JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **137**, 477–484 (1989)

On Existence of Solutions for a Class of Fully Nonlinear Noncoercive Problems

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Submitted by E. Stanley Lee

Received August 7, 1987

1. INTRODUCTION

We study existence of solutions for a fully nonlinear noncoercive elliptic problem

$$\begin{aligned} &u_y - Fu_{xx} = \rho(x, z, u, u_x), && y = 1, \\ &f(x, y, z, u, Du, D^2u) = g(x, y, z), && 0 < y < 1, \\ &u = 0, && y = 0. \end{aligned} \tag{1.1}$$

Here the functions f, g, and ρ are assumed to be 2π periodic in x and z, F is a positive constant, and we are looking for a solution $u(x, y\, z)$, which is also 2π periodic in x and z.

The boundary condition at $y = 1$ (with $F < 0$) arises in the three-dimensional water wave theory (see M. Shinbrot [10]), and also in the engineering problem of "hydraulic fracturing" of oil wells, see J. R. Canon and G. H. Meyer [2, 7]. Mathematically, the model problem (1) is of interest, since it represents one of the simplest noncoercive elliptic problems (i.e., the Lopatinski–Shapiro condition fails, see [4]) for which there is no general theory.

The field of fully nonlinear elliptic equations has recently attracted a great deal of attention (see, e.g., a recent paper of G. M. Lieberman and N. S. Trudinger [6] and the references therein to other work of the same authors as well as that of L. Evans, N. Krylov, P. L. Lions, and others). It appears that the case of second order boundary operators is not well studied, particularly for noncoercive problems.

We present an existence result for (1), which extends the one in [3], where we had considered the equation $\Delta u = f(x, y, z, u, Du)$ with the same boundary conditions. The main difficulty for the fully nonlinear case lies in

0022-247X/89 \$3.00

the derivation of suitable a priori bounds, which had to be performed in a way completely different from [3].

2. Notation and the Preliminary Results

We consider functions of three variables x, y, z which are 2π periodic in x, z and $0 \leqslant y \leqslant 1$. By V we denote the domain $0 \leqslant x, z \leqslant 2\pi$, $0 < y < 1$; its boundary we denote by ∂V and the top $(y = 1)$ part of the boundary by V_t.

We shall also denote

$$\int f = \int_0^{2\pi} \int_0^1 \int_0^{2\pi} f(x, y, z)\, dx\, dy\, dz,$$

$$\int_t f = \int_0^{2\pi} \int_0^{2\pi} f(x, 1, z)\, dx\, dz.$$

It is often convenient to denote $x_1 = x$, $x_2 = y$, $x_3 = z$, $u_i = \partial u / \partial x_i$, $u_{ij} = \partial^2 u / \partial x_i\, \partial x_j$.

By $\|\cdot\|_m$ we denote the norm in the Sobolev space $H^m(V)$, and by $\overline{\|\cdot\|}_m$ the one in $H^m(V_t)$. We shall also need the norms

$$|f|_N = \sum_{|\alpha| \leqslant N} |D^\alpha f|_{L^\infty(V)}, \qquad N = \text{integer} \geqslant 0.$$

All irrelevant positive constants independent of unknown functions we denote by c; $Du \equiv \nabla u$, $D^2 u$ the Hessian of u.

We shall need the following relations between our norms (see [4]).

Lemma 2.1. *For any integer $m \geqslant 0$ and any $\varepsilon > 0$ one has*

(i) $\overline{\|v\|}_m \leqslant \|v\|_{m+1}$,

(ii) $\|v\|_m \leqslant \varepsilon\, \|v\|_{m+1} + c(\varepsilon)\, \|v\|_0$,

(iii) $\overline{\|v\|}_m \leqslant \varepsilon\, \|v\|_{m+1} + c(\varepsilon)\, \|v\|_0$,

(iv) $\overline{\|v\|}_m \leqslant \varepsilon\, \|v_y\|_m + c(\varepsilon)\, \|v\|_0$.

The following lemma is taken from [8].

Lemma 2.2. *Let $\phi(x, y, z, v)$ be defined in $V \times \{v = (v_1, ..., v_m)$ in $|v|_0^2 = \sum_{i=1}^m v_i^2 < c^2\}$. Assume that ϕ possesses continuous derivatives up to order m which are bounded by c. Then we have*

$$\|\phi(x, y, z, v)\|_m \leqslant c(\|v\|_m + 1),$$

provided that $v \in H^m$ and $|v|_0 \leqslant c$. (We denote $\|v\|_m^2 = \sum_{i=1}^m \|v_i\|_m^2$.)

We define a linear normed space G^m as a subset of functions in $H^m(V)$ for which $u(x, 0, z) = 0$ and

$$|||u||| \equiv \|u\|_m + \overline{\|u_x\|}_{m-1} < \infty.$$

LEMMA 2.3. *The space G^m with the norm $|||\cdot|||_m$ is a Banach space, provided $m \geqslant 3$.*

Proof. To prove completeness, let $\{u^r\}$ be a Cauchy sequence in G^m, i.e., $\|u^r - u^p\|_m + \overline{\|u_x^r - u_x^p\|}_{m-1} \to 0$ as $r, p \to \infty$. It follows that $u^r \to u$ in $H^m(V)$ and $u_x^r \to v$ in $H^{m-1}(V_t)$ for some $u \in H^m(V)$ and $v \in H^{m-1}(V_t)$. It remains to show that $v = u_x$. Indeed, both functions are continuous and

$$\overline{\|v - u_x\|}_0 \leqslant \overline{\|v - u_x^r\|}_0 + \|u_x^r - u_x\|_1 \to 0 \qquad \text{as} \quad r \to \infty.$$

3. A PRIORI ESTIMATES AND EXISTENCE RESULTS

We now present the existence result for fully nonlinear equations. In the following lemmas we first derive a priori estimates and discuss the solvability for the linearized problem. Then we use the contractive mapping principle to prove existence for the nonlinear problem. Throughout this section summation in $i, j = 1, 2, 3$ is implied.

LEMMA 3.1. *Consider the problem*

$$\begin{aligned} &u_y - F u_{xx} - r_1 u_x - r_2 u = g, && y = 1, \\ &Lu \equiv a_{ij} u_{ij} + a_i u_i + au = f, && 0 < y < 1, \\ &u = 0, && y = 0. \end{aligned} \tag{3.1}$$

Assume that $a_{ij}(x, y, z)$, $a_i(x, y, z)$, $a(x, y, z)$, $r_i(x, z)$ are $C^0(V)$, $a \leqslant 0$, $r_2 \leqslant 0$, $F = \text{const} > 0$, and $a_{ij}\xi_i\xi_j \geqslant c_0 |\xi|^2$ for all $\xi \in R^3$, $\xi \neq 0$, $c_0 > 0$ $(i, j = 1, 2, 3)$. Then

$$|u|_0 \leqslant c(|f|_0 + |g|_0). \tag{3.2}$$

Proof. Let $v = \pm u + |f|_0 \psi_1(y) + |g|_0 \psi_2(y)$, where $\psi_1(y) = b(1 - e^{-\lambda y} - \lambda y e^{-\lambda})$, $\psi_2(y) = d(1 - e^{-\mu y})$, with constants b, d, λ, μ to be determined. (Note that $\psi_1, \psi_2 \geqslant 0$.) Compute

$$\begin{aligned} Lv &= \pm f + |f|_0 L\psi_1 + |g|_0 L\psi_2, \\ L\psi_1 &= -b\lambda e^{-\lambda y}(a_{22}\lambda - a_2 + a_2 e^{\lambda(y-1)}) + a\psi_1, \\ L\psi_2 &= -d\mu e^{-\mu y}(a_{22}\mu - a_2) + a\psi_2. \end{aligned}$$

Since $a_{22} \geqslant c_0 > 0$, we can fix λ so large that $a_{22}\lambda - a_2 + a_2 e^{\lambda(y-1)} > 0$, and then take b large, so that $L\psi_1 \leqslant -1$. Also we can pick μ so large that $L\psi_2 \leqslant 0$. Then

$$Lv \leqslant \pm f - |f|_0 \leqslant 0, \qquad 0 < y < 1. \tag{3.3}$$

At the boundaries $y=0$ and $y=1$ we have

$$v = 0 \qquad \text{at} \quad y = 0, \tag{3.4}$$

$$\begin{aligned} v_y - Fv_{xx} - r_1 v_x - r_2 v = \pm g + |g|_0\, d\mu\, e^{-\mu y} \\ - r_2\, |f|_0 \psi_1(1) - r_2\, |g|_0\, \psi_2(1) \geqslant 0 \qquad \text{at} \quad y = 1, \end{aligned} \tag{3.5}$$

provided d is taken sufficiently large.

From (3.3) to (3.5) it follows that $v \geqslant 0$. (By (3.3) v cannot assume a negative minimum except possibly on the boundary of the strip $0 \leqslant y \leqslant 1$. The point of negative minimum P cannot lie on the $y=1$ part of the boundary, since then $v_y(P) < 0$ by Hopf's lemma, and $v_{xx}(P) \geqslant 0$, which contradicts (3.5). Hence the minimum of v is zero.) Then $|u|_0 \leqslant b\, |f|_0 + d\, |g|_0$.

Remark 3.1. The same result holds for the problem (3.13) below.

LEMMA 3.2. *In addition to the conditions of Lemma* 3.1 (*without the condition* $a \leqslant 0$) *assume that* $|a_{ij}|_2$, $|r_i|_2$, $\|a_{ij}\|_m$, $\|a_i\|_m$, $\|a\|_m$, $\|r_i\|_{m+1} \leqslant c$, *with integer* $m \geqslant 0$ $(i, j = 1, 2, 3)$. *Then*

$$\|u\|_{m+2} + \overline{\|u_x\|}_{m+1} \leqslant c(\|f\|_m + \|g\|_{m+1} + \|u\|_0). \tag{3.6}$$

Proof. To simplify the presentation we shall assume that $a_i = a = r_2 = 0$, $F = 1$ $(i = 1, 2, 3)$.

Step 1. Multiply the equation in (3.1) by u and consider two cases.

(i) $a_{ij} \neq a_{22}$, e.g., $i \neq 2$. Then $\int a_{ij} u u_{ij} = -\int a_{ij} u_i u_j - \int a_{ij,i} u u_j$ $(a_{ij,i} \equiv (\partial/\partial x_i) a_{ij})$.

(ii) $\int a_{22} u u_{yy} = -\int a_{22} u_y^2 - \int a_{22,y} u u_y + \int_t a_{22} u u_y$; $\int_t a_{22} u u_y = \int_t a_{22} u(u_{xx} + r_1 u_x + g) = -\int_t a_{22} u_x^2 + \frac{1}{2} \int_t a_{22,xx} u^2 - \frac{1}{2} \int_t (\partial/\partial x)(a_{22} r_1) u^2 + \int_t a_{22} u g$.

Combining both cases (i) and (ii), we get

$$\begin{aligned} -\sum_{i,j=1}^{3} \int a_{ij} u_i u_j - \sum_{i,j=1}^{3} \int a_{ij,i} u u_j - \int_t a_{22} u_x^2 + \frac{1}{2} \int_t a_{22,xx} u^2 \\ -\frac{1}{2} \int_t \frac{\partial}{\partial x} (a_{22} r_1) u^2 + \int_t a_{22} u g = \int f u. \end{aligned}$$

By our assumptions and Lemma 2.1 we easily conclude

$$\int |\nabla u|^2 + \int_t u_x^2 \leqslant c\left(\int f^2 + \int_t g^2 + \int u^2\right). \tag{3.7}$$

Step 2. Multiply the equation in (3.1) by u_{xx} and consider two cases.

(i) $a_{ij} \neq a_{22}$, *e.g.*, $i \neq 2$. Then $\int u_{xx} a_{ij} u_{ij} = -\int u_{xxi} a_{ij} u_j - \int u_{xx} a_{ij,i} u_j = \int a_{ij} u_{xi} u_{xj} + \int u_{xi} a_{ij,x} u_j - \int u_{xx} a_{ij,i} u_j$.

(ii) $\int u_{xx} a_{22} u_{yy} = -\int u_{xxy} a_{22} u_y - \int a_{22,y} u_{xx} u_y + \int_t a_{22} u_{xx} u_y$; $-\int u_{xxy} a_{22} u_y = \int a_{22} u_{xy}^2 + \int a_{22,x} u_{xy} u_y$; $\int a_{22} u_{xx} u_y = \int_t a_{22} u_{xx}(u_{xx} + r_1 u_x + g) = \int_t a_{22} u_{xx}^2 - \frac{1}{2}\int_t (\partial/\partial x)(a_{22} r_1) u_x^2 - \int_t a_{22,x} u_x g - \int_t a_{22} u_x g_x$.

Combining (i) and (ii) we get

$$\sum_{i,j=1}^{3} \int a_{ij} u_{xi} u_{xj} - \sum_{i,j=1}^{3} \int a_{ij,i} u_{xx} u_j + \sum_{i,j=1}^{3} \int a_{ij,x} u_{xi} u_j$$
$$+ \int_t a_{22} u_{xx}^2 - \frac{1}{2}\int_t \frac{\partial}{\partial x}(a_{22} r_1) u_x^2 - \int_t a_{22,x} u_x g - \int_t a_{22} u_x g_x = \int f u_{xx}$$

from which we easily conclude, using (3.7),

$$\int |\nabla u_x|^2 + \int_t u_{xx}^2 \leqslant c(\|g\|_1^2 + \|f\|_0^2 + \|u\|_0^2). \tag{3.8}$$

Similarly (multiplying the equation by u_{zz} and using Lemma 2.1(iv)), we get

$$\int |\nabla u_z|^2 + \int_t u_{xz}^2 \leqslant c(\|g\|_1^2 + \|f\|_0^2 + \|u\|_0^2). \tag{3.9}$$

(The only significant difference is in the treatment of term

$$\begin{aligned}\int_t a_{22} u_{zz} u_{xx} &= -\int_t a_{22,x} u_x u_{zz} - \int_t a_{22} u_x u_{xzz} \\ &= \int_t a_{22,xz} u_x u_z - \tfrac{1}{2}\int_t a_{22,xx} u_z^2 + \int_t a_{22} u_{xz}^2 \\ &\quad - \tfrac{1}{2}\int_t a_{22,zz} u_x^2. \Big)\end{aligned}$$

From Eq. (3.1) we express $u_{yy} = -(a_{ij}/a_{22}) u_{ij} + f/a_{22}$ which allows us to estimate $\int u_{yy}^2$. Combining this with (3.8) and (3.9) we conclude (3.6) with $m = 0$.

Step 3. Differentiate (3.1) in x or z, denoting the derivative by a prime:

$$\begin{aligned} &u'_y - u'_{xx} - r_1 u'_x = g' + r'_1 u_x, && y=1,\\ &a_{ij}u'_{ij} = f' - a'_{ij}u_{ij}, && 0<y<1, \qquad (3.10)\\ &u' = 0, && y=0. \end{aligned}$$

Applying the estimate (3.6) with $m=0$, we get

$$\|u'\|_2 + \overline{\|u'_x\|}_1 \leqslant c(\|f\|_1 + \|g\|_2 + \|u\|_0) \tag{3.11}$$

(since $\|a'_{ij}u_{ij}\|_0 \leqslant |a_{ij}|_1\ \|u\|_2$, $\overline{\|r'_1 u_x\|}_1 \leqslant c\ |r_1|_2\ \overline{\|u_x\|}_1$). From (3.1) we express $u_{yyy} = (\partial/\partial y)(f/a_{22}) - (\partial/\partial y)((a_{ij}/a_{22})u_{ij})$, which together with (3.11) gives $\|u_{yyy}\|_0 \leqslant c(\|f\|_1 + \|g\|_2 + \|u\|_0)$. Combining this with (3.11), we conclude the estimate (3.6) with $m=1$.

Step 4. We proceed with an induction proof, assuming that (3.6) holds up to an index $m \geqslant 1$. We use that $\|fg\|_m \leqslant \|f\|_m\ \|g\|_m$ for $m > n/2$, where n is dimension of the space ($n=3$ for V, $n=2$ for V_t). Applying (3.6) to (3.10) we estimate for $m \geqslant 2$,

$$\begin{aligned} \|u'\|_{m+2} + \overline{\|u'_x\|}_{m+1} &\leqslant c(\|f'\|_m + \|a'_{ij}\|_m\ \|u_{ij}\|_m \\ &\quad + \overline{\|r'\|}_{m+1}\ \overline{\|u_x\|}_{m+1} + \|g'\|_{m+1}) \leqslant c(\|f\|_{m+1} + \|g\|_{m+2} + \|u\|_0). \end{aligned} \tag{3.12}$$

For $m=1$ the same inequality (3.12) is true, since

$$\|a'_{ij}u_{ij}\|_1 \leqslant |a_{ij}|_2\ \|u\|_3 \leqslant c(\|f\|_1 + \|g\|_2 + \|u\|_0).$$

From (3.1) we express $D_y^{m+3}u = D_y^{m+1}(f/a_{22}) - D_y^{m+1}((a_{ij}/a_{22})u_{ij})$. Then using (3.12) and Lemma 2.2 (note that all derivatives of $\phi = 1/u$ are bounded for $u \geqslant c_0$) we estimate

$$\begin{aligned} \|D_y^{m+3}u\|_0 &\leqslant \left\|\frac{f}{a_{22}}\right\|_{m+1} + \left\|\frac{a_{ij}}{a_{22}}u_{ij}\right\|_{m+1} \\ &\leqslant c\ \|a_{22}\|_{m+1}\ \|f\|_{m+1} + c\ \|a_{ij}\|_{m+1}\ \|a_{22}\|_{m+1}\ \|u_{ij}\|_{m+1} \\ &\leqslant c(\|f\|_{m+1} + \|g\|_{m+2} + \|u\|_0). \end{aligned}$$

Combining this with (3.12), we obtain the estiate (3.6) for $m+1$, completing the proof.

LEMMA 3.3. *In addition to the conditions of Lemma* 3.2 *assume that* $a \leqslant 0$, $m \geqslant 2$, *then*

$$\|u\|_{m+2} + \overline{\|u_{xx}\|}_m \leqslant c(\|f\|_m + \|g\|_{m+1}).$$

Proof. By Lemmas 3.1, 3.2, and the Sobolev's imbedding theorem, we conclude $\|u\|_0 \leqslant c\,|u|_0 \leqslant c(|f|_0 + |g|_0) \leqslant c(\|f_m + \|g\|_m)$.

LEMMA 3.4. *In addition to the conditions of Lemma* 3.3 *assume that* $f \in H^2(V)$, $g \in H^3(V_t)$. *Then the problem* (3.1) *has a unique solution of class* G^4.

Proof. For $0 \leqslant t \leqslant 1$ consider the problem

$$\begin{aligned} &u_y - Fu_{xx} - tr_1 u_x - tr_2 u = g, && y = 1, \\ &ta_{ij}u_{ij} + (1-t)\,\Delta u + ta_i u_i + tau = f, && 0 < y < 1, \\ &u = 0, && y = 0. \end{aligned} \tag{3.13}$$

For $t = 0$ the problem (3.13) has a solution of class G^4, as can be seen by simple Fourier analysis. We show next that the set of t for which the problem (3.13) is solvable is both open and closed in $[0, 1]$, which will imply existence for (3.1) by setting $t = 1$ (uniqueness follows by Remark 3.1). The openness part easily follows by the contractive mapping principle. Assume now that for a sequence $\{t_n\}$ with $t_n \to t_0$, problem (3.13) has a solution $u_n \in G^4$. By Lemma 3.3 $\|u_n\|_4 \leqslant c$, and hence we can extract a subsequence which converges in $H^3(V)$ to some $u \in H^3(V)$. Passing to the limit in (3.13) (with $t = t_n$, $u = u_n$), we see that u is a solution of (3.13) corresponding to $t = t_0$. Moreover, $u \in G^4$ by Lemma 3.3.

THEOREM 3.1. *For the problem* (1.1) *assume that* $F > 0$, f, q, *and* ρ *are* 2π *periodic in* x *and* z, *and the following conditions.*

(i) *In* $V \times \{|u|_2 < r\}$ *we have* $f(x, y, z, 0, 0, 0) \equiv 0$, $f \in C^3(V \times R^3)$, $f_u(x, y, z, 0, 0, 0) \leqslant 0$, *and* $f_{u_{ij}}\xi_i\xi_j \geqslant c_0\,|\xi|^2$ *for all* $\xi \in R^3$, $\xi \neq 0$, *for some constants* r, $c_0 > 0$.

(ii) *In* $V_t \times \{|u|_1 < r_0\}$ *we have* $\rho(x, z, 0, 0) = 0$, $\rho \in C^4(V_t \times R^2)$, $\rho_u(x, z, 0, 0) \leqslant 0$ *for some constant* $r_0 > 0$.

Then for $\|q\|_2$ *sufficiently small the problem* (1.1) *has a solution of class* $C^2(\bar{V})$.

Proof. Define a map $T\colon G^4 \to G^4$ by solving $(v = Tu)$

$$\begin{aligned} &v_y - Fv_{xx} - r_1 v_x - r_2 v = \rho(x, z, u, u_x) - r_1 u_x - r_2 u, && y = 1, \\ &a_{ij}v_{ij} + a_i v_i + av \\ &\qquad = q - f(x, y, z, u, Du, D^2u) + a_{ij}u_{ij} + a_i u_i + au, && 0 < y < 1, \\ &v = 0, && y = 0, \end{aligned}$$

where $r_1 = \rho_{u_x}(x, z, 0, 0)$, $r_2 = \rho_u(x, z, 0, 0)$, $a_{ij} = f_{u_{ij}}(x, y, z, 0, 0, 0)$, $a_i = f_{u_i}(x, y, z, 0, 0, 0)$, $a = f_u(x, y, z, 0, 0, 0)$. By Lemmas 3.3, 3.4, and 2.2 the map T is well defined. It is straightforward to show that T is a contraction on sufficiently small balls around the origin in G^4, provided $\|q\|_2$ is small enough (see [3] for a similar argument).

Remark 3.2. Under the additional assumption that $f_u \leqslant 0$ and $\rho_u \leqslant 0$ for $|u|_2$ small, it is easy to prove uniqueness of the sufficiently small solution.

Remark 3.3. Theorem 3.1 implies existence of a solution for the problem $Lu \equiv a_{ij}u_{ij} + a_i u_i + au = \varepsilon f(x, y, z, u, Du, D^2u)$, $0 < y < 1$, with the boundary conditions of (1.1), provided ε is sufficiently small (with a_{ij}, a_i, a satisfying conditions of Lemma 3.3, f that of Theorem 3.1, $f_u \geqslant 0$). Indeed, writing $g(x, y, z, u, Du, D^2u) \equiv Lu - \varepsilon f + \varepsilon f(x, y, z, 0, 0, 0) = \varepsilon f(x, y, z, 0, 0, 0)$, we see that $g(x, y, z, 0, 0, 0) = 0$, and Theorem 3.1 applies.

Acknowledgments

I thank L. Nirenberg for posing the problem and for his useful discussions, and June Anderson for her efficient typing.

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