MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2002; **25**:3–20 (DOI: 10.1002/mma.273)
MOS subject classification: 35 J 60; 34 B 15

Global solution curves for semilinear systems

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Communicated by W. Sprößig

SUMMARY

We study semilinear elliptic systems in two different directions. In the first one we give a simple constructive proof of existence of solutions for a class of sublinear systems. Our main results are in the second direction, where we use bifurcation theory to study global solution curves. Crucial to our analysis is proving positivity properties of the corresponding linearized systems. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: solution curves; positivity for the linearized problem

1. INTRODUCTION

We study semilinear elliptic systems in two different directions. The first one involves a class of sublinear systems, for which we give an example next. On a bounded smooth set $\Omega \subset R^n$ we consider a system

$$\begin{aligned} \Delta u + v^p = 0 \quad \text{on } \Omega, \quad u = 0 \text{ on } \partial\Omega \\ \Delta v + u^q = 0 \quad \text{on } \Omega, \quad v = 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

We assume that the system is sublinear, i.e. the positive constants p and q satisfy $pq < 1$. It turns out that just by scaling alone one can get an a priori estimate for (1), which usually provides a crucial step in proving existence of solutions. Indeed, we make a change of variables $u = \alpha U$, $v = \beta V$, with constant α and β. We choose β to equalize the coefficients in front of the nonlinear terms of the resulting system, i.e. $\beta^p/\alpha = \alpha^q/\beta$. So that we set $\beta = \alpha^{(q+1)/(p+1)}$. System (1) transforms as

$$\begin{aligned} \Delta U + \alpha^\theta V^p = 0 \quad \text{on } \Omega, \quad U = 0 \text{ on } \partial\Omega \\ \Delta V + \alpha^\theta U^q = 0 \quad \text{on } \Omega, \quad V = 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

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Contract/grant sponsor: Taft Faculty Grant

Received 19 December 2000

with $\theta = (pq-1)/(p+1)$. Observe that sublinearity assumption implies that $\theta < 0$. We claim that there is a positive constant c, depending on p and q so that any solution of (1) satisfies an *a priori* estimate

$$|u|_{L^\infty}, |v|_{L^\infty} < c \tag{3}$$

Indeed, set $\alpha = |u|_{L^\infty}$, so that

$$|U|_{L^\infty} = 1 \tag{4}$$

If α were to be large, then using (4), we would conclude from the second equation in (2) that V is small, and then from the first equation in (2) U would have to be small, contradicting (4).

By a similar argument we prove convergence of monotone iterations for a class of sublinear systems, obtaining both existence of solutions and a constructive way to approximate the solution. After completing this work, we became aware that a similar existence result was proved by Dalmasso [1]. He used the Leray–Schauder degree. Our proof appears to be a little simpler, it is constructive, and it provides an extra piece of information: the existence of a minimal solution.

Our main results are in another direction. In recent years Y. Li, T. Ouyang, J. Shi and the present author have used bifurcation analysis to study global curves of positive solutions for the two-point Dirichlet problem (λ is a positive parameter)

$$u''(x) + \lambda f(u(x)) = 0 \quad x \in (-1,1), \quad u(-1) = u(1) = 0 \tag{5}$$

see e.g. References [2–4]. The emphasis was for every λ to obtain the *exact* number of positive solutions, and to draw the global bifurcation diagrams. Let us briefly recall the steps of that analysis. It revolved around the properties of the linearized problem

$$w''(x) + \lambda f'(u(x))w(x) = 0 \quad x \in (-1,1), \ w(-1) = w(1) = 0 \tag{6}$$

The first two properties are very easy to prove, and they hold for any $f(u) \in C^1(R_+)$. Namely, if the problem (6) admits non-trivial solutions, then the solution space is one-dimensional (since it can be parameterized by $w'(1)$), and $w(x)$ does not change sign, so that we may choose $w(x) > 0$ on $(-1,1)$ (just observe that $u'(x)$ also satisfies the equation in (6), and that $x = 0$ is the only root of $u'(x)$). The main work is then in studying the direction of bifurcation at the turning points.

In the present paper we extend this analysis to systems of equations. It turns out that for the system case even the first two properties of the linearized system (mentioned above) cannot be taken for granted. We show by an example that a linear system may have a two-dimensional solution set, even in case of constant coefficients. Proving positivity of non-trivial solutions of the linearized system turned out to be a non-trivial task, and it required us to place some considerable restrictions on the nonlinearities. We were able to handle several classes of systems. One of them is

$$\begin{aligned} u''(x) + \lambda f(v(x)) = 0 \quad x \in (0,1), \quad u(0) = u(1) = 0 \\ v''(x) + \lambda g(u(x)) = 0 \quad x \in (0,1), \quad v(0) = v(1) = 0 \end{aligned} \tag{7}$$

with positive and increasing $f(t)$ and $g(t)$. After proving that the null-space of the linearized problem (at a degenerate solution) is one-dimensional, with both components positive, we were able to conclude several properties of the solution curves of (7). We show that the solution curves are global, i.e. they can be always continued. We also give conditions under which curves do not turn, and when there is only one curve. Our work culminates in Section 5, where we give an exact multiplicity result for (7), which to the best of our knowledge is the only such result for an elliptic system.

To continue the solutions, our main tool is the Crandall–Rabinowitz bifurcation theorem, [5], which we recall next.

Theorem 1.1 (Reference [5]). Let X and Y be Banach spaces. Let $(\bar{\lambda},\bar{x})\in\mathbf{R}\times X$ and let F be a continuously differentiable mapping of an open neighbourhood of $(\bar{\lambda},\bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda},\bar{x}))=\text{span }\{x_0\}$ be one-dimensional and $\operatorname{codim} R(F_x(\bar{\lambda},\bar{x}))=1$. Let $F_\lambda(\bar{\lambda},\bar{x})\notin R(F_x(\bar{\lambda},\bar{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda,x)=F(\bar{\lambda},\bar{x})$ near $(\bar{\lambda},\bar{x})$ form a curve $(\lambda(s),x(s))=(\bar{\lambda}+\tau(s),\bar{x}+sx_0+z(s))$, where $s\to(\tau(s), z(s))\in\mathbf{R}\times Z$ is a continuously differentiable function near $s=0$ and $\tau(0)=\tau'(0)=0$, $z(0)=z'(0)=0$.

2. EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A CLASS OF SUBLINEAR SYSTEMS

On a bounded smooth domain $\Omega\subset R^n$ we consider a system

$$\begin{aligned} &\Delta u+f(v)=0 \quad \text{on } \Omega \quad u=0 \text{ on } \partial\Omega \\ &\Delta v+g(u)=0 \quad \text{on } \Omega \quad v=0 \text{ on } \partial\Omega \end{aligned} \tag{8}$$

We assume that the functions $f(v)$ and $g(v)$ are Hölder continuous on R_+, and satisfy:

$$\begin{aligned} &c_1v^p\leqslant f(v)\leqslant c_2+c_3v^p \\ &c_4u^q\leqslant g(u)\leqslant c_5+c_6u^q \end{aligned} \tag{9}$$

for some positive constants c_1, c_3, c_4 and c_6, and non-negative constants c_2 and c_5; we assume that p and q are positive constants with

$$pq<1 \tag{10}$$

Theorem 2.1. In addition to conditions (9) and (10), assume that the functions $f(v)$ and $g(u)$ are strictly increasing on R^+. Then the problem (8) admits a positive solution.

Proof. We assume first that $p<1<q$ and (10) holds. We shall construct a monotone sequence converging to the solution from below. We search for a subsolution of (8) in the form $(\bar{u},\bar{v})=(\varepsilon\phi_1(x),\varepsilon^{(q+1)/(p+1)}\phi_1^q(x))$, where $\phi_1(x)>0$ is the principal eigenvalue of $-\Delta$ on Ω, corresponding to the principal eigenvalue λ_1, and a constant $\varepsilon>0$ is to be chosen. With

$\theta = (pq-1)/(p+1) < 0$ as before, we compute using our conditions (9) and (10)

$$\Delta \bar{u} + f(\bar{v}) \geqslant \varepsilon(-\lambda_1 \phi_1 + \varepsilon^{\theta} c_1 \phi_1^{pq}) > 0 \quad \text{for all } x \in \Omega \tag{11}$$

provided $\varepsilon > 0$ is sufficiently small. Similarly, using that $q > 1$,

$$\Delta \bar{v} + g(\bar{u}) \geqslant \varepsilon^{(q+1)/(p+1)}(q(q-1)\phi_1^{q-2}|\nabla \phi_1|^2 - q\lambda_1 \phi_1^q + \varepsilon^{\theta} c_4 \phi_1^q) > 0 \tag{12}$$

for all $x \in \Omega$, if ε small enough. It follows that $(\bar{u}, \bar{v})$ is a subsolution of (8) for ε small.

We now construct a monotone increasing sequence of iterates $(u_0, v_0) < (u_1, v_1) < \cdots < (u_n, v_n) < \ldots$, by setting $(u_0, v_0) = (\bar{u}, \bar{v})$ and solving inductively the system

$$\begin{aligned} &\Delta u_{n+1} + f(v_n) = 0 \quad \text{on } \Omega, \quad u_{n+1} = 0 \text{ on } \partial\Omega \\ &\Delta v_{n+1} + g(u_n) = 0 \quad \text{on } \Omega, \quad v_{n+1} = 0 \text{ on } \partial\Omega \end{aligned} \tag{13}$$

The fact that (u_n, v_n) is an increasing sequence (componentwise, i.e. $u_{n+1} \geqslant u_n$ and $v_{n+1} \geqslant v_n$) follows easily by the maximum principle, see e.g. Reference [6]. We claim that (u_n, v_n) are uniformly bounded, i.e. there is a constant $c > 0$ so that $u_n, v_n \leqslant c$ for all n and all $x \in \Omega$. Then the limit of this sequence will give us a classical solution of (2.1), see e.g. [6]. Assume on the contrary that $\alpha_n \equiv |u_n|_{L^\infty} \to \infty$. Denote $\beta_n = \alpha_n^{(q+1)/(p+1)}$. Then both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone increasing. We now make a change of variables $u_n = \alpha_n U_n$ and $v_n = \beta_n V_n$. Notice that

$$|U_n|_{L^\infty} = 1 \quad \text{for all } n \tag{14}$$

After the change of variables (13) becomes

$$\begin{aligned} -\Delta U_{n+1} &= \frac{1}{\alpha_{n+1}} f(\beta_n V_n) \quad \text{on } \Omega, \quad U_{n+1} = 0 \text{ on } \partial\Omega \\ -\Delta V_{n+1} &= \frac{1}{\beta_{n+1}} g(\alpha_n U_n) \quad \text{on } \Omega, \quad V_{n+1} = 0 \text{ on } \partial\Omega \end{aligned} \tag{15}$$

Using our conditions (9) and monotonicity of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$,

$$\begin{aligned} \frac{1}{\alpha_{n+1}} f(\beta_n V_n) &\leqslant \frac{c_2}{\alpha_{n+1}} + \frac{c_3 \alpha_n^{p(q+1)/(p+1)}}{\alpha_{n+1}} V_n^p \\ &\leqslant \frac{c_2}{\alpha_n} + c_3 \frac{\alpha_n^{p(q+1)/(p+1)}}{\alpha_n} V_n^p = \frac{c_2}{\alpha_n} + c_3 \alpha_n^{\theta} V_n^p \end{aligned} \tag{16}$$

Similarly,

$$\frac{1}{\beta_{n+1}} g(\alpha_n U_n) \leqslant \frac{c_5}{\beta_n} + \frac{c_6 \alpha_n^q}{\beta_n} U_n^p = \frac{c_5}{\beta_n} + c_6 \alpha_n^{\theta} U_n^p \tag{17}$$

Since $\theta < 0$, using (14), we see that the right hand side of (17) tends to zero with n. Using maximum principle, we see from the second equation in (15) that $|V_n|_{L^\infty}$ tends to zero. Using (16), we see from the first equation in (15) that $|U_n|_{L^\infty} \to 0$, which contradicts (14).

We proved that u_n are uniformly bounded, but then from the second equation in (13) the same is true for v_n.

In the case $p, q < 1$ we verify that $(\varepsilon\phi_1, \varepsilon\phi_1)$ is a subsolution of (8) for ε small, and proceed as before.

Remark. An uniqueness result under a slight additional condition is available in Dalmasso [1].

3. POSITIVITY FOR LINEAR SYSTEMS

We consider classical solutions for a class of linear systems on the interval $(-1, 1)$

$$
\begin{aligned}
w'' + b(x)z = 0 \quad &\text{on } (-1,1), \quad w(-1) = w(1) = 0\\
z'' + c(x)w = 0 \quad &\text{on } (-1,1), \quad z(-1) = z(1) = 0
\end{aligned} \tag{18}
$$

Lemma 3.1. Assume that the coefficient functions $b(x)$ and $c(x)$ are continuous on $[-1, 1]$. Then for any non-trivial solution of (18) at any $x \in [-1, 1]$ at least one of the quantities $w(x)$, $z(x)$, $w'(x)$ and $z'(x)$ must be non-zero.

Proof. Follows from the uniqueness for the initial value problems in case of systems.

Lemma 3.2. Assume that the coefficient functions $b(x)$ and $c(x)$ are positive and continuous on $[-1, 1]$. Then if the null space of (18) is non-trivial, it is one-dimensional. Moreover, if the functions $b(x)$ and $c(x)$ are even on $(-1, 1)$, then $w(x)$ and $z(x)$ are also even.

Proof. First, we observe that the null-space of (18) is at most two-dimensional, since its elements are determined by the vector (p, q), where $p = w'(1)$ and $q = z'(1)$. The null-space will be one-dimensional if all possible vectors (p, q) lie on one line, and two-dimensional if (p, q) span the R^2. Assuming the null-space to be two-dimensional, consider its element (w, z) with $p = w'(1) = 1$ and $q = z'(1) = -1$. Then near $x = 1$ we have $z(x) > 0$ and $w(x) < 0$, and consequently $w'' = -bz < 0$ and $z'' = -cw > 0$. As we go left from $x = 1$, we see that $w(x)$ is an increasing function, so long as $z > 0$, while $z(x)$ is decreasing so long as $w < 0$. Clearly, this situation persists for all $x < 1$, contradicting the boundary conditions at the left end $x = -1$.

If either $w(x)$ or $z(x)$ failed to be even in the case of even coefficients, the pair $(w(-x), z(-x))$ would give us another solution of (18), which is not a multiple of $(w(x), z(x))$, contradicting to the fact that the null space is one-dimensional.

Remark. In general the null-space may fail to be one-dimensional even for a system with constant coefficients, as the following example shows:

$$
\begin{aligned}
w'' + \lambda_1 w + bz = 0 \quad &\text{on } (-1,1), \quad w(-1) = w(1) = 0\\
z'' + \lambda_2 z = 0 \quad &\text{on } (-1,1), \quad z(-1) = z(1) = 0
\end{aligned} \tag{19}
$$

where $\lambda_1 = \pi^2/4$ is the principal eigenvalue of $-w''$ on $(-1,1)$, with the corresponding eigenfunction $\phi_1(x) = \cos \pi/2x$, and $\lambda_2 = \pi^2$ is the second eigenvalue of the same operator, corresponding to $\phi_2(x) = \sin \pi x$. Then solution set of (19) includes $(\phi_1, 0)$ and (θ, ϕ_2), where $\theta(x)$ is a non-trivial solution of

$$\theta'' + \lambda_1\theta + b\phi_2 = 0 \quad \text{on } (-1,1), \quad \theta(-1) = \theta(1) = 0$$

which exists by the Fredholm alternative.

Lemma 3.3. In addition to the conditions of Lemma 3.2 assume that $b(x)$ and $c(x)$ are even functions on $(-1,1)$. If $w(x) > 0$ (or $z(x) > 0$) on $(0,1)$ then $w(x) > 0$ and $z(x) > 0$ on the full interval $(-1,1)$.

Proof. By our conditions and the previous lemma the functions $w(x)$ and $z(x)$ are even, and hence $w(x) \geqslant 0$ on $(-1,1)$ (since we do not know the value of $w(0)$, we cannot assert a strict inequality). But applying the strong maximum principle to the second equation in (18), and then to the first one, we conclude the lemma.

The following lemma provides a Sturm comparison type result for systems.

Lemma 3.4. Consider the system

$$\begin{aligned} &U'' + B(x)V = 0 \quad \text{on } (-1,1), \; U(-1) = U(1) = 0 \\ &V'' + C(x)U = 0 \quad \text{on } (-1,1), \; V(-1) = V(1) = 0 \end{aligned} \tag{20}$$

Assume that

$$B(x) \geqslant b(x) \quad \text{and} \quad C(x) \geqslant c(x) \quad \text{for all } x \in (-1,1) \tag{21}$$

with at least one of the inequalities being strict on a set of positive measure. If the problem (18) admits a positive solution (i.e. both components are positive), then the problem (20) cannot have a positive solution.

Proof. Assume on the contrary that the problem (20) has a positive solution. From the first equations in (21) and (18) we obtain

$$-\int_{-1}^{1} U'v' \, \mathrm{d}x + \int_{-1}^{1} u'V' \, \mathrm{d}x + \int_{-1}^{1} (B-b)Vv \, \mathrm{d}x = 0$$

Similarly from the second equations we have

$$-\int_{-1}^{1} u'V' \, \mathrm{d}x + \int_{-1}^{1} U'v' \, \mathrm{d}x + \int_{-1}^{1} (C-c)Uu \, \mathrm{d}x = 0$$

Adding, we have

$$\int_{-1}^{1} (B-b)Vv \, \mathrm{d}x + \int_{-1}^{1} (C-c)Uu \, \mathrm{d}x = 0$$

which is a contradiction, since both terms on the left are non-negative, with at least one strictly positive.

We will be studying positive solutions for non-linear systems of the form

$$\begin{aligned} u'' + f(v) &= 0 \quad \text{on } (-1,1),\ u(-1)=u(1)=0 \\ v'' + g(u) &= 0 \quad \text{on } (-1,1),\ v(-1)=v(1)=0 \end{aligned} \tag{22}$$

The linearized problem for (22) is

$$\begin{aligned} w'' + b(x)z &= 0 \quad \text{on } (-1,1),\ w(-1)=w(1)=0 \\ z'' + c(x)w &= 0 \quad \text{on } (-1,1),\ z(-1)=z(1)=0 \end{aligned} \tag{23}$$

with $b(x)=f'(v)$ and $c(x)=g'(u)$.

Theorem 3.1. Let (u,v) be a positive solution of (22). Assume that the functions $f,g\in C^1(\bar{R}_+)$ satisfy

$$f(v)>0,\ f'(v)>0,\ g(u)>0,\ g'(u)>0 \quad \text{for all } u,v>0 \tag{24}$$

Then any non-trivial solution of the linearized problem (23) can be chosen to be positive on $(-1,1)$.

Proof. Since our system is of cooperative type, by the result of W. Troy [7], both u and v are even functions, with $u'(x)<0$ and $v'(x)<0$ on $(0,1)$. Denoting $p=u'(x)$ and $q=v'(x)$, we obtain

$$\begin{aligned} p'' + b(x)q &= 0 \quad \text{on } (-1,1) \\ q'' + c(x)p &= 0 \quad \text{on } (-1,1) \end{aligned} \tag{25}$$

with $b(x)=f'(v)$ and $c(x)=g'(u)$, i.e. (p,q) is also a solution of the linearized problem. Since $p'=u''=-f(v)<0$, and similarly $q'<0$, we see that the functions p and q are negative and strictly decreasing on $(0,1)$.

Assume contrary to what we want to prove that w vanishes on $(0,1)$. By Lemma 3.3 z also vanishes on $(0,1)$. Let ξ denote the largest root of w on $(0,1)$ and let α be largest root of z on $(0,1)$. Assume first that $\alpha<\xi$. We may assume that $z>0$ on $(\alpha,1)$. From the first equation in (23) we then see that w is concave on $(\alpha,1)$. Since $w(\xi)=0$, this implies that the function $w(x)$ is positive on $(\xi,1)$, and it is negative and increasing on (α,ξ). In particular we have (see Figure 1)

$$w(\alpha)<0 \quad \text{and} \quad w'(\alpha)>0 \tag{26}$$

We now multiply the first equation in (23) by q, and subtract from it the first equation in (25) multiplied by z. We then integrate over $(\alpha,1)$, obtaining

$$(w'q - p'z)|_\alpha^1 - \int_\alpha^1 w'q'\,\mathrm{d}x + \int_\alpha^1 z'p'\,\mathrm{d}x = 0 \tag{27}$$

Similarly from the second equations in (23) and (25) we obtain

$$(z'p - q'w)|_\alpha^1 - \int_\alpha^1 z'p'\,\mathrm{d}x + \int_\alpha^1 w'q'\,\mathrm{d}x = 0 \tag{28}$$

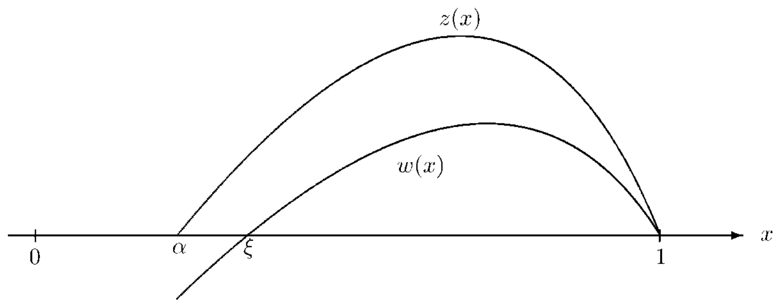

Figure 1.

Adding (27) and (28), we have

$$(w'q - p'z + z'p - q'w)|_{\alpha}^{1} = 0$$

which is

$$w'(1)q(1) + z'(1)p(1) - w'(\alpha)q(\alpha) - z'(\alpha)p(\alpha) + q'(\alpha)w(\alpha) = 0 \tag{29}$$

Using (26) and the fact that p, p' and q, q' are negative, we see that all five terms on the left in (29) are non-negative and by Lemma 3.1 either the first or the second one is strictly positive, a contradiction. The case $\alpha > \xi$ is similar, and in case $\alpha = \xi$ the fifth term in (29) is zero, while the other four terms are still positive, resulting in the same contradiction.

We need the following elementary lemmas.

Lemma 3.5. Assume $w(x) \in C^1[0, \gamma]$ satisfies $w(0) = 0$. Then

$$\int_0^\gamma w'^2 \, \mathrm{d}x \geqslant \frac{\pi^2}{4\gamma^2} \int_0^\gamma w^2 \, \mathrm{d}x$$

Proof. Minimizing $\int_0^\gamma w'^2 \, \mathrm{d}x$ subject to $\int_0^\gamma w^2 \, \mathrm{d}x = 1$, and the value $w(\gamma)$ being free, we obtain an extremal $w = \sin(\pi/2\gamma)x$, and the lemma follows.

Lemma 3.6. Let $w(x) \in C^2[\alpha, \xi]$ for some $0 < \alpha < \xi < 1$ satisfy the inequalities

$$w'' + a(x)w < 0 \quad \text{and} \quad w < 0 \text{ on } (\alpha, \xi) \tag{30}$$

Assume also that $a(x) \leqslant \pi^2/4$ for all $x \in [0, 1]$, and $w(\xi) = 0$. Then

$$w'(\alpha) > 0 \tag{31}$$

Proof. Multiply (30) by $w < 0$ and integrate over (α, ξ)

$$-w(\alpha)w'(\alpha) - \int_\alpha^\xi w'^2 \, \mathrm{d}x + \frac{\pi^2}{4} \int_\alpha^\xi w^2 \, \mathrm{d}x > 0 \tag{32}$$

If we assume (31) to be violated, we obtain a contradiction in (32), since the first term is non-positive, while the second term dominates the third one by the Lemma 3.5.

We shall now consider non-linear systems of the form

$$\begin{aligned} u'' + f(u) + bv &= 0 \quad \text{on } (-1,1),\ u(-1)=u(1)=0 \\ v'' + cu + g(v) &= 0 \quad \text{on } (-1,1),\ v(-1)=v(1)=0 \end{aligned} \tag{33}$$

with positive constants b and c, and the corresponding linearized system

$$\begin{aligned} w'' + a(x)w + bz &= 0 \quad \text{on } (-1,1),\ w(-1)=w(1)=0 \\ z'' + cw + d(x)z &= 0 \quad \text{on } (-1,1),\ z(-1)=z(1)=0 \end{aligned} \tag{34}$$

where we have denoted $a(x)=f'(u(x))$ and $d(x)=g'(v(x))$.

Lemma 3.7. Assume that the coefficient functions $a(x)$ and $d(x)$ are continuous, and $a(x)<\pi^2/4$, $d(x)<\pi^2/4$ on $(-1,1)$, while b and c are positive constants. Then if the solution space of (34) is non-trivial, it is one-dimensional. Moreover, if the functions $a(x)$ and $d(x)$ are even on $(-1,1)$, then $w(x)$ and $z(x)$ are also even. In such a case, if $w(x)>0$ (or $z(x)>0$) on $(0,1)$ then $w(x)>0$ and $z(x)>0$.

Proof. We proceed as in proof of Lemma 3.2. Assuming the null-space to be two-dimensional, consider its element (w,z) with $p=w'(1)=1$ and $q=z'(1)=-1$. Then near $x=1$ $w(x)$ is negative, while $z(x)$ is positive. Both functions vanish at $x=-1$. Let $-1\leqslant\alpha<1$ be the largest root of $w(x)z(x)$. Assume for definiteness that $w(\alpha)=0$, while $z(x)>0$ on $(\alpha,1)$. From the first equation in (34)

$$w'' + a(x)w = -bz < 0 \quad \text{on } (\alpha,1)$$

Since $a(x)$ is less than the principal eigenvalue of Laplacian on $(-1,1)$ (and hence on the smaller interval $(\alpha,1)$), it follows by the well-known negativity of the Green's function that $w(x)$ is positive on $(\alpha,1)$, which is a contradiction. This proves that the solution space of (34) is one-dimensional, and the rest of the proof is similar to that of Lemma 3.2.

Theorem 3.2. Let (u,v) be a positive solution of (33). Assume that the functions $f,g\in C^1$ $(\bar{R}_+)$ satisfy

$$f(t)>0,\ f'(t)<\frac{\pi^2}{4},\ g(t)>0,\ g'(t)<\frac{\pi^2}{4} \quad \text{for all } t>0 \tag{35}$$

Then any non-trivial solution for the linearized problem (34) can be chosen to be positive on $(-1,1)$.

Proof. We proceed as in the Theorem 3.1. Since our system (33) is of co-operative type, by the result of Troy [7], both u and v are even functions, with $u'(x)<0$ and $v'(x)<0$ on

$(0,1)$. Denoting $p=u'(x)$ and $q=v'(x)$, we obtain differentiating (33)

$$\begin{aligned} p''+a(x)p+bq &= 0 \quad \text{on } (-1,1) \\ q''+cp+d(x)q &= 0 \quad \text{on } (-1,1) \end{aligned} \tag{36}$$

Assume contrary to what we want to prove that w vanishes on $(0,1)$. By Lemma 3.7 z also vanishes on $(0,1)$. Let ξ denote the largest root of w on $(0,1)$ and let α be the largest root of z on $(0,1)$. We may assume as before that $\alpha<\xi$. We may assume that $z>0$ on $(\alpha,1)$. From the first equation in (34) using the well-known negativity of the Green's function on $(\xi,1)$, we conclude that $w(x)$ is positive on $(\xi,1)$. By Lemma 3.6 $w(x)$ is negative and increasing on (α,ξ). In particular we have (26). From the first equations in (34) and (36) we obtain

$$(w'p-wp')'+b(pz-qw)=0 \tag{37}$$

Similarly from the second equations:

$$(z'q-zq')'+c(wq-pz)=0 \tag{38}$$

Multiplying (37) by c, (38) by b and adding

$$c(w'p-wp')'+b(z'q-zq')'=0 \tag{39}$$

Observe that

$$p'(\alpha)=u''(\alpha)<0 \tag{40}$$

since we assumed $f(u)$ to be positive. We now integrate (39) over $(\alpha,1)$, obtaining

$$c[w'(1)p(1)+w(\alpha)p'(\alpha)-w'(\alpha)p(\alpha)]+b[z'(1)q(1)-z'(\alpha)q(\alpha)]=0 \tag{41}$$

We have a contradiction, since all five terms on the left in (41) are non-negative, and the second one is strictly positive.

4. CURVES OF POSITIVE SOLUTIONS FOR A CLASS OF SYSTEMS

We consider positive solutions of the system

$$\begin{aligned} u''+\lambda f(v) &= 0 \quad \text{on } (-1,1),\ u(-1)=u(1)=0 \\ v''+\lambda g(u) &= 0 \quad \text{on } (-1,1),\ v(-1)=v(1)=0 \end{aligned} \tag{42}$$

depending on a positive parameter λ. Throughout this section we assume that

$$f(t),f'(t)>0 \quad \text{and} \quad g(t),g'(t)>0 \quad \text{for all } t\geqslant 0 \tag{43}$$

By a result of Troy [7] positive solutions of (42) are even functions, with $u'(x)<0$ and $v'(x)<0$ for $x>0$. We shall then use the maximal values $(u(0),v(0))$ to represent the solution, obtaining curve(s) in (u,v) plane depending on the parameter λ.

The following lemma extends a fact, which is well known in the case of one equation.

Lemma 4.1. The point $(u(0), v(0))$ uniquely identifies the solution. I.e. if $(U(x), V(x))$ is another positive solution of (42) corresponding to a parameter μ, then $\mu=\lambda$, and $(U(x), V(x)) \equiv (u(x), v(x))$.

Proof. Assuming that $\lambda \neq \mu$, we observe that both $(u((1/\sqrt{\lambda})x), v((1/\sqrt{\lambda})x))$ and $(U((1/\sqrt{\mu})x), V((1/\sqrt{\mu})x))$ satisfy the same equations in (42), with $\lambda=1$, and they have the same values at $x=0$. Since these solutions are different (they have different first roots), we have a contradiction by the uniqueness result for initial value problems. It follows that $\lambda=\mu$, and then applying uniqueness result for initial value problems again, we conclude that $(U(x), V(x)) \equiv (u(x), v(x))$.

In view of the above lemma we can identify any positive solution of (42) with a point in the first quadrant of the $(u(0), v(0))$ plane. When λ varies we obtain curves in the $(u(0), v(0))$ plane.

Theorem 4.1. Assume conditions (43) are satisfied. Then the solutions of (42) lie on global continuous non-intersecting curves in the $(u(0), v(0))$ plane.

Proof. We show that at any solution of (42) either the implicit function theorem or the Crandall–Rabinowitz bifurcation theorem applies, and hence we can always continue to the neighbouring solutions (that is what we mean by 'global curves'). Indeed, define a map

$$F(u,v)=\begin{pmatrix} u''+\lambda f(v) \\ v''+\lambda g(u) \end{pmatrix} : C_0^2(-1,1)\times C_0^2(-1,1)\to C(-1,1)\times C(-1,1)$$

Clearly, our system (42) can be recast in the form $F(u,v)=0$, and the system (23) gives the linearized problem. If (23) has only the trivial solution then the implicit function theorem applies, allowing us to continue the solution to nearby λ's.

Assume now that at some $(\bar{\lambda}, \bar{u}, \bar{v})$ system (23) has a non-trivial solution. Then by Lemma 3.2 the solution set is one-dimensional, and by Theorem 3.1 we may assume that $w(x)>0$ and $z(x)>0$ for all $x\in(-1,1)$. The range of the map $F_{(u,v)}(\bar{u},\bar{v})$ has codimension 1, as can be seen by recasting (23) as a system of two integral equations, them defining a compact operator on the product space $C(-1,1)\times C(-1,1)$, and applying the Riesz–Schauder theory. To apply the Crandall–Rabinowitz theorem 1.1 it remains to show that $F_\lambda(\bar{\lambda},\bar{x}) \notin R(F_x(\bar{\lambda},\bar{x}))$, with $\bar{x}=(\bar{u},\bar{v})$. Assuming otherwise, there exist $(W(x), Z(x))$ in $C^2(-1,1)$ so that

$$\begin{aligned} W''+b(x)Z &= f(v) \quad \text{on } (-1,1),\ W(-1)=W(1)=0 \\ Z''+c(x)W &= g(u) \quad \text{on } (-1,1),\ Z(-1)=Z(1)=0 \end{aligned} \tag{44}$$

with $b(x)=f'(v(x))$, and $c(x)=g'(u(x))$. Multiply the first equation in (44) by z, the first equation in (23) by Z, subtract and integrate over $(-1,1)$. Similarly, multiply the second equation in (44) by w, the second equation in (23) by W, subtract and integrate over $(-1,1)$. Finally, add both results. After appropriate cancelations we have

$$\int_{-1}^{1} [f(v)z+g(u)w]\,\mathrm{d}x=0$$

This is a contradiction, since the integrand is positive.

The following result gives conditions for the solution curves to have no turns in λ (i.e. λ can then be used as a global parameter).

Theorem 4.2. Assume that conditions (43) hold, and assume also that $f'(t) > f(t)/t$ and $g'(t) > g(t)/t$ for all $t > 0$. Then problem (42) has at most one positive solution on any solution curve, for all $\lambda > 0$ (i.e., any solution curve intersects at most once the line $\lambda = \lambda_0$, for any $\lambda_0 > 0$).

Proof. Observe that the Theorem 3.1 applies here. We claim that when continued in λ none of the solution curves may turn. Indeed, if a solution curve were to turn, then near the turning point we would have a second solution (U, V), with $U > u$ and $V > v$ (since by the Crandall–Rabinowitz Theorem 1.1 the difference of two solutions is asymptotically proportional to the positive solution of the linearized system). We have

$$\begin{aligned} U'' + \lambda f(V) &= 0 \quad \text{on } (-1,1),\ U(-1) = U(1) = 0 \\ V'' + \lambda g(U) &= 0 \quad \text{on } (-1,1),\ V(-1) = V(1) = 0 \end{aligned} \tag{45}$$

Multiply the first equation in (42) by V, the first equation in (45) by v, subtract and integrate over $(-1, 1)$. Similarly, multiply the second equation in (42) by U, the second equation in (45) by u, subtract and integrate over $(-1, 1)$. Finally, add both results. After cancelations we have

$$\int_{-1}^{1} vV \left(\frac{f(V)}{V} - \frac{f(v)}{v} \right) \mathrm{d}x + \int_{-1}^{1} uU \left(\frac{g(U)}{U} - \frac{g(u)}{u} \right) \mathrm{d}x = 0$$

Since our conditions imply that the functions $f(t)/t$ and $g(t)/t$ are increasing for $t > 0$, we have both integrands positive, which is a contradiction.

We can conclude uniqueness of solution if there is an a priori estimate.

Theorem 4.3. In addition to the conditions of the Theorem 4.2 assume that there are constants $\bar{\lambda} > 0$ and $c > 0$, so that for all $\lambda \leqslant \bar{\lambda}$ we have

$$u(x) + v(x) \leqslant c \quad \text{for all } x \in (-1, 1) \text{ uniformly in } \lambda \tag{46}$$

Assume also that $f(t)$ and $g(t)$ are continuously differentiable for all $t \geqslant 0$. Then for every $0 < \lambda \leqslant \bar{\lambda}$, problem (42) has at most one positive solution.

Proof. The *a priori* estimate (46) implies that any solution curve must enter $(0, 0)$ as $\lambda \to 0$. We claim that there is at most one solution curve. Since by the previous theorem solution curves cannot turn, uniqueness will follow. Indeed, assume there is a second solution curve $(U(x, \lambda), V(x, \lambda))$. Consider the equation satisfied by $U - u$, use the mean-value theorem, then multiply by $U - u$ and integrate. Do the same for $V - v$, and then add the results. We have

$$\|U - u\|_{L^2}^2 + \|V - v\|_{L^2}^2 \leqslant \lambda \left(\max_{[0,c]} |f'(t)| + \max_{[0,c]} |g'(t)| \right) (\|U - u\|_{L^2}^2 + \|V - v\|_{L^2}^2)$$

which implies a contradiction for λ sufficiently small.

Remark. The *a priori* estimate (46) will hold, for example, if either $f(t)$ or $g(t)$ is a bounded function, or in the case system (42) is sublinear.

5. EXACT MULTIPLICITY RESULTS

We now derive a complete description of the solution set for the problem

$$\begin{aligned} u'' + \lambda f(v) &= 0 \quad \text{on } (-1,1),\ u(-1)=u(1)=0 \\ v'' + \lambda g(u) &= 0 \quad \text{on } (-1,1),\ v(-1)=v(1)=0 \end{aligned} \tag{47}$$

for a class of convex f and g.

Theorem 5.1. Assume that the functions $f(t),\ g(t) \in C^2(\bar{R}_+)$ satisfy condition (24), and in addition

$$f(0)+g(0)>0 \tag{48}$$

$$f''(t)\geqslant 0, \quad \text{and } g''(t)\geqslant 0 \quad \text{for all } t>0 \tag{49}$$

with at least one of the inequalities being strict on a set of positive measure,

$$f(t)>\alpha t, \quad \text{and } g(t)>\alpha t \quad \text{for some } \alpha>0 \quad \text{and all } t>0 \tag{50}$$

$$\text{either } \lim_{t\to\infty} \frac{f(t)}{t}=\infty, \quad \text{or } \lim_{t\to\infty} \frac{g(t)}{t}=\infty \tag{51}$$

Then there is a critical $\lambda_0>0$ such that problem (47) has exactly 2, 1 or 0 positive solutions, depending on whether $\lambda<\lambda_0$, $\lambda=\lambda_0$ or $\lambda>\lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda<\lambda_0$ has two branches denoted by $0<(u,v)^-(x,\lambda)<(u,v)^+(x,\lambda)$, with $(u,v)^-(x,\lambda)$ strictly monotone increasing (componentwise) in λ. We also have $\lim_{\lambda\to 0+}(u,v)^-(x,\lambda)=(0,0)$ and $\lim_{\lambda\to 0+}(u,v)^+(x,\lambda)=(\infty,\infty)$.

Proof. We begin by proving existence of positive solutions for small $\lambda>0$. Since the problem is quasi-monotone we can use monotone iterations as in Reference [6]. Clearly $(0,0)$ is a strict subsolution. Let $s(x)$ denote the solution of the problem

$$s''=-1 \quad \text{on } (-1,1),\ s(-1)=s(1)=0 \tag{52}$$

Then for λ small $(s(x),s(x))$ is a supersolution of (47). Existence of a minimal positive solution then follows by Reference [6]. (Alternatively, we could use the implicit function theorem to continue the trivial solution at $\lambda=0$, and then observe that by the maximum principle solutions are positive for small $\lambda>0$.)

Next, we show that no positive solution exists for λ large. Assuming existence of solutions, we multiply each equation in (47) by $\cos(\pi/2)x$, integrate and add the results. Using the condition (50), we conclude

$$-\frac{\pi^2}{4}\int_{-1}^{1}(u(x)+v(x))\cos\frac{\pi}{2}x\,\mathrm{d}x+\lambda\alpha\int_{-1}^{1}(u(x)+v(x))\cos\frac{\pi}{2}x\,\mathrm{d}x<0$$

which results in a contradiction in case λ is large.

Our next step is to show that a positive solution of (47) cannot become unbounded at any $\bar{\lambda}>0$. The idea of the proof is that solution can become large only on a small interval, which is inconsistent with its concavity. Assume contrary to what we want to prove that either $u(x)$ or $v(x)$ becomes unbounded as $\lambda\to\bar{\lambda}$, say $u(x)$. Since $u(x)$ is concave, it follows that it has to become large on any proper subinterval of $(-1,1)$, say on $(0,1/2)$. Using the condition (50) it easily follows that $v(x)$ must also become unbounded on $(0,1/2)$ (by using the Green's function, for example). Assume for definiteness that the first condition in (51) holds. Then for any $M>0$ we have

$$f(v)>Mv \quad \text{on } (0,1/2) \tag{53}$$

provided that λ is close to $\bar{\lambda}$. We now multiply the first equation in (47) by $\sin 2\pi x$, and integrate over $(0,1/2)$. Using (53), we have (dropping two positive terms on the left)

$$-4\pi^2\int_0^{1/2}u(x)\sin 2\pi x\,\mathrm{d}x+\lambda M\int_0^{1/2}v(x)\sin 2\pi x\,\mathrm{d}x<0 \tag{54}$$

Similarly from the second equation in (47),

$$-4\pi^2\int_0^{1/2}v(x)\sin 2\pi x\,\mathrm{d}x+\lambda\alpha\int_0^{1/2}u(x)\sin 2\pi x\,\mathrm{d}x<0 \tag{55}$$

Since M is large the inequalities (54) and (55) contradict each other.

We now consider the minimal positive solution of (47) at some small λ. If the corresponding linearized problem (23) has only a trivial solution $w=0$, $z=0$, then using the implicit function theorem we can locally continue this solution for increasing λ. Since for large λ the problem (47) has no positive solutions, we cannot continue the curve indefinitely with λ increasing. Let λ_0 be the supremum of λ for which we can continue the curve to the right. By above, solutions on this curve remain bounded as $\lambda\to\lambda_0$. Using elliptic estimates we pass to the limit, and conclude that at λ_0 the problem (47) has a solution $(u_0,v_0)\in C^2[-1,1]\times C^2[-1,1]$. By the definition of λ_0 it follows that the map $F(u,v)$, defined in the proof of the Theorem 4.1, is singular at (u_0,v_0), i.e. the problem (23) has a non-trivial solution, (w,z). By Theorem 3.1 we have

$$w(x)>0 \quad \text{and} \quad z(x)>0 \quad \text{for all } x\in(-1,1) \tag{56}$$

As we checked in the proof of Theorem 4.1, the Crandall–Rabinowitz Theorem 1.1 applies at (u_0,v_0), and hence we can continue the solution curve through this point. We now compute the direction of bifurcation at (u_0,v_0) (and at any other singular point).

Near (λ_0,u_0,v_0) we represent $\lambda=\lambda(s),(u,v)=(u(s),v(s))$, with $\lambda(0)=\lambda_0$ and $(u(0),v(0))=(u_0,v_0)$. Notice that $\lambda'(0)=0$, since the solution curve does not extend beyond λ_0. Also, by the Crandall–Rabinowitz theorem $(u_s(0),v_s(0))=(w,z)$. Differentiating the first equation in (47) twice in s, and then letting $s=0$, we obtain

$$u''_{ss}+\lambda_0 f'v_{ss}+\lambda_0 f''z^2+\lambda''(0)f=0, \quad u_{ss}(-1)=u_{ss}(1)=0 \tag{57}$$

Similarly, from the second equation in (47) we obtain

$$v_{ss}'' + \lambda_0 g' u_{ss} + \lambda_0 g'' w^2 + \lambda''(0) g = 0, \quad v_{ss}(-1) = v_{ss}(1) = 0 \tag{58}$$

We now multiply (57) by z, the first equation in (23) by v_{ss}, subtract and integrate

$$-\int_{-1}^{1} z' u_{ss}' \,\mathrm{d}x + \int_{-1}^{1} w' v_{ss}' \,\mathrm{d}x + \lambda_0 \int_{-1}^{1} f''(v_0) z^3 \,\mathrm{d}x + \lambda''(0) \int_{-1}^{1} f(v_0) z \,\mathrm{d}x = 0 \tag{59}$$

Similarly, using (58) and the second equation in (23), we obtain

$$-\int_{-1}^{1} w' v_{ss}' \,\mathrm{d}x + \int_{-1}^{1} z' u_{ss}' \,\mathrm{d}x + \lambda_0 \int_{-1}^{1} g''(u_0) w^3 \,\mathrm{d}x + \lambda''(0) \int_{-1}^{1} g(u_0) w \,\mathrm{d}x = 0 \tag{60}$$

Adding (59) and (60), we have using (49) and Theorem 3.1

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^{1} f''(v_0) z^3 \,\mathrm{d}x + \int_{-1}^{1} g''(u_0) w^3 \,\mathrm{d}x}{\int_{-1}^{1} f(v_0) z \,\mathrm{d}x + \int_{-1}^{1} g(u_0) w \,\mathrm{d}x} < 0 \tag{61}$$

It follows from (61) that only 'turns to the left' are possible at any singular solution of (47).

We see that after turning to the left at (λ_0, u_0, v_0) the solution curve continues to travel to the left (since turns to the right are impossible). Near the turning point (λ_0, u_0, v_0) solutions on the upper branch are decreasing in λ (or increasing when following the curve), as follows by the Theorems 1.1 and 3.1. By Lemma 4.1, we see that the maximum values $(u(0), v(0))$ keep increasing, as we follow the curve for decreasing λ. Since solutions cannot become unbounded at $\lambda > 0$, and there are no non-trivial solutions at $\lambda = 0$, it follows that solutions on the curve become unbounded as $\lambda \downarrow 0$.

Finally we prove monotonicity of the lower branch. As we have just observed, near the turning point (λ_0, u_0, v_0) on the lower branch we have

$$(u_\lambda(x,\lambda), v_\lambda(x,\lambda)) > 0 \quad \text{for all } x \in (-1,1) \tag{62}$$

We claim that (62) holds for all $0 < \lambda < \lambda_0$ on the lower branch. Assuming otherwise, let $\lambda_1 > 0$ denote the infimum of λ's for which the inequality (62) is true. Then $(u_\lambda(x,\lambda), v_\lambda(x,\lambda))$ is still non-negative, but either $u_\lambda(x,\lambda_1)$ has a root inside $(-1,1)$ or a zero slope at either end, $x = -1$ or $x = 1$ (or the same situation holds for $v_\lambda(x,\lambda_1)$). But

$$u_\lambda''(x,\lambda_1) = -\lambda_1 f'(v) v_\lambda - f(v) < 0 \quad \text{for all } x \in (-1,1)$$

which rules out both of the above possibilities. It follows that (62) holds for all $0 < \lambda < \lambda_0$, and the lower branch is increasing in λ. As $\lambda \downarrow 0$ the lower branch then has to enter $(0,0)$, which completes the proof.

In case $f(v) = v$, we obtain an exact multiplicity result for the fourth-order equation, which we state explicitly next, given its physical significance (it describes deflections of a freely supported elastic beam).

Theorem 5.2. Consider the problem

$$u'''' = \lambda f(u) \quad \text{on } (-1,1), \; u(-1) = u(1) = u''(-1) = u''(1) = 0 \tag{63}$$

Assume that $f(t) \in C^2(\bar{R}_+)$ satisfies condition (24), and in addition the following conditions:

$$f(0) > 0$$

$$f''(t) \geqslant 0, \quad \text{for all } t > 0$$

with the inequality being strict on a set of positive measure,

$$f(t) > \alpha t \quad \text{for some } \alpha > 0 \quad \text{and all } t > 0$$

$$\lim_{t \to \infty} \frac{f(t)}{t} = \infty$$

Then there is a critical $\lambda_0 > 0$ such that the problem (63) has exactly 2, 1 or 0 positive solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$.

We shall now present a uniqueness result for another class of systems

$$\begin{aligned} u'' + f(u) + bv = 0 \quad \text{on } (-1,1),\ u(-1) = u(1) = 0 \\ v'' + cu + g(v) = 0 \quad \text{on } (-1,1),\ v(-1) = v(1) = 0 \end{aligned} \tag{64}$$

But first we need an elementary lemma. We present it in the form it is used later (the second condition in (67) below can be relaxed).

Lemma 5.1. Consider the problem

$$u'' + f(u) = 0 \quad \text{on } (-1,1),\ u(-1) = u(1) = 0 \tag{65}$$

Assume that the function $f \in C(\bar{R}_+)$ satisfies $f(u) > 0$ for all $u > 0$, and in addition

$$f(0) > 0 \tag{66}$$

$$\frac{f(u)}{u} \text{ is decreasing for all } u > 0 \text{ and } \lim_{u \to \infty} \frac{f(u)}{u} = 0 \tag{67}$$

Then problem (65) has a unique positive solution.

Proof. By our conditions one can find $b > 0$ so that $f(u) < \frac{1}{2}u + b$ for all $u > 0$. Let $s(x)$ again denote the solution of (52). Then it is easy to check that $Ms(x)$ will be a supersolution of (65), provided we choose M sufficiently large. Since zero is a subsolution of (65), we conclude existence of a positive solution. If there were more than one solution, we would have a maximal solution, and easily obtain a contradiction, using the first condition in (67).

As before, we denote by $\lambda_1 = \pi^2/4$ the principal eigenvalue of $-u''$ on $(-1,1)$, with the corresponding eigenfunction $\phi_1(x) = \cos \pi/2x$.

Theorem 5.3. Assume that conditions (64) hold, b and c are positive constants, satisfying

$$bc < {\lambda_1}^2 \tag{68}$$

and in addition both f and g satisfy (66) and (67). Then the system (64) has a unique positive solution. Moreover, when either b or c is varied, solutions of (64) lie on continuous global curves.

Proof. We imbed our problem (64) into a family of problems

$$
\begin{aligned}
&u'' + f(u) + \lambda b v = 0 \quad \text{on } (-1,1),\ u(-1) = u(1) = 0 \\
&v'' + \lambda c u + g(v) = 0 \quad \text{on } (-1,1),\ v(-1) = v(1) = 0
\end{aligned}
\tag{69}
$$

with a parameter $\lambda \in [0,1]$. When $\lambda = 0$ by Lemma 5.1 there exists a unique positive solution. Similarly to the Theorem 4.1 we show that when λ is varied solutions of (69) lie on smooth curves, i.e. at each point either the implicit function theorem or the Crandall–Rabinowitz theorem applies. By a small modification of the proof of the Theorem 4.2 we conclude that these curves do not turn. Indeed, assuming otherwise, as in the Theorem 4.2 we conclude existence of two positive solutions of (69), with $(U,V) > (u,v)$, and then from the corresponding equations we obtain

$$
\frac{1}{b}\int_{-1}^{1}\left(\frac{f(u)}{u} - \frac{f(U)}{U}\right) uU\,\mathrm{d}x + \frac{1}{c}\int_{-1}^{1}\left(\frac{g(v)}{v} - \frac{g(V)}{V}\right) vV\,\mathrm{d}x = 0
$$

which is impossible, since both f and g satisfy (67).

To conclude the existence and uniqueness of solution at $\lambda = 1$ it remains to show that no solution curve of (69) can become unbounded at some $\lambda \in (0,1)$. Assume on the contrary that $|u|_{L^\infty}$ and hence also $|v|_{L^\infty}$, as can be seen from the second equation in (69), becomes unbounded as $\lambda \to \bar{\lambda}$. (If v were to remain bounded, the same would be true for $g(v)$, leading to a contradiction.) Set $U(\lambda) = u/(|u|_{L^\infty} + |v|_{L^\infty})$ and $V(\lambda) = v/(|u|_{L^\infty} + |v|_{L^\infty})$. Then using the elliptic estimates and our condition (67), we see that along some sequence of λ's converging to $\bar{\lambda}$, $(U(\lambda), V(\lambda))$ converges to some (U,V), which is a classical solution of

$$
\begin{aligned}
&U'' + \bar{\lambda} b V = 0 \quad \text{on } (-1,1),\ U(-1) = U(1) = 0 \\
&V'' + \bar{\lambda} c U = 0 \quad \text{on } (-1,1),\ V(-1) = V(1) = 0
\end{aligned}
\tag{70}
$$

Multiplying each equation in (70) by $\phi_1(x)$ and integrating, we obtain a contradiction by our condition (68).

Remark. The condition (68) cannot be removed. For example, if $b = c = \lambda_1$ the problem (64) has no positive solution, as can be seen by multiplying either of the equations by ϕ_1 and integrating.

ACKNOWLEDGEMENTS

It is a pleasure to thank the referee for careful reading of the paper, and for a long list of useful suggestions. Supported in part by the Taft Faculty Grant at the University of Cincinnati.

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