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# On Existence of Periodic Solutions for a Class of Quasilinear Non-Coercive Problems

By

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## 1. Introduction

We study nonlinear boundary value problems in $n+1$ dimensional space $(x_1, \ldots, x_n, y)$ of the type

$$(1.1) \qquad \begin{aligned} u_y - \sum_{k=1}^{\ell} a_k(x) D^{\alpha_k} u &= g(x, D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) \qquad y = 1\,, \\ \Delta u &= f(x, y, u, Du) \qquad 0 < y < 1\,, \\ u &= 0 \qquad y = 0\,, \end{aligned}$$

where multi-index $\alpha_k = (\alpha_{k1}, \ldots, \alpha_{kn}, 0)$, the given functions $f$ and $g$ and the unknown function $u$ are $2\pi$ periodic in each variable $x_i$. In [2–4] we had studied the case $n = 2$, and the boundary condition $u_y - Fu_{xx} = g(x, u)$ at $y = 1$. When the constant $F < 0$, the problem comes from three-dimensional water wave theory in the absence of surface tension, see [8]. We are interested in the problem primarily since it represents a model non-coercive problem (i.e. the Lopatinski-Schapiro condition fails at $y = 1$, see [3], and hence one cannot use the standard elliptic theory). In [2–4] we had considered only the case $F > 0$, as in the physical case $F < 0$ one has a difficulty caused by presence of small divisors.

In this paper we extend the results of [2, 3] in two directions. In section 3 we consider non-coercive problems of type (1.1) with nonlinear boundary conditions of arbitrarily high order. We introduce a notion of dominating derivatives, which plays a role similar to that of the derivatives of the highest order in the coercive case. We state conditions allowing establishment of a priori estimates, and prove existence results for nonlinear problems.

In Section 4, we consider the case $F < 0$, which leads to small divisors. To see the difficulty, let us consider the problem (4.7) with $f = 0$. Look for a solution in the form $u = \sum_{j,k=-\infty}^{\infty} u_{jk}(y) e^{ijx+ikz}$, then $u_{jk}(y) = c_{jk} \sinh \sqrt{j^2+k^2}\, y$. If $g(x, z) = \sum_{j,k=-\infty}^{\infty} g_{jk} e^{ijx+ikz}$, then to satisfy the boundary condition at $y = 1$ we must solve $c_{jk}(\sqrt{j^2+k^2} \cosh \sqrt{j^2+k^2} - Fj^2 \sinh \sqrt{j^2+k^2}) = g_{jk}$, which involves division by possibly arbitrarily small numbers.

We distinguish between two and three dimensional cases. For $n = 2$, it turns out there are really no "small divisors", i.e. for $F = F_j \equiv (\coth j)/j$, $j = 1, 2, \ldots$, we have zero divisors, while for $F \neq F_j$ divisors are bounded away from zero. We are then able to derive a priori estimates and establish existence results for nonlinear problems. For $n = 3$ the situation is more involved. We restrict ourselves to rational $F$, and discover that for $F > 1/2$ it is again either zero divisors or divisors bounded away from zero, while for $F \leqslant 1/2$ one can get arbitrarily small divisors. The a priori estimates which we derive for $F > 1/2$ allow us to prove existence for the linear problem, which is nontrivial in the presence of everywhere dense set of zero divisors.

## 2. Notation and the preliminary results

We consider functions of $n + 1$ variables $(x_1, \ldots, x_n, y)$ which are $2\pi$ periodic in each variable $x_i$, and $0 \leqslant y \leqslant 1$. By $V$ we denote the domain $0 \leqslant x_i \leqslant 2\pi$, $i = 1, \ldots, n$, $0 \leqslant y \leqslant 1$; its boundary we denote by $\partial V$, and the top $(y = 1)$ part of the boundary by $V_t$. By $D^\alpha u$ we understand the derivative corresponding to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1} \geqslant 0$. Also,

$$u_i = \partial u/\partial x_i\,, \qquad u_{ij} = \partial^2 u/\partial x_i \partial x_j\,.$$

We shall denote

$$\int f = \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 f(x_1, \ldots, x_n, y)\,dx_1 \ldots dx_n dy\,;$$

$$\int_t f = \int_0^{2\pi} \cdots \int_0^{2\pi} f(x_1, \ldots, x_n, 1)\,dx_1 \ldots dx_n\,.$$

By $\|\cdot\|_m$ we denote the norm in the Sobolev space $H^m(V)$, and by $\overline{\|\cdot\|}_m$ the one in $H^m(V_t)$. We shall also need the norms

$$|f|_N = \sum_{|\alpha| \leqslant N} |D^\alpha f|_{L^\infty(V)}\,, \qquad N = \text{integer} \geqslant 0\,.$$

All irrelevant positive constants independent of unknown functions we denote by $c$; $Du = \nabla u$, $i = 1, \ldots, n + 1$.

We shall need the following relations between our norms, see [3].

**Lemma 2.1.** *Assume that $v \in H^{m+1}(V)$. For any integer $m \geqslant 0$ and any $\varepsilon > 0$ one can find a constant $c(\varepsilon)$ so that*

(i) $\overline{\|v\|}_m \leqslant \|v\|_{m+1}$

(ii) $\|v\|_m \leqslant \varepsilon \|v\|_{m+1} + c(\varepsilon) \|v\|_0$

(iii) $\overline{\|v\|}_m \leqslant \varepsilon \|v\|_{m+1} + c(\varepsilon) \|v\|_0$ .

The following lemma is taken essentially from [5].

**Lemma 2.2.** *Suppose that the functions $w_1, \ldots, w_s \in C^m(V)$ or $C^m(V_t)$. Suppose that $\phi = \phi(w_1, \ldots, w_s)$ possesses continuous derivatives up to order $m \geqslant 1$ bounded by c on $\max_i |w_i| < 1$. Then*

(i) $$\|\phi(w_1, \ldots, w_s)\|_m \leqslant c(\|w\|_m + 1) \quad \text{for } \max_i \|w_i\|_{L^\infty} < 1 .$$

*(We denote $\|w\|_m = \max_i \|w_i\|_m$). If in addition we assume $\phi(0, \ldots, 0) = 0$, $m \geqslant 1$ then*

(ii) $$\|\phi(w_1, \ldots, w_s)\|_m = \delta(\|w\|_m) ,$$

*where $\delta(t) \to 0$ as $t \to 0$.*

*Remark.* The lemma is also true for functions $\phi = \phi(x, y, w_1, \ldots, w_s)$ with $\phi \in C^m$ in all variables. Conclusion (i) is as before, and for (ii) the corresponding assumption is $\phi(x, y, 0, \ldots, 0) = 0$ for all $(x, y) \in V$ or $V_t$.

**Lemma 2.3.** *Let $\alpha_1, \ldots, \alpha_\ell$ be some collection of multi-indices, $k_0 = \max_{1 \leqslant k \leqslant l} |\alpha_k|$. Consider the subset $G^m$ of functions in $H^m(V)$ such that*

$$|||u|||_m \equiv \|u\|_m + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k} u\|_{m-1}} < \infty .$$

*Then $G^m$ with norm $|||\cdot|||_m$ is a Banach space, provided that $m \geqslant k_0 + [(n+1)/2] + 1$.*

*Proof.* To prove completeness, let $\{u^r\}$ be a Cauchy sequence in $G^m$, i.e. $\|u^r - u^p\|_m + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k} u^r - D^{\alpha_k} u^p\|_{m-1}} \to 0$, as $r, p \to \infty$. Since $H^m(V)$ and $H^{m-1}(V_t)$ are Banach spaces, $u^r \to u$ in $H^m(V)$, and $D^{\alpha_k} u^r \to v_{\alpha_k}$ in $H^{m-1}(V_t)$. It remains to show that $v_{\alpha_k} = D^{\alpha_k} u(x, 1)$. Indeed, both functions are continuous and $\overline{\|v_{\alpha_k} - D^{\alpha_k} u\|_0} \leqslant \overline{\|v_{\alpha_k} - D^{\alpha_k} u^r\|_{m-1}} + \overline{\|D^{\alpha_k} u^r - D^{\alpha_k} u\|_{m-k_0}} \to 0$ as $r \to \infty$.

## 3. A priori estimates and existence results

Consider the problem (non-coercive in general)

$$
\begin{aligned}
u_y + \sum_{k=1}^{\ell} r_k(x) D^{\alpha_k} u &= g(x) & y &= 1 , \\
\Delta u &= f(x, y) & 0 < y &< 1 , \qquad (3.1) \\
u &= 0 & y &= 0 .
\end{aligned}
$$

Here $u = u(x, y)$, $x = (x_1, \ldots, x_n)$, $0 \leqslant y \leqslant 1$, $D^{\alpha_k}$ denotes derivatives in $x$

variables only, $|\alpha_k| = \alpha_{k1} + \cdots + \alpha_{kn}$; $r_k(x) = a_k + \rho_k(x)$, $a_k = \text{const}$, $\ell = \text{integer} \geqslant 1$. Throughout this section $u$, $f$, $g$, $\rho_k$ are assumed to be $2\pi$ periodic in each $x_i$. We shall study solvability and derive a priori estimates for the problem (3.1) without restricting $\max|\alpha_k|$, the order of the boundary operator, and without requiring it to be coercive. Then we consider nonlinear problems.

**Definition.** We say that derivative $D^\beta u$ is subordinate to $D^\alpha u$ if $\alpha_i \geqslant \beta_i$, $i = 1, \ldots, n$, and $\beta_i \neq 0$ if $\alpha_i \neq 0$. We say that derivative $D^{\alpha_k} u$ is a dominating derivative in a set $S = \{D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u\}$, if it is not subordinate to any other derivative in that set.

Clearly, there can be several dominating derivatives in a set, and of different orders.

**Lemma 3.1.** *For the problem* (3.1) *assume that* $(-1)^{|\alpha_k|/2} a_k \geqslant 0$ *for all even* $|\alpha_k|$, *and either* $(-1)^{(|\alpha_k|+1)/2} a_k \geqslant 0$ *holds for all odd* $|\alpha_k|$, *or the opposite inequality does. The above inequalites are assumed to be strict for all k corresponding to the dominating derivatives in the set S. Then for* $\max_k |\rho_k|_m$ *sufficiently small the following a priori estimate holds* ($m = \textit{integer} \geqslant 0$)

$$(3.2) \qquad \|u\|_{m+1} + \sum_{k=1}^{\ell} \overline{\|D^{\alpha_k}u\|}_m \leqslant c\left(\sum_{k=1}^{\ell} \|D^{\alpha_k}f\|_m + \|f\|_m + \|g\|_m\right).$$

*Proof.* We begin by assuming that $\rho_k(x) \equiv 0$ for all $k$. Let $u = \sum_j u_j(y)e^{ij\cdot x}$, $f = \sum_j f_j(y)e^{ij\cdot x}$, $g = \sum_j g_j e^{ij\cdot x}$, $\rho = \sqrt{j_1^2 + \cdots + j_n^2}$. Substituting these into (3.1) and suppressing the multi-index $j$, we get:

$$(3.3a) \qquad u'(1) + \sum_{k=1}^{\ell} a_k(ij)^{\alpha_k} u(1) = g\,,$$

$$(3.3b) \qquad u''(y) - \rho^2 u = f(y) \qquad 0 < y < 1\,,$$

$$(3.3c) \qquad u(0) = 0\,.$$

For $\rho \neq 0$ solution of (3.3b) and (3.3c) is

$$(3.4) \qquad u(y) = \gamma \sinh \rho y + \frac{1}{\rho}\int_0^y f(t) \sinh \rho(y-t)\,dt\,.$$

To find $\gamma$ we substitute this into (3.3a)

$$(3.5) \qquad \gamma\left(\rho \cosh \rho + \sum_{k=1}^{l} a_k(ij)^{\alpha_k} \sinh \rho\right) + \int_0^1 f(t)\left[\cosh \rho(1-t) + \frac{1}{\rho}\left(\sum_{k=1}^{l} a_k(ij)^{\alpha_k}\right) \sinh \rho(1-t)\right] dt = g\,.$$

Denote $A = \rho \cosh \rho + \sum_{k=1}^{\ell} a_k(ij)^{\alpha_k} \sinh \rho$. Multiplying (3.4) by $A$, using (3.5) and standard identities for hyperbolic functions, we get:

$$
\begin{aligned}
(3.6) \quad Au(y) &= g \sinh \rho y - \int_0^1 f(t)\Bigg[\cosh \rho(1-t) \sinh \rho y \\
&\qquad + \frac{1}{\rho}\left(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\right) \sinh \rho(1-t) \sinh \rho y\Bigg] dt \\
&\quad + \int_0^y f(t)\Bigg[\sinh \rho(y-t) \cosh \rho \\
&\qquad + \frac{1}{\rho}\left(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\right) \sinh \rho(y-t) \sinh \rho\Bigg] dt \\
&= g \sinh \rho y - \int_0^1 f(t)\Bigg[\cosh \rho(1-t) \sinh \rho y \\
&\qquad + \frac{1}{\rho}\left(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\right) \sinh \rho(1-t) \sinh \rho y\Bigg] dt \\
&\quad - \int_0^y f(t)\Bigg[\sinh \rho t \cosh \rho(y-1) \\
&\qquad + \frac{1}{\rho}\left(\sum_{k=1}^{\ell} a_k(ij)^{\alpha_k}\right) \sinh \rho t \sinh \rho(1-y)\Bigg] dt\,.
\end{aligned}
$$

Notice that by our assumptions $|A| \geqslant ce^{\rho}(\rho + j^{\alpha_k})$, for all $1 \leqslant k \leqslant \ell$. Then from (3.6) we easily estimate (see [2, p. 876] for a similar argument)

$$
(3.7) \qquad |u(y)| \leqslant c\left(\frac{|g|}{\rho + j^{\alpha_k}} + \frac{1}{\rho}\left(\int_0^1 |f(t)|^2 dt\right)^{1/2}\right).
$$

This implies (restoring the subscripts)

$$
(3.8) \qquad \int_0^1 |u_j(y)|^2 dy \leqslant c\left(\frac{|g_j|^2}{\rho^2} + \frac{1}{\rho^2}\int_0^1 |f_j|^2 dt\right).
$$

In the case $\rho = 0$ we easily get from (3.3)

$$
(3.9) \qquad \int_0^1 |u_0|^2 dy \leqslant c\left(|g_0|^2 + \int_0^1 |f_0(t)|^2 dt\right).
$$

Differentiating (3.6) and going through the same steps as in derivation of (3.7), we get

$$
(3.10) \qquad |u_j'|^2 \leqslant c\left(|g_j|^2 + \int_0^1 |f_j|^2 dt\right).
$$

From (3.8–10) and from (3.7) with $y = 1$ we conclude the estimate (3.2) with $m = 0$. Higher estimates are easily proved by induction, differentiating (3.1) in $x$.

Turning to the general case, we write the boundary condition at $y = 1$ in the form

$$u_y + \sum_{k=1}^{\ell} a_k D^{\alpha_k} u = g + \sum_{k=1}^{\ell} \rho_k(x) D^{\alpha_k} u ,$$

and apply the estimate (3.2) for the constant coefficient case.

Since

$$\|\rho_k(x) D^{\alpha_k} u\|_m \leqslant c|\rho_k|_m \|D^{\alpha_k} u\|_m ,$$

with $|\rho_k|_m$ small, the proof follows.

*Remark* 1. Notice that the above argument establishes existence of solution $u(x, y) \in G^{m+1}$ to the problem (3.1) in the case $\rho_k(x) \equiv 0$ for all $k$, provided $f \in H^{m+k_0}(V)$, $g \in H^m(V_t)$, $m \geqslant 0$. For the general case existence follows in the same way as in the theorem 3.1 below, under the additional condition $m \geqslant k_0 + [(n+1)/2] + 1$.

*Remark* 2. All conditions on $a_k$ which correspond to $|\alpha_k|$ odd can be dropped if none of the corresponding $D^{\alpha_k}$ is a dominating derivative in the set $S$.

*Remark* 3. A partition of unity argument (which allows one to remove the smallness conditions on $\rho_k$) produces the estimate (3.2) with an extra term $\|u\|_0$ on the right. We do not know how to remove this term (notice, there is no apparent maximum principle).

Sharper estimates can be obtained in the following special case.

**Lemma 3.2.** *Consider the problem* (3.1) *with* $\ell = 1$. *Assume that* $(-1)^{|\alpha_1|/2} a_1 > 0$ *in case* $|\alpha_1|$ *is even and* $a_1 \neq 0$ *in case* $|\alpha_1|$ *is odd. Then for* $|\rho_1|_m$ *sufficiently small we have* (*integer* $m \geqslant 0$)

$$\|u\|_{m+1} + \overline{\|D^{\alpha_1} u\|}_m \leqslant c(\|f\|_m + \|g\|_m) .$$

*Proof.* It is sufficient to consider the case $\rho_1 \equiv 0$, from which the general case will follow as before. Follow the proof of lemma 3.1. From (3.10) it follows (by taking derivatives of (3.1) in $x$ and setting $y = 1$)

$$\overline{\|u_y\|}_m \leqslant c(\|f\|_m + \|g\|_m) ,$$

and hence

$$\overline{\|D^{\alpha_1} u\|}_m \leqslant \overline{\|u_y\|}_m + \|g\|_m \leqslant c(\|f\|_m + \|g\|_m) .$$

Combining this with (5.8–9), we conclude the proof.

**Theorem 3.1.** *Consider the problem*

$$
(3.11)\qquad
\begin{aligned}
u_y &= \rho(x, D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) && y = 1\,,\\
\Delta u &= \varepsilon f(x, y) && 0 < y < 1\,,\\
u &= 0 && y = 0\,.
\end{aligned}
$$

*Assume that* $\rho(x, 0, \ldots, 0) \equiv 0$. *Denote* $r_k = -(\partial\rho/\partial D^{\alpha_k}u)(x, 0, \ldots, 0)$, *and assume that* $r_k(x)$ *satisfy the same conditions as in lemma* 3.1. *Let* $k_0 = \max_{1 \leqslant k \leqslant \ell}|\alpha_k| \geqslant 1$, $m_0 = k_0 + [(n+1)/2] + 1$, $f \in C^{m_0+k_0-1}$, $\rho \in C^{m_0}$ *for* $(x, y) \in V$ *and in small balls around the origin for other variables. Then for* $\varepsilon$ *and* $\max_k|\rho_k|_{m_0}$ *sufficiently small the problem* (3.11) *has a solution* $u \in C^2(V) \cap C^{k_0}(V_t)$.

*Proof.* Define a map $T\colon u \in G^{m_0} \to v \in G^{m_0}$ by solving (see lemma 2.3)

$$
\begin{aligned}
v_y + \sum_{k=1}^{\ell} a_k D^{\alpha_k} v &= \rho(D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) + \sum_{k=1}^{\ell} a_k D^{\alpha_k}u && y = 1\,,\\
\Delta v &= \varepsilon f(x, y) && 0 < y < 1\,,\\
v &= 0 && y = 0\,.
\end{aligned}
$$

Using lemma 3.1 it is easy to see that $T$ is a contraction on sufficiently small balls around the origin in $G^{m_0}$.

*Remarks.* 1. It is easy to see that a similar perturbation result will hold for $f = f(x, y, u, Du, D^2u)$, provided $\max_{1 \leqslant k \leqslant \ell}|\alpha_k| \leqslant 1$.

2. Clearly the smoothness of solution increases with that of $\rho$ and $f$. In particular if $\rho, f \in C^\infty$ so does $u$.

*Example.* Let $u = u(x, y, z)$. The non-coercive problem

$$
\begin{aligned}
u_y &= u_{xx}^2 - u_{xxxx} + u_{xzz} && y = 1\,,\\
\Delta u &= \varepsilon \sin x \sin z && 0 < y < 1\,,\\
u &= 0 && y = 0
\end{aligned}
$$

verifies conditions of the theorem 3.1, and hence it has a $C^\infty$ solution, $2\pi$ periodic in $x$ and $z$, provided $\varepsilon$ is small enough.

**Theorem 3.2.** *Consider the problem*

$$
(3.12)\qquad
\begin{aligned}
u_y &= \rho(x, D^{\alpha_1}u, \ldots, D^{\alpha_\ell}u) && y = 1\,,\\
\Delta u &= \varepsilon f(x, y, u, Du) && 0 < y < 1\,,\\
u &= 0 && y = 0\,.
\end{aligned}
$$

*Assume that* $\rho(x, 0, \ldots, 0) \equiv 0$. *Assume that one of the derivatives in the set* $S$, *say* $D^{\alpha_1}$, *dominates all others. With* $r_k(x)$ *as defined in the theorem* 3.1, *we assume that* $r_1(x)$ *satisfies the conditions of lemma* 3.2, *and* $r_k \equiv 0$ *for* $k = 2, \ldots, \ell$. *With* $k_0$ *and* $m_0$ *as above assume that* $f, \rho \in C^{m_0}$ *for* $(x, y) \in V$ *and in small balls around the origin for other variables. Then for* $\varepsilon$ *and* $|\rho_1|_{m_0}$ *sufficiently small the problem* (3.12) *has a solution* $u \in C^2(V) \cap C^{k_0}(V_t)$.

The proof is similar to that of the theorem 3.1.

## 4. Small divisors in dimensions two and three

We show first that the situation is rather simple for the two-dimensional case, i.e. $u = u(x, y)$. Namely, except for $F = (\coth j)/j$ where zero divisors appear, for other $F$ the divisors are bounded away from zero.

**Lemma 4.1.** *Consider the problem* ($u$, $f$ *and* $g$ *are* $2\pi$ *periodic in* $x$)

$$(4.1)\qquad \begin{aligned} u_y + F u_{xx} &= g(x) & y &= 1\,, \\ \Delta u &= f(x, y) & 0 < y &< 1\,, \\ u &= 0 & y &= 0\,. \end{aligned}$$

*Assume that* $F \neq (\coth j)/j$, $j = 1, 2, \ldots$, *and* $F \neq 0$. *Then*

$$(4.2)\qquad \|u\|_{m+2} + \overline{\|u\|}_{m+2} \leqslant c(\|f\|_{m+1} + \|g\|_m)\,.$$

*Proof.* Look for solution in the form $u(x, y) = \sum_{j=-\infty}^{\infty} u_j(y) e^{ijx}$, and follow the proof of lemma 3.1. This time $A = j \cosh j - F j^2 \sinh j$. Notice that if $F \neq (\coth j)/j$ and $F \neq 0$, then $|A| \geqslant c_0 j^2 e^j$ for some $c_0 > 0$. The rest of the proof is similar to that of lemma 3.1.

**Theorem 4.1.** *Consider the problem*

$$(4.3)\qquad \begin{aligned} u_y &= \rho(u, u_x, u_{xx}) & y &= 1\,, \\ u_{xx} + u_{yy} &= f(x, y, u, u_x, u_y) & 0 < y &< 1\,, \\ u &= 0 & y &= 0\,. \end{aligned}$$

*Assume that* $f$ *is* $2\pi$ *periodic in* $x$, *and the following*

(i) $\rho(0, 0, 0) = \rho_u(0, 0, 0) = \rho_{u_x}(0, 0, 0) = 0$;

$$F_0 = -\rho_{u_{xx}}(0, 0, 0) \neq \frac{1}{j} \coth j, \qquad j = 1, 2, \ldots, F_0 \neq 0\,.$$

(ii) $\rho \in C^3$, $f$, $f_u$, $f_{u_x}$, $f_{u_y} \in C^3$ *in all arguments* (*for* $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant 1$ *and in small balls around the origin for other variables*).

*Then for* $\|f(x, y, 0, 0, 0)\|_3$, $\|f_u(x, y, 0, 0, 0)\|_3$, $\|f_{u_x}(x, y, 0, 0, 0)\|_3$ *and* $\|f_{u_y}(x, y, 0, 0, 0)\|$ *sufficiently small the problem* (4.3) *has a* $C^2$ *solution,* $2\pi$ *periodic in* $x$.

*Proof.* Let $G^m(m = \text{integer} \geqslant 1)$ be a subset of $H^m(V)$ consisting of functions $u \in H^m(V)$ such that in addition $u \in H^m(V_t)$. By lemma 2.3 $G^m$ with the norm $|||u|||_m = \|u\|_m + \overline{\|u\|}_m$ is a Banach space (notice that by lemma 2.1 this norm is equivalent to $\|u\|_m + \overline{\|u_x\|}_{m-1}$ for $n = 1$). Define a map $T$ of $G^m$ into itself by solving ($v = Tu$)

$$
\begin{aligned}
v_y + F_0 v_{xx} &= \rho(u, u_x, u_{xx}) + F_0 u_{xx} && y = 1\,, \\
\Delta v &= f(x, y, u, u_x, u_y) && 0 < y < 1\,, \\
v &= 0 && y = 0\,.
\end{aligned}
$$

Using lemma 4.1 it is straightforward to show that the map $T$ takes a sufficiently small ball around the origin in $G^4$ into itself and is a contraction (see [2] for a similar argument).

Next we turn to the three-dimensional case, i.e., $u = u(x, y, z)$, where the situation is more involved. Notice first that the set of $F$ corresponding to the set $\mathscr{F} = \{F_{jk}\}$ of zero divisors $F_{j,k} = (\sqrt{j^2 + k^2/j^2}) \coth \sqrt{j^2 + k^2}$ is everywhere dense on the positive real axis, as we showed in [2].

We restrict now to the rational $F = p/q$, and see that the situation changes depending on whether $F > 1/2$ or $F \leqslant 1/2$. For $F = p/q > 1/2$ and $F \notin \mathscr{F}$ we show that the denominators are bounded away from zero, which allows us to derive a priori estimates. For $F = p/q \leqslant 1/2$ it is possible that $F \notin \mathscr{F}$, but the denominators can get arbitrarily small. We also show that for each rational $F$ condition $F \in \mathscr{F}$ can be decided by a finite number of computations.

**Lemma 4.2.** *Let* $F = p/q > 1/2$ *be an irreducible fraction. Then there exists a constant* $c_0 > 0$, *such that*

$$d \equiv |\sqrt{(j^2 + k^2)} - Fj^2| \geqslant c_0\,,$$

*for all integers* $j$, $k$, *possibly with the exception of finitely many pairs* $(j, k)$.

*Proof.* Without loss of generality we restrict to positive $j$ and $k$. We consider three cases.

(i) $k \geqslant Fj^2$. Then

$$d \geqslant \sqrt{(j^2 + F^2 j^4)} - Fj^2 = \frac{j^2}{\sqrt{(j^2 + F^2 j^4)} + Fj^2} \geqslant \frac{1}{\sqrt{(1 + F^2)} + F}$$

(ii) $k = Fj^2 - \ell$, $\ell \geqslant 2$. Then one easily gets:

$$d = \frac{k(2\ell - 1/F) + \ell(\ell - 1/F)}{\sqrt{(1/F(k+\ell) + k^2)} + k + \ell} \geqslant \frac{2k + \ell(\ell - 1/F)}{\sqrt{(2(k+\ell) + k^2)} + k + \ell} \geqslant \frac{c_1'(k+\ell)}{c_1''(k+\ell)}$$
$$\equiv c_1 > 0\,.$$

(iii) It is easy to see that it remains to consider the case $k = Fj^2 - \ell$ with $1/q < \ell < 2 - 1/q$, where $\ell = m/q$ (for some $m =$ integer $\geqslant 0$) may be a reducible fraction. Then

$$d = \frac{|F^2 j^4 - j^2 - (Fj^2 - \ell)^2|}{\sqrt{(j^2 + k^2)} + Fj^2} = \frac{|-j^2 + 2F\ell j^2 - \ell^2|}{\sqrt{(j^2 + k^2)} + Fj^2}$$

Notice that $2F\ell = 2(p/q)(m/q) \neq 1$, for otherwise we would have $2m = q^2/p$ with $p$ and $q$ being mutually prime a contradiction. Denote $|2F\ell - 1| = \bar{c}_1 > 0$. Then for $j \geqslant j_0$ – large,

$$d \geqslant \frac{c_1 j^2 - \ell^2}{\sqrt{j^2 + (Fj^2 - \ell)^2} + Fj^2} \geqslant c_0 > 0.$$

*Remark* 4.1. If $F = 1$, then we easily estimate $d > 3/8$, with the exception of $j = k = 0$ and $j = \pm 1$, $k = 0$.

**Lemma 4.3.** *Let $F > 1/2$ be rational, and $F \neq F_{j,k}$. Then there exists a constant $c_3 > 0$, such that for any pair of integers $j$, $k$ we have*

$$|\Delta| \equiv |\sqrt{j^2 + k^2}\cosh\sqrt{j^2 + k^2} - Fj^2 \sinh\sqrt{j^2 + k^2}| \geqslant c_3 e^{\sqrt{j^2+k^2}}\,.$$

*Proof.* Write

$$|\Delta| = \frac{e^{\sqrt{j^2+k^2}}}{2}|\sqrt{j^2 + k^2} - Fj^2 + e^{-2\sqrt{j^2+k^2}}(\sqrt{j^2 + k^2} + Fj^2)|$$
$$\equiv \frac{e^{\sqrt{j^2+k^2}}}{2}|d + d_1|\,.$$

By Lemma 4.2, $|d| \geqslant c_0$ for $|j| \geqslant j_0$. Also $|d| \geqslant c_0$ for $|j| < j_0$ and $|k| \geqslant k_0$, $k_0$ – large. By increasing $j_0$ and $k_0$, if necessary, we can also assume that $|d_1| \leqslant c_0/2$ for $|j| \geqslant j_0$ or $|k| \geqslant k_0$. Let now $\bar{c}_2 = \min_{|j|<j_0, |k|<k_0} |d + d_1|$. Notice that $\bar{c}_2 > 0$, since $F \neq F_{j,k}$. The lemma now follows with $c_3 = \min(c_0/4, \bar{c}_2/2)$.

**Lemma 4.4.** *For each rational $F > 1/2$, condition $F \in \mathscr{F}$ can be decided by a finite number of computations.* (*Recall that the set $\mathscr{F}$ is everywhere dense*).

*Proof.* Condition $F \in \mathscr{F}$ implies that for some $j$ and $k$

$$e^{-2\sqrt{j^2+k^2}} = \frac{Fj^2 - \sqrt{j^2 + k^2}}{Fj^2 + \sqrt{j^2 + k^2}}\,. \tag{4.4}$$

By Lemma 4.2,

$$\frac{|Fj^2 - \sqrt{j^2+k^2}|}{Fj^2 + \sqrt{j^2+k^2}} \geqslant \frac{c_0}{F(j^2+k^2) + \sqrt{j^2+k^2}},$$

and the left hand side of (4.4) is less than that quantity for large $j$, $k$, e.g. for

$$\sqrt{j^2+k^2} > \ln \sqrt{\frac{F+1}{c_0}(j^2+k^2)}, \tag{4.5}$$

which concludes the proof.

*Remark* 4.2. $1 \neq \mathscr{F}$. Indeed, by Remark 4.1 we can take $c_0 = 3/8$. Then in view of (4.5) we have only to check $j = \pm 1$, $k = 0$ and $j = 0$, $k = \pm 1$, which is easily done.

**Lemma 4.5.** *Condition $F > 1/2$ in lemmas 4.2, 4.3 cannot be removed.*

*Proof.* Namely, we show that $1/2 \notin \mathscr{F}$, but $d = \sqrt{j^2+k^2} - j^2/2$ gets arbitrarily small for large $j$, $k$. Indeed, take $k = j^2/2 - 1$, $j$ – even. Then $d = 1/(\sqrt{j^2+k^2} + j^2/2)$. Condition $1/2 \notin \mathscr{F}$ is equivalent to checking impossibility for any $j$ and $k$ of

$$e^{-2\sqrt{j^2+k^2}} = \frac{j^2/2 - \sqrt{j^2+k^2}}{j^2/2 + \sqrt{j^2+k^2}}. \tag{4.6}$$

For this, one first notices that the left hand side of (4.6) is less than the absolute value of the right hand side if $j^2 + k^2 \geqslant 8$, and then one eliminates all remaining possibilities.

We can now obtain the following a priori estimates.

**Theorem 4.2.** *Consider the problem* ($u = u(x, y, z)$)

$$\begin{aligned} u_y + Fu_{xx} &= g(x, z) & y &= 1, \\ \Delta u &= f(x, y, z) & 0 &< y < 1, \\ u &= 0 & y &= 0. \end{aligned} \tag{4.7}$$

*Assume that $F > 1/2$ is rational, $F \notin \mathscr{F}$; $f$ and $g$ are $2\pi$ periodic in $x$ and $z$. Then we have the following estimate*

$$\|u\|_m + \overline{\|u\|}_m \leqslant c(\|f\|_{m+1} + \|g\|_m). \tag{4.8}$$

*Proof.* Look for solution in the form $u(x, y, z) = \sum_{j,k=-\infty}^{\infty} u_{jk}(y) e^{ijx+ikz}$. This time $|\Delta| = |\sqrt{j^2+k^2} \cosh \sqrt{j^2+k^2} - Fj^2 \sinh \sqrt{j^2+k^2}| \geqslant c_0 e^{\sqrt{j^2+k^2}}$ by lemma 3.3. Then proceed as in lemma 5.1.

**Corollary.** *If* $f \in H^5(V)$, $g \in H^4(V_t)$ *then the problem* (4.7) *has a unique* $C^2(V)$ *solution.*

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