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On the oscillations of the solution curve for a class of semilinear equations

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Abstract

We consider positive solutions of the Dirichlet problem

$$\Delta u+\lambda(u+\sin u)=0, \quad x\in B, \qquad u=0 \quad \text{for } x\in\partial B,$$

where B is unit ball in R^n, λ is a positive parameter. Let λ_1 denote the principal eigenvalue of the Laplacian on B with zero boundary conditions. We show that for $1\leqslant n\leqslant 5$ the problem has infinitely many positive solutions at $\lambda=\lambda_1$, while for $n\geqslant 6$ the problem has at most finitely many solutions at any λ.

Keywords: Multiplicity of solutions; Bifurcation; Oscillatory integrals

1. Introduction

It has been observed that complexity of the solution curve for the one-dimensional boundary value problem

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$$u'' + \lambda f(u) = 0 \quad \text{for } 0 < x < L, \qquad u(0) = u(L) = 0 \tag{1}$$

seems to mirror that of the nonlinearity $f(u)$. For example, if $f(u)$ is convex, $f(u)$ can have at most one critical point, and correspondingly the solution curve of (1) admits at most one turn, see, e.g., P. Korman et al. [9]. Similarly, for some functions with one inflection points one can show that the solution curve admits either exactly one, or exactly two turns, see [9] and also P. Korman and Y. Li [7]. In the case when $f(u)$ changes concavity infinitely many times the solution curve may have infinitely many turns. This case was considered in a number of papers, see, e.g., R. Schaaf and K. Schmitt [13], D. Costa et al. [2], H. Kielhoffer and S. Maier [6], Y. Cheng [1], S.-H. Wang [15], P. Korman and Y. Li [8].

Namely, in D. Costa et al. [2] the following resonant boundary value problem was considered:

$$\begin{cases} Lu + \lambda_1 u + g(u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, L is a second order self-adjoint uniformly elliptic operator, and λ_1 is the principal eigenvalue of $-L$ on Ω with zero boundary conditions. For certain types of bounded domains Ω it was shown that the problem has infinitely many positive solutions, bifurcating from infinity. The condition imposed on the domain Ω turned out to be somewhat restrictive, since for the important special case, on a unit ball,

$$\begin{cases} \Delta u + \lambda_1 u + \sin u = 0 & \text{in } |x| < 1, \\ u = 0 & \text{on } |x| = 1, \end{cases} \tag{3}$$

the result applies only if $n = 1, 2$. In case $n > 2$ the paper [2] reports on some numerical computations, which suggest a different behavior of solutions. However, these computations were somewhat inconclusive, since their conclusions changed with the number of mesh points. It will follow from our results that the problem (3) has infinitely many positive solutions for $1 \leqslant n \leqslant 5$, but not for $n \geqslant 6$.

We consider first a little different problem

$$\begin{cases} \Delta u + \lambda(u + a\sin u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{4}$$

where B is unit ball in R^n, $n \geqslant 1$, and a is a positive constant. The advantage of working with the problem (4) is that we get more detailed results, by using the bifurcation theory, as explained in, e.g., [10]. In particular, all positive solutions lie on a unique smooth curve of solutions, whose one end bifurcates from zero, and the other one from infinity. (For the problem (3) a continuum of solutions bifurcating from infinity was considered in [2].) The question is: how many times does this curve cross the line $\lambda = \lambda_1$? We will show that for $1 \leqslant n \leqslant 5$ the problem (4) has infinitely many positive solutions at $\lambda = \lambda_1$, while at any other value of λ the number of positive solutions is at most finite. If $n \geqslant 6$ the problem (4) has at most finitely many positive solutions for all λ. In case $a = \frac{1}{\lambda_1}$, we see that the problem (3) has infinitely many positive solutions, when $1 \leqslant n \leqslant 5$, and finitely many when $n \geqslant 6$.

It is clear that our approach can handle more general oscillating nonlinearities, but for simplicity we restricted to the model problem (4). Notice that by elliptic regularity any solution of the problem (4) $u \in C^\infty(B)$, since $\sin x$ is a C^∞ function, i.e., throughout the paper we are dealing with the classical solutions. We state our main result next.

Theorem 1.

(i) *If* $1 \leqslant n \leqslant 5$ *then the problem* (4) *has infinitely many positive solutions at* $\lambda = \lambda_1$*, while at any other value* λ *the number of positive solutions is at most finite. Moreover, all solutions lie on a unique smooth curve, whose one end bifurcates from zero at* $\lambda = \lambda_1/2$*, and the other one from infinity at* $\lambda = \lambda_1$.

(ii) *If* $n \geqslant 6$ *then the problem* (4) *has only finitely many positive solutions for all* λ*. All solutions lie on a unique smooth curve, whose one end bifurcates from zero at* $\lambda = \lambda_1/2$*, and the other one from infinity at* $\lambda = \lambda_1$.

For the proof we extend the approach of [8], where the one-dimensional version of this problem was considered. By the classical theorem of B. Gidas et al. [5] any solution of the problem (4) is radially symmetric. By the theory of bifurcation from infinity, see, e.g., [16], large solutions approach a constant multiple of $\varphi_1(r)$, the principal eigenfunction of Δ on B, with zero boundary condition. This allows us to express $\lambda - \lambda_1$ as a certain integral, see formula (14) below. We study the oscillations of this integral about zero, by using some tools from complex analysis, see [14], which we review in the next section.

Finally, we mention that the study of resonant problems has been initiated by the classical paper of E.M. Landesman and A.C. Lazer [11].

2. Asymptotics of the oscillatory integrals

The following two lemmas are taken from [14]. We present their short proofs for completeness.

Lemma 1. *Let* $f(x) \in C^2[0,a]$ *and* $\alpha \neq 0$*. Then if* $\mu \to \infty$,

$$\Phi(\mu) \equiv \int_0^a f(x) e^{(i/2)\alpha\mu x^2}\,dx = \frac{1}{2}\sqrt{\frac{2\pi}{|\alpha|\mu}}\, e^{i(\pi/4)\delta(\alpha)} f(0) + O\left(\frac{1}{\mu}\right), \tag{5}$$

where $\delta(\alpha) = \operatorname{sgn}\alpha$.

Proof. We take $\alpha > 0$ for definiteness, and we assume first that $f(x) \equiv 1$. After a change of variable $\sqrt{\alpha\mu}\,x = t$, we obtain

$$\int_0^a e^{(i/2)\alpha\mu x^2}\,dx = \frac{1}{\sqrt{\alpha\mu}}\left[\int_0^\infty e^{it^2/2}\,dt - \int_{a\sqrt{\alpha\mu}}^\infty e^{it^2/2}\,dt\right].$$

The first integral is a Fresnel's integral and it is equal to $\frac{1}{2}e^{i\pi/4}\sqrt{2\pi}$.

In the second integral, we denote $y = a\sqrt{\alpha\mu}$ and integrate by parts

$$\int_y^\infty e^{it^2/2}\,dt = \int_y^\infty \frac{1}{2it}\,d\big(e^{it^2}\big) = -\frac{e^{iy^2}}{2yi} + \frac{1}{2i}\int_y^\infty \frac{e^{it^2/2}}{t^2}\,dt.$$

Both terms on the right are $O(1/y) = O(1/\sqrt{\mu})$, and so we obtain

$$\int_0^a e^{(i/2)\alpha\mu x^2}\,dx = \frac{1}{2}e^{i\pi/4}\sqrt{\frac{2\pi}{\alpha\mu}} + O\left(\frac{1}{\mu}\right). \tag{6}$$

In case $\alpha < 0$, we make a change of variables $-\sqrt{|\alpha|\mu}\,x = t$, use the Fresnel's integral $\int_0^\infty e^{-it^2/2}\,dt = (1/2)e^{-i\pi/4}\sqrt{2\pi}$, and arrive at a formula similar to (6), with a minus in the exponent. Combining, we have for any α

$$\int_0^a e^{(i/2)\alpha\mu x^2}\,dx = \frac{1}{2}e^{i\frac{\pi}{4}\delta(\alpha)}\sqrt{\frac{2\pi}{\alpha\mu}} + O\left(\frac{1}{\mu}\right).$$

For an arbitrary function $f(x)$, we write

$$f(x) = f(0) + \big[f(x) - f(0)\big] = f(0) + xh(x),$$

where $h(x) = \frac{f(x)-f(0)}{x} \in C^1[0,a]$. Then

$$\Phi(\mu) = \frac{1}{2}f(0)\sqrt{\frac{2\pi}{\alpha\mu}}e^{i\frac{\pi}{4}\delta(\alpha)} + \int_0^a e^{(i/2)\alpha\mu x^2}xh(x)\,dx + O\left(\frac{1}{\mu}\right). \tag{7}$$

Integrating by parts, similarly to the above, it is easy to see that

$$\int_0^a e^{(i/2)\alpha\mu x^2}xh(x)\,dx = O\left(\frac{1}{\mu}\right) \quad (\mu \to +\infty). \tag{8}$$

(Here is an intuitive derivation of (8). We have fast oscillations in the integral, except for a small interval near $x = 0$. The length of this interval is $O(1/\sqrt{\mu})$, and the same asymptotic formula holds for $xh(x)$.) Combining (8) with (7), we obtain (5). $\square$

Lemma 2. *Assume that the $C^2[0,1]$ functions $f(x)$ and $g(x) > 0$ satisfy*

$$g'(x) < 0 \quad \textit{for all } x \in (0,1] \quad \textit{and} \quad g'(0) = 0, \quad g''(0) < 0.$$

Then as $\mu \to \infty$,

$$\int_0^1 f(x)e^{i\mu g(x)}\,dx = e^{i(\mu g(0)-\pi/4)}\sqrt{\frac{\pi}{\mu|g''(0)|}}f(0) + O\left(\frac{1}{\mu}\right).$$

Proof. On the $[0,1]$ we change the variables $x \to t$, $x = \psi(t)$, so that

$$g(x) - g(0) = -t^2. \tag{9}$$

Since $g(x)$ is a decreasing function, g^{-1} exists. Thus

$$\psi(t) = g^{-1}\big(g(0) - t^2\big).$$

Note, that

$$\text{if } x = 0, \quad \text{then } t = 0; \qquad \text{if } x = 1, \quad \text{then } t = \sqrt{g(0) - g(1)} = b > 0.$$

Therefore, after the change of variables we obtain

$$I=\int_0^1 f(x)e^{i\mu g(x)}\,dx=e^{i\mu g(0)}\int_0^b f\bigl(\psi(t)\bigr)\psi'(t)e^{-i\mu t^2}\,dt.$$

To apply Lemma 1, we need to calculate $\psi'(0)$. Differentiating (9), we have

$$g'\bigl(\psi(t)\bigr)\psi'(t)=-2t.$$

We cannot calculate $\psi'(0)$ from here, since $g'(0)=0$. So we differentiate this formula again and set $t=0$, obtaining

$$\psi'(0)=\sqrt{-\frac{2}{g''(0)}}.$$

Applying Lemma 1 to the integral I, we conclude the proof. □

Corollary 1. *Under the conditions of the Lemma* 2*, as $\mu\to\infty$ we have*

$$\int_0^1 f(x)e^{i\mu g(x)}\,dx=O\left(\frac{1}{\sqrt{\mu}}\right).$$

3. Reduction to the oscillatory integrals

Without restriction of generality we assume that $a=1$ in (4), for the rest of the paper. Our first lemma shows that positive solutions of (4) lie in a bounded in λ strip. A similar result for $n=1$ case was proved in [8]. Recall, that we denote by λ_1 the principal eigenfunction of $-\Delta$ on the unit ball B, with zero boundary conditions, and by $\varphi_1=\varphi_1(r)$ the corresponding eigenfunction, normalized so that $\varphi_1(0)=1$.

Lemma 3. *If the problem* (4) *has a positive solution, then*

$$\frac{\lambda_1}{2}<\lambda<\frac{\pi}{\pi-1}\lambda_1. \tag{10}$$

Proof. Multiplying both sides of the equation in (4) by u and integrating, we obtain

$$\int_B u\Delta u\,dx+\lambda\int_B\bigl(u^2+u\sin u\bigr)\,dx=0.$$

Since $u^2+u\sin u\leqslant 2u^2$, then after integration by parts and using the Poincaré inequality, we have

$$2\lambda\int_B u^2\,dx>\lambda\int_B\bigl(u^2+u\sin u\bigr)\,dx=\int_B|\nabla u|^2\,dx\geqslant\lambda_1\int_B u^2\,dx,$$

from which the left-hand side of the inequality (10) follows.

Multiplying the equation in (4) by φ_1 and integrating, we obtain

$$\int_B\bigl[\lambda(u+\sin u)-\lambda_1 u\bigr]\varphi_1\,dx=0. \tag{11}$$

Denote $g(u) = \frac{\lambda-\lambda_1}{\lambda}u + \sin u$. The second part of the inequality (10) we will prove by contradiction. Assume on the contrary that at some solution

$$\lambda \geqslant \frac{\pi}{\pi - 1}\lambda_1 > \lambda_1. \tag{12}$$

From (12) we obtain immediately that $g(u) > 0$ on $(0, \pi)$. Note that $g(\pi) = \frac{\lambda-\lambda_1}{\lambda}\pi$. It is clear, that if $g(\pi) \geqslant 1$, then $g(u)$ is positive for all $u > 0$. But from (12) we have

$$\lambda(\pi - 1) \geqslant \pi\lambda_1 \quad \Longrightarrow \quad \pi(\lambda - \lambda_1) \geqslant \lambda \quad \Longrightarrow \quad \frac{\pi(\lambda - \lambda_1)}{\lambda} \geqslant 1.$$

Thus $g(u)$ is positive. Since $\varphi_1 > 0$, we obtain contradiction in (11). Thus $\lambda < \frac{\pi}{\pi-1}\lambda_1$. □

It is well known that there is a curve of positive solutions of (4) bifurcating off the trivial one (to the right) at $\lambda_1/2$, see [3]. Similarly, there is a curve of positive solutions bifurcating from infinity at $\lambda = \lambda_1$, see, e.g., in E. Zeidler [16, p. 673]. Let us consider the branch bifurcating from infinity. This branch extends globally, since at each point the implicit function theorem or a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [4] applies, see [9,12] for details. By Lemma 3 this branch is constrained to a strip $\frac{\lambda_1}{2} < \lambda < \frac{\pi}{\pi-1}\lambda_1$, so it has to go to zero at $\lambda_1/2$. By the uniqueness of the solution, bifurcating from zero, this branch has to link up with the lower one. We thus obtain a solution curve connecting $(\lambda_1/2, 0)$ to (λ_1, ∞). It is known that this curve exhausts the set of all possible solutions of (4), and it may have only finitely many turns, while $u(0)$ belongs to a bounded set, see [9,12] for details. It follows that the number of solutions at any $\lambda \neq \lambda_1$ is at most finite.

Since we are interested in the behavior of solution $u(x)$ at infinity in L^∞, it is appropriate to use the scaling

$$u = \mu v, \quad \text{so that} \quad |v|_{L^\infty} \equiv 1, \quad \text{while } \mu \to \infty.$$

Then (4) becomes

$$\Delta v + \lambda v + \frac{\lambda}{\mu}\sin(\mu v) = 0 \quad \text{in } B, \qquad v = 0 \quad \text{on } \partial B. \tag{13}$$

Multiplying (13) by φ_1 and integrating by parts, we obtain

$$\lambda - \lambda_1 = -\frac{\lambda}{\mu}\frac{\int_B \sin(\mu v)\varphi_1\,dx}{\int_B v\varphi_1\,dx}. \tag{14}$$

By the classical theorem of B. Gidas et al. [5] any solution of the problem (4) is radially symmetric. Therefore, the problem (13) becomes:

$$v'' + \frac{n-1}{r}v' + \lambda v + \frac{\lambda}{\mu}\sin(\mu v) = 0, \quad 0 < r < 1, \qquad v'(0) = v(1) = 0. \tag{15}$$

By the theory of bifurcation from infinity, $v \to \varphi_1$ in the norm C^2, as $\mu \to \infty$, see, e.g., [16], or Lemma 4 below. Hence, the integral $\int_B v\varphi_1\,dx$ is positive for large μ. So, the issue is whether the integral $\int_B \sin(\mu v)\varphi_1\,dx$ changes sign infinitely many or finitely many times, as $\mu \to \infty$. Correspondingly, the solution curve will cross the line $\lambda = \lambda_1$ either infinitely many or finitely many times, as follows from (14).

We have

$$\int\limits_B \sin(\mu v)\varphi_1\,dx = \omega_n \int\limits_0^1 \varphi_1(r)\sin\bigl(\mu v(r)\bigr)r^{n-1}\,dr = \omega_n \operatorname{Im}\int\limits_0^1 \varphi_1(r)r^{n-1}e^{i\mu v(r)}\,dr,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$. We shall study the integral

$$I \equiv \operatorname{Im} \int_0^1 \varphi_1(r) r^{n-1} e^{i\mu v(r)}\, dr. \tag{16}$$

Depending on the dimension, we will need up to three integrations by parts in (16). With $f_1(r) \equiv \frac{\varphi_1(r) r^{n-1}}{v'(r)}$, we have

$$\begin{aligned}
\int_0^1 \varphi_1(r) r^{n-1} e^{i\mu v(r)}\, dr &= \frac{1}{i\mu} \int_0^1 \frac{\varphi_1(r) r^{n-1}}{v'}\, d\big(e^{i\mu v(r)}\big) \\
&= \frac{1}{i\mu} \int_0^1 f_1(r)\, d\big(e^{i\mu v(r)}\big) \\
&= \frac{1}{i\mu} f_1(r) e^{i\mu v(r)} \Big|_0^1 - \frac{1}{i\mu} \int_0^1 f_1'(r) e^{i\mu v(r)}\, dr.
\end{aligned} \tag{17}$$

Writing $e^{i\mu v(r)}\, dr = \frac{1}{i\mu v'(r)}\, d(e^{i\mu v(r)})$ again and denoting $f_2 = f_1'/v'$, we have

$$\begin{aligned}
\int_0^1 \varphi_1(r) r^{n-1} e^{i\mu v(r)}\, dr = {} & \frac{1}{i\mu} f_1(r) e^{i\mu v(r)} \Big|_0^1 + \frac{1}{\mu^2} f_2(r) e^{i\mu v(r)} \Big|_0^1 \\
& - \frac{1}{\mu^2} \int_0^1 f_2'(r) e^{i\mu v(r)}\, dr.
\end{aligned} \tag{18}$$

With $f_3 = f_2'/v'$ we integrate by parts again, obtaining

$$\begin{aligned}
\int_0^1 \varphi_1(r) r^{n-1} e^{i\mu v(r)}\, dr = {} & \frac{1}{i\mu} f_1(r) e^{i\mu v(r)} \Big|_0^1 + \frac{1}{\mu^2} f_2(r) e^{i\mu v(r)} \Big|_0^1 \\
& - \frac{1}{i\mu^3} f_3(r) e^{i\mu v(r)} \Big|_0^1 + \frac{1}{i\mu^3} \int_0^1 f_3'(r) e^{i\mu v(r)}\, dr.
\end{aligned} \tag{19}$$

Lemma 4. *Let $v(r)$ be a positive solution of the problem* (13). *Then as $\mu \to \infty$, $v(r) \to \varphi_1(r)$ in $C^2[0,1]$.*

Proof. If we write $v(r) = \alpha\varphi_1(r) + w(r)$, then as $\mu \to \infty$ we have $\alpha \to 1$ and $w(r) \to 0$, uniformly in B. We have

$$-\Delta w = \lambda w + (\lambda - \lambda_1)\alpha\varphi_1 + \frac{\lambda}{\mu} \sin(\mu\alpha\varphi_1 + \mu w) \quad \text{in } B, \qquad w = 0 \quad \text{on } \partial B.$$

Since the right-hand side tends to zero in L^p for any $p > 1$, as $\mu \to \infty$, we conclude that $w \to 0$ in $W^{2,p}(B)$, and hence in $C^{1,\beta}(\bar{B})$ for any $\beta \in (0,1)$. Hence $v(r) \to \varphi_1(r)$ in $C^1[0,1]$. We then see from the corresponding equations that $v''(r) \to \varphi_1''(r)$. This is straightforward for $r \in (0, 1]$, while at $r = 0$ the corresponding equations become

$$nv''(0) + \lambda v(0) + \frac{\lambda}{\mu}\sin\bigl(\mu v(0)\bigr) = 0,$$

and

$$n\varphi_1''(0) + \lambda_1\varphi_1(0) = 0.$$

Since $\lambda \to \lambda_1$ and $v(0) = \varphi_1(0) = 1$, we conclude that $v''(0) \to \varphi_1''(0)$, as $\mu \to \infty$. □

Lemma 5. *Assume μ is large.*

(i) *For $n \geqslant 2$ the function $f_1(r)$ is a C^∞ function.*
(ii) *For $n \geqslant 4$ the function $f_2(r)$ is a C^∞ function.*
(iii) *For $n \geqslant 6$ the function $f_3(r)$ is a C^∞ function.*

Proof. We express from (15)

$$(n-1)\frac{v'}{r} = -v'' - \lambda v - \frac{\lambda}{\mu}\sin(\mu v).$$

Since $v \in C^\infty$, then v'/r is a C^∞ function. This function is negative for $r \neq 0$, while at $r = 0$ we have

$$\lim_{r\to 0}\frac{v'}{r} = v''(0) \simeq \varphi_1''(0) < 0.$$

Hence, v'/r is in C^∞ and it never vanishes. Now it is easy to prove the lemma. We have

$$f_1(r) = \frac{r^{n-1}\varphi_1(r)}{v'(r)} = \frac{r^{n-2}\varphi_1(r)}{v'(r)/r}. \tag{20}$$

For $n \geqslant 2$ the function $f_1(r)$ is a quotient of two C^∞ functions, and the function in the denominator is never equal to zero. Thus $f_1(r)$ is a C^∞ function.

Observe that from (20), we can write $f_1(r) = r^{n-2}\tilde{f}_1(r)$, where $\tilde{f}_1(r) \in C^\infty$. Therefore,

$$f_2(r) = \frac{f_1'(r)}{v'(r)} = \frac{(r^{n-2}\tilde{f}_1(r))'}{v'(r)} = \frac{(n-2)r^{n-4}\tilde{f}_1(r) + r^{n-3}\tilde{f}_1'(r)}{v'(r)/r}. \tag{21}$$

Thus for $n \geqslant 4$, $f_2(r)$ is a C^∞ function.

From (21), $f_2(r) = r^{n-4}\tilde{f}_2(r)$, where $\tilde{f}_2(r) \in C^\infty$. We then have

$$f_3(r) = \frac{f_2'(r)}{v'(r)} = \frac{(r^{n-4}\tilde{f}_2(r))'}{v'(r)} = \frac{(n-4)r^{n-6}\tilde{f}_2(r) + r^{n-5}\tilde{f}_2'(r)}{v'(r)/r}.$$

Thus for $n \geqslant 6$, $f_3(r)$ is a C^∞ function. □

4. Proof of Theorem 1

Using Eq. (15) to express $v''(r)$ through the lower order terms, we express

$$f_2(r) = \frac{r^{n-1}f(v(r))\varphi_1(r) + 2(n-1)r^{n-2}\varphi_1(r)v'(r) + r^{n-1}v'(r)\varphi_1'(r)}{\varphi_1'^3(r)}, \tag{22}$$

where we denote $f(v(r)) = \lambda v(r) + \frac{\lambda}{\mu}\sin(\mu v(r))$. Similarly, expressing $v''(r)$ from (15), and $\varphi''(r)$ from the corresponding equation

$$f_2'(r) = \frac{3r^{n-1} f^2(v(r))\varphi_1(r) + 8(n-1)r^{n-2} f(v(r))v'(r)\varphi_1(r)}{v'^4(r)} + \frac{r^{n-3}(8 - 14n + 6n^2 - r^2\lambda_1 + r^2 f'(v(r)))\varphi_1(r)v'^2(r) + 4(n-1)r^{n-2}\varphi_1'(r)v'^2(r)}{v'^4(r)}. \tag{23}$$

The case of $n = 1$ has already been covered in [8], so we start with $n = 2$.

(i) $n = 2$. We use (17). Here $f_1(r) = \frac{r\varphi_1(r)}{v'(r)}$. Observe that $f_1(1) = 0$, $f_1(0) = \frac{\varphi_1(0)}{v''(0)} \simeq \frac{\varphi_1(0)}{\varphi_1''(0)} < 0$ for large μ. By Lemma 5, $f_1 \in C^\infty$, and then using Corollary 1 of Lemma 2 we obtain

$$\begin{aligned} I &= \operatorname{Im} \frac{1}{i\mu}\left[-f_1(0)e^{i\mu v(0)} - \int_0^1 f_1'(r)e^{i\mu v(r)}\,dr\right] \\ &= \frac{f_1(0)}{\mu}\cos\mu v(0) + O\left(\frac{1}{\mu^{3/2}}\right) \quad \text{as } \mu \to \infty. \end{aligned}$$

It follows that I changes sign infinitely many times as $\mu \to \infty$. (Here, of course, $v(0) = 1$.)

(ii) $n = 3$. We use (17) again. Here $f_1(r) = \frac{r^2\varphi_1(r)}{v'(r)}$. This time we have $f_1(0) = f_1(1) = 0$, while by the L'Hopital rule $f_1'(0) \neq 0$. Hence by Lemma 2

$$I = \operatorname{Im}\left[\frac{i}{\mu}\int_0^1 f_1'(r)e^{i\mu v(r)}\,dr\right] = \operatorname{Re}\left[\frac{c_1}{\mu^{3/2}} f_1'(0)e^{i(\mu v(0) - \pi/4)}\right] + O\left(\frac{1}{\mu^2}\right), \tag{24}$$

where $c_1 = \sqrt{\frac{\pi}{|v''(0)|}} \simeq \sqrt{\frac{\pi}{|\varphi_1''(0)|}}$. As above, we see that I changes sign infinitely many times as $\mu \to \infty$.

(iii) $n = 4$. Here $f_1(r) = \frac{r^3\varphi_1(r)}{v'(r)}$, and By Lemma 5, $f_2 \in C^\infty$. We have again $f_1(0) = f_1(1) = 0$, but now $f_1'(0) = 0$, and hence the principal term in (24) is zero. We have to integrate by parts twice, i.e., we use (18)

$$I = \operatorname{Im}\left[\frac{1}{\mu^2} f_2(r)e^{i\mu v(r)}\Big|_0^1 - \frac{1}{\mu^2}\int_0^1 f_2'(r)e^{i\mu v(r)}\,dr\right].$$

We have $v'(r) \sim c_2 r$ near $r = 0$ for some constant c_2. ($c_2 = v''(0) \simeq \varphi_1''(0) = -\lambda_1/4$; here and later on we denote by c_i various non-zero constants.) Similarly, $f_1 \sim c_3 r^2$ and $f_1'(r) \sim c_4 r$ in the neighborhood of zero, therefore $f_2(0) = \lim_{r\to 0} \frac{f_1'(r)}{v'(r)} \neq 0$. (Here, and later on, we use asymptotic formulas only as a heuristic tool, while we use (22) and (23) for rigorous justifications. Indeed, using the L'Hopital's rule, we see from (22) that $f_2(0) \neq 0$.) Applying Corollary 1 of Lemma 2, we obtain (since $\operatorname{Im}[(1/\mu^2) f_2(1)] = 0$)

$$I = \operatorname{Im} \frac{1}{\mu^2}\left[f_2(1) - f_2(0)e^{i\mu v(0)} - \int_0^1 f_2'(r)e^{i\mu v(r)}\,dr \right]$$
$$= -\frac{1}{\mu^2} f_2(0)\sin\bigl(\mu v(0)\bigr) + O\left(\frac{1}{\mu^{5/2}}\right) \quad \text{as } \mu \to \infty.$$

Again, I changes sign infinitely many times as $\mu \to \infty$.

(iv) $n = 5$. Again we use (18). Here $f_1(r) = \frac{r^4\varphi_1(r)}{v'(r)}$, and $f_2(r) = \frac{f_1'(r)}{v'(r)}$. Then $f_1 \sim c_5 r^3$ near zero, therefore $f_2(0) = 0$, but $f_2'(0) \neq 0$. (We have used the L'Hopital's rule in (23) together with the observation that $v''(0) = -\frac{1}{5} f(v(0))$.) Thus, using Lemma 2, we obtain

$$I = -\frac{1}{\mu^2}\operatorname{Im}\int_0^1 f_2'(r)e^{i\mu v(r)}\,dr = -\frac{c_6}{\mu^{5/2}} f_2'(0)\sin\left(\mu v(0) - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu^3}\right)$$
as $\mu \to \infty$.

Thus, for $2 \leqslant n \leqslant 5$ the integral

$$I = \int_B \sin(\mu v)\varphi_1\,dx$$

changes sign infinitely many times as $\mu \to \infty$, so does $\lambda - \lambda_1$ as well. The first part of the theorem is proved.

(v) $n = 6$. Here $f_1(r) = \frac{r^5\varphi_1(r)}{v'(r)}$. We have $f_1(0) = f_1(1) = 0$, as before, and by (22) and (23), $f_2(0) = f_2'(0) = 0$. We need to integrate by parts three times, i.e., we use (19). Since

$$\operatorname{Im}\left[\frac{1}{\mu^2} f_2(1)\right] = 0,$$

then for sufficiently large μ we have

$$I = \operatorname{Im}\left[-\frac{1}{i\mu^3} f_3(r)e^{i\mu v(r)}\Big|_0^1 + \frac{1}{i\mu^3}\int_0^1 f_3'(r)e^{i\mu v(r)}\,dr \right]$$
$$= \frac{1}{\mu^3}\bigl[f_3(1) - \operatorname{Im}\bigl(i f_3(0)e^{i\mu v(0)}\bigr)\bigr] + O\left(\frac{1}{\mu^{7/2}}\right)$$
$$= \frac{1}{\mu^3}\bigl[f_3(1) - f_3(0)\cos\bigl(\mu v(0)\bigr)\bigr] + O\left(\frac{1}{\mu^{7/2}}\right).$$

Whether this quantity has infinitely many zeros depends on the relative sizes of $|f_3(1)|$ and $|f_3(0)|$. By Lemma 5, $f_3 \in C^\infty$, and since $f_3 = \frac{f_2'}{v'}$, we have in our case $n = 6$ (in view of (23))

$$f_3(r) = \frac{1}{\varphi_1'^5}\bigl[3r^5 f^2\bigl(v(r)\bigr)\varphi_1(r) + 40r^4 f\bigl(v(r)\bigr)\varphi_1(r)\varphi_1'(r) + 140r^3\varphi_1(r)v'^2(r)$$
$$- r^5\lambda_1\varphi_1(r)v'^2(r) + r^5\varphi_1(r)f'\bigl(v(r)\bigr)v'^2(r) + 3r^5 f\bigl(v(r)\bigr)\varphi_1'(r)v'(r)$$
$$+ 20r^4\varphi_1'(r)v'(r)^2\bigr].$$

At $r = 1$ we have

$$f_3(1) = \frac{20\varphi_1'(1)}{v'^3(1)} \sim \frac{20}{(\varphi_1'(1))^2}, \quad \text{as } \mu \to \infty.$$

Using the L'Hopital's rule and the relation $f(v(0)) = -6v''(0)$, we obtain

$$f_3(0) = \frac{8\varphi_1(0)}{(v''(0))^3} \sim \frac{8\varphi_1(0)}{(\varphi_1''(0))^3}, \quad \text{as } \mu \to \infty.$$

Since $\varphi_1''(0) = -\frac{\lambda_1}{6}\varphi_1(0)$, we finally express for large μ

$$f_3(0) \sim -\frac{8 \cdot 6^3}{\lambda_1^3 \varphi_1^2(0)} = -\frac{1728}{\lambda_1^3}.$$

We use the representation of the principal eigenfunction of the Laplace operator by means of the Bessel functions, which is easily derived,

$$\varphi_1(r) = c_n r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(b_1 r), \tag{25}$$

where $J_{\frac{n-2}{2}}(x)$ is the Bessel function of order $\frac{n-2}{2}$, b_1 is the first zero of $J_{\frac{n-2}{2}}(x)$, and c_n is a suitable constant, chosen so that $\varphi_1(0) = 1$. One evaluates (using *Mathematica*)

$$f_3(1) \approx 71.4, \quad \text{while } f_3(0) \approx -0.09.$$

Hence $I > 0$ for large μ. The solution curve crosses the line $\lambda = \lambda_1$ only finitely many times.

(vi) $n \geqslant 7$. Here $f_1(r) = \frac{r^{n-1}\varphi_1(r)}{v'(r)}$. As before, $f_1(0) = f_1(1) = f_2(0) = f_2'(0) = 0$. So again the first two terms in (19) are zero. (Observe that $\mathrm{Im}[\frac{1}{\mu^2} f_2(r) e^{i\mu v}|_0^1] = 0$.)

In view of (23), $f_3(0) = 0$, while for large μ

$$f_3(1) = \frac{4(n-1)\varphi_1'(1)}{(v_1'(1))^3} \sim \frac{4(n-1)}{(\varphi_1'(1))^2} > 0.$$

Hence from (19), we obtain

$$I \sim \frac{1}{\mu^3} \frac{4(n-1)}{(\varphi_1'(1))^2} > 0 \quad \text{for large } \mu.$$

Thus $\lambda - \lambda_1$ does not change sign for large μ, for all $n \geqslant 7$, i.e., the solution curve crosses the line $\lambda = \lambda_1$ only finitely many times (since $I > 0$, the solution curve stays to the left of $\lambda = \lambda_1$ as $\mu \to \infty$, in view of (14)). $\square$

Remark. We have evaluated the integral $I = I(\mu)$,

$$I(\mu) = \int_0^1 \sin\bigl(\mu\varphi_1(r)\bigr)\varphi_1(r) r^{n-1}\, dr,$$

by using the formula (25) and *Mathematica* software. The results for $n = 2$ and $n = 3$ clearly showed oscillations of $I(\mu)$, and these oscillations are evident even for relatively small μ, see Fig. 1. Here we scaled φ_1, so that $\varphi_1(0) = 1$.

For $n = 4$ we also see oscillations around zero, but one would have to go considerably further in μ to get a symmetric picture, as in Fig. 1. For $n = 5$ one has to go still further in μ for

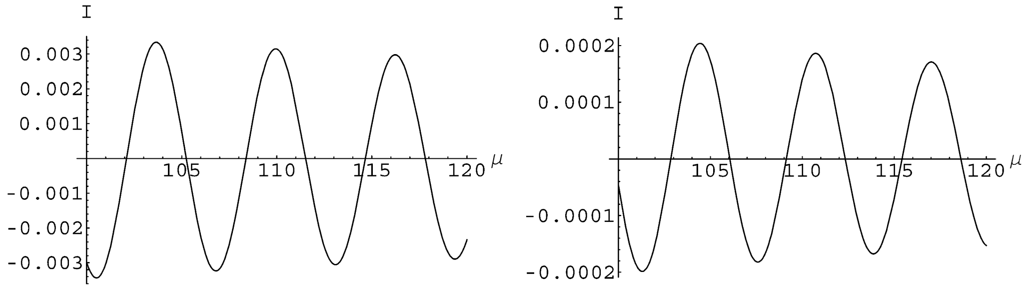

Fig. 1. $n = 2$, $n = 3$.

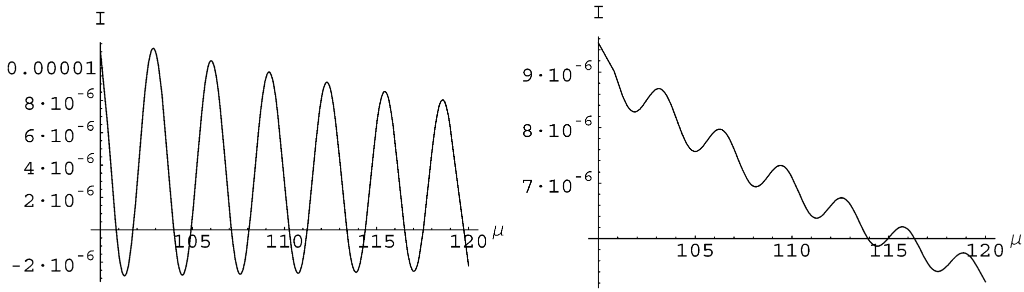

Fig. 2. $n = 4$, $n = 5$.

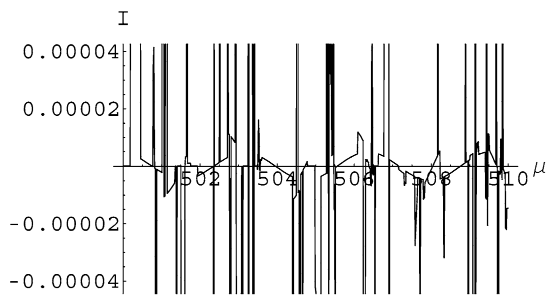

Fig. 3. $n = 5$, in case of larger μ.

oscillations to cross zero. This did not yet happen in Fig. 2. In Fig. 2 we scaled φ_1, so that $\varphi_1(0) = 2$. Doubling φ_1 is of course almost the same as multiplying μ by 2.

The reason we did not increase μ in Fig. 2 is that then we ran into highly oscillatory integrals, which *Mathematica* cannot handle well. In Fig. 3 we show a typical picture for larger μ (here again $\varphi_1(0) = 2$).

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