Differential
Equations
& Applications
[0137] First Galley Proofs

COMPUTING THE LOCATION AND THE DIRECTION OF BIFURCATION FOR SIGN CHANGING SOLUTIONS

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Abstract. We consider sign-changing solutions of the Dirichlet problem

$$u'' + \lambda f(u) = 0,\ 0 < x < 1,\ u(0) = u(1) = 0,$$

with $n \geqslant 0$ interior roots. We give a necessary and sufficient condition that a turn occurs at the solution $(\lambda, u(x))$, depending only on the maximum value of the solution $u(x)$. If a turn does occur, we give another formula allowing to compute the direction of the turn. Our results generalize those in P. Korman, Y. Li and T. Ouyang [6], where positive solutions were considered. We give similar results for Neumann problem.

1. Introduction

We consider the equation (for $u = u(x)$)

$$u'' + \lambda f(u) = 0,\ 0 < x < 1 \tag{1.1}$$

depending on a positive parameter λ, with either Dirichlet or Neumann boundary conditions. To continue the solutions in λ, one needs to consider the corresponding linearized equation

$$w'' + \lambda f'(u)w = 0,\ 0 < x < 1\,. \tag{1.2}$$

Observe that u' is a solution of (1.2). Let x_0 be any point on the interval $(0,1)$, such that $u'(x_0) \neq 0$, then one verifies that $u' \int_{x_0}^{x} [1/u'^2(t)]\,dt$ gives the second solution of (1.2), and the general solution, on any interval of monotonicity of $u(x)$, is then

$$w = c_1 u' + c_2 u' \int_{x_0}^{x} \frac{1}{u'^2(t)}\,dt\,. \tag{1.3}$$

Patching up general solutions on all intervals of monotonicity of $u(x)$, we can construct non-trivial solutions of (1.2), or try to prove that no such non-trivial solution $w(x)$ solution exists (in which case the solution $u(x)$ of (1.1) is called *non-singular*, and we can continue solutions of (1.1) by the implicit function theorem, see [3] for an

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exposition). In [6], P. Korman, Y. Li and T. Ouyang used similar ideas to give computer assisted proofs of two long standing conjectures on positive solutions of the Dirichlet problem, particularly proving S.-H. Wang's conjecture [9] on S-shaped curves from combustion theory. Here we consider the Neumann problem, and the Dirichlet problem with sign-changing solutions.

We are interested in sign-changing solutions of (1.1), having exactly $n \geqslant 0$ interior roots. Such solutions lie on global smooth solution curves, i.e., at each solution $(\lambda, u(x))$ either the implicit function theorem, or the M. G. Crandall and P. H. Rabinowitz theorem [1] applies, see e.g. [3] for the details. This means that either solutions can be continued in λ, or a simple turn occurs on the solution curve, providing multiple solutions at the same λ. Also, it is known that the number of the interior roots is preserved along the solution curves. Unlike the case of positive solutions, few *exact* multiplicity results are known for sign-changing solutions. It turns out that the shape of the solution curve may depend in a interesting way on the number of interior roots. Consider the sign-changing solutions of the Dirichlet problem

$$u'' + \lambda (e^u - 1) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0. \tag{1.4}$$

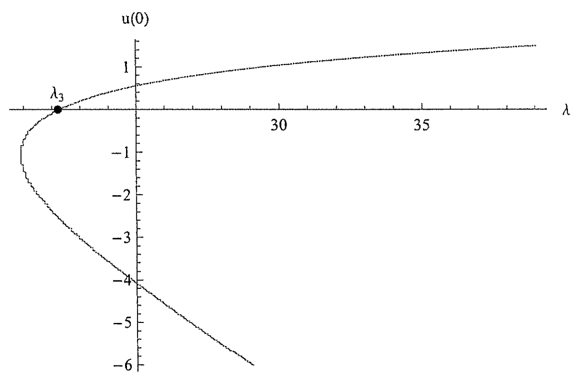

Figure 1: Solution curve of the problem (1.4) with two interior roots

Then $\lambda_m = (m^2\pi^2)/4$ are the Dirichlet eigenvalues of $-u''$ on the interval $(-1,1)$. It is well known that a curve of solutions with $m-1$ interior roots bifurcates from the trivial solution at $\lambda = \lambda_m$. In Figure 1 we present the solution curve with two interior roots, computed numerically using *Mathematica*. This curve bifurcates from zero, left to right, at $\lambda_3 = (9\pi^2)/4$, the curve is parabola-like, and the turn occurs where $u(0) < 0$. In Figure 2 the solution curve with four interior roots is computed. Now bifurcation from zero occurs at $\lambda_5 = (25\pi^2)/4$, it is right to left, the curve makes exactly one turn, and $u(0) > 0$ at the turning point. Using the results of the present paper, as well as those in R. Schaaf [8], we provide a computer assisted proof that the bifurcation diagram for the problem (1.4) in Figure 2 is correct (the shape of other solution curves can be justified in the same way). Our computations suggest that all sign-changing solution curves with $n = 4k$ interior roots (with integer $k > 1$), are similar to Figure 2 (i.e., $u(0) > 0$ at the turning point), while for $n = 4k+2$ the picture is like in Figure 1 ($u(0) < 0$ at the turning point). For n odd, solution curves are all parabola-like curves, with a turn at the trivial solution, as follows from the results of R. Schaaf [8].

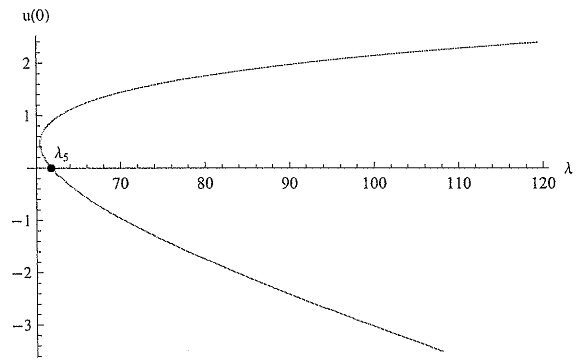

Figure 2: Solution curve of the problem (1.4) with four interior roots

2. Location of bifurcation for Neumann Problem

We consider the problem

$$u'' + f(u) = 0, \ \ 0 < x < 1, \ \ u'(0) = u'(1) = 0. \tag{2.1}$$

Here the parameter λ is absorbed into $f(u)$. The corresponding linearized problem is

$$w'' + f'(u)w = 0, \ \ 0 < x < 1, \ \ w'(0) = w'(1) = 0. \tag{2.2}$$

If the problem (2.2) admits only the trivial solution, we say that $u(x)$ is a *non-singular* solution of (2.1), otherwise we call $u(x)$ *singular*. It suffices to study solutions that increase on $(0,1)$, since the solution curves with arbitrary number of monotonicity changes are all similar, see R. Schaaf [8] or P. Korman [3]. We shall consider solutions that change sign. We denote $u(0) = \alpha$, $u(1) = \beta$, with $\alpha < 0 < \beta$. Since $u(x)$ is increasing, $u''(0) \geqslant 0$. The case $u''(0) = 0$ is not possible (then $u'(x)$ would satisfy a linear equation, with zero initial conditions, i.e., $u'(x) \equiv 0$, but $u(x)$ is increasing). So $u''(0) > 0$, i.e., $f(\alpha) < 0$. Similarly, $u''(1) < 0$ and $f(\beta) > 0$. So, we make the following natural assumption: $f(u) \in C^2(\mathbb{R})$ satisfies

$$f(\alpha) < 0, \ \ f(\beta) > 0, \ \ f(u) \text{ changes sign exactly once on } (\alpha, \beta). \tag{2.3}$$

Since the energy $E = \frac{1}{2}u'^2(x) + F(u(x))$ is constant for solutions of equation (2.1), it follows that

$$F(\alpha) = F(\beta). \tag{2.4}$$

We see that the value of α uniquely determines that of β ($F(u)$ is monotone by (2.3)). Hence the following function depends on α only

$$H(\alpha) \equiv -f(\alpha)\int_\beta^0 \frac{f(u)-f(\beta)}{[F(\beta)-F(u)]^{3/2}}\,du + f(\beta)\int_\alpha^0 \frac{f(u)-f(\alpha)}{[F(\alpha)-F(u)]^{3/2}}\,du + 2\frac{f(\alpha)-f(\beta)}{\sqrt{F(\alpha)}}.$$

THEOREM 2.1. *Let $u(x)$ be an increasing solution of* (2.1), *with $u(0)=\alpha$ and $u(1)=\beta$, and assume that the condition* (2.3) *holds. Then $u(x)$ is a singular solution (i.e., the problem* (2.2) *has a non-trivial solution) if and only if*

$$H(\alpha)=0. \tag{2.5}$$

We postpone the proof to Section 4.

3. Curves of sign-changing solutions

We study solutions of the Dirichlet problem

$$u''+\lambda f(u)=0,\ \ -1<x<1,\ \ u(-1)=u(1)=0, \tag{3.1}$$

with k interior zeros, depending on a positive parameter λ. For any odd k the solution curve has the same shape as the solution curve of the Neumann problem (2.1), considered in the preceding section, see e.g. R. Schaaf [8], or P. Korman [3].

Therefore, we shall now consider the case when k is even, $k=2n$. For this case solutions of (3.1) are symmetric with respect to $x=0$ (see e.g. [3]), and we can replace (3.1) by the Dirichlet-Neumann problem (absorbing the parameter λ into $f(u)$)

$$u''+f(u)=0,\ \ 0<x<1,\ \ u'(0)=u(1)=0, \tag{3.2}$$

with n interior roots on the interval $(0,1)$. We denote by $\beta=u(0)>0$ and by $\alpha<0$ the maximum and the minimum values of $u(x)$ respectively, and we shall assume that $f(u)\in C^2(\mathbb{R})$ satisfies the condition (2.3). The assumption $\beta=u(0)>0$ does not restrict the generality, since we can always change $u\to -u$. The linearized problem for (3.2) is

$$w''+f'(u)w=0,\ \ 0<x<1,\ \ w'(0)=w(1)=0. \tag{3.3}$$

We shall construct solutions of (3.3), by using the functions

$$r(x,x_1)=r(x)\equiv u'\int_x^{x_1}\frac{1}{u'^2(t)}\,dt,$$

whose properties we now study.

LEMMA 3.1. *Let $u(x)$ be a solution of the equation in* (3.2), *which is increasing on some interval* $(x_0,x_2) \subseteq (0,1)$, *with* $u'(x_0)=u'(x_2)=0$, *and* $u(x_1)=0$ *at some* $x_1 \in (x_0,x_2)$. *Denote* $L_+ = \lim_{x\to x_0+} r'(x,x_1)$, *and* $U_- = \lim_{x\to x_2-} r'(x,x_1)$. *Then*

$$\begin{aligned} L_+ &= \int_\alpha^0 \frac{f(u)-f(\alpha)}{2^{3/2}\left[F(\alpha)-F(u)\right]^{3/2}}\,du - \frac{1}{\sqrt{2F(\alpha)}}\,; \\ U_- &= \int_\beta^0 \frac{f(u)-f(\beta)}{2^{3/2}\left[F(\beta)-F(u)\right]^{3/2}}\,du - \frac{1}{\sqrt{2F(\alpha)}}\,. \end{aligned} \tag{3.4}$$

Also

$$\lim_{x\to x_0} r(x,x_1) = -\frac{1}{f(\alpha)}, \quad \textit{and} \quad \lim_{x\to x_2} r(x,x_1) = -\frac{1}{f(\beta)}\,. \tag{3.5}$$

Proof. Compute

$$r'(x) = u''(x)\int_x^{x_1} \frac{1}{u'^2(t)}\,dt - \frac{1}{u'(x)}\,. \tag{3.6}$$

As $x \downarrow x_0$, the first term tends to ∞, and the second one to $-\infty$. Therefore, we need to combine them. Write

$$-\frac{1}{u'(x)} = \int_x^{x_1} \frac{d}{dt}\frac{1}{u'(t)}\,dt - \frac{1}{u'(x_1)} = \int_x^{x_1} \frac{f(u(t))}{u'^2(t)}\,dt - \frac{1}{u'(x_1)}\,,$$

and use this in (3.6), to obtain

$$r'(x) = \int_x^{x_1} \frac{f(u(t))-f(u(x))}{u'^3(t)}\,u'(t)\,dt - \frac{1}{u'(x_1)}\,. \tag{3.7}$$

We can now calculate the limit from the right at x_0. From the energy relation

$$\frac{1}{2}u'^2(t) + F(u(t)) = F(\alpha) = F(\beta),$$

we express

$$u'(t) = \sqrt{2\left[F(\alpha)-F(u(t))\right]} \ \text{ and } \ u'(x_1) = \sqrt{2F(\alpha)}.$$

In the integral above we make a change of variables $t \to u$, by letting $u = u(t)$, obtaining

$$r'(x) = \int_{u(x)}^0 \frac{f(u)-f(u(x))}{2^{3/2}\left[F(\alpha)-F(u)\right]^{3/2}}\,du - \frac{1}{\sqrt{2F(\alpha)}}\,; \tag{3.8}$$

Taking the limit $x \to x_0+$, we obtain the first formula in (3.4). Taking the limit $x \to x_2-$, we obtain the second formula in (3.4).

The formula (3.5) follows by the L'Hospital's rule. □

Similarly, we prove the next lemma (begin with the formula (3.7)).

LEMMA 3.2. *Let $u(x)$ be a solution of the equation in* (3.2)*, which is decreasing on some interval* $(x_0,x_2) \subseteq (0,1)$*, with* $u'(x_0) = u'(x_2) = 0$*, and* $u(x_1) = 0$ *at some* $x_1 \in (x_0,x_2)$*. Denote* $U_+ = \lim_{x \to x_0+} r'(x,x_1)$*, and* $L_- = \lim_{x \to x_2-} r'(x,x_1)$*. Then*

$$\begin{aligned} U_+ &= -U_- ; \\ L_- &= -L_+ . \end{aligned} \tag{3.9}$$

REMARK. Our condition (2.3) implicitly rules out the possibility that

$$u(x_1) = u'(x_1) = 0 \text{ at some } x_1 \in (0,1) , \tag{3.10}$$

for sign-changing solutions. (This possibility would make $r(x,x_1)$ undefined.) Indeed, considering energy, condition (3.10) would have to hold at all roots of $u(x)$. Observe that $u''(x_1) \neq 0$, since otherwise $u'(x)$ would satisfy a linear equation with zero initial conditions at x_1, leading to a contradiction. If $u''(x_1) > 0$ (< 0), then $\alpha = \min u(x) = 0$ ($\beta = 0$), contrary to the assumption (2.3). (By the way, the cases just described were considered in P. Korman [2], in connection with symmetry breaking.) For positive solutions we shall assume that $f(0) \geqslant 0$, which will rule out (3.10) by the Hopf boundary lemma.

We now consider sign changing solutions of (3.2). Recall that we have denoted by $\alpha \leqslant 0$ their minimum value, and by $\beta > 0$ the maximal value.

THEOREM 3.1. *Let $u(x)$ be a solution of* (3.2)*, with* $u(0) = \beta > 0$*, and having exactly* $n \geqslant 0$ *interior roots. Assume that* $f(u) \in C^2(\mathbb{R})$ *satisfies* (2.3) *in case* $n \geqslant 1$*, and* $f(0) \geqslant 0$ *for* $n = 0$*. Let* U_+ *and* L_+ *be the numbers computed by* (3.4) *and* (3.9)*. Then $u(x)$ is a singular solution (i.e., the problem* (3.3) *has a non-trivial solution) if and only if one of the following two conditions, depending on the parity of n holds: if n is even, then*

$$\frac{nL_+ f(\beta)}{f(\alpha)} + (n+1)U_+ = 0; \tag{3.11}$$

if n is odd, then

$$\frac{(n+1)L_+ f(\beta)}{f(\alpha)} + nU_+ = 0. \tag{3.12}$$

Proof. The case $n = 0$ was considered in [6].
Case $n = 1$. Assume $u(x)$ is a singular solution of (3.2), with one interior root denoted by x_1. By x_2 we denote the point of minimum in $(x_1,1)$, $u(0) = \beta > 0$, $u(x_2) = \alpha < 0$. We shall construct a solution of (3.3) in the form

$$w(x) = \begin{cases} au' + r(x,x_1) & \text{for } x \in (0,x_2], \\ r(x,1) & \text{for } x \in (x_2,1), \end{cases} \tag{3.13}$$

with a constant a to be chosen. Clearly, both functions in (3.13) solve the equation in (3.3) on the corresponding sub-intervals. By ~~L~~emma 3.1, one-sided limits $w(x_2-) = w(x_2+) = -1/f(\alpha)$ are the same, and so we may define $w(x_2) = -1/f(\alpha)$. We now V

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choose the constant a to match the one-sided derivatives of $w(x)$ at x_2. We compute the one-sided derivatives, using Lemmas 3.1 and 3.2

$$-af(\alpha)+L_- = L_+ ,$$

i.e., $a = -2L_+/f(\alpha)$. With this value of a we compute, using Lemmas 3.1 and 3.2

$$w'(0) = \frac{2L_+ f(\beta)}{f(\alpha)} + U_+ .$$

To obtain a solution of the linearized problem (3.3) it is necessary and sufficient that $w'(0) = 0$, giving us (3.12).

Case $n=2$. Assume $u(x)$ is a singular solution of (3.2), with two interior roots. We be denote by x_1 the first root of $u(x)$, by x_2 we denote the point of minimum that follows. Let x_3 the second root of $u(x)$, by x_4 we denote the point of maximum that follows, $u(0) = u(x_4) = \beta > 0$, $u(x_2) = \alpha < 0$. We construct a solution of (3.3) in the form

$$w(x) = \begin{cases} au' + r(x,x_1) & \text{for } x \in (0,x_2], \\ bu' + r(x,x_3) & \text{for } x \in (x_2,x_4), \\ r(x,1) & \text{for } x \in (x_4,1), \end{cases} \tag{3.14}$$

with the constants a and b to be chosen. Using Lemmas 3.1 and 3.2, we equate one-sided derivatives at x_2 and at x_4:

$$\begin{aligned} -af(\alpha)+L_- &= -bf(\alpha)+L_+ \\ -bf(\beta)+U_- &= U_+ , \end{aligned}$$

i.e.,

$$\begin{aligned} b-a &= \frac{2L_+}{f(\alpha)}, \\ -b &= \frac{2U_+}{f(\beta)} . \end{aligned} \tag{3.15}$$

This gives $-a = \frac{2L_+}{f(\alpha)} + \frac{2U_+}{f(\beta)}$, i.e.,

$$w'(0) = \left(\frac{2L_+}{f(\alpha)} + \frac{2U_+}{f(\beta)} \right) f(\beta) + U_+ .$$

Setting $w'(0) = 0$, we obtain (3.11), in case $n=2$.

Case $n=3$. Assume $u(x)$ is a singular solution of (3.2), with three interior roots, $x_1 < x_3 < x_5$. Let $x_2 \in (x_1,x_3)$ and $x_6 \in (x_5,1)$ be points of minimum, and $x_4 \in (x_3,x_5)$ a point of maximum. This time

$$w(x) = \begin{cases} au' + r(x,x_1) & \text{for } x \in (0,x_2], \\ bu' + r(x,x_3) & \text{for } x \in (x_2,x_4), \\ cu' + r(x,x_5) & \text{for } x \in (x_4,x_6), \\ r(x,1) & \text{for } x \in (x_6,1) . \end{cases}$$

Equating one-sided derivatives at x_2, x_4 and x_6, we obtain similarly to (3.15):

$$b-a=\tfrac{2L_+}{f(\alpha)},$$
$$c-b=\tfrac{2U_+}{f(\beta)},$$
$$-c=\tfrac{2L_+}{f(\alpha)}.$$

This gives $-a=\frac{4L_+}{f(\alpha)}+\frac{2U_+}{f(\beta)}$, and then

$$w'(0)=\left(\frac{4L_+}{f(\alpha)}+\frac{2U_+}{f(\beta)}\right)f(\beta)+U_+.$$

Setting $w'(0)=0$, we obtain (3.12).

For n arbitrary we proceed similarly. If $n=2s$, then $-a=\frac{2sL_+}{f(\alpha)}+\frac{2sU_+}{f(\beta)}$, while in case $n=2s+1$, we have $-a=\frac{(2s+2)L_+}{f(\alpha)}+\frac{2sU_+}{f(\beta)}$, and the proof follows. □

4. Direction of bifurcation for Neumann problem

Proof of the Theorem 2.1 We consider again increasing solutions $u(x)$ of the Neumann problem (2.1). We use the definitions of $\alpha<0<\beta$ from Section 2. Let $x_1\in(0,1)$ be the root of $u(x)$. We construct a solution of the linearized problem (2.2) in the form

$$w(x)=c_0u'(x)+r(x,x_1), \tag{4.1}$$

with the constant c_0 to be chosen. Compute

$$w'(1)=-c_0f(\beta)+U_-=0,$$
$$w'(0)=-c_0f(\alpha)+L_+=0,$$

i.e.,

$$c_0=-\frac{U_+}{f(\beta)}=\frac{L_+}{f(\alpha)}. \tag{4.2}$$

Hence a non-trivial solution $w(x)$ exists iff

$$f(\alpha)U_++f(\beta)L_+=0,$$

which is the condition (2.5). □

Recall that the direction of bifurcation at a singular solution is governed by the integral $I=\int_0^1 f''(u)w^3\,dx$ (see the Theorem 5.2 below for the explanation).

THEOREM 4.1. *Let $u(x)$ be a singular increasing solution of* (2.1), *with* $u(0)=\alpha$, $u(1)=\beta$. *Then, with c_0 defined by* (4.2),

$$I=2\int_\alpha^\beta f''(u)\,[F(\alpha)-F(u)]\left[c_0+\int_0^u\frac{1}{2^{3/2}\,[F(\alpha)-F(s)]^{3/2}}\,ds\right]^3du. \tag{4.3}$$

Proof. At a singular solution we have $w(x) = c_0 u'(x) + u'(x) \int_{x_1}^{x} \frac{1}{u'^2}\, dt$, where x_1 is the root of $u(x)$. We then have

$$I = \int_0^1 f''(u(x)) u'^2(x) \left[c_0 + \int_{x_1}^{x} \frac{1}{u'^2(t)}\, dt \right]^3 u'(x)\, dx. \tag{4.4}$$

As before, letting $s = u(t)$,

$$\int_{x_1}^{x} \frac{1}{u'^2(t)}\, dt = \int_{x_1}^{x} \frac{1}{u'^3(t)} u'(t)\, dt = \int_0^{u(x)} \frac{1}{2^{3/2} \left[F(\alpha) - F(s) \right]^{3/2}}\, ds \tag{4.5}$$

Using this in (4.4), and making a change of variables $x \to u$, by letting $u = u(x)$, we conclude (4.3). $\square$

THEOREM 4.2. *In the conditions of the Theorem 4.1, a turn to the right occurs on the solution curve if $I > 0$, and a turn to the left, if $I < 0$.*

We postpone the proof until the next section.

5. Direction of bifurcation for sign-changing solutions of the Dirichlet problem

The argument leading to (4.5) gives us the following lemma.

LEMMA 5.1. *Define a function*

$$J(u) = \int_u^0 \frac{1}{2^{3/2} \left[F(\beta) - F(s) \right]^{3/2}}\, ds.$$

Let x_i be a root of $u(x)$, and x is taken from the adjacent to x_i interval of monotonicity of $u(x)$. Then

$$\int_x^{x_1} \frac{1}{u'^2(t)}\, dt = \begin{cases} J(u(x)) & \text{on the intervals where } u'(x) > 0 \\ -J(u(x)) & \text{on the intervals where } u'(x) < 0. \end{cases}$$

Assume now that $u(x)$ is a singular solution of (3.2) with exactly n interior roots, i.e., the corresponding linearized problem (3.3) admits a non-trivial solution $w(x)$. The direction of bifurcation is governed by the sign of the integral

$$I \equiv \int_0^1 f''(u(x)) w^3(x)\, dx. \tag{5.1}$$

Next we give a formula for the integral I, depending on the parity of n.

THEOREM 5.1. *Assume n is odd. Define the numbers $a_1, a_2, \ldots, a_n$ as follows:*

$$a_1 = -\frac{2L_+}{f(\alpha)},\ a_2 = -\frac{2L_+}{f(\alpha)} - \frac{2U_+}{f(\beta)},\ a_3 = -\frac{4L_+}{f(\alpha)} - \frac{2U_+}{f(\beta)},$$

and so on (*to get* a_{k+1}, *we alternatively add either* $-\frac{2L_+}{f(\alpha)}$ *or* $-\frac{2U_+}{f(\beta)}$ *to* a_k). *Define* $n+1$ *integrals as follows:*

$$I_0 = 2\int_\alpha^0 f''(u)\,[F(\beta)-F(u)]\,J^3(u)\,du,$$
$$I_1 = 2\int_\beta^\alpha f''(u)\,[F(\beta)-F(u)]\,(a_1-J(u))^3\,du,$$
$$I_2 = 2\int_\alpha^\beta f''(u)\,[F(\beta)-F(u)]\,(a_2+J(u))^3\,du,$$
$$I_3 = 2\int_\beta^\alpha f''(u)\,[F(\beta)-F(u)]\,(a_3-J(u))^3\,du,\ \ldots,$$
$$I_n = 2\int_\beta^\alpha f''(u)\,[F(\beta)-F(u)]\,(a_n-J(u))^3\,du.$$

Then

$$I = I_0 + I_1 + \ldots + I_n. \tag{5.2}$$

Assume n *is even (including* $n=0$ *case). Define the numbers* $b_1, b_2, \ldots, b_n$ *as follows:*

$$b_1 = -\frac{2U_+}{f(\beta)},\ b_2 = -\frac{2U_+}{f(\beta)} - \frac{2L_+}{f(\alpha)},\ b_3 = -\frac{4U_+}{f(\beta)} - \frac{2L_+}{f(\alpha)},$$

and so on (to get b_{k+1}, *we alternatively add either* $-\frac{2U_+}{f(\beta)}$ *or* $-\frac{2L_+}{f(\alpha)}$ *to* b_k*). Define* $n+1$ *integrals as follows:*

$$K_0 = 2\int_\beta^0 f''(u)\,[F(\beta)-F(u)]\,(-J(u))^3\,du,$$
$$K_1 = 2\int_\alpha^\beta f''(u)\,[F(\beta)-F(u)]\,(b_1+J(u))^3\,du,$$
$$K_2 = 2\int_\beta^\alpha f''(u)\,[F(\beta)-F(u)]\,(b_2-J(u))^3\,du,$$
$$K_3 = 2\int_\alpha^\beta f''(u)\,[F(\beta)-F(u)]\,(b_3+J(u))^3\,du,\ \ldots,$$
$$K_n = 2\int_\beta^\alpha f''(u)\,[F(\beta)-F(u)]\,(b_n-J(u))^3\,du.$$

Then

$$I = K_0 + K_1 + \ldots + K_n. \tag{5.3}$$

Proof. The case $n=0$ was considered in [6].

Case $n=1$. Assume $u(x)$ is a singular solution of (3.2), with one interior root. The corresponding solution of the linearized problem was constructed in (3.13). Accordingly (with x_2 defined in the construction of $w(x)$), we decompose

$$I = \int_{x_2}^1 f''(u(x))w^3(x)\,dx + \int_0^{x_2} f''(u(x))w^3(x)\,dx.$$

In view of Lemma 5.1, the first integral is I_0, and the second one is I_1.
Case $n=2$. Assume $u(x)$ is a singular solution of (3.2), with two interior roots. The corresponding solution of the linearized problem was constructed in (3.14), with $a=b_2$ and $b=b_1$. Accordingly, we decompose

$$I=\int_{x_4}^{1} f''(u(x))w^3(x)\,dx+\int_{x_2}^{x_4} f''(u(x))w^3(x)\,dx+\int_{0}^{x_2} f''(u(x))w^3(x)\,dx.$$

In view of Lemma 5.1, the first integral is K_0, the second one is K_1, and the third one is K_2.

We proceed similarly for other n. □

We consider sign-changing solutions of the Dirichlet problem

$$u''+\lambda f(u)=0,\ \ 0<x<1,\ \ u(0)=u(1)=0. \tag{5.4}$$

depending on a positive parameter λ.

THEOREM 5.2. *Assume $u(x)$ is a singular sign-changing solution of* (5.4)*, and let I be the integral defined in* (5.1)*. Then a turn to the right occurs on the solution curve if $I>0$, and a turn to the left, if $I<0$.*

Proof. One shows that all solutions of (5.4) with n interior roots lie on smooth curves $(\lambda,u(x))$, i.e., at each solution either the implicit function theorem, or the bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [1] applies, see e.g. [3] for the details. In particular, this theorem implies that near a critical point $(\lambda_0,u_0(x))$, we have $\lambda=\lambda_0+\tau(s)$ on the solution curve, where s is a parameter with $\tau(0)=0, \tau'(0)=0$. Hence, the direction of the turn is governed by the sign of $\tau''(0)$. Recall (see e.g. [3]) that

$$\tau''(0)=-\lambda_0\frac{\int_0^1 f''(u(x))w^3(x)\,dx}{\int_0^1 f(u(x))w(x)\,dx}, \tag{5.5}$$

and $\int_0^1 f(u(x))w(x)\,dx=\frac{1}{2\lambda_0}u'(1)w'(1)$. By construction, $w(x)=u'\int_x^1\frac{1}{u'^2(t)}\,dt$ near $x=1$, and so $w'(1)=-\frac{1}{u'(1)}$. It follows that $-\int_0^1 f(u(x))w(x)\,dx=\frac{1}{2\lambda_0}$, and then $\tau''(0)=\frac{I}{2}$, and the result follows. □

Proof of the Theorem 4.2 As in the preceding case, increasing solutions of Neumann problem lie on smooth curves $(\lambda,u(x))$, i.e., at each solution either the implicit function theorem, or the bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [1] applies, and the formula (5.5) holds in this case too, see e.g. [3]. It follows from P. Korman [4] that

$$-\int_0^1 f(u(x))w(x)\,dx=\frac{1}{2\lambda_0}w(1)u''(1).$$

We have observed earlier that $u''(1)<0$ for increasing solutions, while the formula (4.1) and Lemma 3.1 imply that $w(1)<0$. Hence the sign of $\tau''(0)$ is the same as that of I, and the proof follows. □

6. A computer assisted exact multiplicity result

THEOREM 6.1. *All solutions of the Dirichlet problem*

$$u'' + \lambda (e^u - 1) = 0, \ -1 < x < 1, \ u(-1) = u(1) = 0,$$

having four interior roots lie on a unique smooth solution curve. This curve bifurcates from zero at $\lambda_5 = \frac{25\pi^2}{4}$, *and one of its branches travels to the left in* $(\lambda, u(0))$ *plane, until a turn to the right occurs at a critical* λ_0. *After the turn the curve travels to the right without any more turns. Another branch, bifurcating from the trivial solution, travels to the right without any turns. Hence, the problem has no solutions with four interior roots for* $\lambda < \lambda_0$, *exactly one such solution at* $\lambda = \lambda_0$, *and exactly two solutions for* $\lambda > \lambda_0$, *see Figure* 2.

Proof. By the result of R. Schaaf [8] there is at most one turn on the solution curve. Our computations will prove that a turn does happen. According to the Theorem 3.1, a turn on the solution curve occurs at a solution with $u(0) = \beta$, if and only if the function (here $F(\beta) = F(\alpha)$)

$$T(\beta) \equiv \frac{2L_+ f(\beta)}{f(\alpha)} + 3U_+$$

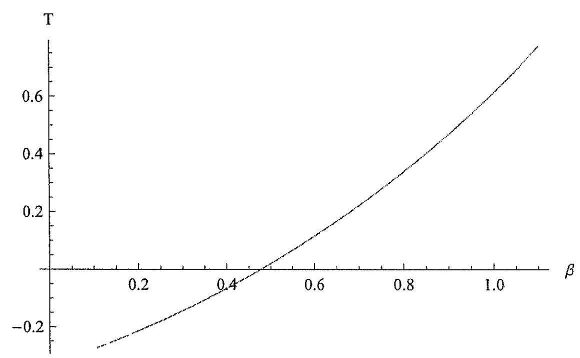

Figure 3: The graph of $T(\beta)$

vanishes (this is the case of two roots inside $(0,1)$). In Figure 3 we give the graph of $T(\beta)$, which leaves no doubt that this function has a root, and hence a turn occurs.

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