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EXISTENCE AND APPROXIMATION OF SOLUTIONS OF INVERSE-POSITIVE PROBLEMS

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1. INTRODUCTION

IN THIS paper we apply monotone techniques to equations other than second order elliptic or parabolic, in particular to wave and biharmonic equations. One of these techniques—the monotone iteration scheme, see [2, 9, 11], is extended in the theorem 1 below. This theorem, which was essentially proved in [7], shows that the monotone scheme can be applied to systems of equations with some gain of derivatives (compactness) for the corresponding linear problem, and with inverse-positivity. Inverse-positivity is a property more general than a weak maximum principle, and it states roughly that nonnegative data and the right-hand side of equation imply nonnegativity of the solution, see e.g. [12] or [8] for a discussion.

Theorem 1 requires that $(\partial/\partial u_i)f_j \geq 0$ for all i and j, i.e. nonlinearities to be increasing in all variables. The case $(\partial/\partial u_i)f_j \leq 0$ is covered by the theorem 2. In a forthcoming paper with Leung we treat all possible cases of "mixed" monotonicity. In the theorem 3 we extend the Serrin's sweeping principle [11] to the case of inverse-positive problems, which allows us to prove some general nonexistence results.

We apply our general results to fourth order ODE, to the Dirichlet problem for biharmonic equations, and to wave equations in space dimensions one, two and three (there is no inverse-positivity in dimensions greater than three). In [7] we applied the theorem 1 to a model noncoercive elliptic problem related to water waves.

Next we discuss the notation. By $\|\cdot\|_m$ we denote the norm in the mth Sobolev space $H^m \equiv W_2^m$, $m =$ integer ≥ 0. We write c for all irrelevant positive constants. If we wish to distinguish a positive constant, we denote it c_0, c_1, etc. Independent variable is usually $x = (x_1, \ldots, x_n)$. The unknown function $u(x)$ is either scalar or vector-valued. In the vector case $u = (u_1, \ldots, u_l)$, we define $\|u\|_m = \sum_{k=1}^{l} \|u_k\|_m$. If $v = (v_1, \ldots, v_l)$, we shall write $u \geq v$ iff $u_i \geq v_i$ for all i; if c is a scalar, $u \geq c$ implies $u_i \geq c$ for all i. Other order relations are defined similarly. If $f(u) : R^k \to R^k$, then $f_u \geq 0$ will imply $(\partial/\partial u_i)f_j \geq 0$ for all i and j. We shall use standard concepts of a cone and of order intervals in Banach spaces, see, e.g., [2]. We consider vector-valued partial differential operators of order m_0, $Lu = (L^1u_1, \ldots, L^lu_l)$. Here L^i is a scalar PDO of order m_i, and $m_0 = \max_i m_i$. We shall say that domain of L is H^m, when referring to $H^m \times H^m \times \ldots \times H^m$. Similarly with the boundary operators, $B_ju = (B_{j1}u_1, \ldots, B_{jl}u_l)$, m_{ji} denotes the order of B_{ji}, and we call $m_j = \max m_{ji}$ the order of B_j, $j = 1, \ldots, k$.

2. GENERAL RESULTS

THEOREM 1. Consider the boundary value problem

$$
\begin{aligned}
Lu &= f(x, u) \quad \text{in } \mathscr{D} \\
B_j u &= g_j(x) \quad \text{on } \partial\mathscr{D}, j = 1, 2, \ldots, k.
\end{aligned} \tag{1}
$$

Here L and B_j are vector-valued partial differential operators of orders $m_0 > 0$ and $m_j \geq 0$ respectively; $\mathscr{D}$ is a domain in R^n, $\partial\mathscr{D}$ is part (or the whole) of its boundary; $u(x) = (u_1, \ldots, u_l)$ is the unknown function.

We make no explicit assumptions on L, B_j and $\mathscr{D}$. Instead, we require the problem (1) to satisfy the following conditions.

(i) *Inverse-positivity.* Conditions

$$Lu \geq 0 \text{ in } \mathscr{D}, B_j u \geq 0 \text{ on } \partial\mathscr{D}$$

imply $u \geq 0$ in $\bar{\mathscr{D}}$.

(ii) *Solvability of the linear problem and compactness.* For any $F(x) \in C^{m_0}$, $G_j(x) \in C^{m_j}$ the problem

$$
\begin{aligned}
Lu &= \mathscr{F}(x) \quad \text{in} \quad \mathscr{D} \\
B_j u &= G_j(x) \quad \text{on} \quad \partial\mathscr{D}, \qquad j = 1, 2, \ldots, k
\end{aligned}
$$

is uniquely classically solvable and (for any $m \geq 0$)

$$\|u\|_{m+1} \leq c\left(\|F\|_m + \sum_j \|G_j\|_m\right). \tag{2}$$

(iii) There exists a supersolution $\phi(x) = (\phi_1(x), \ldots, \phi_l(x))$, such that

$$
\begin{aligned}
L\phi &\geq f(x, \phi) \quad \text{in } \mathscr{D} \\
B_j \phi &\geq g_j(x) \quad \text{on } \partial\mathscr{D}, j = 1, 2, \ldots, k,
\end{aligned}
$$

and a subsolution $\psi(x) = (\psi_1(x), \ldots, \psi_l(x))$, defined by reversing the above inequalities.

(iv) $\psi(x) \leq \phi(x)$ and $f_u(x, u) \geq 0$ for $x \in \mathscr{D}$, $\psi \leq u \leq \phi$.

Denote $m = \max_j(m_0, m_j) + [n/2] + 1$, $\mu = \max_j m_j$, and assume finally that $\psi, \phi \in C^{m_0}(\mathscr{D}) \cap C^{\mu}(\partial\mathscr{D})$, $f \in C^m(\mathscr{D})$, $g \in C^{\mu}(\partial\mathscr{D})$. Then the problem (1) has a classical solution $u(x)$ with $\psi \leq u \leq \phi$ in $\mathscr{D}$. (More precisely, the problem (1) has minimal and maximal solutions obtained by monotone iteration scheme.)

This theorem was proved in [7] in a more general setting for the case of one equation. Generalization to the system case is immediate. Also, examining the proof, we see that an estimate weaker than (2) suffices:

$$\|u\|_{m+1} \leq c(\|F\|_m + \Sigma\|G_j\|_m + \|u\|_0). \tag{2'}$$

THEOREM 2. For the problem (1) assume conditions (i) and (ii) of the theorem 1. In addition assume the following.

(iii)′ Starting with some $u_0(x)$ define inductively

$$\begin{aligned} Lu_{n+1} &= f(x, u_n), n = 0, 1, 2, \ldots \\ B_j u_{n+1} &= g_j(x), j = 1, \ldots, k, \end{aligned} \tag{3}$$

and assume that $u_0 \leqslant u_1$, $u_0 \leqslant u_2$.

(iv)′ $f_u(x, u) \leqslant 0$ for $x \in \mathfrak{D}$ and $u_0 \leqslant u \leqslant u_1$.

Finally, with m as in the theorem 1 assume that f, $u_0 \in C^m(\mathfrak{D})$, $g \in C^m(\partial\mathfrak{D})$. Then (1) has a classical solution $u(x)$, and moreover

$$u_0 \leqslant u_2 \leqslant u_4 \leqslant \ldots \leqslant u \leqslant \ldots \leqslant u_5 \leqslant u_3 \leqslant u_1. \tag{4}$$

(Convergence of $\{u_n\}$ to u requires additional assumptions.)

Proof. Rewrite (3) as

$$u_{n+1} = L^{-1}f(x, u_n) \equiv Mu_n,$$

and notice that M is a decreasing operator with respect to positive cone in H^m. Then (4) easily follows by induction (see [14]). Since H maps the order interval $[u_0, u_1]$ into itself, existence of solutions follows from Schauder's fixed point theorem as stated in [2, p. 660]. (Since the order interval $[u_0, u_1]$ is not bounded in H^m, we need this sharper version of Schauder's theorem). The smoothness of solution follows by Sobolev imbedding.

Remark. Differentiability conditions in the theorems 1 and 2 usually can be considerably reduced. In particular, if the problem admits a weak formulation, one can first produce a weak solution, and then use "boot-strap" arguments to show that it is classical. Or, if the problem has a bounded Green's function, one can recast it in equivalent integral form and pass to the limit by dominated convergence theorem.

The following theorem is an extension of Serrin's sweeping principle [11].

THEOREM 3. Assume the following.

(i) The problem (1) is strongly inverse-positive, i.e. $Lu \geqslant 0$, $Lu \not\equiv 0$, $B_j u \geqslant 0$ imply $u > 0$ in $\mathfrak{D}$.

(ii) $f(x, u)$ is strictly increasing in u for $x \in \mathfrak{D}$.

(iii) Let $\{v_\lambda\}$, $\lambda \geqslant \lambda_0$ be an increasing and continuous in λ family of subsolutions, none of which is a solution, and $u > v_{\lambda_0}$ in $\mathfrak{D}$, $u \geqslant v_\lambda$ on $\partial\mathfrak{D}$ for all $\lambda \geqslant \lambda_0$.

Then $u > v_\lambda$ in $\mathfrak{D}$ for any $\lambda \geqslant \lambda_0$.

Proof. We shall get a contradiction at the first $\bar\lambda > \lambda_0$, where $v_{\bar\lambda}(x)$ touches $u(x)$, i.e. $v_{\bar\lambda}(x_0) = u(x_0)$ for some $x_0 \in \mathfrak{D}$ and $v_{\bar\lambda}(x) \leqslant u(x)$, $v_{\bar\lambda}(x) \not\equiv u(x)$. Let $w = u - v_{\bar\lambda}$. Then

$$Lw = f(u) - f(v_{\bar\lambda}) \geqslant 0, Lw \not\equiv 0 \quad \text{in} \quad \mathfrak{D}; B_j w \geqslant 0 \quad \text{on} \quad \partial\mathfrak{D}.$$

Hence $w > 0$ in $\mathfrak{D}$, $w(x_0) > 0$ a contradiction.

This theorem provides a tool for proving nonexistence results for (1), and the strategy is as follows. Assume one can find a family of subsolutions as described above $\{v_\lambda\}$, $\lambda_0 \leqslant \lambda < \infty$, so that $v_{\lambda_0} < u$ in $\mathfrak{D}$, and for any $M > 0$ there is $\lambda' > \lambda_0$, such that $\sup_{\mathfrak{D}} v_{\lambda'} > M$. Then by theorem 3 the problem (1) cannot have a bounded solution (solution is "sweeped out" to infinity).

Notice that strong inverse positivity of the problem (1) and the estimates (2) imply that the solution operator L^{-1} of (1) extends to a strongly positive and compact operator in $C(\mathfrak{D})$. Then by the Krein–Rutman theorem it has a positive in $\mathfrak{D}$ eigenfunction $\phi_0(x)$, i.e. for some $\lambda_0 > 0$

$$L\phi_0 = \lambda_0 \phi_0 \quad \text{in} \quad \mathfrak{D}, B_j\phi_0 = 0, \quad j = 1, \ldots, k \quad \text{on} \quad \partial\mathfrak{D}. \tag{5}$$

THEOREM 4. Consider the problem

$$\begin{aligned} Lu &= \lambda f(x, u) \quad \text{in} \quad \mathfrak{D} \\ B_j u &= g_j(x) \quad \text{on} \quad \partial\mathfrak{D}, j = 1, 2, \ldots, k. \end{aligned} \tag{6}$$

Assume conditions (i), (ii) of the theorem 1, and moreover that the problem is strongly inverse-positive. Let f, g_j be as smooth as in the theorem 1, and $(c_0, c_1 > 0)$

(iii)′ $f(x, u) \geq c_0$, $f(x, u) \geq c_1 u$, $f_u \geq 0$ for $x \in \mathfrak{D}$ and $u \geq 0$.
(iv)′ $g_j(x) \geq 0$, $j = 1, \ldots, k$, $x \in \partial\mathfrak{D}$; one of B_j is an identity operator.

Then there is a critical number $\lambda_c > 0$, such that for $0 < \lambda < \lambda_c$ the problem (6) has a positive solution, while for $\lambda > \lambda_c$ there is no positive solution.

Proof. The argument is divided into three parts. First we show that (6) has a positive solution for λ sufficiently small. Then we show that if (6) has a positive solution for $\bar{\lambda}$, then it has a positive solution for all $0 < \lambda < \bar{\lambda}$. Finally, we show that for λ sufficiently large (6) has no positive solution.

(i) For small λ we construct super- and subsolutions, and apply the theorem 1. Define ϕ by solving $L\phi = 1$ in $\mathfrak{D}$, $B_j\phi = g_j(x)$ on $\partial\mathfrak{D}$. Then ϕ is a supersolution for λ sufficiently small. For a subsolution we take $\psi > 0$, defined by $L\psi = \lambda c_0/2$ in $\mathfrak{D}$, $B_j\psi = g_j(x)$ on $\partial\mathfrak{D}$.

(ii) Assume that $L\bar{u} = \bar{\lambda} f(x, \bar{u})$, $B_j\bar{u} = g_j(x)$. Then obviously $\bar{u}$ is a supersolution of (6) for $\lambda < \bar{\lambda}$. Taking subsolution ψ as in (i), we conclude that (6) has a positive solution for $0 < \lambda < \bar{\lambda}$.

(iii) We use Serrin's sweeping principle (theorem 3), and take $v_\mu = \mu\phi_0$(ϕ_0 as defined by (5)), $\mu > 0$. For μ sufficiently small and fixed $\lambda > 0$

$$L(u - \mu\phi_0) = \lambda f(x, u) - \mu\lambda_0\phi_0 > 0, B_j(u - \mu\phi_0) = g_j \geq 0,$$

i.e. $u > \mu\phi_0$ in $\mathfrak{D}$. On the other hand, fix $\lambda > \lambda_0/c_1$, then for any $\mu > 0$

$$\begin{aligned} L\mu\phi_0 &= \mu\lambda_0\phi_0 < c_1\lambda\mu\phi_0 \leq \lambda f(x, \mu\phi_0), \\ B_j\mu\phi_0 &= 0 \leq g_j(x), \end{aligned}$$

i.e. v_μ is a family of subsolutions as in the theorem 3. Hence (6) has no positive solution for $\lambda > \lambda_0/c_1$, completing the proof.

Remark. In the conditions of the preceding theorem consider the problem

$$\begin{aligned} Lu &= u^p + \lambda \quad \text{in} \quad \mathfrak{D}, p > 1 \\ B_j u &= g_j(x) \quad \text{on} \quad \partial\mathfrak{D}. \end{aligned} \tag{7}$$

By an argument similar to the above, it follows that for

$$\lambda > \left(\frac{\lambda_0}{p}\right)^{p/(p-1)} (p-1) \tag{8}$$

the problem (7) has no positive solution.

Next we consider some applications of our results.

3. STATIC DEFLECTION OF A BEAM

Consider the problem (one-dimensional)

$$\begin{aligned} &u'''' = f(x, u), 0 < x < l \\ &u(0) = \alpha, u'(0) = \beta, u(1) = \gamma, -u'(1) = \delta. \end{aligned} \tag{9}$$

If the constants α, β, γ, δ are all zero, it describes deflection of a clamped beam under the load $f(x, u)$.

THEOREM 5. Assume the following for $0 < x < l$.

(i) There exists a supersolution $\phi(x) \in C^4$, i.e.

$$\phi'''' \geq f(x, \phi),\ \phi(0) \geq \alpha,\ \phi'(0) \geq \beta,\ \phi(1) \geq \gamma,\ -\phi'(1) \geq \delta,$$

and a subsolution ψ defined by reversing the above inequalities. Moreover $\psi \leq \phi$.

(ii) f is continuous, increasing in u for $\psi \leq u \leq \phi$. Then the problem (9) has a C^4 solution $u(x)$, and $\psi \leq u \leq \phi$.

Proof. If $f = f(x)$, then (unique) solution of (9) can be written as

$$\begin{aligned} u(x) = \alpha\left(1 - \frac{3}{l^2}x^2 + \frac{2}{l^3}x^3\right) + \beta\frac{x}{l^2}(l-x)^2 + \gamma\frac{x^2}{l^3}(3l - 2x) + \delta\frac{x^2}{l^2}(1-x) \\ + \int_0^l G(x, \xi) f(\xi)\, d\xi, \end{aligned} \tag{10}$$

where $G(x, \xi)$ is the Green's function of (9) with $\alpha = \beta = \gamma = \delta = 0$, and is given by (see [3])

$$G(x, \xi) = \frac{1}{6l^3}x^2(\xi - l)^2(3l\xi - 2\xi x - lx) \quad \text{for} \quad x \leq \xi, G(x, \xi) = G(\xi, x).$$

It is easy to check that $G > 0$ for $0 < x$, $\xi < l$, so that our problem (9) is strongly inverse positive (i.e. $f \geq 0$ implies $u > 0$). In the usual way one obtains an increasing and a decreasing sequences of iterates. To pass to the limit it suffices to rewrite (9) in the equivalent integral form (see (10)), and use the monotone convergence theorem.

Similarly, we prove the following theorem.

THEOREM 6. For the problem (9) define a sequence of iterations as in the theorem 2, and again assume that $u_0 \leq u_1$, $u_0 \leq u_2$, and that $f(x, u)$ is a continuous function, decreasing in u for $0 \leq x \leq l$, $u_0 \leq u \leq u_1$.

Then (9) has a C^4 solution and (4) holds. It is also easy to interpret theorem 3 for the problem (9). Next we give some simple examples.

Example 1. $u'''' = u^2 + 1$, $u(0) = u'(0) = u(l) = u'(l) = 0$. Seek super- and subsolutions in the form $\varphi = \alpha x^2(l-x)^2$, $\psi = \beta x^2(l-x)^2$. Simple computations show that we can take $\beta = 1/24$, and α the smaller root of $(l^8/256)\,\alpha^2 - 24\alpha - 1 = 0$. For $l = 1$, α agrees with $1/24$ to six decimal places, and we conclude existence of solutions with the estimates

$$\frac{1}{24}x^2(l-x)^2 \leqslant u(x) \leqslant \left(\frac{1}{24}+\varepsilon\right)x^2(l-x)^2,\ \varepsilon < 10^{-6}.$$

For l large there is no solution, as can be seen by scaling $x = l\xi$ and applying the theorem 4.

Example 2. $u'''' = e^{-u}$, $u(0) = u'(0) = u(l) = u'(l) = 0$. Taking $u_0 \equiv 0$, we compute

$$u_1 = \frac{x^2(l-x)^2}{24}.$$

Since $u_2 \geqslant 0$, the theorem 6 applies, giving existence of solutions with

$$0 \leqslant u(x) \leqslant \frac{x^2(l-x)^2}{24}.$$

It is easy to see that the solution is unique.

Notice that in both previous examples one can easily compute further monotone iterates.

Example 3. $u'''' = u^2 + \lambda$, $u(0) = u'(0) = u(l) = u'(l) = 0$. By (8), the problem has no solution, provided $\lambda > \lambda_0^2/4$, where λ_0 is the principal eigenvalue of the corresponding linear problem ($\lambda_0 \simeq 500$ if $l = 1$, see [3]).

Remark. All considerations of this paragraph depended on positivity of the Green's function, and hence carry over to other important boundary conditions, e.g. $u(0) = u''(0) = u(l) = u''(l) = 0$ and $u(0) = u'(0) = u(l) = u''(l)$. It is easy to check that the Green's functions for these problems are positive (see [3]).

4. BIHARMONIC EQUATION

Consider the problem

$$\begin{aligned} &\Delta^2 u = f(x,u) \quad \text{in} \quad \mathfrak{D} \\ &u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\mathfrak{D}\ \left(\frac{\partial}{\partial n}\text{-normal derivative}\right). \end{aligned} \tag{11}$$

$\mathfrak{D}$ is a bounded domain in R^n. We assume that its smoothness and shape are such that

(i) Green's function exists and is positive;

(ii) for any $F(x) \in C^\alpha$ the problem $\Delta^2 u = F(x)$ in $\mathfrak{D}$,

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\mathfrak{D}$$

has (unique) solution and Schauder's and L_p estimates hold (see [1, theorems 7.3, 12.7, 15.2]).

THEOREM 7. In addition to the above conditions on $\mathfrak{D}$, assume the following.

(i) There exists a supersolution $\phi(x) \in C^4$, defined by

$$\Delta^2\phi \geqslant f(x, \phi) \quad \text{in} \quad \mathfrak{D}, \phi = \frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \partial\mathfrak{D},$$

and a subsolution $\psi(x)$, defined by reversing the above inequalities, and $\psi \leqslant \phi$ in $\mathfrak{D}$.

(ii) $f \in C^\alpha$ and increasing in u for $x \in \mathfrak{D}$, $\psi \leqslant u \leqslant \phi$. Then the problem (11) has a C^4 solution, and $\psi \leqslant u \leqslant \phi$.

Proof. Positivity of Green's function implies that the problem is inverse-positive, which allows us to define two standard sequences of monotone iterates. Passage to the limit is like that in [11], using Schauder's and L_p estimates.

It is easy to interpret theorems 2 and 4 for the problem (11). Next we give two simple examples (here $u = u(x, y)$, and the Green's function is known to be positive, see [8]).

Example 4.

$$\Delta^2 u = \sqrt{u + 1} \text{ in } x^2 + y^2 < R^2, u = \frac{\partial u}{\partial r} = 0 \text{ on } x^2 + y^2 = R^2.$$

One easily sees existence of solutions with the estimates

$$\frac{1}{64}(x^2 + y^2 - R^2)^2 \leqslant u \leqslant \left(\frac{R^2 + \sqrt{(R^4 + 256)}}{128}\right)^2 (x^2 + y^2 - R^2)^2.$$

Example 5.

$$\Delta^2 u = \frac{1}{c_0 u + 1} \quad \text{in} \quad x^2 + y^2 < R^2, u = \frac{\partial u}{\partial r} = 0 \quad \text{on} \quad x^2 + y^2 = R^2, c_0 > 0.$$

Taking $u_0 \equiv 0$, we compute $u_1 = (1/64)(x^2 + y^2 - R^2)^2$ and $u_2 \geqslant 0$. Hence a (unique) solution exists, and $0 \leqslant u(x, y) \leqslant (1/64)(x^2 + y^2 - R^2)^2$.

Remark. Notice that definitions of super and subsolutions are more restrictive than in one-dimensional case. This is because conditions $\Delta^2 u \geqslant 0$ in $\mathfrak{D}$, $u \geqslant 0$, $-(\partial u/\partial n) \geqslant 0$ on $\partial\mathfrak{D}$ do not imply $u \geqslant 0$ in $\mathfrak{D}$ even in the two-dimensional case, see [4]. Also, we considered only one set of boundary conditions. There are others with inverse-positivity, see [8], however, then it seems possible to use results of Tsai [13], after converting the problem to a system of second order equations.

5. BOUNDED STRING WITH FREE ENDS

Consider the problem ($n = 1$)

$$\begin{aligned} &u_{tt} - u_{xx} - pu_t = f(x, t, u), 0 < x < l, t > 0, p = \text{const} \geqslant 0 \\ &u(x, 0) = g(x), u_t(x, 0) = h(x), u_x(0, t) = u_x(l, t) = 0, \end{aligned} \tag{12}$$

and also the problem (12′) with f changed to $f(x, t)$.

THEOREM 8. Assume the following for $0 < x < l$, $0 < t < T$.
(i) there exists a supersolution $\phi(x, t) \in C^2$, defined by

$$\phi_{tt} - \phi_{xx} - p\phi_t \geqslant f(x, t, \phi)$$
$$\phi(x, 0) = g(x),\ \phi_t(x, 0) \geqslant h(x),\ \phi_x(0, t) = \phi_x(l, t)\,(\phi(x, 0) \geqslant g(x) \quad \text{if} \quad p = 0),$$

and a subsolution $\psi(x) \in C^2$, defined by reversing the above inequalities and $\psi \leqslant \phi$.
(ii) $pT < 1/2$.
(iii) $g'(0) = g'(l) = h'(0) = h'(l) = 0$.
(iv) f, $h \in C^4$, $g \in C^5$; $f_u \geqslant 0$ for $\psi \leqslant u \leqslant \phi$.
Then the problem (12) has a C^2 solution.

Proof. Uniqueness for (12)′ follows from the estimate (14) below. Let f_e, g_e, h_e denote even $2l$ periodic in x extensions of f, g, h. Solution of the problem (12)′ is equal to that of the Cauchy's problem

$$u_{tt} - u_{xx} - pu_t = f_e(x, t),\ u(x, 0) = g_e(x),\ u_t(x, 0) = h_e(x).$$

It follows from [10, p. 199] that $f_e \geqslant 0$, $g_e \equiv 0$ ($g_e \geqslant 0$ if $p = 0$), $h_e \geqslant 0$ imply $u \geqslant 0$. This shows that our problem (12)′ is inverse-positive. Hence the proof will follow from the theorem 1, once we establish the following lemma.

LEMMA 1. For the problem (12)′ we have the estimate (m = integer $\geqslant 0$)

$$\|u\|_{m+1} \leqslant c(\|f\|_m + \|g\|_{m+1} + \|h\|_m + \|u\|_0),\ c = c(T). \tag{13}$$

Here we use Sobolev norms in x, t variables for u and f, and in x variable only for g and h.

We need the following lemma, whose standard proof we omit.

LEMMA 2. $\int_0^l (\mathscr{D}_x^r \mathscr{D}_t^s f(x, 0))^2 \,\mathrm{d}x \leqslant c\|f\|_{r+s+1}^2$, $c = c(T)$.

Proof of lemma 1. Multiply (12)′ by u_t and integrate $\int_0^l \mathrm{d}x$ by parts. Get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^l (u_t^2 + u_x^2)\,\mathrm{d}x - p\int_0^l u_t^2\,\mathrm{d}x = \int_0^l u_t f\,\mathrm{d}x.$$

Integrate this $\int_0^t \mathrm{d}t$, $0 < t \leqslant T$.

$$\frac{1}{2}\int_0^l (u_t^2 + u_x^2)\,\mathrm{d}x - \frac{1}{2}\int_0^l (h^2 + g'^2)\,\mathrm{d}x - p\int_0^l\int_0^T u_t^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^l\int_0^T |u_t f|\,\mathrm{d}x\,\mathrm{d}t.$$

Integrating again $\int_0^t \mathrm{d}t$ and using (ii), we easily get:

$$\int_0^l\int_0^T (u_t^2 + u_x^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant c(T)\left(\int_0^l (h^2 + g'^2)\,\mathrm{d}x + \int_0^l\int_0^T f^2\,\mathrm{d}x\,\mathrm{d}t\right), \tag{14}$$

which is the desired estimate (13) for $m = 0$. Higher estimates are proved by induction. Using the equation (12)′, one easily shows by induction that

$$\mathscr{D}_t^m u(x, 0) = c_1 \mathscr{D}_t^k g + c_2 \mathscr{D}_t^{k-1} h + c_3 \mathscr{D}_t^r \mathscr{D}_x^s f(x, 0) + \dots \tag{15}$$

where $k \leqslant m$, $r + s \leqslant m - 2$, and the remaining terms are of lower order. Let $\mathscr{D}_t^m u = w$. Then

$$\begin{aligned} &w_{tt} - w_{xx} - w_t = \mathscr{D}_f^m f \\ &w(x, 0) = \mathscr{D}_t^m u(x, 0),\ w_t(x, 0) = \mathscr{D}_t^{m+1} u(x, 0),\ w_x(0, t) = w_x(l, t) = 0. \end{aligned} \tag{16}$$

Applying to (16) the estimate (13) with $m = 0$, using (15), lemma 1, and estimating all remaining derivatives from the equation (12)′, we establish our estimate (13).

6. CAUCHY'S PROBLEM FOR NONLINEAR WAVE EQUATION IN DIMENSIONS ONE, TWO AND THREE

THEOREM 9. Consider the problem ($n = 1$)

$$u_{tt} - u_{xx} = f(x, t, u),\ u(x, 0) = g(x),\ u_t(x, 0) = h(x). \tag{17}$$

For $-\infty < x < \infty$ and $0 \leqslant t \leqslant T$ we assume the following.

(i) There exists a supersolution $\phi(x, t) \in C^2$, i.e. $\phi_{tt} - \phi_{xx} \geqslant f(x, t, \phi)$, $\phi(x, 0) \geqslant g(x)$, $\phi_t(x, 0) \geqslant h(x)$, and a subsolution $\psi(x)$, defined by reversing the above inequalities, and $\psi \leqslant \phi$.

(ii) $f_u \geqslant 0$ for $\psi \leqslant u \leqslant \phi$; f, $h \in C^1$, $g \in C^2$. Then the problem (17) has a C^2 solution, and $\psi \leqslant u \leqslant \phi$ for $0 \leqslant t \leqslant T$.

Proof. From D'Alembert's formula we see that the problem is inverse-positive. i.e. f, g, $h \geqslant 0$ imply $u \geqslant 0$, and also that f, $h \in C^1$ and $g \in C^2$ imply that $u \in C^2$. In the usual way we obtain two C^2 sequences of monotone iterates. Passage to the limit is by bounded convergence theorem in the D'Alemberts formula. This produces a measurable solution of the corresponding integral equation, and then by "boot-strap" we see that it is in C^2.

In dimensions two and three solution $u(x)$ of the linear problem is in general no smoother (in C^k) than h and f, and may be less smooth than g (see e.g. [6]). Hence we have to rely on the theorem 1 to prove the convergence.

THEOREM 10. Consider the problem

$$u_{tt} - \Delta u = f(x, t, u),\ u(x, 0) = g(x),\ u_t(x, 0) = h(x). \tag{18}$$

For $x \in R^n$, $n = 2, 3$ and $0 \leqslant t \leqslant T$ assume the following.

(i) There exists a supersolution $\phi(x, t) \in C^2$, i.e.

$$\phi_{tt} - \Delta\phi \geqslant f(x, t, \phi),\ \phi(x, 0) = g(x),\ \phi_t(x, 0) \geqslant h(x),$$

and a subsolution $\psi(x, t) \in C^2$, defined by reversing the above inequalities, and $\psi \leqslant \phi$.

(ii) $f_u \geqslant 0$ for $\psi \leqslant u \leqslant \phi$; f, $h \in C^4$, $g \in C^5$. Then the problem (18) has a C^2 solution.

Proof. First we consider the special case, when functions ϕ, ψ, g, h, f are of compact support in x. The problem (18) is inverse-positive, i.e. $f \geqslant 0$, $g \equiv 0$, $h \geqslant 0$ imply $u \geqslant 0$ [10]. The rest of the proof follows from the theorem 1 in view of the following standard lemma, whose proof we omit.

LEMMA 3. In the above assumptions we have ($f = f(x, t)$):

$$\|u\|_{m+1} \leq c(T)(\|f\|_m + \|g\|_{m+1} + \|h\|_m),$$

where we use Sobolev norm in $\mathcal{D} \times [0, T]$ for u and f, and the one in $\mathcal{D}$ for g and h; $\mathcal{D}$ is any bounded domain in R^n, containing the domain of influence of the data.

For the general case, we notice that the domain of dependence of any compact set is another compact set, outside of which we can modify our functions to be of compact support in x, without changing the solution.

We conclude by remarking, that there is no inverse-positivity for wave equations in space dimensions $n > 3$. This is clear from the explicit representation of solution [6, p. 223].

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