A computer assisted study of uniqueness of nodal ground state solutions

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Abstract

We show that the global solution curves for both Dirichlet and Neumann problems provide numerical evidence of uniqueness of nodal ground state solutions for equations describing standing waves for a class of nonlinear Schroedinger's equations

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1 Introduction

Let $v(x,t)$ be a complex-valued solution of a nonlinear Schroedinger's equation ($x \in R^n$, $t > 0$)

$$iv_t + \Delta v + v|v|^{p-1} = 0\,.$$

Here $p > 1$ is a constant, and $|v|$ denotes the modulus of v. Looking for the standing waves, one substitutes $v(x,t) = e^{imt}u(x)$, with a real valued $u(x)$, and a constant $m > 0$. Then $u(x)$ satisfies

$$\Delta u - mu + u|u|^{p-1} = 0\,.$$

For a more general equation, where $v(x,t)$ is a complex-valued solution of

$$iv_t + \Delta v + f(v) = 0$$

a similar reduction works for any complex valued function $f(v)$, satisfying

$$f(e^{imt}u) = e^{imt} f(u)\,, \text{ for any real } m \text{ and } u\,,$$

and it leads to the equation

$$\Delta u - mu + f(u) = 0\,.$$

We shall study the radially symmetric solutions of this equation, with $r = |x|$, satisfying

$$u'' + \frac{n-1}{r}u' - mu + f(u) = 0\,. \tag{1.1}$$

Of particular interest will be sign-changing ground state solutions, with arbitrary many roots. These are solutions satisfying $\lim_{r\to\infty} u(r) = 0$.

Let us discuss the solutions of the initial value problem

$$u'' + \frac{n-1}{r}u' + f(u) = 0\,, \;\; r > 0\,, \;\; u(0) = a, \;\; u'(0) = 0\,, \tag{1.2}$$

modeled on $f(u) = -u + u|u|^{p-1}$, for all initial values $a > 1$ (for which $f(a) > 0$), and all $r > 0$, with a sub-critical p, $1 < p < \frac{n+2}{n-2}$. Since $u''(0) = -\frac{1}{n}f(a) < 0$, the solution $u(r)$ goes down for small $r > 0$. (In the model case, $u''(0) = \frac{1}{n}(a - a^p) < 0$.) Define the "energy" $E(r) = \frac{1}{2}u'^2(r) + F(u(r))$, where $F(u) = \int_0^u f(t)\,dt$. We have

$$\frac{d}{dr}E(r) = -\frac{n-1}{r}u'^2 < 0\,, \tag{1.3}$$

and so the energy is decreasing. The initial energy at $r = 0$ is equal to $F(a)$. The energy at any root r_0 is equal to $\frac{1}{2}u'^2(r_0) > 0$. Since energy is decreasing along solutions, it follows that the initial energy $F(a)$ must be positive and high enough, in order for the solution to have roots, and higher energy is needed to have more roots. After the initial energy is dissipated, the solutions gets captured by either one of the stable roots of $f(u)$, tending to either 1 or -1 as $r \to \infty$. Ground states are the exceptional unstable solutions tending to 0 as $r \to \infty$, and separating different stable behaviors.

When the initial energy is low, solutions turn around while staying positive, and then tend to 1 as $r \to \infty$. At the next higher energy level solutions have enough energy to develop one root, and then they tend to -1 as $r \to \infty$. Computations suggest that there is a unique positive ground state solution, separating the above two regimes. Uniqueness of positive ground state solution was proved for a class of $f(u)$ for which $\frac{uf'(u)}{f(u)}$ is decreasing, see M.K. Kwong [6]. This class includes the model case $f(u) = -u + u|u|^{p-1}$, but not other similar equations, like $f(u) = -2u + u^3 + u^4$. Other results for both the Dirichlet and the ground state problems, under the condition that the function $\frac{uf'(u)}{f(u)}$ is decreasing, can be found in M.K. Kwong [6], L. Zhang [12], M.K. Kwong and Y. Li [7], P. Korman, Y. Li, and T. Ouyang [5], T. Ouyang and J. Shi [9]). Radial sign-changing solutions were studied for a related equation in E. Yanagida and S. Yotsutani [11].

At the next higher energy level solutions have two roots, and then they tend to 1 as $r \to \infty$. This level is separated from the preceding one by a ground state solution with one root. Proving its uniqueness appears to be out of the question at present. The latter energy level is separated by the ground state solution with two roots from the next level, involving the solutions with 3 roots, tending to -1, and so on.

We approach the initial value problem (1.2) by studying the solution curves of the parameter dependent boundary value problems

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0\,, \quad r > 0, \;\; u'(0) = 0\,, \tag{1.4}$$

with either the Dirichlet

$$u(1) = 0\,, \tag{1.5}$$

or the Neumann

$$u'(1) = 0 \tag{1.6}$$

boundary condition. Any solution of the problem (1.2) with the first root at some ξ corresponds to a positive solution of the Dirichlet problem (1.4), (1.5) at $\lambda = \xi^2$, after rescaling. As λ varies, we have a curve of positive solutions of the Dirichlet problem, denoted by D_0. Any solution of the problem (1.2) with the first point of zero slope at some η corresponds to a solution of the Neumann problem (1.4), (1.6) at $\lambda = \eta^2$, after rescaling. We denote by N_0 the curve of positive solutions of the Neumann problem (1.4), (1.6).

We also performed computations for $f(u) = -2u + u^3 + u^4$, and $n = 3$, which is a problem considered in [7]. It differs from our model case, since $f(u)$ has only one positive root but no negative ones. The model for that case is $f(u) = -u + |u|^p$. For small values of the energy the solutions in both cases oscillate around 1. On the other when the solution has developed a root then the solution in the second case will tend to minus infinity since $f(u(r)) < 0$ for $u < 0$. In Figure 1 we draw D_0 (the higher curve) and N_0 for the model case $f(u) = -u + u^2$. Since the picture remains the same for related functions, it provides strong evidence of the uniqueness of the positive ground state, which occurs at the value of $u(0)$ where D_0 and N_0 come together.

When it comes to sign-changing ground state solutions, no uniqueness or multiplicity results are proved even for the model case $f(u) = -u + u|u|^{p-1}$, which we will consider from now on. We compute similarly D_k's, the Dirichlet curves with k interior roots on $(0, 1)$, and the properly defined Neumann curves N_k. Figure 2 shows that each Neumann curve tends to two adjacent Dirichlet curves, and it provides strong numerical evidence of the uniqueness of nodal ground states, where D_k and N_k come together. Notice the very sharp turns to the right that each Neumann curves makes, which presented considerable computational challenge.

2 Sign changing ground state solutions

We assume that $f(u)$ has three roots $f(b_1) = f(0) = f(b_2) = 0$, with $b_1 < 0 < b_2$, and

(2.1) $f(u) > 0$ on $(b_1, 0) \cup (b_2, \infty)$, $f(u) < 0$ on $(-\infty, b_1) \cup (0, b_2)$,

(2.2) $f(u) = g(u) + u|u|^{p-1}$, with $\displaystyle\lim_{u \to \infty} \frac{g(u)}{u|u|^{p-1}} = 0$, and $1 < p < \frac{n+2}{n-2}$.

The following lemma is well-known, see e.g., J. Shi [10].

Lemma 2.1 *Assume that a solution of (1.2) satisfies* $\lim_{r \to \infty} u(r) = c$. *Then* $f(c) = 0$.

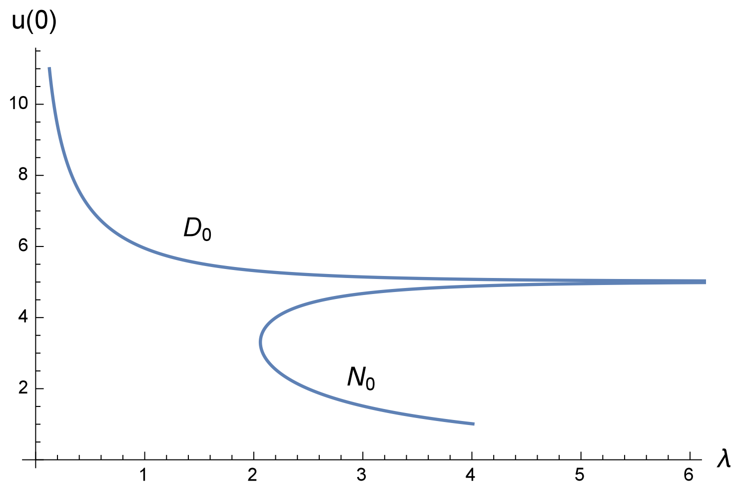

Figure 1: The curves of positive solutions for the Dirichlet and Neumann problems for $f(u) = -u + u^2$, and $n = 3$

Proof: Writing (1.2) in the form $\left(r^{n-1}u'\right)' = -f(u)r^{n-1} \approx -f(c)r^{n-1}$, and integrating twice, we see that for large r, $u(r)$ is asymptotic to $c - f(c)\frac{r^2}{2n}$, so that $f(c) = 0$. ♢

In case $f(u) = -u + u|u|^{p-1}$ the possible values of c are 0 and ± 1.

Lemma 2.2 *Assume that a solution of (1.2) has a point of local minimum at some $r_1 > 0$. Then $u(r) > u(r_1)$, for $r > r_1$. If $r_2 > 0$ is a point of local maximum, then $u(r) < u(r_2)$, for $r > r_2$.*

Proof: Follows immediately from energy being decreasing. ♢

Corollary 1 *Let r_1 be a point of local minimum, and $u(r_1) > 0$. Then $\lim_{r\to\infty} u(r) = b_2$. If r_2 be a point of local maximum, and $u(r_2) < 0$, then $\lim_{r\to\infty} u(r) = b_1$.*

Lemma 2.3 *Assume that a solution of (1.2) satisfies $\lim_{r\to\infty} u(r) = 0$. Then this solution is eventually monotone, and $\lim_{r\to\infty} E(r) = 0$.*

Proof: When $|u(r)|$ is small, $u(r)$ cannot have points of local minimum where it is negative, and points of local maximum where it is positive. It follows that $u(r)$ is eventually monotone, and then $u'(r) \to 0$ as $r \to \infty$. ♢

We now make more precise the global picture outlined in the Introduction. Our assumptions imply the existence of $u_0 > b_2$ such that $F(u) < 0$ on $(0, u_0)$, and $F(u) > 0$ on (u_0, ∞) (For the model case $f(u) = -u + u|u|^{p-1}$, $u_0 = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}$.) If $b_2 < a < u_0$, then the initial energy $E(a) = F(a) < 0$, and the solution cannot have any roots (at which the energy is positive). Hence, the solution turns around at some ρ_1 ($u'(\rho_1) = 0$, $u(\rho_1) > 0$), and then (by our lemmas) tends to $u = b_2$, making infinitely many oscillations around $u = b_2$. To see that there are infinitely many oscillations, one solves the linearized equation at $u = b_2$, using Bessel's functions. The point ρ_1 is the point of global minimum of $u(r)$. The same behavior persists for $a > u_0$, but a close to u_0.

There is a critical u_1, so that when $a > u_1 > u_0$, the solution has just enough energy to develop a root at some r_1, and then it turns around at some ρ_2 ($u'(\rho_2) = 0$, $b_1 < u(\rho_2) < 0$), and then tends to $u = b_1$, making infinitely many oscillations around $u = b_1$. When $a = u_1$, the corresponding solution $u(r, u_1)$ is a positive ground state. When $a > u_2$, the next critical value, the solution has energy for two roots, after which it turns around, and tends to $u = b_2$, making infinitely many oscillations around $u = b_2$. When $a = u_2$, the corresponding solution $u(r, u_2)$ is a ground state, with exactly one root. For increasing a we will have more and more roots, and we expect a unique ground state with exactly k roots for any integer $k \geq 0$.

We shall consider positive and sign-changing solutions of the Dirichlet problem

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0\,, \quad 0 < r < 1\,, \quad u'(0) = u(1) = 0\,. \tag{2.3}$$

For positive solutions one has $u'(r) < 0$ for all $r \in (0, 1)$, so that $u(0)$ gives the maximum value of $u(r)$. Recall that the set of positive solutions of (2.3) can be represented by curves in the $(\lambda, u(0))$ plane, see e.g., [3]. The following result was proved in [5], [9], see also [3].

Theorem 2.1 *Assume that $f(u) = -u + u|u|^{p-1}$. Then all positive solutions of (2.3) lie on a unique, monotone, hyperbola-like solution curve in the $(\lambda, u(0))$ plane, with $u(0) \to \infty$ as $\lambda \to 0$, and $u(0) \to \beta$ as $\lambda \to \infty$, where $\beta > 0$ satisfies $F(\beta) > 0$.*

Let $\rho = \rho(a)$ denote the first root of the solution of (1.2). The last result, after rescaling, implies that $\rho(a) \to \infty$ as $a \to \beta$, $\rho(a) \to 0$ as $a \to \infty$.

By D_k we shall denote the solution set $(\lambda, u(r))$ of (2.3) with $k \geq 0$ interior roots on $(0,1)$. In case $k \geq 1$, no detailed results like the Theorem 2.1 above is available. However, we have the following result.

Theorem 2.2 *For the problem (2.3), the value of $u(0)$ uniquely identifies the solution pair $(\lambda, u(r)) \in D_k$ (i.e., the value of $u(0)$ is a global parameter on D_k). D_k consists of a single smooth solution curve. On one side of D_k, $\lambda \to 0$ and $u(0) \to \infty$, and on the other side, $\lambda \to \infty$ and $u(0) \to g_k$ (the initial conditions $u(0) = g_k$ and $u'(0) = 0$ produce the ground state solution with k roots).*

Proof: Assume, on the contrary, that we have two solution pairs in D_k $(\lambda, u(r))$ and $(\mu, v(r))$, with $u(0) = v(0) = \alpha$. Clearly, $\lambda \neq \mu$, since otherwise we have a contradiction with uniqueness of initial value problems, which is known for this type of problems, see e.g., [3]. (Recall that $u'(0) = v'(0) = 0$.) The change of variables $r = \frac{1}{\sqrt{\lambda}} t$ takes (2.3) into

$$u''(t) + \frac{n-1}{t} u'(t) + f(u) = 0, \quad u(0) = \alpha, \;\; u'(0) = 0\,. \tag{2.4}$$

The change of variables $r = \frac{1}{\sqrt{\mu}} t$ takes the equation for $v(r)$, at μ, also into (2.4). By uniqueness, $u(t) \equiv v(t)$, but that is impossible, since $u(t)$ has its k-th root at $t = \sqrt{\lambda}$, while the k-th root of $v(t)$ is $t = \sqrt{\mu}$.

Similarly to the case of positive solutions, one shows that D_k extends globally, i.e., at each point of this solution curve either the implicit function theorem or the Crandall-Rabinowitz bifurcation theorem [1] applies, see e.g., P. Korman [3]. By Lemma 2.4 below, $u(0)$ extends to infinity along D_k. By the a priori estimates of B. Gidas and J. Spruck [2], this may only happen for $\lambda \to 0$. Since the value of $u(0)$ is a global parameter on D_k, $u(0)$ is decreasing as $\lambda \to \infty$, and by continuity it tends to g_k. $\diamondsuit$

Lemma 2.4 *Assume that $f(u)$ satisfies the conditions (2.1) and (2.2). Then all sufficiently large values of $u(0) = a$ belong to the Dirichlet range D_k, i.e., the corresponding solution of (1.2) has at least k roots.*

Proof: Recall that $f(u) = g(u) + u|u|^{p-1}$, with $\lim_{u \to \infty} \frac{g(u)}{u|u|^{p-1}} = 0$. In (1.2) we set $u = az$, $r = \frac{1}{a^\beta}\xi$, $a > 0$. Letting $2\beta = p - 1$, and using primes for the derivatives of $z(\xi)$, we get

$$z'' + \frac{n-1}{\xi} z' + \frac{g(az)}{a^p} + z|z|^{p-1} = 0, \;\; z(0) = 1, \;\; z'(0) = 0\,. \tag{2.5}$$

For the problem

$$z'' + \frac{n-1}{\xi} z' + z|z|^{p-1} = 0, \quad z(0) = 1, \quad z'(0) = 0$$

it is well known that the solution is changing sign infinitely many times. By continuity, for a large, the solution of (2.5) has arbitrary many roots (the term $\frac{g(az)}{a^p}$ is uniformly small). $\diamondsuit$

Turning to the Neumann branches, we notice that there is an abundance of solutions of the Neumann problem (1.4), (1.6). For example, let $v(r)$ be a solution with no roots, tending to b_2. There are infinitely many r_k with $v'(r_k) = 0$. The scaling $r \to t$, given by $r = r_k t$, will provide infinitely many solutions of the Neumann problem (1.4), (1.6). We need to select the Neumann branches, which tend to the ground state solutions.

We define N_0 to be the set of positive and decreasing solutions of the Neumann problem (1.4), (1.6) (the lower curve in Figure 1). The upper branch of N_0 tends to the positive ground state.

We define N_1 to be the set of solutions of (1.4), (1.6) with exactly one root, exactly one monotonicity change, and ending (when $r = 1$) at the second point of zero slope. These are precisely the type of solutions of the initial value problem (1.2) which lie just below the ground state with one root, and so the upper branch of N_1 tends to this ground state. (Assume that $u(0) = g_1$, and $u'(0) = 0$, produces the ground state with one root. Then $u(0) = g_1 + \epsilon$, with $\epsilon > 0$ small, produces a solution with exactly two roots, and tending to b_2. As ϵ tends to zero, the second root tends to infinity. The point $u(0) = g_1 - \epsilon$ produces a solution with exactly one root, tending to b_1. As ϵ tends to zero, the second point of zero slope tends to infinity, and so λ on N_1 tends to infinity.) Remarkably, the lower branch of N_1 tends to the positive ground state, as our computations show.

We define N_2 to be the set of solutions of (1.4) and (1.6) with exactly two roots, exactly two monotonicity changes, and ending at the third point of zero slope. The upper branch of N_2 tends to a ground state solution with two roots, while the lower branch of N_2 tends to a ground state solution with one root.

We define N_k, $k \geq 3$, similarly, and our computations show similar results. Our computations indicate that there are no gaps between N_k's and D_k's, so that there is a unique ground state with exactly k roots.

In Figure 2 we computed the Dirichlet and Neumann curves for $f(u) = -u+u^3$ in the dimension $n = 3$. Notice the very sharp turns to the right that

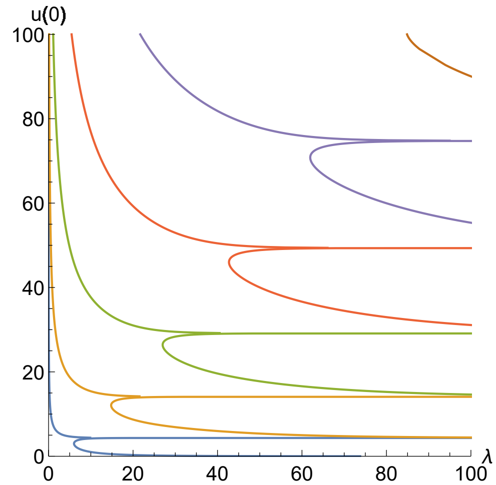

Figure 2: The Dirichlet and Neumann curves for $f(u) = -u + u^3$, and $n = 3$

the Neumann curves make, which presented some computational challenges to detect. When we changed the dimension to $n = 2$, the picture was similar, except for the scales of the axis. For $u(0) < 10$ we encounter already 16 Dirichlet curves and the Neumann curves are shifted considerably to the right (they only appear for large λ). In the dimensions $n \geq 4$ no Dirichlet curve is found and only the Neumann curve N_0 exists, which tends to zero as $u(0)$ increases. This result is not surprising, since $p = 3$ is critical for $n = 4$, and super-critical for $n > 4$.

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