Exact Multiplicity Results for a Class of Boundary-Value Problems with Cubic Nonlinearities

PHILIP KORMAN*

Institute for Dynamics and Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221-0025

AND

TIANCHENG OUYANG

Department of Mathematics, Brigham Young University, Provo, Utah 84602

We give an exact multiplicity result for a class of boundary-value problems for cubic nonlinearities with an explicit x dependence. Moreover, we provide a detailed analysis of global solution curves. © 1995 Academic Press, Inc.

1. INTRODUCTION

We present exact multiplicity results and a detailed study of the solution branches for the Dirichlet problem

$$u'' + \lambda u(u - a(x))(b(x) - u) = 0 \quad \text{on } (-1, 1), \ u(-1) = u(1) = 0. \quad (1.1)$$

Here λ is a positive parameter and the functions $a(x)$ and $b(x)$ are assumed to be even with $0 < a(x) < b(x)$ for all x. Under additional assumptions on $a(x)$ and $b(x)$ we prove that there is a critical $\lambda_0 > 0$ so that the problem (1.1) has either zero, one, or two nontrivial solutions depending on whether λ is smaller, equal to, or larger than λ_0. Note that by the maximum principle any nontrivial solution is in fact positive. Moreover, we show that all solu-

* E-mail address: korman@ucbeh.san.uc.edu.

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tions of (1.1) lie on a single smooth solution curve, and study behavior of both branches of this curve as $\lambda \to \infty$. We remark that for constant a and b the multiplicity result is known; see S.-H. Wang [11], where one can find references to earlier papers, and [5], where a simpler proof is given.

We use techniques from bifurcation theory, particularly a theorem of M. G. Crandall and P. H. Rabinowitz [1], which is stated in the next section. Our assumptions can be roughly summarized as follows. We assume that the even functions $a(x)$ and $b(x)$ for x positive satisfy $a'(x) > 0$, $b'(x) < 0$, $b''(x) < 0$, $a'''(x) + b'''(x) \leq 0$, $a' + b' < 0$, $(ab)' > 0$ and that the variations of $a(x)$ and $b(x)$ are not large relative to $b - a$. There are many functions satisfying these conditions, and the result should be contrasted with a nontrivial case of constant a and b.

2. PRELIMINARY RESULTS

We consider boundary-value problems of the type

$$u'' + f(x, u) = 0 \quad \text{for } x \in (-1, 1), \ u(-1) = u(1) = 0. \quad (2.1)$$

We shall need the following lemma from P. Korman and T. Ouyang [3]. (Except for the last assertion, this lemma is also included in B. Gidas, W.-M. Ni, and L. Nirenberg [2].)

LEMMA 2.1. *Assume that the function $f \in C^1([0, 1] \times R_+)$ is such that*

$$f(-x, u) = f(x, u) \quad \text{for all } x \in (-1, 1) \text{ and } u > 0, \quad (2.2)$$

$$xf_x(x, u) < 0 \quad \text{for } x \in (-1, 1)\backslash\{0\} \text{ and } u > 0. \quad (2.3)$$

Then any positive solution of (2.1) is an even function with $u'(x) < 0$ on $(0, 1)$. Moreover any two positive solutions of (2.1) cannot intersect on $(-1, 1)$ (and hence they are strictly ordered on $(-1, 1)$).

Remark 2.2. If in addition $f(x, 0) = 0$ for all $x \in (-1, 1)$, then $u'(x) < 0$ on $(0, 1]$. Indeed, since $u \equiv 0$ is also a solution of (2.1), the possibility that $u'(1) = 0$ is excluded by the uniqueness theorem for initial-value problems.

A linearized problem corresponding to (2.1),

$$w'' + f_u(x, u)w = 0 \quad \text{on } (-1, 1), \ w(-1) = w(1) = 0, \quad (2.4)$$

will be used often $(w = w(x))$. Clearly $w(x)$ is an even function (since $w(-x)$ is also a solution of (2.4)).

LEMMA 2.3. *Under the conditions (2.2) and (2.3), if a nontrivial solution*

of (2.4) *exists, it does not change sign on* $(-1, 1)$, *i.e., we can choose it so that* $w(x) > 0$ *on* $(-1, 1)$.

Proof. Assume that $w(x)$ changes sign on $(-1, 1)$. Assume that $w(x)$ has a zero on $(-1, 0]$, and let $\xi \in (-1, 0]$ be the smallest root of $w(x)$ (the case when $w(x)$ has a zero on $(0, 1)$ is similar).

Differentiating (2.1), we obtain

$$u''_x + f_u u_x + f_x = 0. \tag{2.5}$$

Lemma 2.1 implies that $u'(x) > 0$ on $(-1, 0)$. Next, we multiply Eq. (2.4) by u' and Eq. (2.5) by w, integrate, and subtract. Obtain

$$u'(\xi)w'(\xi) - u'(-1)w'(-1) - \int_{-1}^{\xi} f_x w \, dx = 0. \tag{2.6}$$

Since all three terms on the left are negative, we have a contradiction.

The following lemma is needed to verify the condition $F_\lambda \notin R(F_u)$ of the Crandall–Rabinowitz theorem, and to compute the direction of bifurcation. It generalizes Lemma 2.1 in [5].

LEMMA 2.4. *Let $u(x)$ be a solution of* (2.1) *and assume there exists a nontrivial solution $w(x)$ of* (2.4), *and let all conditions of Lemma 2.1 be satisfied. Then*

$$\int_0^1 f(x, u)w(x) \, dx > 0. \tag{2.7}$$

Proof. Multiply Eq. (2.4) by u_x and Eq. (2.5) by w and subtract. Obtain

$$(u'w' - wu'')' = wf_x < 0 \qquad \text{for all } x \in (0, 1).$$

Since the function $u'w' - wu''$ is decreasing

$$u'(x)w'(x) - w(x)u''(x) > u'(1)w'(1) \geq 0 \qquad \text{for all } x \in (0, 1). \tag{2.8}$$

Integrating (2.8) yields

$$0 < \int_0^1 (u'w' - wu'') \, dx = -2 \int_0^1 wu'' \, dx = 2 \int_0^1 f(x, u)w \, dx.$$

A word on notation. We shall denote derivatives of $u(x)$ by either $u'(x)$

or u_x and mix both notations to make our proofs more transparent (u'_x will denote the second derivative of $u(x)$, when convenient).

Next we list some background results. The following result is standard.

LEMMA 2.5. *Let $\gamma(x)$ and $\psi(x)$ be respectively super- and subsolutions of (2.1), and $\gamma(x) \geq \psi(x)$ on $(-1, 1)$, with $\gamma(x) \not\equiv \psi(x)$; then $\gamma(x) > \psi(x)$ on $(-1, 1)$.*

We shall often use this lemma with either $\gamma(x)$ or $\psi(x)$ or both as solutions of (2.1). The following lemma is a consequence of the first.

LEMMA 2.6. *Let $u(x)$ be a nontrivial solution of (2.1) with $f(x, 0) \equiv 0$. If $u(x) \geq 0$ on $(-1, 1)$, then $u > 0$ on $(-1, 1)$.*

Next we state a bifurcation theorem of Crandall and Rabinowitz [1].

THEOREM 2.1 [1]. *Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \mathrm{span}\{x_0\}$ be one-dimensional and $\mathrm{codim}\, R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\mathrm{span}\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

3. MULTIPLICITY RESULTS FOR CUBIC EQUATIONS WITH VARIABLE ROOTS

We study exact multiplicity results for a class of Dirichlet problems with cubic nonlinearities,

$$u'' + \lambda u(u - a(x))(b(x) - u) = 0 \qquad \text{for } x \in (-1, 1) \tag{3.1}$$

$$u(-1) = u(1) = 0, \tag{3.2}$$

depending on a parameter $\lambda > 0$. We assume that

$a(x)$ and $b(x)$ are even functions of class $C^2(-1, 1) \cap C^0[-1, 1]$, with

$$0 < a(x) < b(x) \qquad \text{for all } x \in (-1, 1). \tag{3.3}$$

We shall denote $f(x, u) = u(u - a(x))(b(x) - u) \equiv -u^3 + \alpha(x)u^2 - \beta(x)u$, with $\alpha = a + b$, $\beta = ab$. We assume that the even functions $\alpha(x)$ and $\beta(x)$ satisfy the conditions

$$\alpha'(x) < 0 \qquad \text{for } x \in (0, 1). \tag{3.4}$$

$$\beta'(x) > 0 \qquad \text{for } x \in (0, 1). \tag{3.5}$$

By Lemma 2.1 any positive solution of (3.1), (3.2) is an even function with $u'(x) < 0$ for $x > 0$.

We shall also need the linearization of (3.1),

$$w'' + \lambda f_u(x, u)w = 0 \qquad \text{on } (-1, 1), w(-1) = w(1) = 0. \tag{3.6}$$

By Lemma 2.3 we can assume that $w(x) > 0$ on $(-1, 1)$.

Since max $b(x)$ is a supersolution of (3.1)–(3.2) it follows that if this problem has positive solutions, it has a maximal solution. We continue the curve of maximal solutions for decreasing λ using the implicit function theorem. When the implicit function theorem does not apply, i.e., the problem (3.6) has a positive solution, we can use the Crandall–Rabinowitz theorem. Indeed, the crucial condition of Theorem 2.1, $F_\lambda \notin R(F_u)$, follows by application of Lemmas 2.3 and 2.4. Once we show that only turns "to the right" are possible, we will obtain an exact multiplicity result. At a turning point (λ_0, u_0) we have $\lambda'(0) = 0$ and

$$\lambda''(0) = -\lambda_0 \frac{\int_0^1 f_{uu}(x, u)w^3(x)\, dx}{\int_0^1 f(x, u)w(x)\, dx}. \tag{3.7}$$

To prove that $\lambda''(0) > 0$ we need, in view of Lemma 2.4, to show that

$$\int_0^1 f_{uu}w^3\, dx < 0. \tag{3.8}$$

Differentiate Eq. (3.1) twice,

$$u_x'' + \lambda f_u u_x + \lambda f_x = 0, \tag{3.9}$$

$$u_{xx}'' + \lambda f_u u_{xx} + \lambda f_{uu} u_x^2 + 2\lambda f_{ux} u_x + \lambda f_{xx} = 0. \tag{3.10}$$

Multiply Eq. (3.10) by w, and subtract from it Eq. (3.6) multiplied by u_{xx}, and then integrate over $(0, 1)$,

$$wu_{xx}'|_0^1 - u_{xx}w'|_0^1 + \lambda \int_0^1 f_{uu}u_x^2 w\, dx$$

$$+ \lambda \int_0^1 (2f_{ux}u_x + f_{xx})w\, dx = 0. \tag{3.11}$$

We have $u'''(0) = 0$, since $u'''(x)$ is odd, and $u''(1) = -\lambda f(1, u(1)) = -\lambda f(1, 0) = 0$. Hence all the boundary terms in (3.12) vanish and then

$$\int_0^1 f_{uu} u_x^2 w\, dx + \int_0^1 (2f_{ux}u_x + f_{xx})w\, dx = 0. \tag{3.12}$$

As in [5] our goal will be to show that the second integral in (3.12), $I \equiv \int_0^1 (2f_{ux}u_x + f_{xx})w\, dx$, is positive. This will imply that

$$\int_0^1 f_{uu} u_x^2 w\, dx < 0, \tag{3.13}$$

and the last inequality will be used to prove (3.8).

LEMMA 3.1. *In addition to the conditions (3.4)–(3.5) assume that*

$$\alpha'''(x) \leq 0 \qquad \text{for all } x \in (0, 1), \tag{3.14}$$

$$\alpha(x) - \sqrt{\alpha^2(x) - 3\beta(x)} < \alpha(1) \qquad \text{for all } x \in (0, 1), \tag{3.15}$$

$$\frac{1}{2}\alpha(0) < \frac{\alpha(x) + \sqrt{\alpha^2 - 3\beta(x)}}{3} \qquad \text{for all } x \in (0, 1). \tag{3.16}$$

Then the solution of (3.1)–(3.2) intersects each of the functions $\frac{1}{2}\alpha(x)$ and $\frac{1}{3}\alpha(x)$ exactly once on (0, 1).

Remark.   It is interesting to examine the conditions (3.15) and (3.16) in the case when $a(x)$ and $b(x)$ are constant functions. The inequality (3.15) then holds trivially, while (3.16) is equivalent to $(b - a)^2 > 0$.

Proof.   We begin by showing that $u(x)$ must intersect the function $\frac{1}{2}\alpha(x)$ (and hence also $\frac{1}{3}\alpha(x)$) at least once. Indeed, assuming otherwise we would have $u(x) < \frac{1}{2}\alpha(x)$ for all $x \in (0, 1)$, and then

$$f_u(x, u) - \frac{f(x, u)}{u} = u(-2u + \alpha) > 0 \qquad \text{for all } x \in (-1, 1).$$

Comparing now the problems (3.1)–(3.2) and (3.6), we see, in view of Sturm's comparison theorem, that it is impossible for both of them to have positive solutions, a contradiction.

To see that $u(x)$ and $\frac{1}{2}\alpha(x)$ intersect exactly once on (0, 1), we introduce $p(x) = \frac{1}{2}\alpha(x) - u$. Compute

$$p'''(x) = \frac{1}{2}\alpha''' - (u'')' = \frac{1}{2}\alpha''' + \lambda f_u u' + \lambda f_x. \tag{3.17}$$

The first and third terms on the right are negative by our assumptions. Let

x_1 and x_2 denote respectively the smallest and largest points of intersection of $\frac{1}{2}\alpha(x)$ and $u(x)$ on $(0, 1)$. On (x_1, x_2)

$$\tfrac{1}{2}\alpha(1) < \tfrac{1}{2}\alpha(x_2) < u(x) < \tfrac{1}{2}\alpha(x_1) < \tfrac{1}{2}\alpha(0).$$

Denote by u_1 and u_2 the roots of f_u, $u_{1,2} = (\alpha(x) \pm \sqrt{\alpha^2(x) - 3\beta(x)})/3$. The function f_u is positive between its roots. Conditions (3.15) and (3.16) imply that $(\frac{1}{2}\alpha(1), \frac{1}{2}\alpha(0)) \subset (u_1, u_2)$ for all $x \in (x_1, x_2)$. Hence the second term in (3.17) is also negative, and we conclude that

$$p'''(x) < 0 \qquad \text{for } x \in (x_1, x_2). \tag{3.18}$$

Note that $p(1) = \frac{1}{2}\alpha(1) > 0$. Assume now that there is exactly one zero of $p(x)$ in (x_1, x_2), i.e., $p(\xi) = 0$ and $\xi \in (x_1, x_2)$. Then somewhere on (x_1, ξ) we would have $p'' < 0$, and somewhere on (ξ, x_2): $p'' > 0$. But this contradicts the fact that $p''(x)$ is a decreasing function, in view of (3.18). If $p(x)$ has more than one zero in (x_1, x_2), one obtains a similar contradiction. The final possibility is that $p(x)$ has no more zeroes inside (x_1, x_2). The function $p(x)$ must change signs at x_2 and then at x_1, since the other possibilities lead to the same contradiction as that above. Hence $p(x) > 0$ on $(0, x_1)$, i.e., $u(x) < \frac{1}{2}\alpha(x) < \frac{1}{2}\alpha(0)$ on $(0, x_1)$. Hence $f_u > 0$ for $x \in (0, x_2)$, and so $p''' < 0$ on $(0, x_2)$. Also, $p'(0) = 0$. Somewhere on $(0, x_1)$, $p(x)$ is concave, while somewhere on (x_1, x_2) it is convex, which again leads to the same contradiction.

To show that $\frac{1}{2}\alpha(x)$ and $u(x)$ intersect exactly once, we denote $p(x) = \frac{1}{2}\alpha(x) - u(x)$ and proceed similarly.

LEMMA 3.2.   *Under the conditions* (3.4)–(3.5) *and* (3.14)–(3.16) *the inequality* (3.13) *holds.*

Proof.   Integrating by parts, we express

$$I = \int_0^1 \left(f_{ux} u_x + \frac{d}{dx} f_x \right) w \, dx = \int_0^1 f_{ux} u' w \, dx - \int_0^1 f_x w' \, dx, \tag{3.19}$$

using the fact that $f_x w |_0^1 = 0$ (because f_x is odd). Since $f_{ux} = 2\alpha' u - \beta'$, $f_x = \alpha' u^2 - \beta' u$, we express

$$I = \int_0^1 \alpha'(2uu'w - u^2w') \, dx - \int_0^1 \beta'(u'w - uw') \, dx \equiv I_1 + I_2. \tag{3.20}$$

We claim that

$$I_1 = \int_0^1 \alpha'(2uu'w - u^2w')\, dx > 0. \tag{3.21}$$

If we denote $J_1 = 2uu'w - u^2w'$, it suffices to show that $J_1 < 0$. Note that

$$J_1(0) = J_1(1) = 0. \tag{3.22}$$

Differentiate J_1 and use (3.1) and (3.6) to express the second derivatives

$$\begin{aligned}
J_1' &= 2u'^2w + 2uu''w - u^2w'' \\
&= 2u'^2w + \lambda w(-u^4 + \beta u^2).
\end{aligned} \tag{3.23}$$

It follows from the previous lemma that the graphs of $u(x)$ and $\sqrt{\beta(x)}$ intersect at least once, and since $\beta(x)$ is increasing, they intersect exactly once. Let ξ be the point where $u(\xi) = \sqrt{\beta(\xi)}$. Then from (3.23) we see that $J_1' > 0$ and $(\xi, 1)$, and hence by (3.22), $J_1(x) < 0$ and $[\xi, 1)$, and in particular

$$J_1(\xi) < 0. \tag{3.24}$$

We now differentiate J_1'/w using (3.23), and again expressing u'' from (3.1),

$$\left(\frac{J_1'}{w}\right)' = \lambda u'u(6\beta - 4\alpha u). \tag{3.25}$$

On the interval $(0, \xi)$ we have $u(x) > \sqrt{\beta(x)}$, and then

$$4\alpha u > 4\alpha\sqrt{\beta} = 4(a + b)\sqrt{ab} > 6ab = 6\beta,$$

which implies in view of Lemma 2.1 that the right hand side of (3.25) is positive. But then

$$\frac{J_1''w - J_1'w'}{w^2} > 0 \qquad \text{on } (0, \xi).$$

Applying the maximum principle on $(0, \xi)$, and using (3.22) and (3.24), we conclude that $J_1(x) < 0$ on $(0, 1)$, and hence $I_1 > 0$.

We now turn to the integral I_2, denoting $J_2 = u'w - uw'$. Clearly,

$$J_2(0) = J_2(1) = 0. \tag{3.26}$$

Differentiate $J_2(x)$, using (3.1) and (3.6),

$$J_2' = u''w - uw'' = \lambda u^2 w(-2u + \alpha).$$

By Lemma 3.1 we know that J_2' is positive near $x = 1$, negative near $x = 0$, and vanishes only once. Together with (3.26) this implies that $J_2(x) < 0$ on $(0, 1)$, which finishes the proof of the lemma.

LEMMA 3.3.   *Under the conditions (3.4)–(3.5) the functions w^2 and u_x^2 intersect exactly once on $(0, 1)$.*

Proof.   It suffices to prove that $w(x)$ and $-u_x$ intersect exactly once. Recall that our conditions imply that $f_x < 0$ for $x \in (0, 1)$ and $u > 0$. It follows that $-u_x$ is a supersolution of (3.6). Assuming that $-u_x$ and $w(x)$ intersect more than once, we conclude existence of two intersection points $x_1 < x_2$, so that $-u_x < w(x)$ on (x_1, x_2) (since $-u_x(0) < w(0)$ and $-u_x(1) > w(1)$). Since $w(x)$ is solution of a linear equation (3.6), any multiple of $w(x)$ is also a solution of (3.6). For $0 < \gamma < 1$ sufficiently small $\gamma w(x) < -u_x$ on (x_1, x_2). It follows that there is a $0 < \gamma_0 < 1$, such that $\gamma_0 w(x) \leq -u_x$ on (x_1, x_2) and $\gamma_0 w(\bar{x}) = -u_x(\bar{x})$ for some $\bar{x} \in (x_1, x_2)$. But that is impossible in view of Lemma 2.5.

Next we remark that the conditions (3.4)–(3.5) imply that

$$a'(x) > 0 \quad \text{and} \quad b'(x) < 0 \quad \text{for } x > 0. \tag{3.27}$$

Indeed, using (3.4) in (3.5) we conclude that

$$0 < a'b + ab' < b'(a - b),$$

which implies that $b' < 0$. But then from (3.5)

$$a'b > -ab' > 0,$$

and so $a' > 0$.

We shall also assume that

$$b''(x) < 0 \quad \text{for } x \in (-1, 1). \tag{3.28}$$

We show next that this condition implies that any solution of (3.1)–(3.2) satisfies

$$u(x) < b(x) \qquad \text{for all } x \in (-1, 1). \tag{3.29}$$

Indeed, denoting $v = b(x) - u$, we express

$$b'' - v'' + (b - v)(b - a - v)v = 0 \qquad \text{on } (-1, 1),$$
$$v(\pm 1) = b(\pm 1) > 0. \tag{3.30}$$

If (3.29) were violated, we would obtain a contradiction in (3.30) at the point of a non-positive minimum of $v(x)$. Also, we note that any nontrivial solution of (3.1)–(3.2) is positive by the maximum principle.

We now state our main result.

THEOREM 3.1. *For the problem* (3.1)–(3.2) *assume that the conditions* (3.4)–(3.5), (3.14)–(3.16), *and* (3.28) *are satisfied. Then only two possibilities can occur:*

(A) *The problem* (3.1)–(3.2) *has no nontrivial solution for any* $\lambda > 0$.

(B) *There is a* $\lambda_0 > 0$ *so that the problem* (3.1)–(3.2) *has either zero, one, or two solutions depending on whether* $\lambda < \lambda_0$, $\lambda = \lambda_0$, *or* $\lambda > \lambda_0$, *respectively. Moreover, all solutions are even functions and lie on a single* $\subset$-*like curve. Solutions on the lower branch tend to zero over* $(-1, 1)\backslash\{0\}$, *and moreover the maximum value of solutions on the lower branch decreases monotonously.*

Proof. If there is a nontrivial (positive) solution of (3.1)–(3.2) then, as described previously, we continue this solution for decreasing λ until a turning point (λ_0, u_0) is reached, i.e., there exists a solution $w(x) > 0$ of (3.6). At this point the Crandall–Rabinowitz theorem applies. Indeed, define a map $F: C^2(-1, 1) \times R_+ \to C^0(-1, 1)$ by $F(\lambda, u) = u'' + \lambda f(x, u)$. Then the crucial condition $F_\lambda \notin R(F_u)$ is equivalent to checking that the problem

$$z'' + \lambda_0 f_u(x, u_0)z = f(x, u_0) \qquad \text{on } (-1, 1), \, z(-1) = z(1) = 0$$

has no solution. Since by Lemma 2.4, $\int_{-1}^{1} f(x, u_0)w \, dx \neq 0$, this is clearly the case by the Fredholm alternative.

Applying the Crandall–Rabinowitz theorem, we conclude that (λ_0, u_0) is a bifurcation point, near which the solutions of (3.1)–(3.2) form a curve $(\lambda_0 + \tau(s), u_0 + sw + z(s))$ with s near $s = 0$, and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^{1} f_{uu}(x, u_0)w^3 \, dx}{\int_{-1}^{1} f(x, u_0)w \, dx} = -\lambda_0 \frac{\int_0^1 f_{uu}w^3 \, dx}{\int_0^1 fw \, dx}. \tag{3.31}$$

For completeness we include next the derivation of (3.31). Differentiate
(3.1) in s twice, set $s = 0$, and use the fact that $\tau'(0) = 0$ and $u_s|_{s=0} = w(x)$,

$$u_{ss}'' + \lambda_0 f_{uu} w^2 + \lambda_0 f_u u_{ss} + \tau''(0)f = 0. \tag{3.32}$$

Multiplying (3.32) by w and (3.6) by u_{ss}, integrating, and subtracting, we
obtain (3.31).

By Lemma 2.4 the denominator in (3.31) is positive. We claim that

$$\int_0^1 f_{uu} w^3 \, dx < 0. \tag{3.33}$$

Since $f_{uu} = -6u + 2\alpha$, it follows by Lemma 3.1 that there is a point $x_0 \in$
$(0, 1)$, such that $f_{uu} < 0$ on $(0, x_0)$ and $f_{uu} > 0$ on $(x_0, 1)$. By Lemma 3.3
the point of intersection of w^2 and u_x^2 is also unique. By scaling $w(x)$ we
can make w^2 and u_x^2 to intersect also at x_0. But then by Lemma 3.2

$$\int_0^1 f_{uu} w^3 \, dx < \int_0^1 f_{uu} w u_x^2 \, dx < 0$$

(f_{uu} is positive where $w^2 < u_x^2$ and negative where the opposite inequality
holds).

We conclude that $\lambda''(0) > 0$, and so at any bifurcation point a "turn to the
right" occurs in the (λ, u) "plane." By the Crandall–Rabinowitz theorem we
have two solution branches in the neighborhood of (λ_0, u_0) which we denote
by $u_+(x, \lambda)$ and $u_-(x, \lambda)$, and $u_-(x, \lambda) < u_+(x, \lambda)$ for λ close to λ_0 and all
$x \in (-1, 1)$. Moreover, $u_+(x, \lambda)$ is increasing in λ for λ close to λ_0, and
$u_-(x, \lambda)$ is similarly decreasing in λ.

To recapitulate, we followed a curve of nontrivial solutions for decreasing
λ until a turning point was reached, and a "turn to the right" in the (λ, u)
"plane" occurred. Since at any critical solution (λ, u) the Crandall–
Rabinowitz theorem applies with a turn to the right, it follows that there
are no more turning points on the branches $u_\pm(x, \lambda)$, and hence both
branches can be continued for all $\lambda > \lambda_0$, giving us a parabola-like curve
of solutions.

Next we justify our claims about the lower branch $u_-(x, \lambda)$. We claim
that $u(0, \lambda)$ is decreasing on the lower branch. We adapt a similar argument
from [5]. We begin by noting that for any $x_1 \in (0, 1)$,

$$\lambda \int_{u(x_1)}^{u(0)} f(x(u), u) \, du = \frac{1}{2} u'^2(x_1) > 0, \tag{3.34}$$

where $x = x(u)$ is the inverse of the solution $u(x)$ (just multiply (3.1) by u' and integrate). Differentiate (3.1) in λ, denoting $(\partial/\partial\lambda)u_-(x, \lambda) = u_\lambda$,

$$u_\lambda'' + \lambda f_u u_\lambda + f(x, u) = 0 \qquad \text{on } (-1, 1), \ u_\lambda(-1) = u_\lambda(1) = 0. \quad (3.35)$$

By the Crandall–Rabinowitz theorem we know that $u_\lambda(x, \lambda) < 0$ for λ close to λ_0 and all $x \in (-1, 1)$. Assuming the claim to be false, let λ_1 be the smallest λ where $u_\lambda(0, \lambda_1) = 0$. From (3.34) we conclude that $f(0, u(0, \lambda_1)) > 0$, and then we see from (3.35) that $u_\lambda''(0, \lambda_1) < 0$. This implies that $x = 0$ is not a point of minimum of $u_\lambda(x, \lambda_1)$, and then we may assume that $u_\lambda(x, \lambda_1)$ is negative for x positive and close to zero (the other case is similar). Let $0 < x_1 \leq 1$ be the zero of $u_\lambda(x, \lambda_1)$ adjacent to $x = 0$, i.e.,

$$u_\lambda'(x_1, \lambda_1) \geq 0. \qquad (3.36)$$

Multiply (3.35) by u_x and (3.9) by u_λ, integrate from 0 to x_1, and subtract

$$(u_\lambda' u_x - u_x' u_\lambda)\big|_0^{x_1} + \int_{u(0)}^{u(x_1)} f(x(u), u) \, du - \lambda \int_0^{x_1} f_x u_\lambda \, dx = 0. \qquad (3.37)$$

The first term in (3.37) is equal to $u_\lambda'(x_1)u'(x_1) \leq 0$, the second is negative by (3.34), and the third is negative by our assumptions. The resulting contradiction proves that $u_-(0, \lambda)$ is decreasing.

We show next that $\lim_{\lambda \to \infty} u_-(x, \lambda) = 0$ for $x \in (-1, 1)\backslash\{0\}$. Let $I_\lambda = \{x : u(x) > a(x)\}$. We claim that $I_\lambda \to \{0\}$ as $\lambda \to \infty$. Indeed, assume that $I_{\lambda_n} \supset (-c, c)$ for some $c > 0$ and some sequence $\lambda_n \to 0$. Since the shapes of $u(x)$ and $a(x)$ are different, we would have $u(x, \lambda_n) > a(x) + \varepsilon$ over $(0, c - \delta)$, and then $u''(x, \lambda_n)$ would tend to $-\infty$ over a fixed interval, which is impossible for a bounded solution. One shows similarly that $\lim_{\lambda \to 0}\{x : \varepsilon < u(x) < a(x)\} = \varnothing$ for any $\varepsilon > 0$, which shows that $u_-(x, \lambda)$ tends to zero for $x \neq 0$.

We show next that the problem (3.1)–(3.2) has no other solutions. Indeed, by the same analysis, any other solution would have to lie on a similar parabola-like curve of solutions, with a lower branch $v_-(x, \lambda)$ tending to zero. Arguing as in [5], we obtain a contradiction, which completes the proof of the theorem.

It is easy to give a condition ensuring existence of a positive solution of (3.1)–(3.2) for some $\lambda > 0$, thus obtaining an exact multiplicity result. With $f(x, u)$ denoting, as before, the nonlinearity in (3.1), define $F(x, u) = \int_0^u f(x, z) \, dz$.

THEOREM 3.2. *In addition to the conditions of Theorem* 3.1 *assume that*

$$\int_0^1 F(x, b(x))\, dx > 0.$$

Then the possibility (B) *of Theorem* 3.1 *holds. If moreover*

$$F(x, b(x)) > 0 \qquad for\ all\ x \in (-1, 1), \tag{3.38}$$

then the upper branch tends to $b(x)$ *over* $(-1, 1)$ *as* $\lambda \to \infty$.

Proof. Solutions of (3.1)–(3.2) are critical points in $H_0^1(-1, 1)$ of the functional

$$J(u) = \int_{-1}^1 \left[\frac{1}{2} u'^2 - \lambda F(x, u) \right] du.$$

By modifying the function $b(x)$ near ± 1 we obtain a function $u_0(x) \in H_0^1(-1, 1)$, such that $\int_0^1 F(x, u_0(x))\, dx > 0$. But then $J(u_0) < 0$ for λ sufficiently large. It follows that the functional J has a negative minimum which gives a nontrivial solution of (3.1)–(3.2) and rules out the possibility (A) of Theorem 3.1.

Turning to the upper branch, we know that it increases in λ near λ_0, and by the same argument that we used for the lower branch, it follows that for $u(x, \lambda) = u_+(x, \lambda)$ we have $u_\lambda(0, \lambda) > 0$ for all $\lambda > \lambda_0$. Assume that $u_+(x, \lambda)$ does not converge to $b(x)$ for $x \in (-1, 1)$ as $\lambda \to \infty$. Arguing as in Theorem 3.1, we conclude existence of $0 < \alpha < 1$, such that $u_+(x, \lambda)$ converges to $b(x)$ over $(-\alpha, \alpha)$ and to zero over $(-1, 1)\backslash(-\alpha, \alpha)$. In view of (3.38), for large λ we would then have $J(u_0(x)) < J(u_+(x, \lambda))$, with the function $u_0(x)$ as defined above. Minimizing $J(u)$ over $H_0^1(-1, 1)$, we would then obtain a nontrivial solution of (3.1)–(3.2) which is different from $u_\pm(x, \lambda)$. But this is impossible, since we have exactly two nontrivial solutions, completing the proof of the theorem.

Remark. If the condition (3.38) is violated, then $u^+(x, \lambda)$ may indeed converge to zero in a neighborhood of $x = \pm 1$, as the following numerical example indicates. We took $a = 2$, and $b(x) = 4.9$ for $|x| \le 0.1$ and $b(x) = 2.5$ for $0.1 < |x| < 1$. The computed solution was converging to zero over $(-1, 1)\backslash[-0.1, 0.1]$ as λ was increasing, and to $b = 4.9$ over $[-0.1, 0.1]$.

EXAMPLE. Let $a(x) = A + Bx^2$, $b(x) = C - Dx^2$ with positive constants A, B, C, and D. Then $\alpha' = 2(B - D)x < 0$ for $x > 0$, provided we assume $D > B$, and $\beta' = 2x(BC - AD) - 4BDx^3 > 0$ for $x > 0$, if we assume

$C > (1/B)[AD + 2BD]$. Assuming that B, D are not large compared to $C - A$ (so that (3.15) and (3.16) are satisfied), we see that Theorem 3.1 applies. If additionally $C - 2A > 2B + D$ then $F(x, b(x)) > 0$ for all $x \in (-1, 1)$, and so Theorem 3.2 also applies.

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