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# Exact Multiplicity Results for Two Classes of Periodic Equations

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For two classes of first-order equations we study the exact multiplicity of periodic solutions. If the nonlinearity is a quadric in the unknown function, we give conditions for existence of exactly four solutions. In another direction, we use bifurcation theory to derive exact multiplicity results for quadratic and cubic equations. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

We present exact multiplicity results for two classes of first-order equations of the form

$$\dot{x} = f(t, x). \tag{1.1}$$

Here $f(t + p, x) = f(t, x)$ for all real $t$ and $x$, and we are looking for p-periodic solutions $x = x(t)$. In Section 2 we consider the case when $f(t, x)$ is a quadric in $x$ polynomial with distinct roots. The nonlinearity here is

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neither concave nor convex, and exact multiplicity results in such a setup are rare. Under some conditions we prove existence of exactly four solutions.

Our main results are in Section 3. We begin with an alternative proof of a well-known result of McKean and Scovel; see [6] and also [1]. Our basic tool is a bifurcation theorem of Crandall and Rabinowitz [2]. Our proof appears to be more elementary and self-contained, and it produces a slight generalization and some extra information. We then consider a class of cubic equations and prove an exact multiplicity result by using similar techniques. We prove that all solutions lie on a unique S-shaped solution curve, giving an exact count of both positive and negative solutions, and we discuss monotonicity of the branches of the solution curve. In [1] Berger has suggested an open problem of studying perodic solutions of

$$\dot{x} + P_N(x, \lambda) = f(t),$$

where $P_N$ is a polynomial of degree $N$ in $x$. Our results for a cubic seems to provide a step in this direction.

We mention that other earlier results on multiplicity of periodic solutions of (1.1), to which we refer later on, include those of Mawhin [5], Lloyd [4], and Nkashama [8]. Next we state the Crandall–Rabinowitz bifurcation theorem.

THEOREM 1.1. [2]. *Let $X$ and $Y$ be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and let $F$ be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into $Y$. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and* codim $R(F_x(\bar{\lambda}, \bar{x})) = 1$. *Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If $Z$ is a complement of* span $\{x_0\}$ *in $X$, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$* near $(\bar{\lambda}, \bar{x})$ *form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

## 2. A QUADRIC EQUATION WITH EXACTLY FOUR SOLUTIONS

First we consider a cubic equation with distinct roots.

THEOREM 2.1. *Consider the equation*

$$\dot{x} = e(t)(x - a(t))(x - b(t))(x - c(t)), \tag{2.1}$$

*with continuous p-periodic functions $a(t)$, $b(t)$, $c(t)$, and $e(t) \in C^0(R)$, such that $e(t) > 0$ for almost all $t$ (or $e(t) < 0$) and*

$$\max_{t\in R} a(t) < \min_{t\in R} b(t), \tag{2.2}$$

$$\max_{t\in R} b(t) < \min_{t\in R} c(t).$$

*Then the problem* (2.1) *has exactly three p-periodic solutions.*

*Proof.* To prove the existence of at least three solutions, we define the Poincaré map $x_0 \in R \to T(x_0) \in R$ by $T(x_0) = x(p, x_0)$, where $x(t, x_0)$ is the solution of (2.1) with $x(0) = x_0$. It is well known that $T$ is a continuous map. Define the intervals $I_1 = [\min_t a(t), \max_t a(t)]$, $I_2 = [\min_t b(t), \max_t b(t)]$ and $I_3 = [\min_t c(t), \max_t c(t)]$. It is clear that $T$ maps $I_2$ into itself, while $T^{-1}$ maps $I_1$ and $I_3$ into itself. By the Brouwer's fixed point theorem the map $T$ has at least three fixed points, which correspond to p-periodic solutions of (2.1).

Existence of at most three solutions follows from the results of [7, 9]. For completeness we present a different proof. It uses the following two lemmas, which will also be needed for the rest of the paper.

LEMMA 2.1. *Consider the problem*

$$\dot{w} = c(t)w, \qquad \text{with } c(t)\text{—p-periodic}. \tag{2.3}$$

*Then* (2.3) *has a non-trivial p-periodic solution if and only if*

$$\int_0^p c(t)\, dt = 0.$$

*Proof.* Obvious.

LEMMA 2.2. *Consider the equation*

$$\dot{x} = f(t, x), \qquad f(t + p, x) = f(t, x), \tag{2.4}$$

*with f convex in x, for $x \in I$, a possibly unbounded interval, i.e.,*

$$f_{xx} > 0 \text{ for all } t \text{ and almost all } x \in I. \tag{2.5}$$

*Then* (2.4) *has at most two p-periodic solutions with values in I.*

*Proof.* Assume that $x_1(t) < x_2(t) < x_3(t)$ are three p-periodic solutions. Let $w_1 = x_2 - x_1$, $w_2 = x_3 - x_2$. Then the p-periodic functions $w_1(t)$ and $w_2(t)$ satisfy

$$\dot{w}_1 = f(t, x_2) - f(t, x_1) = \int_0^1 \frac{d}{d\theta} f(t, \theta x_2 + (1-\theta)x_1)\, d\theta$$

$$= \int_0^1 f_x(t, \theta x_2 + (1-\theta)x_1)\, d\theta w_1 \equiv c_1(t) w_1.$$

$$\dot{w}_2 = f(t, x_3) - f(t, x_2) = \int_0^1 f_x(t, \theta x_3 + (1-\theta)x_2)\, d\theta w_2$$

$$\equiv c_2(t) w_2.$$

Since $c_2(t) > c_1(t)$, we have a contradiction with Lemma 2.1.

*Remark* 1. Lemma 2.2 holds when the sign in (2.5) is reversed.

*Remark* 2. Lemma 2.2 appeared earlier in [5].

Next we note that all p-periodic solutions of (2.1) lie entirely in the strips $S_1 = I_1 \times R$, $S_2 = I_2 \times R$, $S_3 = I_3 \times R$. Making a change of variables $x = y - \alpha$, with a constant $\alpha$, we get

$$\dot{y} = (y - (a(t) + \alpha))(y - (b(t) + \alpha))(y - (c(t) + \alpha)), \tag{2.6}$$

which is of the same type as (2.1) and which has the same number of p-periodic solutions as (2.1). This allows us to shift all the strips $S_1$, $S_2$, and $S_3$ by the same amount. In particular, we can assume that $x(t) \neq 0$ for all $t$ in all three strips $S_i$.

Divide (2.1) by $x^2$ and denote $u = 1/x$. Obtain

$$\dot{u} = e[-1/u + a + b + c - (ab + ac + bc)u + abcu^2]. \tag{2.7}$$

Equation (2.7) has the same number of positive (negative) p-periodic solutions as (2.1). If $f(t, u)$ denotes the right-hand side of (2.7) then

$$f_{uu} = e\left(-\frac{2}{u^3} + 2abc\right).$$

By shifting the strips we can assume that $a(\mathrm{t}) < 0$, while $b(t) > 0$ and $c(t) > 0$ for all $t$. Then $f_{uu} < 0$ for $u > 0$, which means by Lemma 2.2 that Eq. (2.7) has at most two positive p-periodic solutions; i.e., there are at most two p-periodic solutions of (2.1) in $S_2 \cup S_3$. By a different shifting of the strips we can arrange that $a(t) < 0$ and $b(t) < 0$, while $c(t) > 0$ for all $t$. Then $f_{uu} > 0$ for $u < 0$. Thus there are at most two p-periodic solutions of (2.1) in $S_1 \cup S_2$. It follows, there is exactly one p-periodic solution in each of the strips $S_i$, completing the proof of the theorem.

EXAMPLE. Consider the equation

$$\dot{x} = x(x - a(t))(b(t) - x), \tag{2.8}$$

with continuous positive p-periodic functions $a(t)$ and $b(t)$, such that

$$\max_t a(t) < \min_t b(t).$$

Then (2.8) has exactly two positive p-periodic solutions. This equation represents a logistic-type population model with a threshold. In this case one could also make a change of variables $u = 1/x^2$ to obtain an equation with $f_{uuu} > 0$ for $u > 0$, and conclude the result by Lemma 2.2.

Next we consider a similar class of quadric equations.

THEOREM 2.2. *Consider the equation*

$$\dot{x} = e(t)(x - a(t))(x - b(t))(x - c(t))(x - d(t)), \tag{2.9}$$

*with continuous p-periodic functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t) \in C^0(R)$, such that $e(t) > 0$ for almost all $t$ (or $e(t) < 0$). We assume conditions* (2.2) *to be satisfied, and, in addition, that*

$$\max_{t\in R} c(t) < \min_{t\in R} d(t). \tag{2.2$'$}$$

*We denote $\mu(t) = [a(t) + b(t) + c(t) + d(t)]/4$ and assume that*

$$\begin{aligned} \tfrac{3}{4}\max_t b(t) + \tfrac{1}{4}\min_t c(t) < \mu(t) < \tfrac{3}{4}\min_{t\in R} c(t) \\ + \tfrac{1}{4}\max_t b(t) \qquad \text{for almost all } t \in R. \end{aligned} \tag{2.10}$$

*Then Eq.* (2.9) *has exactly four p-periodic solutions.*

*Proof.* To simplify the presentation we assume that $e(t) \equiv 1$. Let $S_1$, $S_2$, $S_3$ be as in the Theorem 2.1 and let $S_4$ be the strip $[\min_{t\in R} d(t), \max_{t\in R} d(t)] \times R$. Then as before we obtain existence of at least four p-periodic solutions, at least one in each of the strips $S_1$, $S_2$, $S_3$, and $S_4$.

We show that there are at most two p-periodic solutions in $S_3 \cup S_4$, i.e., when $x \geq \min_{t\in R} c(t)$. Denote $y = x - \max_{t\in R} b(t)$. In $S_3 \cup S_4$ we have

$$y \geq \min_{t\in R} c(t) - \max_{t\in R} b(t).$$

Equation (2.9) becomes

$$\dot{y} = y^4 + \left(4 \max_{t \in R} b(t) - 4\mu\right) y^3 + \alpha(t) y^2 + \beta(t) y + \gamma(t), \tag{2.11}$$

with certain p-periodic functions $\alpha(t)$, $\beta(t)$, and

$$\gamma(t) = \left(\max_t b(t) - a(t)\right)\left(\max_t b(t) - b(t)\right)\left(\max_t b(t) - c(t)\right) \tag{2.12}$$
$$\left(\max_t b(t) - d(t)\right) \geq 0 \qquad \text{for all } t.$$

Dividing (2.11) by $y^2$ and denoting $u = 1/y$, we obtain

$$-\dot{u} = \frac{1}{u^2} + \left(4 \max_{t \in R} b(t) - 4\mu\right) \frac{1}{u} + \alpha(t) + \beta(t) u + \gamma(t) u^2. \tag{2.13}$$

Solutions in the strips $S_2$ and $S_3$ correspond to $u > 0$. Denoting by $f(t, u)$ the right-hand side of (2.13), we estimate, using (2.10) and (2.12),

$$\begin{aligned} f_{uu} &= \frac{2}{u^3}\left[3y + 4\left(\max_t b(t) - \mu\right)\right] + 2\gamma(t) \\ &\geq \frac{2}{u^3}\left[3\left(\min_t c - \max_t b\right) + 4 \max_t b - 4\mu\right] > 0 \qquad \text{for } u > 0. \end{aligned}$$

Hence there are at most two solutions in $S_3 \cup S_4$, which implies that there is exactly one p-periodic solution in each of $S_3$ and $S_4$.

Similarly, considering $y = \min_t c(t) - x$, we prove that there is exactly one p-periodic solution in both $S_1$ and $S_2$, and the theorem follows.

*Remark.* This theorem should be compared with a result by Neto [7], which shows that there exist equations of the type

$$\frac{dx}{dt} = \sum_{j=0}^{4} a_j(t) x^j,$$

with periodic $a_j(t)$, which have an arbitrary number of p-periodic solutions. It is natural to ask whether our conditions (2.2), (2.2)′ rule out such a possibility, and whether our condition (2.10) can be dropped.

## 3. Multiplicity for Operators of Riccati's Type and for a Cubic

We begin this section with an alternative proof of McKean–Scovel result on multiplicity of p-periodic solution of

$$A(y) = y' + a(t)h(y) = f(t). \tag{3.1}$$

Our result is slightly more general than in [6], since we do not assume strict convexity of $h(y)$, and that $h''(y) \geq c > 0$ near $y = \pm\infty$. We assume that $f(t) \in L^2$, which will denote the space of square integrable and p-periodic functions, and that $a(t) \in C^{\circ}(R)$ is a positive p-periodic function. We assume that the function $h(y)$ is convex (see (3.2)) and assumes its global minimum on $R$. By subtracting the minimum value of $h(y)$ from both sides of (3.1), and by shifting $y$, we can assume that $h(y)$ takes its global minimum at $y = 0$ and $h(0) = 0$. So we assume that

$$h''(y) > 0 \qquad \text{for almost all } y \in R,\ h(0) = h'(0) = 0. \tag{3.2}$$

We are looking for p-periodic solutions $y = y(t)$. The solutions will lie in the Sobolev space $H^1$, consisting of p-periodic absolutely continuous functions with the norm $\|y\|^2 = \int_0^p [y'^2(t) + y^2]\, dt$.

As in [6], we denote by $M$ the set of points $y \in H^1$, where the linearized equation

$$u' + a(t)h'(y)u = 0 \tag{3.3}$$

has nontrivial solutions $u \in H^1$. Clearly, this happens iff

$$\int_0^p a(t)h'(y(t))\, dt = 0. \tag{3.4}$$

We will show that the set $A(M)$ separates $H^1$ into two connected pieces, one above $A(M)$ and one below it, and that the number of preimages of $A^{-1}(f)$ is 0, 1, or 2, according to $f \in L^2$ lying below, on, or above $A(M)$.

Lemma 3.1. *Let $f \in A(M)$. Then Eq.* (3.1) *has a unique solution $y \in H^1$.*

*Proof.* The existence of at least one solution $y \in M$ follows by the definition of $A(M)$. Assuming the existence of another solution $x \in H^1$ and denoting $z = x - y$, we express

$$z' + (a(t)h'(y) + \tfrac{1}{2}a(t)h''(c)z)z = 0, \tag{3.5}$$

with some $c = c(t)$. Since we can regard (3.5) as a linear equation in $z$ it follows that $z(t)$ is never zero, so that we may assume for definiteness that $z > 0$. Dividing (3.5) by $z$ and integrating

$$\int_0^p a(t)h'(y)\,dt = -\frac{1}{2}\int_0^p a(t)h''(c)\,z\,dt < 0,$$

which contradicts (3.4).

Next we recall the Fredholm alternative for a linear equation

$$x' + b(t)x = f(t), \tag{3.6}$$

with p-periodic functions $b(t) \in C^\circ(R)$ and $f(t) \in L^2(R)$. One considers the adjoint homogeneous equation

$$\mu' - b(t)\mu = 0. \tag{3.7}$$

If (3.7) has no nontrivial p-periodic solutions (i.e., $\int_0^p b(\tau)d\tau \neq 0$) then (3.5) has a unique p-periodic solution of class $H^1$. Moreover, if $\int_0^p b(\tau)\,d\tau > 0$ and $f(t) > 0$, then $x(t) > 0$. If (3.7) has a nontrivial p-periodic solution $\mu(t)$ then (3.5) has a $H^1$ solution iff

$$\int_0^p f(t)\mu(t)\,dt = 0.$$

The proofs of these assertions follow by a direct integration.

LEMMA 3.2. *Let $f_0 \in A(M)$. Then for any $f \in L^2$ with $f \leq f_0$ in $L^2$ and $f \neq f_0$ in $L^2$, Eq.* (3.1) *has no $H^1$ solution.* (*We will say that $f$ is below $f_0$ when the above inequalities are satisfied*).

*Proof.* Assume, on the contrary, there exists $y \in H^1$ with $A(y) = f$. Let $x \in M$ be such that $A(x) = f_0$. Denoting $z = y - x$, we obtain as before

$$z' + a(t)h'(x)z = -a(t)\int_x^y (y-\xi)h''(\xi)\,d\xi + f - f_0 < 0. \tag{3.8}$$

Since by the Fredholm alternative the right-hand side of (3.8) must be orthogonal to the nontrivial solutions of the corresponding adjoint homogeneous equation (which is of one sign), we have a contradiction.

COROLLARY 3.1. *Two elements of $A(M)$ cannot be ordered, and hence the surface $A(M)$ divides $L^2$ into two pieces: the one above $A(M)$, and the one below.*

THEOREM 3.1. *Consider the problem* (3.1) *with* $h(y)$ *satisfying* (3.2), *and* $a(t)$ *being positive p-periodic function of class* $C^\circ(R)$. *Then* $A(M)$ *is a surface dividing* $L^2$ *into two connected components, one above* $A(M)$—*denoted by* $A^+$—*and one below* $A(M)$—*denoted by* $A^-$. *Equation* (3.1) *has exactly 0, 1, or 2 solutions in* $H^1$, *depending on whether* $f \in A^-$, $f \in A(M)$, *or* $f \in A^+$, *respectively. Moreover, the set* $A^+$ *is convex.*

*Proof.* We claim that for any $f \in A^+$ Eq. (3.1) has at least one solution in $H^1$. Indeed, let $f_0 \in A(M)$ be below $f$ and let $A(y_0) = f_0$. Then $y_0$ is a subsolution of (3.1), while a large constant $P$ is a supersolution. The claim follows by the well-known method of monotone iterations. (Let $y_{n+1} \in H^1$ be solution of $y'_{n+1} + cy_{n+1} = cy_n - a(t)h(y_n) + f(t)$, where the constant $c$ is chosen so that the function on the right is increasing in $y$ for $y \in (0, P)$, $n = 0, 1, \ldots$. Then $y_n(t)$ form an increasing sequence, bounded by $P$. This method has been used previously in a similar context; see, e.g., [8].)

Next we remark that condition (3.2) implies that

$$h(y) - yh'(y) < 0 \qquad \text{for all } y \neq 0. \tag{3.9}$$

Indeed,

$$\begin{aligned} h(y) - yh'(y) &= \int_0^y (y - \xi)h''(\xi)\, d\xi - y\int_0^y h''(\xi)\, d\xi \\ &= -\int_0^y \xi h''(\xi)\, d\xi < 0. \end{aligned}$$

We embed (3.1) into a one-parameter family of problems

$$y' + a(t)h(y) = \lambda f(t) + (1 - \lambda)f_0(t), \tag{3.10}$$

where $f_0(t)$ is an element of $A(M)$ lying below $f(t)$, and we study its p-periodic solutions as the parameter $\lambda \geq 0$ varies. Define a map $F: H^1 \to L^2$ by

$$F(\lambda, y) = y' + a(t)h(y) - \lambda f(t) - (1 - \lambda)f_0(t) \tag{3.11}$$

and rewrite (3.1) as

$$F(\lambda, y) = 0. \tag{3.12}$$

We know that Eq. (3.10) is solvable for $\lambda = 1$. We will continue this solution for decreasing $\lambda$. The linearized equation is given by

$$F_y(\lambda, y)u = u' + a(t)h'(y)u = 0. \tag{3.13}$$

At any point $(\lambda_0, y_0)$, where the map $F$ is nonsingular (i.e., the only p-periodic solution of (3.13) is $u \equiv 0$) one can apply the implicit function theorem to continue the solution in $\lambda$. We show next that at any point $(\lambda_0, y_0)$ where $F$ is singular, the Crandall–Rabinowitz theorem applies. Indeed, at such a point

$$y_0' + a(t)h(y_0) = \lambda_0 f(t), \tag{3.14}$$

and Eq. (3.13) has a nontrivial p-periodic solution; i.e., by Lemma 2.1, $\int_0^p a(t)h'(y_0)\,dt = 0$ and, hence, there is a p-periodic $\mu(t) > 0$ satisfying

$$\mu' - a(t)h'(y_0)\mu = 0. \tag{3.15}$$

Clearly, we only need to check that

$$F_\lambda(\lambda_0, y_0) \notin R(F_y(\lambda_0, y_0)). \tag{3.16}$$

Assuming (3.16) to be violated, let $u(t)$ be a p-periodic solution of

$$u' + a(t)h'(y_0)u = f(t). \tag{3.17}$$

By the Fredholm alternative (or by multiplying (3.15) by $u$, (3.17) by $\mu$, adding, and integrating from 0 to $p$),

$$\int_0^p f(t)\mu(t)\,dt = 0. \tag{3.18}$$

Multiply (3.14) by $\mu$, integrate by parts, using (3.15) and (3.18):

$$\int_0^p a(t)(h(y_0) - y_0 h'(y_0))\mu\,dt = 0.$$

This is a contradiction in view of (3.9).

The point $\lambda = 0$, $y = f_0$ is clearly a singular one for $F(\lambda, y)$, since $f_0 \in A(M)$. In the neighborhood of $(0, f_0)$, by the Crandall–Rabinowitz theorem, solutions of (3.10) form a curve $(\lambda(s), y(s)) = (\tau(s), f_0 + s\mu^{-1} + z(s))$, with $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$. Since for negative $\lambda$ there are no p-periodic solutions of (3.10) (as can be seen by Lemma 3.2), it follows that $\tau(s) > 0$, and so for $s$ (and $\lambda$) small there are exactly two p-periodic solutions of (3.10), one strictly positive, and one strictly negative.

We now continue both branches of p-periodic solutions for increasing

$\lambda$. At regular points $(\lambda_0, y_0)$ each branch is continued by the implicit function theorem. At singular points $(\lambda_0, y_0)$ the Crandall–Rabinowitz theorem applies. If a branch were to turn at a singular $(\lambda_0, y_0)$, we would have at least three solutions of (3.10) in contradiction with Lemma 2.2.

Next we observe that both branches stay bounded in $H^1$ for any finite $\lambda$. Indeed, multplying Eq. (3.10) by $h(y)$ and integrating from 0 to $p$, we estimate $\int_0^p h^2(y)\, dt$ in terms of $\|f\|_{L^2}$. The estimate of $\int_0^p y^2\, dt$ follows, since $h(y) \geq c_1|y| - c_2$ for some positive constants $c_1$ and $c_2$. Then we estimate $\int_0^p \dot{y}^2\, dt$ from (3.10). It follows that both branches continue for all $\lambda > 0$. Setting $\lambda = 1$, we obtain two solutions for $f \in A^+$.

Next we remark that any $f$ for which Eq. (3.1) is solvable belongs to $A^- \cup A(M)$. Indeed, assuming $f \notin A(M)$, we can consider the equation $A(y) = f - \lambda$ and continue solutions for $\lambda \geq 0$. Integrating the equation, we see that this process cannot be continued for all $\lambda > 0$. Hence at some $\lambda_0 > 0$, $f - \lambda_0 \in A(M)$, and hence $f \in A^+$.

Turning to the convexity of $A^+$, assume that $f, g \in A^+ \cup A(M)$. We will show that for any $0 < \mu < 1$, $w \equiv \mu f + (1 - \mu)g \in A^+$. Indeed, defining $\overline{w} = \mu A^{-1}(f) + (1 - \mu)A^{-1}(g)$ and using convexity of $h(y)$, we see that $A(\overline{w}) \leq \mu f + (1 - \mu)g$. It follows that $\overline{w}$ is a subsolution of

$$A(u) = \mu f + (1 - \mu)g, \tag{3.19}$$

but not a solution of this equation. It follows that (3.19) has a $H^1$ solution above $\overline{w}$, so that $\mu f + (1 - \mu)g \in A^+ \cup A(M)$. It follows from the proof of Lemma 3.2 that $\mu f + (1 - \mu)g \notin A(M)$, completing the proof of the theorem.

*Remark.* If $f \in A(M)$ and $f \not\equiv 0$, then clearly $\int_0^p f(t)\, dt > 0$. On the other hand, multiplying (3.1) by $\mu(t) > 0$, as defined by (3.15), and using (3.9), we see that $\int_0^p f(t)\mu(t)\, dt < 0$; i.e., $f(t)$ cannot be a positive function. So that for positive $f(t)$ Eq. (3.1) has either none or two solutions, and in the case when there are two solutions, one is positive and increasing in $f$, while the other is negative and decreasing in $f$.

*Remark.* Under slightly more restrictive conditions McKean and Scovel [6] proved that the operator $A$ is diffeomorphic to a global fold of the form $(x_1, x_2, x_3, \ldots) \to (x_1^2, x_2, x_3, \ldots)$.

*Remark.* One can give a similar result for any quadratic nonlinearity by completing the square and changing the variables.

*Remark.* A different generalization of the McKean–Scovel theorem was given by Mawhin [5].

Turning to the cubic equations, we note that the quadratic term can be eliminated by a change of variables. We shall then consider the problem

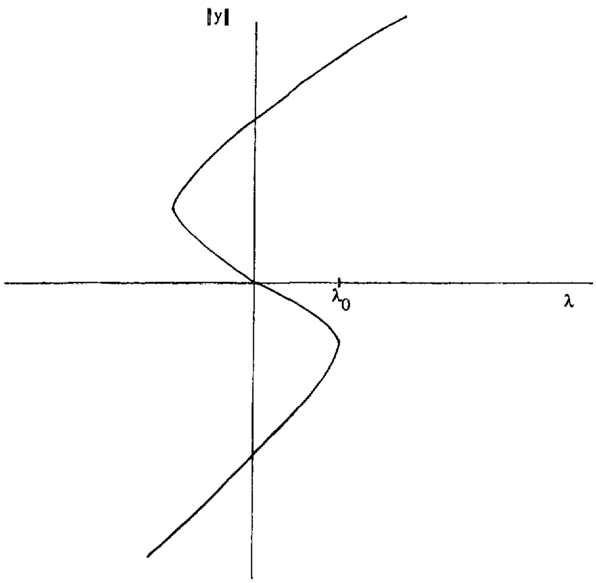


FIGURE 1

$$y' + y^3 - a(t)y = \lambda f(t). \tag{3.20}$$

Here $a(t)$ and $f(t)$ are continuous p-periodic functions and $\lambda$ is a real parameter. Since large positive (negative) constants are supersolutions (subsolutions) of (3.20) it follows that for any real $\lambda$ there is at least one p-periodic solution of (3.20). If $\int_0^p a(\tau)\, d\tau \leq 0$, then using Lemma 2.1, it is easy to show that such a solution is unique. We shall therefore assume that

$$\int_0^p a(\tau)\, d\tau > 0. \tag{3.21}$$

THEOREM 3.2. *For Eq.* (3.20) *assume that* (3.21) *holds and that*

$$f(t) > 0 \qquad \textit{for all } t. \tag{3.22}$$

*Then all p-periodic solutions of* (3.20) *lie on a unique S-shaped solution curve in the* $(y, \lambda)$ *"plane"* (*Fig.* 1). *Namely there is a* $\lambda_0 > 0$ *so that* (3.20) *has a unique p-periodic solution for* $|\lambda| > \lambda_0$, *exactly two p-periodic solutions for* $\lambda = \pm\lambda_0$, *and exactly three p-periodic solutions for* $|\lambda| < \lambda_0$. *Moreover, all p-periodic solutions do not change sign, and solution branches are mono-*

*tone in* $\lambda$ *(the exact number of positive (negative) solutions, and whether the branch is increasing (decreasing) in* $\lambda$ *can be read-off from Fig.* 1.)

*Proof.* We have a positive solution for any $\lambda > 0$, since we can take zero for a subsolution and large positive constants as supersolutions. Starting at some $\lambda > 0$ we can continue the positive solution for decreasing $\lambda$ using the implicit function theorem, provided the linearized equation

$$w' + (3y^2 - a(t))w = 0 \tag{3.23}$$

has only the trivial solution. The solutions of (3.20) stay bounded in the $H^1$ norm. Indeed, multiplying (3.20) by $y^3$ and integrating, we estimate $\int_0^p y^6\, d\tau$ and hence $\int_0^p y^2\, d\tau$. Then we estimate $\int_0^p y'^2\, d\tau$ from Eq. (3.20). Let $\lambda_0$ denote the infimum of $\lambda$'s for which the curve of positive solutions can be continued for decreasing $\lambda$. Passing to the limit in the integral form of (3.20), we establish the existence of a positive solution $y_0(t)$ corresponding to $\lambda_0$. Clearly $(\lambda_0, y_0)$ is a singular point of the map $F(\lambda, y) = y' + y^3 - a(t)y - \lambda f(t)$; i.e., the equation (3.23) at $y = y_0$ has nontrivial solutions. We show next that at $(\lambda_0, y_0)$ the Crandall–Rabinowitz theorem applies, i.e., $F_\lambda(\lambda_0, y_0) \notin R(F_y(\lambda_0, y_0))$. Assuming the last condition to be violated we would have $u(t) \in H^1$ solving

$$u' + (3y_0^2 - a(t))u = f(t). \tag{3.24}$$

If $\mu(t) > 0$ is a solution of

$$\mu' - (3y_0^2 - a(t))\mu = 0, \tag{3.25}$$

then by the Fredholm alternative

$$\int_0^p f(\tau)\mu(\tau)\, d\tau = 0. \tag{3.26}$$

Multiply Eq. (3.20) at $(\lambda_0, y_0)$ by $\mu(t)$ and integrate. Using (3.25) and (3.26) one obtains that $\int_0^p (-2y_0^3)\mu d\tau = 0$, which is a contradiction.

By the Crandall–Rabinowitz theorem near $(\lambda_0, y_0)$ solutions of (3.20) form a curve $(\lambda_0 + \tau(s), y_0 + sw + z(s))$ with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, and the parameter $s$ defined in some neighborhood of $s = 0$. We show next that

$$\tau''(0) > 0, \tag{3.27}$$

so that when $y_0(t) > 0$ only turns to the right in the $(\lambda, y)$ "plane" are possible. Differentiating Eq. (3.20) twice in $s$, we obtain at $y = y_0$

$$y'_{ss} + (3y_0^2 - a(t))y_{ss} + 6y_0 y_s^2 = \tau''(0) f. \tag{3.28}$$

Multiplying (3.25) by $y_{ss}$ and (3.28) by $\mu'$, adding these equations, and integrating, we obtain

$$\tau''(0) = \frac{\int_0^p 6y_0 y_s^2 \mu \, d\tau}{\int_0^p f\mu \, d\tau},$$

and the claim (3.27) follows.

The curve of positive solutions cannot be continued to the left for all $\lambda$. Indeed, for $\lambda < 0$ and $|\lambda|$ large, the integral over $(0, p)$ of the left-hand side of (3.20) is bounded from below, while the same integral of the right-hand side of (3.20) is large in absolute value and negative. Hence the curve of solutions will reach a critical point and a turn to the right occurs. Near $(\lambda_0, y_0)$ we have, by the Crandall–Rabinowitz theorem, two branches of solutions $y_-(t, \lambda) < y_+(t, \lambda)$, with the lower branch $y_-(t, \lambda)$ decreasing in $\lambda$ and the upper one $y_+(t, \lambda)$ increasing in $\lambda$ for $\lambda$ close to $y_0$ $(\lambda > \lambda_0)$. We claim that $y_-(t, \lambda)$ is decreasing in $\lambda$ for all $\lambda$ (a similar argument shows that $y_+(t, \lambda)$ is increasing in $\lambda$ for all $\lambda$). Differentiate (3.20) in $\lambda$:

$$y'_\lambda + (3y^2 - a)y_\lambda = f(t). \tag{3.29}$$

Let $\lambda_1$ be the first $\lambda$, where the monotonicity of $y_-(t, \lambda)$ is violated, i.e., at $y = y_-(t, \lambda_1)$, $y_\lambda \le 0$ for all $t$, but $y_\lambda(\bar{t}) = 0$ for some $\bar{t}$. Then at $t = \bar{t}$ the left-hand side of (3.29) is zero, while $f(\bar{t}) > 0$, a contradiction. A similar argument shows that the only way the branch $y_-(t, \lambda)$ can loose its positivity is by becoming identically zero at some $\lambda$.

We now return to the turning point $(\lambda_0, y_0)$. We claim that $\lambda_0 < 0$. Indeed if $\lambda_0 \ge 0$ (and $y_0 > 0$) then we will show that the branch $y_-(t, \lambda)$ has no place to go for increasing $\lambda > \lambda_0$. Since $y_-(t, \lambda)$ is decreasing in $\lambda$, it has to either become zero at some $\bar{\lambda} > 0$ or to tend to a non-negative solution of (3.20) as $\lambda \to \infty$. Both cases are clearly impossible. So that the turn occurs at some $\lambda_0 < 0$ and, by the argument just given, the branch $y_-(t, \lambda)$ has to enter $(0, 0)$. The rest of the S-shaped curve is obtained by symmetry: If $(\lambda, y(t))$ is a solution of (3.20), so is $(-\lambda, -y(t))$.

Next we exclude the possibility that (3.20) has solutions not lying on the S-shaped curve described above, which we call $S$. Note first that there can be no bifurcation off the S-shaped curve $S$. Indeed, this curve has only two critical (turning) points near which there are precisely two solutions, according to the Crandall–Rabinowitz theorem. Next we appeal to the theorem of Neto and Smale [7], which implies that Eq. (3.20) has at most three p-periodic solutions. Hence any solution of (3.20) which is not on $S$ has to lie in the region $|\lambda| \geq \lambda_0$. Assume for definiteness that there is such a solution $z(t)$ for $\lambda > \lambda_0$, with the other case being similar. If $z(t)$ is positive, then repeating our previous arguments, we will have another S-shaped curve of solutions passing through $(\lambda, z(t))$ and through $(0, 0)$. But by the implicit function theorem we have local uniqueness of solutions near $(0, 0)$, a contradiction. If $z(t)$ is negative or changing sign, we continue this solution for decreasing $\lambda$, using the implicit function theorem. Since this curve of solutions cannot enter the region $|\lambda| < \lambda_0$, it must reach a critical (turning) point, where the Crandall–Rabinowitz theorem applies, producing two branches of solutions. The lower branch, denoted by $z_-(t, \lambda)$ is decreasing in $\lambda$ for all $\lambda$. Let $\bar{t}$ be its point of minimum, $\bar{t} = \bar{t}(\lambda)$. Then for $\lambda$ large the right-hand side of (3.20) is large at $t = \bar{t}$, while the left-hand side is not. This contradiction finishes the proof of the theorem.

*Remark.* A result of similar nature appears in Lloyd [4]. However, as the author himself pointed out, that result had a serious drawback: a condition on a certain set $D$, defined in that paper. Moreover, the result in [4] did not discuss how solutions for different $\lambda$ are connected and the monotonicity of the branches.

## 4. Equations with at Most Three Solutions

In [9] Sandqvist and Andersen proved that if $f(t, x)$ is p-periodic in $t$, and the following condition is satisfied,

$$f_{xxx}(t, x) > 0 \quad \text{for all real } t \text{ and } x, \tag{4.1}$$

then the equation

$$\dot{x} = f(t, x) \tag{4.2}$$

has at most three p-periodic solutions (see also [4]). The purpose of this short section is to give a generalization and a simple proof of this result. We begin with a special case,

$$f(t, 0) \equiv 0 \quad \text{for all } t \in R, \tag{4.3}$$

then we generalize.

LEMMA 4.1. *Assume that the function $f(t, x)$ in* (4.2) *is twice differentiable in $x$ and continuous in $t$; it satisfies* (4.3) *and*

$$Q(t, z) \equiv 2f(t, z) - 2zf_z(t, z) + z^2 f_{zz}(t, z) > 0 \quad \text{for all } t \in R \text{ and } z > 0, \text{ and the same or the opposite inequality holds for } z < 0. \tag{4.4}$$

*Then the problem* (4.2) *has at most two positive* (*or negative*) *p-periodic solutions.*

*Proof.* Setting $x = 1/y$, we transform (4.2) into

$$-\dot{y} = y^2 f\left(t, \frac{1}{y}\right) \equiv g(t, y). \tag{4.5}$$

There is clearly a one-to-one correspondence between the positive (and negative) solutions of (4.2) and (4.5). Compute

$$g_{yy}(t, y) = 2f\left(t, \frac{1}{y}\right) - \frac{2}{y} f_x\left(t, \frac{1}{y}\right) + \frac{1}{y^2} f_{xx}\left(t, \frac{1}{y}\right) > 0,$$

by the assumption (4.4). Applying Lemma 2.2, we conclude that both (4.5) and (4.2) have at most two positive (negative) p-periodic solutions.

*Remark.* Condition (4.4) (and also condition (4.8) below) is more general than (4.1). Indeed, assuming (4.1) to hold we see that

$$Q(t, 0) \equiv 0 \tag{4.6}$$

and

$$\frac{\partial}{\partial z} Q(t, z) = z^2 f_{zzz}(t, z) > 0 \qquad \text{for } z \neq 0, \tag{4.7}$$

and, hence, $zQ(t, z) > 0$ for $z \neq 0$, and (4.4) follows. On the other hand, there are many functions satisfying (4.4) but not (4.1). Indeed, given a positive $Q(t, z)$, we can easily solve (4.4) for $f(t, z)$, and $f_{zzz}$ will have as many sign changes as $Q_z(t, z)$, as follows from (4.7).

THEOREM 4.1. *Assume that the function $f(x, t) \in C^{2,0}(R \times R)$ is p-periodic in $t$ and satisfies*

$$2[f(t, z + \xi) - f(t, \xi)] - 2zf_z(t, z + \xi) + z^2 f_{zz}(t, z + \xi) > 0 \quad \text{for all real } t \text{ and } \xi \text{ and } z > 0, \text{ and the same or the opposite inequality holds when } z < 0. \tag{4.8}$$

*Then the problem* (4.2) *has at most three p-periodic solutions.*

*Proof.* Let $\xi = \xi(t)$ be any solution of (4.2). Setting $z(t) = x(t) - \xi(t)$, we obtain an equation

$$\dot{z} = f(t, z + \xi) - f(t, \xi) \equiv F(t, z),$$

to which the previous lemma applies. It follows that any periodic solution of (4.2) cannot have more than two other periodic solutions either below it or above. The theorem follows.

*Remark.* Similarly to [9], we can generalize the condition (4.1) by requiring only that $Q(t, x) \geq 0$ for all real $x$ and $t$ and $Q(t_0, x) > 0$ for all $x \in R$ and some $t_0$, and the same result holds for the opposite inequalities. Also, our results translate word for word for closed solutions, i.e., when one requires that $x(0) = x(\ell)$ on some interval $(0, \ell)$, without assuming periodicity of $f$.

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