1. Introduction

In this paper, we consider the Dirichlet problem for the elliptic system

$$\begin{gathered}\Delta u_i + f_i(x, u_1, \ldots, u_m) = 0, \quad x = (x_1, \ldots, x_n) \in \mathscr{D} \\ u_i = g_i \quad \text{in } \delta\mathscr{D},\end{gathered} \tag{1.1}$$

for $i = 1, \ldots, m$. Here $\Delta = \sum_{k=1}^{n} \partial^2/\partial x_k^2$ and $\mathscr{D}$ is a bounded domain in R^n with smooth boundary (more specific conditions will be given in Section 2). We shall assume that each of the functions f_i depends monotonically on the variables $u = (u_1, \ldots, u_m)$, i.e. $\partial f_i/\partial u_j \geqq 0$ or $\partial f_i/\partial u_j \leqq 0$, for u varying in some order interval. In Section 2, under rather general conditions we prove existence theorems for (1.1) and provide an alternating sequence of approximations. In [**16**], Sattinger considered the problem when $m = 2$, with $\partial f_i/\partial u_j$ having the same sign for all i, j, $i \neq j$. The sequences constructed in [**16**] are purely monotone (non-alternating). In [**9**], [**10**] Leung considered the cases when $m = 2$ and 3, with $\partial f_i/\partial u_j$ having different signs for various i, j; the sequences constructed are all alternating. The present idea, of constructing a scheme so that each component forms an alternating sequence as in [**10**], is initiated by Leung. Here, the method for utilising previous iterates for off-diagonal components agrees with that in [**10**] when $m = 3$. The diagonal components in [**10**] are used "implicitly", while in the present paper they are delayed one or two steps. Our present result is substantially more general. Moreover the functions f_i's do not have to satisfy $f_i(x, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) = 0$.

The schemes can be applied to study more general systems of partial differential equations. Examining Section 2, we see that it applies as long as the problem has "inverse-positivity" property and the solution operator is compact (see Korman [**5**], [**6**] for background and discussions).

THEOREM 1. *Let u^{-1} and u^0 be two given vector functions with each component in $C^{\alpha}(\bar{\mathscr{D}})$; and define $u^1, u^2, \ldots$ according to the rules described above. Assume that*

$$u^0 \leqq u^2 \leqq u^1 \leqq u^{-1} \quad \text{in } \bar{\mathscr{D}}. \tag{2.2}$$

Then problem (1.1) has a solution u with each component in $C^{2+\alpha}(\bar{\mathscr{D}})$, $u^0 \leqq u \leqq u^{-1}$; moreover, the iteration procedure (2.1) produces two monotone iteration sequences (alternating) satisfying, for each integer r,

$$u^0 \leqq u^2 \leqq \ldots \leqq u^{2r} \ldots \leqq u \leqq \ldots \leqq u^{2r-1} \leqq \ldots \leqq u^1 \leqq u^{-1} \text{ in } \mathscr{D}. \tag{2.3}$$

Proof. The proof is by induction.

(A) First, suppose k is odd and we assume that

$$[H^0 k] \qquad u^0 \leqq u^2 \leqq \ldots \leqq u^{k-1} \leqq u^{k-2} \leqq \ldots \leqq u^1 \leqq u^{-1}$$

9 is true. (Here the superscript 0 in H^0 designates oddness, and note that (2.2) is same as $[H^0 3]$.) We now show that

$$u^{k-1} \leqq u^k \leqq u^{k-2}. \tag{2.4}$$

The proof of (2.4) is by induction on rows.

(i) First we show that

$$u_1^{k-1} \leqq u_1^k \leqq u_1^{k-2}. \tag{2.5}$$

Indeed, we have

$$\begin{aligned} -\Delta u_1^k &= f_1(x, u_j^{k+\ell_1(j)}); \\ -\Delta u_1^{k-1} &= f_1(x, u_j^{k-1+\ell_1(j)}); \\ -\Delta u_1^{k-2} &= f_1(x, u_j^{k-2+\ell_1(j)}). \end{aligned} \tag{2.6}$$

If $\partial f_1/\partial u_j \geqq 0$, then $k + \ell_1(j) = k-2$ and hence, by $[H^0 k]$,

$$u_j^{k-1+\ell_1(j)} \leqq u_j^{k+\ell_1(j)} \leqq u_j^{k-2+\ell_1(j)}, \quad \text{each} \quad j \geqq 1. \tag{2.7}$$

If $\partial f_1/\partial u_j \leqq 0$, then $k + \ell_1(j) = k-1$ and hence, by $[H^0 k]$,

$$u_j^{k-2+\ell_1(j)} \leqq u_j^{k+\ell_1(j)} \leqq u_j^{k-1+\ell_1(j)}, \quad \text{each} \quad j \geqq 1. \tag{2.8}$$

Applying the maximum principle to the differences of the equations in (2.6), we obtain (2.5).

(ii) Assuming that (2.4) holds for all components up to $i-1$, we show that

$$u_i^{k-2} \leqq u_i^k \leqq u_i^{k-1}. \tag{2.9}$$

Indeed, we have

$$\begin{aligned} -\Delta u_i^k &= f_i(x, u_j^{k+\ell_i(j)}); \\ -\Delta u_i^{k-1} &= f_i(x, u_j^{k-1+\ell_i(j)}); \\ -\Delta u_i^{k-2} &= f_i(x, u_j^{k-2+\ell_i(j)}). \end{aligned} \tag{2.10}$$

It is sufficient to consider the case when $i > j$, since for $i \leqq j$ (on and above the diagonal) considerations are the same as in case (i) above. If $\partial f_i/\partial u_j \geqq 0$, then

$k+\ell_i(j)=k$ and (2.7) holds with $\ell_1(j)$ replaced by $\ell_i(j)$, for each $j<i$. If $\partial f_i/\partial u_j \leqq 0$, then $k+\ell_i(j)=k-1$ and (2.8) holds with $\ell_1(j)$ replaced by $\ell_i(j)$, for each $j<i$. By the maximum principle again, (2.9) follows from (2.10).

(B) Next, suppose k is even and we assume that

$$[H^e k] \qquad u^0 \leqq u^2 \leqq \ldots \leqq u^{k-2} \leqq u^{k-1} \leqq \ldots \leqq u^1 \leqq u^{-1}.$$

(Note that $[H^e 4]$ follows from $[H^0 3]$ and (2.4) with $k=3$; e designates evenness.) We now show that

$$u^{k-2} \leqq u^k \leqq u^{k-1}. \tag{2.11}$$

The proof of (2.11) is by induction on rows.

(i) First we show that

$$u_1^{k-2} \leqq u_1^k \leqq u_1^{k-1}. \tag{2.12}$$

These iterates are again determined by (2.6). If $\partial f_1/\partial u_j \geqq 0$, then $k+\ell_1(j)=k-2$, and hence, by $[H^e k]$,

$$u_j^{k-2+\ell_1(j)} \leqq u_j^{k+\ell_1(j)} \leqq u_j^{k-1+\ell_1(j)}, \quad \text{each} \quad j \geqq 1. \tag{2.13}$$

If $\partial f_1/\partial u_j \leqq 0$, then $k+\ell_1(j)=k-1$, and hence, by $[H^e k]$,

$$u_j^{k-1+\ell_1(j)} \leqq u_j^{k+\ell_1(j)} \leqq u_j^{k-2+\ell_1(j)}, \quad \text{each} \quad j \geqq 1. \tag{2.14}$$

In both cases (2.12) follows from (2.6).

(ii) Assume that (2.11) holds for all components up to $i-1$. We show that

$$u_i^{k-2} \leqq u_i^k \leqq u_i^{k-1}. \tag{2.15}$$

These iterates are determined by (2.10) and again we may assume $i>j$. If $\partial f_i/\partial u_j \geqq 0$, then $k+\ell_i(j)=k$ and hence (2.13) holds with $\ell_1(j)$ replaced by $\ell_i(j)$, for each $j<i$. If $\partial f_i/\partial u_j \leqq 0$, then $k+\ell_i(j)=k-1$ and hence (2.14) holds with $\ell_1(j)$ replaced by $\ell_i(j)$, for each $j<i$. In both cases (2.15) follows from (2.10).

Successive applications of parts (A) and (B) lead to the validity of $[H^0 k+2]$ and $[H^e k+2]$, and the ordering of the iterates in (2.3) follows.

The existence of a solution to (1.1) follows by the application of Schauder's fixed point theorem to the order interval $[u^0, u^{-1}]$ in $C^\alpha(\bar{D}) \times \ldots \times C^\alpha(\bar{D})$. (By standard elliptic theory the operator defined by (2.1) is compact. Hence we may use Schauder's fixed point theorem as stated in e.g. [**1**, p. 660].) The ordering of u in (2.3) is proved in the same way as above. This completes the proof of Theorem 1. □

Note. To show that u^k fits right into the sequences displayed in $[H^0 k]$ or $[H^e k]$, we use the property that (k say even) $u_{k-4} \leqq u_{k-2} \leqq u_{k-1} \leqq u_{k-3}$, i.e. the previous four iterates of the sequence. Hence the induction step proved in Theorem 1 has to be supplemented with hypothesis (2.2).

In the next theorem, we will see that (2.2) follows from the existence of upper (i.e. super-) and lower (i.e. sub-) solutions defined as follows. For convenience, we will write $u \leqq v$ if $u_i \leqq v_i$ for each $i=1, \ldots, m$.

Definition. Two vector functions $\bar{u}=(\bar{u}_1, \ldots, \bar{u}_n)$ and $\underline{u}=(\underline{u}_1, \ldots, \underline{u}_n)$ in $C^2(\bar{D}) \times \ldots \times C^2(\bar{D})$ are called a pair of upper (or super-) and lower (or sub-)

solutions to (1.1), if they satisfy

$$\begin{aligned}
\text{(i)}\quad & -\Delta\bar{u}_1 \geqq f_1(x, \bar{u}_1, u_2, \ldots, u_m), \quad \underline{u}_j \leqq u_j \leqq \bar{u}_j, \quad j \neq 1, \quad x \in \mathscr{D}, \\
& \cdots \\
& -\Delta\bar{u}_m \geqq f_m(x, u_1, \ldots, u_{m-1}, \bar{u}_m), \quad \underline{u}_j \leqq u_j \leqq \bar{u}_j, \quad j \neq m, \quad x \in \mathscr{D}, \qquad (2.16) \\
& \bar{u}_i(x) \geqq g_i(x), \quad x \in \delta\mathscr{D}, \quad i = 1, \ldots, m. \\
\text{(ii)}\quad & -\Delta\underline{u}_1 \leqq f_1(x, \underline{u}_1, u_2, \ldots, u_m), \quad \underline{u}_j \leqq u_j \leqq \bar{u}_j, \quad j \neq m, \quad x \in \mathscr{D}, \\
& \cdots \\
& -\Delta\underline{u}_m \leqq f_m(x, u_1, \ldots, u_{m-1}, \underline{u}_m), \quad \underline{u}_j \leqq u_j \leqq \bar{u}_j, \quad j \neq m, \quad x \in \mathscr{D}, \qquad (2.17) \\
& \underline{u}_i \leqq g_i(x), \quad x \in \delta\mathscr{D}, \quad i = 1, \ldots, m. \\
\text{(iii)}\quad & \qquad \underline{u} \leqq \bar{u}, \quad \text{for} \quad x \in \bar{\mathscr{D}}.
\end{aligned}$$

Note. In the next theorem, the monotonicity of the functions f_i's is only assumed between the pair $\underline{u}$ and $\bar{u}$. Moreover, no monotonicity is assumed for the dependence of f_i on the component u_i. The scheme for constructing the sequence of functions is slightly modified.

THEOREM 2. *Assume that problem* (1.1) *has a pair of upper* (*i.e. super-*) *and lower* (*i.e. sub-*) *solutions, and* $\partial f_i/\partial u_j \geqq 0$ *or* $\partial f_i/\partial u_j \leqq 0$ *for* $x \in \mathscr{D}$, $\underline{u} \leqq u \leqq \bar{u}$, *all* i,j *with* $i \neq j$. *Let* $u^{-1} = \bar{u}$, $u^0 = \underline{u}$ *and* u^k, $k = 1, 2, \ldots$, *be defined recursively as solutions of*

$$\begin{aligned}
& -\Delta u_1^k + \Omega_1 u_1^k = \Omega_1 u_1^{k-2} + f_1(x, u_1^{k-2}, u_2^{k+\ell_1(2)}, \ldots, u_m^{k+\ell_1(m)}), \\
& -\Delta u_2^k + \Omega_2 u_2^k = \Omega_2 u_2^{k-2} + f_2(x, u_1^{k+\ell_2(1)}, u_2^{k-2}, \ldots, u_m^{k+\ell_2(m)}), \\
& \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.18) \\
& -\Delta u_m^k + \Omega_m u_m^k = \Omega_m u_m^{k-2} + f_m(x, u_1^{k+\ell_m(1)}, u_2^{k+\ell_m(2)}, \ldots, u_m^{k-2}), \\
& u_i^k = g_i \quad \text{on} \quad \delta\mathscr{D}, \quad i = 1, \ldots, m,
\end{aligned}$$

where $\Omega_i = \sup\{|\partial f_i/\partial u_i| : x \in \mathscr{D},\ u \in [u^0, u^{-1}]\}$, *and* $\ell_i(j)$, $i \neq j$, *satisfy the same rules* (I) *and* (II), *following* (2.1). *Then* (1.1) *has a* $C^{2+\alpha}$ *solution* u *with* $\underline{u} \leqq u \leqq \bar{u}$, *and we have the order approximations* (2.3).

Proof. With the additional terms $\Omega_i u_i$ on the right hand side of (2.18), the dependence of the expression on the right hand side of the ith equation on u_i is nondecreasing. Comparing with (2.1) and Theorem 1, we find that it suffices to prove (2.2) is valid in the present case. The rest of the proof is the same as in Theorem 1.

(A) We start by showing that

$$u^0 \leqq u^1 \leqq u^{-1}. \qquad (2.19)$$

The proof is by induction on rows.

(i) First, we show that

$$u_1^0 \equiv \underline{u}_1 \leqq u_1^1 \leqq \bar{u}_1 \equiv u_1^{-1}. \qquad (2.20)$$

We have

$$-\Delta u_1^1+\Omega_1 u_1^1=\Omega\bar{u}_1+f_1(x,\bar{u}_1,u_j^{1+\ell_1(j)}) \quad \text{in } \mathscr{D}, \quad u_1^1=g_1 \quad \text{on } \delta\mathscr{D}; \tag{2.21}$$

$$-\Delta \underline{u}_1+\Omega_1\underline{u}_1\leqq\Omega_1\underline{u}_1+f_1(x,\underline{u}_1,u_j^{1+\ell_1(j)}) \quad \text{in } \mathscr{D}, \quad \underline{u}_1\leqq g_1 \quad \text{on } \delta\mathscr{D}; \tag{2.22}$$

$$-\Delta \bar{u}_1+\Omega_1\bar{u}_1\geqq\Omega_1\bar{u}_1+f_1(x,\bar{u}_1,u_j^{1+\ell_1(i)}) \quad \text{in } \mathscr{D}, \quad \bar{u}_1\geqq g_1 \quad \text{on } \delta\mathscr{D}. \tag{2.23}$$

Note. The right hand side of (2.21) is the same as that of (2.23), and is greater than or equal to that of (2.22). Inequalities (2.20) follow from the maximum principle applied to the differences of pairs of (2.21) to (2.23).

(ii) Assume (2.19) holds for all rows $j\leqq i-1$. We now show that

$$u_i^0\equiv\underline{u}_i\leqq u_i^1\leqq\bar{u}_i\equiv u_i^{-1}. \tag{2.24}$$

We have

$$-\Delta u_i^1+\Omega_i u_i^1=\Omega_i\bar{u}_i+f_i(x,\bar{u}_i,u_j^{1+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad u_i^1=g_i \quad \text{on } \delta\mathscr{D}; \tag{2.25}$$

$$-\Delta \underline{u}_i+\Omega_i\underline{u}_i\leqq\Omega_i\underline{u}_i+f_i(x,\underline{u}_i,u_j^{1+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad \underline{u}_i\leqq g_i \quad \text{on } \delta\mathscr{D}; \tag{2.26}$$

$$-\Delta \bar{u}_i+\Omega_i\bar{u}_i\geqq\Omega_i\bar{u}_i+f_i(x,\bar{u}_i,u_j^{1+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad \bar{u}_i\geqq g_i \quad \text{on } \delta\mathscr{D}. \tag{2.27}$$

Note. The terms $u_j^{1+\ell_i(j)}$ in (2.26) and (2.27) are substituted in the same way as in (2.25). For $j<i$, $u_j^{1+\ell_i(j)}$ can possibly be u_j^1; however (2.26) and (2.27) are still valid because (2.19) is assumed valid for $j\leqq i-1$. As in (i), we now conclude that (2.24) holds.

(B) It remains to show that

$$u^0\leqq u^2\leqq u^1. \tag{2.28}$$

Again we use induction on rows.

(i) We first show that

$$u_1^0\leqq u_1^2\leqq u_1^1. \tag{2.29}$$

We have

$$-\Delta u_1^1+\Omega_1 u_1^1=\Omega_1\bar{u}_1+f_1(\bar{u}_1,u_j^{1+\ell_1(j)}) \quad \text{in } \mathscr{D}, \quad u_1^1=g_1 \quad \text{on } \delta\mathscr{D}; \tag{2.30}$$

$$-\Delta u_1^2+\Omega_1 u_1^2=\Omega_1\underline{u}_1+f_1(\underline{u}_1,u_j^{2+\ell_1(j)}) \quad \text{in } \mathscr{D}, \quad u_1^2=g_1 \quad \text{on } \delta\mathscr{D}; \tag{2.31}$$

$$-\Delta \underline{u}_1+\Omega_1\underline{u}_1\leqq\Omega_1\underline{u}_1+f_1(\underline{u}_1,u_j^{2+\ell_1(j)}) \quad \text{in } \mathscr{D}, \quad \underline{u}_1\leqq g_1 \quad \text{on } \delta\mathscr{D}. \tag{2.32}$$

If $\partial f_1/\partial u_j\geqq 0$, then $1+\ell_1(j)=-1$ and $u_j^{2+\ell_1(j)}\leqq u_j^{1+\ell_1(j)}$.
If $\partial f_1/\partial u_j\leqq 0$, then $1+\ell_1(j)=0$ and $u_j^{1+\ell_1(j)}\leqq u_j^{2+\ell_1(j)}$, by (A). In both cases (2.29) follows.

(ii) Assume that (2.28) holds for all rows $j\leqq i-1$. We show that

$$u_i^0\leqq u_i^2\leqq u_i^1. \tag{2.33}$$

We have

$$-\Delta u_i^1+\Omega_i u_i^1=\Omega_i\bar{u}_i+f_i(\bar{u}_i,u_j^{1+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad u_i^1=g_i \quad \text{on } \delta\mathscr{D}; \tag{2.34}$$

$$-\Delta u_i^2+\Omega_i u_i^2=\Omega_i\underline{u}_i+f_i(\underline{u}_i,u_j^{2+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad u_i^2=g_i \quad \text{on } \delta\mathscr{D}; \tag{2.35}$$

$$-\Delta \underline{u}_i+\Omega_i\underline{u}_i\leqq\Omega_i\underline{u}_i+f_i(\underline{u}_i,u_j^{2+\ell_i(j)}) \quad \text{in } \mathscr{D}, \quad \underline{u}_i\leqq q_i \quad \text{on } \delta\mathscr{D}. \tag{2.36}$$

Note. The terms $u_j^{2+\ell_i(j)}$ in (2.36) can be substituted in the same way as in (2.35). For $j<i$, $u_j^{2+\ell_i(j)}$ can possibly be u_j^2; however, (2.36) is still valid by the assumption that (2.28) is valid for $j \leqq i-1$. Following part (B)(i) above, it suffices to consider $i>j$. If $\partial f_i/\partial u_j \geqq 0$, then $2+\ell_i(j)=2$ and $u_j^{2+\ell_i(j)} \leqq u_j^{1+\ell_i(j)}$ by induction on (2.28), $j \leqq i-1$. If $\partial f_i/\partial u_j \leqq 0$, then $2+\ell_i(j)=1$, and $u_j^{1+\ell_i(j)} \leqq u_j^{2+\ell_i(j)}$ by (A). As in (i), we now conclude that (2.33) holds. This completes the proof of Theorem 2. $\square$

Remark. Consider the problem (1.1) in the case of one equation ($i=1$)

$$\begin{aligned} -\Delta u &= f(x,u) \quad \text{in } \mathscr{D}, \\ u &= g(x) \quad \text{on } \partial\mathscr{D}. \end{aligned}$$

Assume that a subsolution ψ and a supersolution ϕ are known. Without loss of generality, we may assume that $f_u \geqq 0$ for x in $\mathscr{D}$, $\psi \leqq u \leqq \phi$. Letting $u^{-1}=\phi$, $u^0=\psi$, we define (with $u^j=g$ on $\partial\mathscr{D}$ for any j) according to rule (2.1)

$$\begin{aligned} -\Delta u^1 &= f(\phi), \quad & -\Delta u^2 &= f(\psi). \\ -\Delta u^3 &= f(u^1), \quad & -\Delta u^4 &= f(u^2), \end{aligned}$$

and so on. We see that these are the usual two sequences of monotone iterates renumbered. Also, the concept of a pair of super- and subsolutions degenerates for one equation.

3. Application to two-species reaction-diffusion systems

(I) Cooperating (mutualistic) species

Consider the boundary value problem

$$\begin{aligned} \Delta u + u(\alpha - au + bv) &= 0 \\ \Delta v + v(\beta + cu - dv) &= 0 \quad \text{in } \mathscr{D}, \\ u = v &= 0 \quad \text{on } \partial\mathscr{D}. \end{aligned} \tag{3.1}$$

Here α, β, a, b, c, d are nonnegative constants, and we are looking for positive (in $\mathscr{D}$) solutions $u(x)$, $v(x)$.

This system describes the steady state interaction of two cooperating species in the Volterra–Lotka ecological model. We shall prove existence by exhibiting a pair of positive super- and subsolutions and applying Theorem 2 (notice that here $\partial f_1/\partial v > 0$ and $\partial f_2/\partial u > 0$ for $u, v > 0$). In addition to existence, Theorem 2 will provide approximations to the solution. By $\lambda_0 > 0$ and $\omega(x) > 0$ we shall denote the principal eigenvalue and eigenfunction of the problem

$$\Delta u + \lambda u = 0 \quad \text{in } \mathscr{D}, \quad u = 0 \quad \text{on } \partial\mathscr{D}.$$

THEOREM 3. *Assume that* $\alpha, \beta > \lambda_0$, $a, d > 0$, $b, c \geqq 0$, *and* $ad > bc$. *Then problem* (3.1) *has a positive solution.*

Proof. Let $\underline{u} = \lambda_1\omega$, $\underline{v} = \lambda_2\omega$, $\bar{u} = M$, $\bar{v} = N$, where λ_1, λ_2, M, N are positive constants. Then it is clear that

$$-\Delta \underline{u} = \lambda_0\lambda_1\omega \leqq \lambda_1\omega(\alpha - a\lambda_1\omega + bv),$$
$$-\Delta \underline{v} = \lambda_0\lambda_2\omega \leqq \lambda_2\omega(\beta + cu - d\lambda_2\omega),$$

and

$$-\Delta M \equiv 0 > M(\alpha - aM + bv),$$
$$-\Delta N \equiv 0 > N(\beta + cu - dN),$$

and also $\lambda_1\omega < M$, $\lambda_2\omega < N$, for all u, v satisfying $\lambda_1\omega \leqq u \leqq M$, $\lambda_2\omega \leqq v \leqq N$, provided the constants λ_1, λ_2 are chosen sufficiently small, and M, N sufficiently large in a suitable manner. Here we make use of the following lemma, whose proof is clear.

LEMMA. *With the conditions of Theorem* 3, *we can choose M, N positive and arbitrarily large so that*

$$\alpha - aM + bN < 0,$$
$$\beta + cM - dN < 0.$$

(II) Predator–prey interaction.

In the following model $u(x)$ denotes population of the prey, $v(x)$ that of the predator.

THEOREM 4. *Consider the problem*

$$\begin{aligned} \Delta u + u(\alpha - au - bv) &= 0 \quad \text{in } \mathcal{D};\\ \Delta v + v(\beta + cu - dv) &= 0 \\ u = v &= 0 \quad \text{on } \partial\mathcal{D}. \end{aligned} \tag{3.2}$$

Here α, β, a, b, c, d are nonnegative constants such that

(i) $$a, d > 0, \quad ad > bc;$$

(ii) $$\alpha > \frac{ad\left(\lambda_0 + \frac{b}{d}\beta\right)}{ad - bc}, \quad \beta > \lambda_0.$$

Then problem (3.2) *has a positive solution.*

Proof. We show that $\underline{u} = \lambda_1\omega$, $\underline{v} = \lambda_2\omega$, and $\bar{u} = M$, $\bar{v} = N$ form a pair of super- and subsolutions, provided λ_1 and λ_2 are chosen small enough, and M, N are chosen suitably. Indeed, we need to satisfy the following conditions;

$$-\Delta M \equiv 0 \geqq M(\alpha - aM - bv), \tag{3.3}$$
$$-\Delta N \equiv 0 \geqq N(\beta + cu - dN), \tag{3.4}$$
$$-\Delta \underline{u} \equiv \lambda_0\lambda_1\omega \leqq \lambda_1\omega(\alpha - a\lambda_1\omega - bv), \tag{3.5}$$
$$-\Delta \underline{v} \equiv \lambda_0\lambda_2\omega \leqq \lambda_2\omega(\beta + cu - d\lambda_2\omega), \tag{3.6}$$

and also $\lambda_1\omega \leqq M$, $\lambda_2\omega \leqq N$, for all u, v such that $\lambda_1\omega \leqq u \leqq M$, $\lambda_2\omega \leqq v \leqq N$.

To satisfy (3.3), the best choice is $M = \alpha/a$ (since v is small near $\partial\mathscr{D}$). This leads to the choice of $N = 1/d(\beta + c\alpha/a)$ to satisfy (3.4). It is easy to check that (3.5) and (3.6) are also satisfied under our conditions.

(III) Competing populations

THEOREM 5. *Consider the problem*

$$\left.\begin{aligned}\Delta u + u(\alpha - au - bv) &= 0\\ \Delta v + v(\beta - cu - dv) &= 0\end{aligned}\right\}\ \text{in } \mathscr{D};\qquad u = v = 0 \quad \text{on } \partial\mathscr{D}.$$

Here α, β, a, b, c, d *are nonnegative constants such that*

(i) $$a, d > 0, \quad ad > bc;$$

(ii) $$\alpha > \frac{\lambda_0(ab + ad)}{ad - bc};$$

(iii) $$\frac{d}{b}\alpha - \frac{d}{b}\lambda_0 > \beta > \frac{c}{a}\alpha + \lambda_0, \quad \text{if} \quad b > 0,$$

$$\beta > \frac{c}{a}\alpha + \lambda_0, \quad \text{if} \quad b = 0.$$

Then problem (3.7) *has a positive solution.*

Proof. Again we look for a pair of super- and subsolutions in the form $\bar{u} = M$, $\bar{v} = N$, $\underline{u} = \lambda_1\omega$, $\underline{v} = \lambda_2\omega$. By a similar argument as for Theorem 4, we see that $\bar{u} = \alpha/a$ and $\bar{v} = \beta/d$ are the best choices. Then we need

$$\begin{aligned}-\Delta\underline{u} &\equiv \lambda_0\lambda_1\omega \leqq \lambda_1\omega(\alpha - a\lambda_1\omega - bv),\\ -\Delta\underline{v} &\equiv \lambda_0\lambda_2\omega \leqq \lambda_2\omega(\beta - cu - d\lambda_2\omega),\end{aligned} \tag{3.8}$$

for all u, v, such that $\lambda_1\omega \leqq u \leqq \alpha/a$, $\lambda_2\omega \leqq v \leqq \beta/d$. Conditions (3.8) will hold for λ_1, λ_2 sufficiently small, provided that

$$\begin{aligned}\alpha - b\frac{\beta}{d} &> \lambda_0,\\ \beta - c\frac{\alpha}{a} &> \lambda_0.\end{aligned} \tag{3.9}$$

Condition (i) of Theorem 5 makes the system (3.9) compatible. The allowable region for α and β is then described by (ii) and (iii), which completes the proof. □

Remarks. (i) Although some existence and stability results concerning positive solutions in this section are known (as indicated in Section 1, see e.g. [8], [9], [11], [12], [13]), they are dispersed throughout other material. In the present paper, they are collected in condensed and simple form.

(ii) In Theorems 3, 4 and 5, in addition to existence as stated, the application of Theorem 2 provides approximations to the solutions. The scheme here is different from those in [10], but the similarity is explained in Section 1.

4. Generalisations and a numerical example

Examining the proof of Theorems 1 and 2, we notice that all we required was the "inverse-positivity" of the problem and compactness of the solution operator. Inverse-positivity is a property more general than the weak maximum principle, and it states loosely that $Lu \geqq 0$ implies $u \geqq 0$ (see e.g. [6]). We can easily state results similar to Theorems 1 and 2 for systems of boundary value problems for ordinary differential equations, biharmonic equations (under some restrictions, see [6]), and for problems combining the above mentioned and Laplace operators. Also we can consider more general boundary operators and non-homogeneous boundary conditions for uniformly elliptic operators of the second order.

We conclude with a simple one-dimensional example describing predator–prey interaction.

$$\left.\begin{aligned} u'' + u(2-u-v) &= 0 \\ v'' + v(3+u-6v) &= 0 \end{aligned}\right\} \quad 0<x<\pi$$

$$u(0)=u(\pi)=v(0)=v(\pi)=0.$$

Here $\lambda_0 = 1$, $\omega = \sin x$. As in Theorem 4, we can easily conclude the existence of a positive solution with bounds

$$\tfrac{1}{6}\sin x \leqq u \leqq 2,$$
$$\tfrac{1}{2}\sin x \leqq v \leqq \tfrac{5}{6}.$$

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In Section 3, the results in Section 2 are applied to reaction diffusion systems in population biology. Similar systems were considered by many authors (see e.g. [2], [3], [8], [11] and [14]). Results in Theorems 4 and 5 agree with those given in [8], [9] and [11]. However, our proof here is different. Moreover, using results in Section 2, we can begin analysing systems of large numbers of interacting species. Large systems of biological interactions are attracting recent attention (see e.g. [4] and [15]). Finally, the schemes will also be adaptable to the study of finite difference approximations (see e.g. [7]), and to the study of stability of associated parabolic systems (see e.g. [10]).

2. General theorems and alternating schemes

Specific assumptions made in this paper are as follows. The domain $\mathscr{D}$ is bounded and its boundary $\delta\mathscr{D}$ is of class $C^{2+\alpha}$, $\alpha \in (0, 1)$. The functions $f_i(x, u_1, \ldots, u_m)$, $i = 1, \ldots, m$ are C^α in x and $C^{1+\alpha}$ in u_j, for each j, $x \in \bar{\mathscr{D}}$, and u in every compact set. For Theorem 1, we assume that each f_i depends monotonically on the variables $(u_1, \ldots, u_m)$ for all $(x, u) \in \mathscr{D} \times R^m$, i.e. $\partial f_i/\partial u_j \geqq 0$ or $\partial f_i/\partial u_j \leqq 0$, each i, j. (These monotonicity assumptions will be slightly less restrictive for Theorem 2.) The boundary functions g_i will be assumed in $C^{2+\alpha}$. Let $u^{-1}(x)$ and $u^0(x)$ be any two given vector functions with each component in $C^\alpha(\bar{\mathscr{D}})$. We define inductively a sequence of functions $u^k(x)$, $k = 1, 2, \ldots$ which will be used in Theorem 1

$$\begin{aligned}
&\Delta u_1^k + f_1(x, u_1^{k+\ell_1(1)}, \ldots, u_j^{k+\ell_1(j)}, \ldots, u_m^{k+\ell_1(m)}) = 0\\
&\vdots\\
&\Delta u_m^k + f_m(x, u_1^{k+\ell_m(1)}, \ldots, u_j^{k+\ell_m(j)}, \ldots, u_m^{k+\ell_m(m)}) = 0 \quad x \in \mathscr{D}\\
&u_i^k(x) = g_i(x), \quad x \in \delta\mathscr{D}, \quad i = 1, \ldots, m,
\end{aligned} \tag{2.1}$$

where $\ell_i(j)$ is determined by the following rules.

Rule (I). For $i \leqq j$ (above and on the diagonal),

$$\ell_i(j) = \begin{cases} -2 & \text{if} \quad \partial f_i/\partial u_j \geqq 0\\ -1 & \text{if} \quad \partial f_i/\partial u_j \leqq 0\end{cases}.$$

Rule (II). For $i > j$ (below the diagonal),

$$\ell_i(j) = \begin{cases} 0 & \text{if} \quad \partial f_i/\partial u_j \geqq 0\\ -1 & \text{if} \quad \partial f_i/\partial u_j \leqq 0.\end{cases}$$

Note that each component of u^k is defined by solving a scalar linear equation.

To simplify notation, an expression of the form $f_i(x, \theta_1, \ldots, \theta_m)$ where θ_j, $1 \leqq j \leqq m$ may depend on i and k, will be abbreviated as $f_i(x, \theta_j)$. An expression of the form $f_i(x, \theta_1, \ldots, \theta_{i-1}, \hat{\theta}_i, \theta_{i+1}, \ldots, \theta_m)$ will be abbreviated as $f_i(x, \hat{\theta}_i, \theta_j)$. For example, the ith equation in (2.1) can be written as:

$$\Delta u_i^k + f_i(x, u_j^{k+\ell_i(j)}) = 0, \quad x \in \mathscr{D}.$$

Two m vectors $z = (z_1, \ldots, z_m)$ and $w = (w_1, \ldots, w_m)$ will be denoted as satisfying $z \leqq w$ if $z_i \leqq w_i$ for each $i = 1, \ldots, m$.