

Further remarks on the non-degeneracy condition

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Abstract

The structure of the set of positive solutions of the semilinear elliptic boundary value problem

$$\Delta u(x) + \lambda f(u(x)) = 0 \quad \text{for } x \in D, \quad u = 0 \text{ on } \partial D$$

depends on a certain non-degeneracy condition, which was proved by K.J. Brown [2] and T. Ouyang and J. Shi [12], with a shorter proof given later by P. Korman [8]. In this note we present a more general result, communicated to us by L. Nirenberg [13]. We also discuss the extensions in cases when the domain D is in R^2 , and it is either symmetric or convex.

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1 Introduction

We consider positive solutions of the semilinear boundary value problem

$$(1.1) \quad \Delta u(x) + \lambda f(u(x)) = 0 \quad \text{for } x \in D, \quad u = 0 \text{ on } \partial D,$$

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depending on the positive parameter λ . We assume throughout the paper that D is a bounded region in R^n , whose boundary is of class C^2 and $f \in C^1(\bar{R}_+)$. To study the solution curves of (1.1), we need to consider the linearized problem for (1.1)

$$(1.2) \quad \Delta w(x) + \lambda f'(u(x))w = 0 \quad \text{for } x \in D, \quad w = 0 \quad \text{on } \partial D.$$

If the problem (1.2) admits a non-trivial solution, we refer to $w(x)$ as a singular solution of (1.1). (We consider only the classical solutions of both (1.1) and (1.2), $u, w \in C^2(\bar{D})$.) At a non-singular solution of (1.1) we can apply the implicit function theorem to continue the solution to nearby λ 's. At a singular solution the following non-degeneracy condition is crucial

$$(1.3) \quad \int_D f(u(x))w(x) dx \neq 0.$$

Condition (1.3) ensures that near singular solution, the solution set of (1.1) is as simple as possible, namely all positive solutions of (1.1) in a neighborhood of (λ, u) lie on a single smooth curve, see P. Korman, Y. Li and T. Ouyang [9], T. Ouyang and J. Shi [12], and K.J. Brown [2].

In the present note we complete the proof of the following theorem, based on L. Nirenberg [13]. ($n(x)$ denotes the unit normal vector at $x \in \partial D$, pointing outside.)

Theorem 1.1 *Suppose that D is a bounded domain with C^2 boundary, such that $x \cdot n(x) > 0$ for all $x \in \partial D$. Assume that at some solution $(\lambda, u(x))$ of (1.1) the linearized problem (1.2) admits a positive solution $w(x)$. Then the non-degeneracy condition (1.3) holds.*

With an extra condition $f(0) \geq 0$ this theorem was proved independently by T. Ouyang and J. Shi [12] and K.J. Brown [2], and a simpler proof was given subsequently by P. Korman [8]. The domain D described in the Theorem 1.1 is called star-shaped.

For balls in R^n the Theorem 1.1 was proved in [6], even without the assumption $w(x) > 0$ (see also [7] for a simpler proof). We present here another simple proof of this fact, based on a recent paper of M. Tang [15]. By rescaling, we may consider the unit ball. Recall that by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [4], on a ball B : $|x| < 1$ in R^n , any positive solution of (1.1) is radially symmetric, and hence it satisfies ($r = |x|$)

$$(1.4) \quad u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0 \quad r \in (0, 1), \quad u'(0) = u(1) = 0.$$

By the result of C.S. Lin and W.-M. Ni [11] any solution of the linearized problem for (1.4) is also radially symmetric, and hence it satisfies

$$(1.5) \quad w''(r) + \frac{n-1}{r}w'(r) + \lambda f'(u)w = 0 \quad r \in (0, 1), \quad w'(0) = w(1) = 0.$$

M. Tang [15] has introduced the function $\zeta(r) = r^n [u'w' + \lambda f(u)w] + (n-2)r^{n-1}u'w$, which in view of the equations (1.4) and (1.5) satisfies

$$(1.6) \quad \zeta'(r) = 2\lambda f(u)wr^{n-1}.$$

Integrating (1.6) over $(0, 1)$, we obtain the non-degeneracy condition (where ω_n denotes the area of the unit sphere in R^n)

$$(1.7) \quad \int_B f(u(x))w(x) dx = \frac{\omega_n}{2\lambda} u'(1)w'(1) \neq 0,$$

by the Hopf's lemma. (No matter if $w(r)$ is positive or sign-changing.)

Observe that if u is a singular solution of (1.1), then zero is an eigenvalue of the corresponding linearized operator, i.e. $\mu = 0$ is an eigenvalue of

$$(1.8) \quad \Delta w(x) + \lambda f'(u(x))w + \mu w = 0 \quad \text{for } x \in D, \quad w = 0 \text{ on } \partial D,$$

and the corresponding eigenfunction w is then a solution of (1.2). The assumption that $w > 0$ in the Theorem 1.1 implies that $\mu = 0$ is the first eigenvalue of (1.8).

Is it possible to get around the condition $w > 0$, i.e. the assumption that $\mu = 0$ is the first eigenvalue of (1.8)? As we mentioned above, this condition is not necessary for balls in R^n . In Section 4 we observe that situation is similar for symmetric domains in R^2 , following M. Holzmann and H. Kielhofer [5] and L. Damascelli, M. Grossi and F. Pacella [3]. In Section 3 we show that an elegant argument of C.S. Lin [10] implies that for convex domains in R^2 the non-degeneracy condition holds if $\mu = 0$ is either first or second eigenvalue. What about domains that are not star-shaped? This appears also to be a hard question. Let us consider positive radial solution on an annulus $A : a < r < b$, i.e.

$$(1.9) \quad u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0 \quad r \in (a, b), \quad u(a) = u(b) = 0.$$

Integrating (1.6), we have

$$(1.10) \quad \int_A f(u(x))w(x) dx = \frac{\omega_n}{2\lambda} (b^n u'(b)w'(b) - a^n u'(a)w'(a)).$$

If $w > 0$, it is not at all clear if this quantity is non-zero. On the other hand, the non-degeneracy condition (1.3) does hold if $\mu = 0$ is either second, fourth or any other even eigenvalue.

2 Proof of the Theorem 1.1.

We may assume that D is not a ball, since on a ball even more general result is true. Recall that the proofs in [12], [2] and [8] have established the following generalization of (1.7)

$$(2.1) \quad \int_D f(u)w dx = -\frac{1}{2\lambda} \int_{\partial D} \frac{\partial w}{\partial n} |\nabla u| x \cdot n(x) ds,$$

where $n(x)$ is the unit normal vector at $x \in \partial D$, pointing outside. We present here another proof of this formula, which is even simpler than the one in [8]. The function $v = x \cdot \nabla u$ satisfies

$$\Delta v(x) + \lambda f'(u(x))v = -2\lambda f(u(x)).$$

Combining this with the linearized problem (1.2),

$$(2.2) \quad \int_D f(u)w dx = \frac{1}{2\lambda} \int_{\partial D} v \frac{\partial w}{\partial n} ds.$$

Since ∂D is a level set of $u(x)$, it follows that $\nabla u = -|\nabla u|n$, and hence $v = -|\nabla u|x \cdot n$. Using this in (2.2), we conclude (2.1).

Clearly on ∂D , $\frac{\partial u}{\partial n} \leq 0$, while by the Hopf's lemma, $\frac{\partial w}{\partial n} < 0$. If $\frac{\partial u}{\partial n}$ is negative somewhere on ∂D , then $|\nabla u| > 0$ there, and hence the integral on the right hand side of (2.1) is positive, and we are done. So suppose $\frac{\partial u}{\partial n} \equiv 0$ on ∂D . But then by the classical results of J. Serrin [14] and of

B. Gidas, W.-M. Ni and L. Nirenberg [4], D is a ball, a contradiction (see also Theorem 2.18 in [1]). \diamond

Remark $\frac{\partial u}{\partial n} \equiv 0$ on ∂D is impossible even for the ball. Indeed, in such a case both $w > 0$ and u_{x_1} are solutions of the linearized problem (1.2). But then 0 is the lowest eigenvalue of the linearized operator (since $w > 0$), and it is a simple eigenvalue, meaning u_{x_1} is a multiple of w . But that is impossible, since u_{x_1} is sign-changing.

3 Convex domains in R^2

We now consider the two-dimensional case, i.e. $D \subset R^2$ and $u = u(x, y)$. We adapt an argument of C.S. Lin [10] to prove the following theorem.

Theorem 3.1 *Let D be a bounded convex domain in two dimensions. Assume $u(x, y)$ is a singular positive solution of (1.1), so that $\mu = 0$ is either first or second eigenvalue of (1.8). Then the non-degeneracy condition (1.3) holds.*

Proof: We may assume that D is not a ball, since on a ball even more general result is true. We may also assume that $\mu = 0$ is the second eigenvalue, since the case of the first eigenvalue is covered in the Theorem 1.1. (Clearly, a convex domain is star-shaped with respect to any of its points. Indeed, if D is not star-shaped, then we can find $P \in D$ and $Q \in \partial D$, so that the segment PQ points inside D at Q , and hence a part of this segment lies outside of D , contradicting the convexity of D .)

Assume that on the contrary

$$(3.1) \quad \int_D f(u)w \, dx dy = 0.$$

The function $v(x, y) = xu_x + yu_y$ satisfies

$$(3.2) \quad \Delta v + \lambda f'(u)v = -2f(u).$$

Combining this with the linearized problem (1.2), we conclude

$$(3.3) \quad \int_{\partial D} v \frac{\partial w}{\partial n} \, ds = 0.$$

By Courant's nodal line theorem, the nodal line $\{x \in D \mid w(x) = 0\}$ divides D into two subdomains. There are two possibilities to consider.

Case (i) The nodal line encloses a region inside D (i.e. either the nodal line lies strictly inside D or it touches ∂D at just one point). We choose as an origin any point inside D . We have $v \leq 0$ on ∂D . If $v \equiv 0$ on ∂D , then as before D is a ball, a contradiction. Hence $v < 0$ on some part of ∂D . By Hopf's lemma $\frac{\partial w}{\partial n}$ is of one sign on ∂D , and so we have a contradiction in (3.3).

Case (ii) The nodal line intersects the boundary ∂D at two points, $p_1, p_2 \in \partial D$. Suppose first that the tangent lines to ∂D at p_1 and p_2 are not parallel. We choose origin O to be at their point of intersection. We decompose ∂D into two parts $\partial D = S_1 \cup S_2$, where S_1 is the part of ∂D lying inside the triangle Op_1p_2 , and S_2 is its complement. Observe that ∇u gives direction of the inside pointing normal on ∂D . On S_1 , $v = (x, y) \cdot \nabla u > 0$, while $v < 0$ on S_2 . By Hopf's lemma, $\frac{\partial w}{\partial n}$ has one sign on S_1 , and the opposite sign on S_2 , again we have a contradiction in (3.3).

Assume now that the tangent lines at p_1 and p_2 are parallel, and go along the x -axis. Differentiating the equation (1.1), we have

$$\Delta u_x + \lambda f'(u)u_x = 0.$$

Combining this with linearized problem (1.2), we conclude

$$(3.4) \quad \int_{\partial D} u_x \frac{\partial w}{\partial n} ds = 0.$$

This time let S_1 be the part of ∂D to the left of the line p_1p_2 , and S_2 the part to the right of the line. On S_2 , $u_x < 0$, while $u_x > 0$ on S_1 . The integrand in (3.4) is of one sign, a contradiction. \diamond

4 Symmetric domains in R^2

In this section we show that the non-degeneracy condition holds for symmetric domains in R^2 , without any restriction on w , similarly to balls in R^n .

Theorem 4.1 *Let D be a bounded domain in two dimensions, which is symmetric with respect to both x and y axes, and convex in both x and y directions, with ∂D of class C^3 . Assume that $f(0) \geq 0$. Let $u(x, y)$ be a singular positive solution of (1.1). Then the non-degeneracy condition (1.3) holds.*

Prior to giving the proof, we recall some known facts. By the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [4], any positive solution is an even function in both x and y , with $u_x < 0$ and $u_y < 0$ in the first quadrant $x, y > 0$. Next we recall that by the Theorem 2.1 of L. Damascelli, M. Grossi and F. Pacella [3] any solution of the linearized equation (1.2) is also even in both x and y . We shall also need the following fundamental lemma of M. Holzmann and H. Kielhofer [5].

Lemma 4.1 ([5]) *There is no nontrivial solution $v(x, y)$ of the linearized equation in (1.2), that is even in both x and y , which satisfies $v(0, 0) = 0$, and such that either $v = 0$ or $v < 0$ on ∂D .*

This lemma and the symmetry of $w(x, y)$ (any solution of the linearized problem (1.2)) imply that $w(0, 0) \neq 0$.

Proof of the Theorem 4.1 Observe that the solution set of the linearized problem (1.2) is one dimensional, since otherwise we could produce a non-trivial solution of (1.2) with $w(0, 0) = 0$, in contradiction with Lemma 4.1. Assume that on the contrary $\int_D f(u)w dx dy = 0$. Since the linearized operator is Fredholm of index one, we can find $v_1(x, y)$ which is a solution of

$$(4.1) \quad \Delta v_1 + \lambda f'(u(x, y))v_1 = -2\lambda f(u) \quad \text{for } x \in D, \quad v_1 = 0 \text{ on } \partial D.$$

We claim that $v_1(x, y)$ is even in both x and y . Indeed, if $v_1(x, y)$ was not even in x , then $v_1(-x, y)$ would be another solution of (4.1), and then $v_1(x, y) - v_1(-x, y)$ would be a solution of the linearized problem (1.2) that is not even in x , a contradiction.

Recall that the function $v(x, y) = xu_x + yu_y$ satisfies (3.2). Assume first that $v_1(0, 0) \neq 0$. Consider $z(x, y) \equiv w + v + v_1$. By scaling w , we may achieve $z(0, 0) = 0$. The function $z(x, y)$ is even in both x and y , and it satisfies the equation in (1.2), and $z < 0$ on ∂D . We have a contradiction with Lemma 4.1. In case $v_1(0, 0) = 0$, we define $z = v + v_1$, and proceed as before. \diamond

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