

Stability and instability of solutions of semilinear problems

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Abstract

For two point problems, and for balls in R^2 , we show how stability or instability of solutions can often be determined, when one knows just the maximum value of solution. As an application, we obtain various multiplicity results.

Key words: Stability or instability of solutions, exact multiplicity of solutions.

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1 Introduction

We consider the question of stability of positive solutions for the problem

$$(1.1) \quad u'' + \lambda f(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

and for a corresponding elliptic problem on balls in R^2 . Here λ is a positive parameter. For convenience we set this problem over the interval $(-1, 1)$ (we could consider any other interval as well, since the problem is autonomous). It is well known that positive solutions of (1.1) are even functions, with $u'(x) < 0$ for $x > 0$. Hence $u(0)$ is the maximum value of solution. It turns out that $\alpha = u(0)$ uniquely identifies the solution pair $(\lambda, u(x))$, see E.N. Dancer [3], or P. Korman [8]. Hence the solution set of (1.1) can be faithfully depicted by planar curves in (λ, α) plane. It is natural to ask: which way the solution curve travels through a given point (λ, α) ? An answer, using

time maps, has been known for a while, see K. Brown et al [2]. Namely, denoting $F(u) = \int_0^u f(t) dt$, and $h(u) = 2F(u) - uf(u)$, one has

$$\frac{d}{d\alpha} \lambda(\alpha)^{1/2} = \frac{1}{\sqrt{2}} \int_0^1 \frac{h(\alpha) - h(\alpha v)}{[F(\alpha) - F(\alpha v)]^{3/2}} dv.$$

We see that $\frac{d\lambda}{d\alpha} < 0$ (> 0) and the curve travels to the left (right) in (λ, α) plane, provided that $h(\alpha) < h(u)$ ($h(\alpha) > h(u)$) for all $u \in (0, \alpha)$. Time maps work only for autonomous ODE's. It is desirable to find a more flexible approach, which could be applied to other situations. One way to do so was given by P. Korman and J. Shi [11]. In the present paper we give another approach, which is both simpler, and it gives more general results. Both [11] and the present paper connect direction of the curves to the question of stability, which is reviewed below. It is known that the solution curve travels northeast in the (λ, α) plane iff the solution is stable, and northwest iff it is unstable (we provide a self-contained proof of this fact below). Hence, a change of stability implies a turn of the solution curve, and vice versa. In particular, we can prove either exact *S*-shapedness or uniqueness for the original equations of combustion theory (see R. Aris [1]), depending on the values of its parameters.

Our instability result holds also for balls in R^2 . As an application, we can easily recover the main result of S. Parter [14].

In the final section, we develop stability and instability results for a class of symmetric non-autonomous problems, extending and simplifying the results of P. Korman and J. Shi [11].

2 The ODE case

We consider positive solutions of

$$(2.1) \quad u'' + f(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

with $f(u) \in C^1(\bar{R}_+)$. The eigenvalue problem for the corresponding linearized equation is

$$(2.2) \quad w'' + f'(u)w + \mu w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0.$$

We will be particularly interested in the principal (the smallest) eigenvalue, which we will call μ_1 . The corresponding eigenfunction can be assumed to satisfy $w(x) > 0$ for $x \in (-1, 1)$. Recall that any positive solution of

(2.1) satisfies $u'(x) < 0$ ($u'(x) > 0$) for $x > 0$ ($x < 0$), see [7]. It follows that $\alpha = u(0)$ is its maximum value. Observe that $w(x)$ is also an even function. Indeed, assuming otherwise, $w(-x)$ would give us another solution to the problem (2.2), contradicting the simplicity of the principal eigenvalue. Recall that the solution $u(x)$ of (2.1) is called stable (unstable) if $\mu_1 > 0$ ($\mu_1 < 0$). (In case $\mu_1 = 0$, $u(x)$ is sometimes called neutrally stable.) Define $h(u) = 2F(u) - uf(u)$, where as usual $F(u) = \int_0^u f(t) dt$.

Theorem 2.1 *Assume that*

$$(2.3) \quad h(\alpha) < h(u), \text{ for all } u < \alpha.$$

Then the positive solution of (2.1), with $u(0) = \alpha$, is unstable.

Proof: Assume on the contrary that $\mu_1 \geq 0$. We claim that

$$(2.4) \quad u'(x)w'(x) - u''(x)w(x) > 0, \text{ for all } x \in (-1, 1).$$

Indeed, denoting $p(x) = u'(x)w'(x) - u''(x)w(x)$, we see that $p(0) = -w(0)u''(0) > 0$, and

$$(2.5) \quad p'(x) = -\mu_1 w(x)u'(x).$$

Since $p(x)$ is increasing (decreasing) for $x > 0$ ($x < 0$), the claim follows.

From the equations (2.1) and (2.2) we have

$$(2.6) \quad \int_{-1}^1 h'(u(x))w(x) dx = \int_{-1}^1 (f(u) - uf'(u)) w dx = \mu_1 \int_{-1}^1 uw dx \geq 0.$$

On the other hand, in view of (2.3) and (2.4),

$$(2.7) \quad \begin{aligned} \int_{-1}^1 h'(u(x))w(x) dx &= \int_{-1}^1 \frac{d}{dx} [h(u) - h(\alpha)] \frac{w}{u'} dx \\ &= - \int_{-1}^1 [h(u) - h(\alpha)] \frac{u'(x)w'(x) - u''(x)w(x)}{u'^2} dx < 0, \end{aligned}$$

which is a contradiction. Observe that the last integral in (2.7) is proper, since $h(u(x)) - h(\alpha)$ is quadratic in x near $x = 0$, and the same is true for the denominator, $u'^2(x)$. $\diamond$

The trick of introducing the $h(u) - h(\alpha)$ term was inspired by R. Schaaf and K. Schmitt [16]. Similarly we prove a stability result.

Theorem 2.2 *Assume that*

$$(2.8) \quad h(\alpha) > h(u), \text{ for all } u < \alpha.$$

Then the positive solution of (2.1), with $u(0) = \alpha$, is stable.

Proof: Assume on the contrary that $\mu_1 \leq 0$. Again, we claim that $p(x) = u'(x)w'(x) - u''(x)w(x)$ is positive on $(-1, 1)$. Indeed, $p(\pm 1) = u'(\pm 1)w'(\pm 1) > 0$, while (2.5) implies that $p(x)$ is decreasing (increasing) for $x > 0$ ($x < 0$), and the claim follows. The rest of the proof is the same as in the Theorem 2.1. The inequality signs are now reversed in both inequalities (2.6) and (2.7), and again we reach a contradiction. $\diamond$

For the problem

$$(2.9) \quad u'' + \lambda f(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

depending on a positive parameter λ , it is known that the solution curve travels northeast in the (λ, α) plane iff the solution is stable, and northwest iff it is unstable. This can be deduced, for example, from the Proposition 4.1.3 in R. Schaaf [15]. We give a self-contained proof, although we use some ideas from [15]. We shall need the following version of Sturm comparison theorem, whose straightforward proof we omit.

Lemma 2.1 *Assume that the functions $v(x)$ and $w(x)$ of class C^2 satisfy*

$$v'' + a(x)v \geq 0, \quad v'(0) = 0, \quad v(0) > 0,$$

$$w'' + a(x)w \leq 0, \quad w'(0) = 0, \quad w(0) > 0,$$

on some interval $(0, \gamma)$, with a continuous $a(x)$. Then $w(x)$ oscillates faster than $v(x)$. Namely, if $w(x) > 0$ on $(0, \gamma)$, then $v(x) > 0$ on $(0, \gamma)$. If, on the other hand, $v(\gamma) = 0$, then $w(x)$ must vanish inside $(0, \gamma)$.

As we mentioned above, the maximal value of positive solution of (2.9) $\alpha = u(0)$ uniquely identifies the pair $(\lambda, u(x))$. I.e. we can write $u = u(x, \alpha)$, $\lambda = \lambda(\alpha)$. It is natural to assume that $u_x(1, \alpha) < 0$, since in case $u_x(1, \alpha) = 0$ we have symmetry breaking, see [8].

Proposition 1 *Let $u(x, \alpha)$ be a positive solution of (2.9), with $u(0, \alpha) = \alpha$. Assume that $u_x(1, \alpha) < 0$. Then $\mu_1 < 0$ ($\mu_1 > 0$) if and only if $\lambda'(\alpha) < 0$ ($\lambda'(\alpha) > 0$).*

Proof: By shifting and stretching of the interval, we can convert the problem (2.9) into

$$(2.10) \quad u'' + f(u) = 0, \quad -\mu < x < \mu, \quad u(-\mu) = u(\mu) = 0,$$

with $\mu = \sqrt{\lambda}$. By “shooting” with $u(0) = \alpha$ and $u'(0) = 0$, we have $\mu = \mu(\alpha)$, and $u = u(x, \alpha)$. We shall calculate $\mu'(\alpha)$, which has the same sign as $\lambda'(\alpha)$. Differentiating the relation $u(\mu(\alpha), \alpha) = 0$, we have

$$u_x(\mu(\alpha), \alpha)\mu'(\alpha) + u_\alpha(\mu(\alpha), \alpha) = 0.$$

Since by our assumption $u_x(\mu(\alpha), \alpha) < 0$, it follows that the sign of $\mu'(\alpha)$ is the same as that of $u_\alpha(\mu(\alpha), \alpha)$. To determine the latter, we differentiate the equation (2.10) in α

$$u_\alpha'' + f'(u)u_\alpha = 0, \quad u_\alpha'(0, \alpha) = 0, \quad u_\alpha(0, \alpha) = 1.$$

If $\mu_1 > 0$, then

$$w'' + f'(u)w = -\mu_1 w < 0,$$

and hence by Lemma 2.1 w oscillates faster than u_α , and then $u_\alpha(\mu(\alpha), \alpha) > 0$. In case $\mu_1 < 0$, u_α has to vanish at some $\theta \in (0, \mu)$, since it oscillates faster than w . We claim that $u_\alpha(\mu(\alpha), \alpha) < 0$. Indeed, the function $u'(x)$ satisfies the same linear equation as u_α , and hence their roots interlace. But $u'(x) < 0$ on $(0, \mu)$, and so u_α cannot have any more roots, in addition to θ .

It is well-known that $\mu'(\alpha) = 0$ iff $\mu_1 = 0$, see e.g. [10]. Hence, conversely, $\mu'(\alpha) > 0$ (< 0) implies that $\mu_1 > 0$ (< 0). $\diamond$

Example The solution curve (in the $(\lambda, umax)$ plane, with $umax = u(1/2)$) for the problem

$$(2.11) \quad u'' + \lambda u(2 + \sin u) = 0, \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

has infinitely many turns (see Figure 1 for the bifurcation diagram computed using Mathematica).

Indeed, there is a curve of positive solutions bifurcating from zero. As we follow this curve for increasing $\alpha = u(1/2)$, then according to the Theorems 2.1 and 2.2 there are infinitely many changes of stability, which implies infinitely many changes in direction by the Proposition 1 above. (Here $h(u) = -u^2 \sin u - 2u \cos u + 2 \sin u$. Clearly, there is a sequence $\{\alpha_n\} \rightarrow \infty$, so that $h(u) > h(\alpha_n)$ for all $u \in (0, \alpha_n)$, and another sequence $\{\beta_n\} \rightarrow \infty$, so that $h(u) < h(\beta_n)$ for all $u \in (0, \beta_n)$. Solutions with $umax = u(1/2) = \alpha_n$ are unstable, and the ones with $u(1/2) = \beta_n$ are stable.) It is easy to show that along the solution curve λ cannot go to either zero or infinity. (To see that λ cannot go to zero, just multiply the equation (2.11) by u , and integrate. If $\lambda \rightarrow \infty$, then by Sturm's comparison theorem the solution would have to become sign-changing, a contradiction.)

Remarks

1. Using oscillating integrals as in A. Galstyan et al [6], we can show that the solution curve intersects infinitely often the line $\lambda = \lambda_1$, and moreover $\lambda \rightarrow \lambda_1$, as $u(0) \rightarrow \infty$. We will include the proof of this, and related results, in a forthcoming paper.

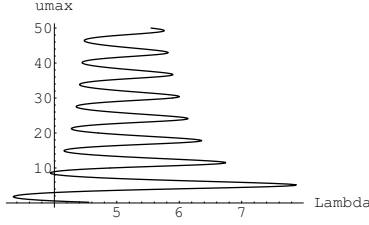


Figure 1: The bifurcation diagram for the problem (2.11)

2. A similar result in case $f(u) = 2u - u \sin u$ was obtained in a recent paper of S.-H. Wang [18].

3 S -shaped curves of combustion theory

We consider positive solutions of the problem

$$(3.1) \quad u'' + \lambda f(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

where λ is a positive parameter. We assume that $f(u) \in C^2[0, \bar{u}]$ for some $\bar{u} \leq \infty$, and it satisfies

$$(3.2) \quad f(u) > 0 \quad \text{for } u \in [0, \bar{u}).$$

The following theorem generalizes the main result in P. Korman and Y. Li [9]. It allows $f(u)$ to change concavity more than once.

Theorem 3.1 *In addition to (3.2) assume that the function $f''(u)$ has a root $u = \alpha_1$, and there is an α_2 , $0 < \alpha_1 < \alpha_2 < \bar{u}$, with*

$$(3.3) \quad f''(u) > 0 \quad \text{for } u \in (0, \alpha_1) \quad f''(u) < 0 \quad \text{for } u \in (\alpha_1, \alpha_2).$$

(No assumptions on $f''(u)$ are made over $(\alpha_2, \bar{u})$.) Denote, as before, $h(u) = 2F(u) - uf(u)$. Assume that

$$(3.4) \quad h(\alpha_1) < 0,$$

$$(3.5) \quad h(\alpha_2) > h(u) \quad \text{for all } u \in (0, \alpha_2),$$

$$(3.6) \quad h'(u) > 0 \quad \text{for all } u \in (\alpha_2, \bar{u}).$$

Then the solution curve for (3.1) is exactly S -shaped.

Proof: By P. Korman and Y. Li [9] there is a curve of solutions, starting at $(\lambda = 0, u = 0)$, and by the time it reaches the level $u(0) = \alpha_1$ the curve has made exactly one turn to the left, and the solution with $u(0) = \alpha_1$ is unstable. By the time the solution curve reaches the level $u(0) = \alpha_2$, it travels to the right, since by the Theorem 2.1, solution with $u(0) = \alpha_2$ is stable. Hence the solution curve has made a turn to the right, when $u(0) \in (\alpha_1, \alpha_2)$. By P. Korman and Y. Li [9], it has made exactly one turn to the right. By the Theorem 2.1, solutions with $u(0) \in (\alpha_2, \bar{u})$ are stable, so there are no more turns on the solution curve. $\diamond$

Remark If $\bar{u} < \infty$, and $f(\bar{u}) = 0$, then, as in P. Korman and Y. Li [9], we see that the curve exist for all $\lambda > 0$, and it tends to $\bar{u}$ as $\lambda \rightarrow \infty$. If $\bar{u} = \infty$, then integrating the inequality $h'(u) \geq 0$, we see that $f(u)$ is below a linear function for large u . If $f(u)$ is asymptotically linear, then the solution curve will go to infinity at some finite λ . If $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 0$, then the solution curve continues for all $\lambda > 0$, see P. Korman and Y. Li [9].

The Theorem 3.1 is applicable to the original equations of combustion theory (see R. Aris [1])

$$(3.7) \quad v'' + \mu(1 - \epsilon v)^m e^{\frac{v}{1+\epsilon v}} = 0, \quad x \in (-1, 1), \quad v(-1) = v(1) = 0,$$

where μ , m and ϵ are positive parameters. (If $m = 0$, we have the so called perturbed Gelfand problem.) For sufficiently small ϵ , S.P. Hastings and J.B. McLeod [5] proved that the solution curve for (3.7) is exactly S-shaped. They used a rather complicated quadrature method. In our experiments, when $\epsilon = \epsilon(m)$ was small, the Theorem 3.1 applied, giving an S-shaped curve, while for larger ϵ the solution curve is monotone, which follows from the results of P. Korman, Y. Li and T. Ouyang [10]. Before we give the examples, it is convenient to rescale $\epsilon v = u$, and $\lambda = \mu\epsilon$, and consider

$$(3.8) \quad u'' + \lambda(1 - u)^m e^{\frac{u}{\epsilon(1+u)}} = 0, \quad x \in (-1, 1), \quad u(-1) = u(1) = 0.$$

We are interested in the solutions satisfying $0 < u(x) < 1$ for all $x \in (-1, 1)$. These examples are also covered by the results of [19], [20] and [21], although our approach appears to be simpler.

Example 1 Let $m = 2$, and $\epsilon = 0.05$, i.e., we consider the problem

$$(3.9) \quad u'' + \lambda(1 - u)^2 e^{\frac{u}{0.05(1+u)}} = 0, \quad x \in (-1, 1), \quad u(-1) = u(1) = 0.$$

Here $\bar{u} = 1$. With the help of *Mathematica* one verifies that the second derivative $f''(u)$ indeed goes plus-minus-plus on the interval $(0, 1)$, with

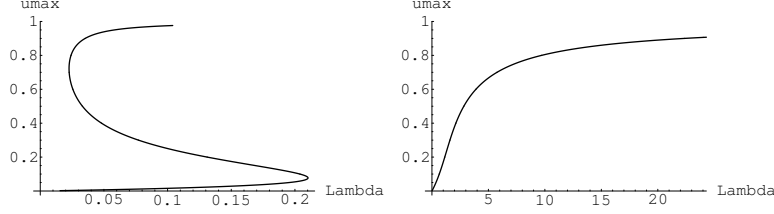


Figure 2: The bifurcation diagrams for $\epsilon = 0.05$, and $\epsilon = 0.2$

the roots $\alpha_1 \simeq 0.52$ and $\alpha_2 \simeq 0.9$. Also $h(\alpha_1) \simeq -49.6 < 0$. One also sees that $h(\alpha_2) > h(u)$, for all $u \in (0, \alpha_2)$, and $h'(u) > 0$ for $u > \alpha_2$. Hence, the Theorem 3.1 applies, and the solution curve of (3.9) is exactly S -shaped (see Figure 2). In particular, for large λ the solution is unique, and $\lim_{\lambda \rightarrow \infty} u(x, \lambda) = 1$ for all $x \in (-1, 1)$.

Example 2 Let $m = 2$, and $\epsilon = 0.2$, i.e., we consider the problem

$$(3.10) \quad u'' + \lambda(1 - u)^2 e^{\frac{u}{0.2(1+u)}} = 0, \quad x \in (-1, 1), \quad u(-1) = u(1) = 0.$$

This time $f(u)$ changes concavity once, at $\alpha \simeq 0.7$, with $f''(u) < 0$ for $u \in (0, \alpha)$ and $f''(u) > 0$ for $u \in (\alpha, 1)$. According to [10] or T. Ouyang and J. Shi [13], the solution curve, which starts at $(\lambda = 0, u = 0)$, can make only turns to the left. But if such a turn had occurred, the solution curve would have nowhere to go, since no more turns are possible, while solutions are bounded by 1. Hence, the solution curve is monotone in λ . In other words, we have existence and uniqueness of positive solution for all $\lambda > 0$ (see Figure 2).

4 An extension of the stability condition

Integrating by parts once more, we can generalize the stability result, the Theorem 2.2, in case $f(u) > 0$. With $h(u) = 2F(u) - uf(u)$ as before, we define $g(u) = \int_0^u h(t) dt$.

Theorem 4.1 Assume that $f(u) > 0$ for all $u > 0$, and

$$(4.1) \quad g(u) > g(\alpha) + g'(\alpha)(u - \alpha), \quad \text{for all } u \in [0, \alpha].$$

Then any positive solution of (2.1), with $u(0) = \alpha$, is stable.

Proof: We proceed similarly to the Theorem 2.2. Assume on the contrary that $\mu_1 \leq 0$. We conclude as before that $p(x) = u'(x)w'(x) - u''(x)w(x)$ is positive on $(-1, 1)$, and also that

$$\int_{-1}^1 h'(u(x))w(x) dx \leq 0.$$

On the other hand, using (2.7),

$$\begin{aligned} \int_{-1}^1 h'(u(x))w(x) dx &= \int_{-1}^1 [h(\alpha) - h(u)] \frac{p(x)}{u'^2(x)} dx = \\ \int_{-1}^1 \frac{d}{dx} [h(\alpha)u - g(u) + g(\alpha) - \alpha h(\alpha)] \frac{p(x)}{u'^3(x)} dx &= 2(g(\alpha) - \alpha g'(\alpha)) \frac{w'(1)}{u'^2(1)} \\ &+ \int_{-1}^1 [g(u) - g(\alpha) - g'(\alpha)(u - \alpha)] \left(\frac{p'}{u'^3} + \frac{3pf(u)}{u'^4} \right) dx > 0. \end{aligned}$$

Here we used that $u'(-1) = -u'(1)$, $w'(-1) = -w'(1)$, and that $g(\alpha) - \alpha g'(\alpha) < 0$ from (4.1). Also observe that in the last integral the quantity in the square bracket is $O(x^4)$ near $x = 0$, and hence the integrand is bounded. Since $p' = -\mu_1 w u'$, the quantity in the second bracket is positive (here we use that $f(u) > 0$). $\diamond$

It is easy to see that the condition (4.1) is more general than (2.8). Indeed, denoting $G(u) = g(u) - g(\alpha) - g'(\alpha)(u - \alpha)$, we see that $G(\alpha) = 0$ and $G'(u) = h(\alpha) - h(u) < 0$, provided that condition (2.8) holds. But then $G(u) > 0$ for $u \in [0, \alpha]$, i.e. (4.1) holds.

5 Positive solutions on a ball in R^2

Even though our instability result holds only in two dimensions, we shall consider the equations on balls in R^n . This does not make the exposition any longer, but allows us to show why this approach seems to fail for $n > 2$. We consider positive solutions of $(x \in R^n)$

$$(5.1) \quad \Delta u + f(u) = 0, \text{ for } |x| < 1, \quad u = 0, \text{ when } |x| = 1.$$

The eigenvalue problem for the corresponding linearized equation is

$$(5.2) \quad \Delta w + f'(u)w + \mu w = 0, \text{ for } |x| < 1, \quad w = 0, \text{ when } |x| = 1.$$

By B. Gidas, W.-M. Ni and L. Nirenberg [7] any positive solution of (5.1) is radially symmetric, i.e. $u = u(r)$, where $r = |x|$, and moreover $u'(r) < 0$ for all $r \in (0, 1)$. If $\mu \leq 0$, it follows from C.S. Lin and W.-M. Ni [12] that

any solution of (5.2) is also radially symmetric, i.e. $w = w(r)$. For radially symmetric solutions the problems (5.1) and (5.2) become respectively

$$(5.3) \quad u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r < 1, \quad u'(0) = u(1) = 0,$$

and

$$(5.4) \quad w'' + \frac{n-1}{r}w' + f'(u)w + \mu w = 0, \quad r < 1, \quad w'(0) = w(1) = 0.$$

Theorem 5.1 *Assume that $n = 2$, and*

$$(5.5) \quad f'(u) > 0, \quad \text{for all } u > 0.$$

Assume we have a positive solution of (5.1) (i.e. of (5.3)) with $u(0) = \alpha$, for which the condition (2.3) holds, and for which the principal eigenfunction of (5.2) is radially symmetric. Then this solution is unstable.

Proof: We need to prove that we have $\mu < 0$, for the principal eigenvalue μ of (5.2) (i.e. of (5.4)). Assume on the contrary that $\mu \geq 0$. We begin by observing that for the principal eigenfunction $w(r)$

$$(5.6) \quad w'(r) \leq 0, \quad \text{for all } r \in [0, 1].$$

Indeed, assuming otherwise, $w(r)$ would have a point of local minimum, at which the left hand side of (5.4) is positive, a contradiction. Next, we claim that in case $n = 2$

$$(5.7) \quad p(r) \equiv 2(n-1)r^{n-1}u'w + r^n u'w' + r^n f(u)w > 0, \quad \text{for all } r \in (0, 1).$$

(It is at this point that our argument fails for $n > 2$.) Indeed, $p(0) = 0$, and we have (expressing u'' and w'' from the corresponding equations)

$$p'(r) = r^{n-1} [(2-n)f(u)w - \mu r w u' + n u' w'] .$$

In case $n = 2$, in view of (5.6), $p(r)$ is increasing, and hence positive.

The rest of the proof is similar to the one dimensional case. From the equations (5.1) and (5.2) we have

$$\int_{|x|<1} [f(u) - u f'(u)] w \, dx = \mu \int_{|x|<1} u w \, dx \geq 0.$$

On the other hand (ω_n denotes the area of unit ball in R^n)

$$\begin{aligned} \int_{|x|<1} [f(u) - uf'(u)] w dx &= \omega_n \int_0^1 \frac{d}{dr} [h(u) - h(\alpha)] \frac{r^{n-1}w}{u'} dr \\ &= -\omega_n \int_0^1 [h(u) - h(\alpha)] \frac{p(r)}{ru'^2} dr < 0, \end{aligned}$$

in case $n = 2$. ◇

Remark One sufficient condition for the principal eigenfunction of (5.2) to be radially symmetric is that $f''(u) > 0$ for $u > 0$. Indeed, denoting $g(w, r) \equiv f'(u(r))w + \mu w$, we see that $g_r(w, r) < 0$, and hence the results of B. Gidas, W.-M. Ni and L. Nirenberg [7] apply. Moreover, their result implies that (5.6) holds (with a strict inequality), and hence the assumption (5.5) is not needed. Hence we have the following result.

Theorem 5.2 *Assume that $n = 2$, and*

$$(5.8) \quad f''(u) > 0, \text{ for all } u > 0.$$

Assume we have a positive solution of (5.1) (i.e. of (5.3)) with $u(0) = \alpha$, for which the condition (2.3) holds. Then this solution is unstable.

We consider again the perturbed Gelfand problem (on a unit ball in R^2)

$$(5.9) \quad \Delta u + \lambda e^{\frac{u}{(1+\epsilon u)}} = 0, \text{ for } |x| < 1, \quad u = 0, \text{ when } |x| = 1.$$

Theorem 5.3 *There exists an ϵ_0 , so that for $\epsilon < \epsilon_0$ the curve of positive solutions of (5.9) makes at least two turns, i.e. there is an interval (λ_1, λ_2) , so that for any $\lambda \in (\lambda_1, \lambda_2)$ the problem (5.9) has at least three positive solutions.*

Proof: Indeed, using *Mathematica*, one shows that the function $h(u)$ is positive for small u , and then becomes negative at some u , if $\epsilon < \epsilon_0$, and our instability result applies (the value of ϵ_0 can be numerically computed). Hence, the solution curve, starting at $(\lambda = 0, u = 0)$ makes at least one turn. The rest of proof is just standard bifurcation theory, thanks to a result of Y. Du and Y. Lou [4] on positivity for the linearized problem, see [4], [10] or [13] for the details. ◇

This theorem corresponds roughly to an old result of S.V. Parter [14], but with a much simpler proof. We remark that for ϵ sufficiently small, Y. Du and Y. Lou [4] have proved that the solution curve is exactly *S*-shaped.

6 A class of symmetric problems

We consider positive solutions of a class of non-autonomous problems

$$(6.1) \quad u'' + a(x)f(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

with even $a(x)$. Namely, we assume that the function $a(x) \in C^1(-1, 1) \cap C[-1, 1]$ satisfies

$$(6.2) \quad a(x) > 0, \quad a(-x) = a(x), \quad a'(x) < 0 \quad \text{for } x \in (0, 1),$$

while $f(u) \in C^2(\bar{R}_+)$ satisfies

$$(6.3) \quad f(u) > 0 \quad \text{for } u > 0.$$

It follows from B. Gidas, W.-M. Ni and L. Nirenberg [7] that under these conditions any positive solution of (6.1) is an even function, with $u'(x) < 0$ for $x > 0$. It is also known that in this case the problem (6.1) has properties similar to those of autonomous problems, see a recent review paper [8]. In particular, in P. Korman and J. Shi [11] an instability result, similar to the Theorem 2.1 above, was given, and it was used to obtain an exact multiplicity result. In this section we add a corresponding stability result, and simplify the proof for the instability part. We again denote by $(\mu_1, w(x) > 0)$ the principal eigenpair of the eigenvalue problem for the corresponding linearized equation

$$(6.4) \quad w'' + a(x)f'(u)w + \mu w = 0, \quad -1 < x < 1, \quad w(-1) = w(1) = 0.$$

As before, $w(x)$ is an even function. Again, we define $h(u) = 2F(u) - uf(u)$, and $p(x) = u'(x)w'(x) - u''(x)w(x) = u'(x)w'(x) + a(x)f(u(x))w(x)$. Observe that $p(x)$ is an even function.

Lemma 6.1 *Assume that either*

$$(6.5) \quad \mu_1 \leq 0,$$

or

$$(6.6) \quad \mu_1 > 0, \quad \text{and } f'(u) \geq 0 \text{ for } u > 0.$$

Then

$$(6.7) \quad p(x) > 0, \quad \text{for all } x \in (-1, 1).$$

Proof: We have $p(1) = u'(1)w'(1) > 0$, and in case $\mu_1 \leq 0$

$$p'(x) = -\mu_1 u'(x)w(x) + a'(x)f(u(x))w(x) < 0, \text{ for } x > 0,$$

and (6.7) follows. In case the condition (6.6) holds, we have as before $w'(x) < 0$ for all $x > 0$, and hence positivity of the even function $p(x)$ follows directly from its definition. $\diamond$

Theorem 6.1 *Assume that the conditions (6.2) and (6.3) hold.*

(i) *If the condition (2.8) holds, then the solution of (2.1), with $u(0) = \alpha$, is stable.*

(ii) *If the condition (2.3) holds, and $f'(u) \geq 0$ for $u \in (0, \alpha)$, then the solution of (2.1), with $u(0) = \alpha$, is unstable.*

Proof: We proceed as before. From the equations (6.1) and (6.4) we have

$$(6.8) \quad I \equiv \int_{-1}^1 a(x) (f(u) - uf'(u)) w dx = \mu_1 \int_{-1}^1 uw dx.$$

We have,

$$(6.9) \quad \begin{aligned} I &= \int_{-1}^1 \frac{d}{dx} [h(u) - h(\alpha)] \frac{a(x)w(x)}{u'(x)} dx \\ &= - \int_{-1}^1 [h(u) - h(\alpha)] \frac{a'(x)u'(x)w(x) + a(x)p(x)}{u'^2} dx. \end{aligned}$$

If case (i) holds, assume on the contrary that the solution is not stable, i.e. $\mu_1 \leq 0$. Then $I \leq 0$ from (6.8), while $I > 0$ from (6.9), since $p(x) > 0$ by the above lemma.

If case (ii) holds, assume on the contrary that the solution fails to be unstable, i.e. $\mu_1 \geq 0$. Then $I \geq 0$ from (6.8), while $I < 0$ from (6.9), still resulting in a contradiction. $\diamond$

We have seen the usefulness of stability and instability results in the previous sections. Here is one example.

Example The solution curve (in the $(\lambda, umax)$ plane, with $umax = u(0)$) for the problem

$$u'' + \lambda a(x)(2u + \sin u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0,$$

has infinitely many turns, assuming that the function $a(x)$ satisfies the conditions (6.2). Indeed, the function $f(u) = 2u + \sin u$ is positive and increasing for $u > 0$, while for $h(u) = 2 - 2\cos u - u \sin u$ there is a sequence $\{\alpha_n\} \rightarrow \infty$, so that $h(u) > h(\alpha_n)$ for all $u \in (0, \alpha_n)$, and another sequence $\{\beta_n\} \rightarrow \infty$, so that $h(u) < h(\beta_n)$ for all $u \in (0, \beta_n)$. Solutions with $u(0) = \alpha_n$ are unstable, and the ones with $u(0) = \beta_n$ are stable.

References

- [1] R. Aris, Mathematical Theory of Diffusion and Reaction in Permeable Catalysts, Oxford University Press, London (1975).
- [2] K. J. Brown, M. M. A. Ibrahim and R. Shivaji, S -shaped bifurcation curves, *Nonlinear Anal.* **5**, no. 5, 475-486 (1981).
- [3] E.N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37**, no. 3, 404-437 (1980).
- [4] Y. Du and Y. Lou, Proof of a conjecture for the perturbed Gelfand equation from combustion theory, *J. Differential Equations* **173**, no. 2, 213-230 (2001).
- [5] S.P. Hastings and J. B. McLeod, The number of solutions to an equation from catalysis, *Proc. Roy. Soc. Edinburgh Sect. A* **101**, no. 1-2, 15-30 (1985).
- [6] A. Galstyan, P. Korman and Y. Li, On the oscillations of the solution curve for a class of semilinear equations, *J. Math. Anal. Applic.* **321** no. 2, 576-588 (2006).
- [7] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.* **68**, 209-243 (1979).
- [8] P. Korman, Global solution branches and exact multiplicity of solutions for two point boundary value problems, Handbook of Differential Equations, Ordinary Differential Equations, Vol. **3**, Edited by A. Canada, P. Drabek and A. Fonda, Elsevier Science, North Holland, 547-606 (2006).
- [9] P. Korman and Y. Li, On the exactness of an S -shaped bifurcation curve, *Proc. Amer. Math. Soc.* **127**, no. 4, 1011-1020 (1999).
- [10] P. Korman, Y. Li and T. Ouyang, Exact multiplicity results for boundary-value problems with nonlinearities generalising cubic, *Proc. Royal Soc. Edinburgh, Ser. A* **126A**, 599-616 (1996).
- [11] P. Korman and J. Shi, Instability and exact multiplicity of solutions of semilinear equations. Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL, 1999), 311-322 (electronic), Electron. J. Differ. Equ. Conf., 5, Southwest Texas State Univ., San Marcos, TX, 2000.

- [12] C.S. Lin and W.-M. Ni, A counterexample to the nodal domain conjecture and related semilinear equation, *Proc. Amer. Math. Soc.* **102**, 271-277 (1988).
- [13] T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, II *J. Differential Equations* **158** no. 1, 94-151 (1999).
- [14] S.V. Parter, Solutions of a differential equation arising in chemical reactor processes, *SIAM J. Appl. Math.* **26**, 687-716 (1974).
- [15] R. Schaaf, Global Solution Branches of Two Point Boundary Value Problems, Lecture Notes in Mathematics, no. **1458**, Springer-Verlag (1990).
- [16] R. Schaaf and K. Schmitt, A class of nonlinear Sturm-Liouville problems with infinitely many solutions, *Trans. Amer. Math. Soc.* **306**, no. 2, 853-859 (1988).
- [17] S.-H. Wang, On S -shaped bifurcation curves, *Nonlinear Anal. TMA* **22**, no. 12, 1475-1485 (1994).
- [18] S.-H. Wang, On the evolution and qualitative behaviors of bifurcation curves for a boundary value problem, To appear in *Nonlinear Anal. TMA*.
- [19] S.-H. Wang and F.P. Lee, Bifurcation of an equation from catalysis theory, *Nonlinear Anal.* **23**, no. 9, 1167-1187 (1994).
- [20] S.-H. Wang, Rigorous analysis and estimates of S -shaped bifurcation curves in a combustion problem with general Arrhenius reaction-rate laws, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **454**, no. 1972, 1031-1048 (1998).
- [21] S.-H. Wang, An S -shaped bifurcation curves theorem for a p -Laplacian Dirichlet problem and its application to the perturbed Gelfand problem, Preprint.