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Uniqueness and exact multiplicity of solutions for a class of Dirichlet problems

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Abstract

We apply the "monotone separation of graphs" technique of L.A. Peletier and J. Serrin [L.A. Peletier, J. Serrin, Uniqueness of positive solutions of semilinear equations in R^n, Arch. Ration. Mech. Anal. 81 (2) (1983) 181–197; L.A. Peletier, J. Serrin, Uniqueness of nonnegative solutions of semilinear equations in R^n, J. Differential Equations 61 (3) (1986) 380–397], as developed further by L. Erbe and M. Tang [L. Erbe, M. Tang, Structure of positive radial solutions of semilinear elliptic equations, J. Differential Equations 133 (2) (1997) 179–202], to the question of exact multiplicity of positive solutions for a class of semilinear equations on a unit ball in R^n. We also observe that using P. Pucci and J. Serrin [P. Pucci, J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, Indiana Univ. Math. J. 47 (2) (1998) 501–528] improvement of a certain identity of L. Erbe and M. Tang [L. Erbe, M. Tang, Structure of positive radial solutions of semilinear elliptic equations, J. Differential Equations 133 (2) (1997) 179–202] produces a short proof of L. Erbe and M. Tang [L. Erbe, M. Tang, Structure of positive radial solutions of semilinear elliptic equations, J. Differential Equations 133 (2) (1997) 179–202] result on the uniqueness of positive solution of ($1 < p, q < \frac{n+2}{n-2}$)

$$\Delta u + u^p + u^q = 0 \quad \text{for } |x| < 1, \qquad u = 0 \quad \text{when } |x| = 1.$$

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1. Introduction

We study uniqueness and exact multiplicity of positive solutions for the Dirichlet problem on a unit ball in R^n ($x \in R^n$)

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < 1, \qquad u = 0 \quad \text{if } |x| = 1, \tag{1.1}$$

with λ a positive parameter. In view of the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5] positive solutions of (1.1) are radially symmetric, i.e. $u = u(r)$, with $r = |x|$, and moreover $u'(r) < 0$ for all $r \in (0, 1)$, and hence they satisfy

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \quad r \in (0,1), \qquad u'(0) = 0, \qquad u(1) = 0. \tag{1.2}$$

Exact multiplicity of positive solutions have been studied extensively in recent years, starting with P. Korman, Y. Li and T. Ouyang [8], and continued by T. Ouyang and J. Shi [13,14] (and in a number of other papers by various authors). In [8] a general scheme for proving such results was developed. It involves several steps: proving positivity of solutions of the linearized problem, studying the direction of bifurcation, showing uniqueness of the solution curve, etc. In the present note we apply the "monotone separation of graphs" technique, introduced by L.A. Peletier and J. Serrin [15,16], and developed further L. Erbe and M. Tang [4], and P. Pucci and J. Serrin [17], to obtain new results on the positivity of solutions of the linearized problem, which in turn imply exact multiplicity and uniqueness results.

By a result of C.S. Lin and W.-M. Ni [12], solutions of the linearized problem for (1.1) are radially symmetric, and hence the linearized problem takes the form ($w = w(r)$)

$$w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0, \quad r \in (0,1), \qquad w'(0) = 0, \qquad w(1) = 0. \tag{1.3}$$

For "most" pairs (λ, u) the eigenvalue problem (1.3) admits only the trivial solution $w(r) \equiv 0$, we call such a solution pair (λ, u) *non-singular*. For the bifurcation approach of [8,13,14] one needs to show that for *singular* pairs (λ, u), i.e. when (1.3) admits non-trivial solutions, these solutions are of one sign, i.e. we may assume that $w(r) > 0$ for all $r \in [0, 1)$. To prove positivity, a method of test functions was used in [8,13,14]. In the present work we observe that the "monotone separation of graphs" technique, originally developed to prove uniqueness of ground states for semilinear equations, can also be used to prove positivity for (1.3), which gives us a new multiplicity result, and also a considerably shorter proof of L. Erbe and M. Tang [4] uniqueness result.

The "monotone separation of graphs" technique allows one to show that different positive solutions of (1.2) do not intersect. We show that this implies that at any turning point any non-trivial solution of the linearized problem (1.3) is of one sign, and then the bifurcation analysis of P. Korman, Y. Li and T. Ouyang [8], and T. Ouyang and J. Shi [13,14] applies. In case $f(u) = u^p + u^q$, with $1 < p, q < \frac{n+2}{n-2}$, we are able to show that the solution curve does not turn, which results in a short proof of uniqueness (and existence) of solutions, which was first proved in L. Erbe and M. Tang [4]. For a class of convex $f(u)$ we show that the solution curve makes exactly one turn, which implies a new exact multiplicity result.

We observe that two more methods for proving positivity of (1.3) are available, in addition to the two methods just described. One of them involves considering the ratio of any two solutions, as in M.K. Kwong and Y. Li [10] or in Adimurthi, F. Pacella and S.L. Yadava [1], and the other one in P. Korman and T. Ouyang [9] is based on certain generalized Wronskians of M. Tang [19].

2. Preliminary results

Since positive solutions of

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad r \in (0,1), \qquad u'(0) = 0, \qquad u(1) = 0 \tag{2.1}$$

are strictly decreasing, see B. Gidas, W.-M. Ni and L. Nirenberg [5], we can consider the inverse function $r = r(u)$, which satisfies

$$r''(u) - (n-1)\frac{r'^2(u)}{r(u)} - f(u)r'^3(u) = 0. \tag{2.2}$$

If $v(r)$ is another solution of (2.1), then its inverse function $r = s(u)$ satisfies

$$s''(u) - (n-1)\frac{s'^2(u)}{s(u)} - f(u)s'^3(u) = 0. \tag{2.3}$$

The following well-known lemma on "monotone separation of graphs" is due to L.A. Peletier and J. Serrin [16].

Lemma 2.1. *(See [16].) Suppose $r(u) - s(u) > 0$ on some interval I. Then $r(u) - s(u)$ can have at most one critical point on I. Moreover, this critical point is a strict maximum.*

Proof. If $\bar{u}$ is a critical point of $r(u) - s(u)$, then $r'(\bar{u}) = s'(\bar{u})$, and hence

$$(r-s)''(\bar{u}) = (n-1)\left(\frac{1}{r(\bar{u})} - \frac{1}{s(\bar{u})}\right)r'^2(\bar{u}) < 0. \qquad \square$$

Assume now there are two intersecting solutions of the Dirichlet problem (2.1), $u_1(r)$ and $u_2(r)$. Denote $\alpha_1 = u_1(0)$ and $\alpha_2 = u_2(0)$. By the uniqueness result of L.A. Peletier and J. Serrin [15], we may assume that

$$\alpha_2 > \alpha_1.$$

Let r_0 be the smallest point of intersection, and let $u_0 = u_1(r_0) = u_2(r_0)$. Let $r_1 \leqslant 1$ be the next point of intersection, with $u_1 = u_1(r_1) = u_2(r_1)$, and $u_1 \geqslant 0$. Clearly, $r_1(u) - r_2(u) > 0$ on (u_1, u_0). In view of Lemma 2.1 we can find a point $\bar{u} \in (u_1, u_0)$ such that

$$r_1'(\bar{u}) = r_2'(\bar{u}), \quad \text{and} \quad r_1'(u) < r_2'(u) \quad \text{for } u \in (\bar{u}, u_0]. \tag{2.4}$$

It was observed by M. Tang [18] that this inequality continues to hold over the interval (u_0, α_1) too.

Lemma 2.2. *(See [18].)*

$$r_1'(u) < r_2'(u) \quad \textit{for } u \in (\bar{u}, \alpha_1). \tag{2.5}$$

Proof. We need to prove (2.5) over (u_0, α_1). On this interval the function $r_2(u) - r_1(u) > 0$ can have by Lemma 2.1 only one critical point. Now, $(r_2 - r_1)(u_0) = 0$ and $(r_2 - r_1)'(u) \to \infty$, as $u \to \alpha_1$. It follows that $r_2 - r_1$ cannot have any critical points at all, i.e. $r_2' - r_1' > 0$ on (u_0, α_1). □

The crucial role in the proof of uniqueness is played by an identity of L. Erbe and M. Tang [4], as generalized by P. Pucci and J. Serrin [17]. Defining [4,17]

$$P(r) = r^n\left(\frac{1}{2}u'^2(r) + F\big(u(r)\big)\right) + nr^{n-1}u'(r)\frac{F(u(r))}{f(u(r))}, \tag{2.6}$$

where as usual, $F(u) = \int_0^u f(t)\,dt$, and $u(r)$ is a solution of (2.1), one shows that [4,17]

$$P'(r) = nr^{n-1}u'^2(r)\Phi\big(u(r)\big), \tag{2.7}$$

where

$$\Phi(u) = \left(\frac{F(u)}{f(u)}\right)' - \frac{1}{2} + \frac{1}{n}. \tag{2.8}$$

We can write $P(r) = r^n E(r) + nr^{n-1}u'(r)\frac{F(u(r))}{f(u(r))}$, where

$$E(r) = \frac{1}{2}u'^2(r) + F\big(u(r)\big).$$

Lemma 2.3. *For any continuous $f(u)$, and any solution $u(r)$ of* (2.1)

$$E(r) > 0 \quad \textit{for all } r \in [0, 1).$$

Proof. We have $E'(r) = -\frac{n-1}{r}u'^2 < 0$, and $E(1) \geqslant 0$. □

As we mentioned, it is often advantageous to work with $r(u)$ instead of $u(r)$. We shall also be switching back and forth between the two representations. For example, consider the expression $Q(r) \equiv r^{n-1}u'(r)$, which occurs in the definition of $P(r)$. It can be written as

$$Q(u) = \frac{r^{n-1}(u)}{r'(u)}. \tag{2.9}$$

However, to compute $Q'(u)$ it seems easier to observe from (2.1) that $Q'(r) = -r^{n-1}f(u)$, and then by the chain rule

$$Q'(u) = -r^{n-1}(u)f(u)r'(u). \tag{2.10}$$

Remark. Of course, the identity (2.7) is straightforward to verify. However, if one writes $P(r) = r^n E(u(r)) + nQ(r)\frac{F(u(r))}{f(u(r))}$, with $E'(r) = -\frac{n-1}{r}u'^2(r)$ and $Q'(r) = -r^{n-1}f(u(r))$, one gets an easy and illuminating proof of the identity (2.7).

Given two solutions $r_1(u)$ and $r_2(u)$ (with $\alpha_2 > \alpha_1$), one defines

$$S(u) = \frac{Q_1(u)}{Q_2(u)}. \tag{2.11}$$

(Here $Q_i(u)$ denotes $Q(u)$ evaluated at $r_i(u)$, $i = 1, 2$.)

Lemma 2.4. *(See [4,17,18].) Assume that $f(u) > 0$ for $u > \gamma \geqslant 0$. Then on (γ, α_1) we have $S'(u) < 0$ ($S'(u) > 0$) if and only if $r_1'(u) < r_2'(u)$ ($r_1'(u) > r_2'(u)$).*

Proof. In view of (2.10), we have

$$S'(u) = \left(\frac{r_1}{r_2}\right)^{n-1} \frac{r_2'}{r_1'} f(u)\left(r_2'^2 - r_1'^2\right) < 0,$$

and the proof follows. □

In terms of $r(u)$ we rewrite $P(r)$ as

$$\begin{aligned} P(u) &= r^n(u)\left[\frac{1}{2}\frac{1}{r'^2(u)} + F(u)\right] + nQ(u)\frac{F(u)}{f(u)} \\ &= r^n(u)E(u) + nQ(u)\frac{F(u)}{f(u)}, \end{aligned} \tag{2.12}$$

where $Q(u)$ is given by (2.9), and then (2.7) takes the form

$$P'(u) = n\Phi(u)Q(u). \tag{2.13}$$

Lemma 2.5. *Assume that $\Phi(u) > 0$ for $u > 0$. Then, with $\alpha = u(0)$, we have $P(\alpha) = 0$, and*

$$P(u) > 0 \quad \textit{for } u \in [0, \alpha).$$

Proof. We see directly that $P(\alpha) = 0$, and from (2.13) $P'(u) < 0$ on $(0, \alpha)$. □

Recall that in case two solutions of (2.1) intersect, we have denoted by (r_0, u_0) the first point of intersection, and there exists a point $\bar{u}$ defined by (2.4). The following theorem is due essentially to L. Erbe and M. Tang [4], see also M. Tang [18], and P. Pucci and J. Serrin [17]. (We have achieved some simplification by introducing the function $\phi(u)$ below.)

Theorem 2.1. *(See [4,17,18].) Assume that $f(u) > 0$ and $\Phi(u) > 0$ for $u > 0$. Then any two positive solutions of* (2.1) *cannot intersect.*

Proof. Given two solutions $r_1(u)$ and $r_2(u)$, one defines

$$\Psi(u) = P_1(u)Q_2(u) - P_2(u)Q_1(u) = r_1^n(u)E_1(u)Q_2(u) - r_2^n(u)E_2(u)Q_1(u),$$

and shows that by two different evaluations $\Psi(\bar{u})$ is both negative and positive. It follows that the point $\bar{u}$ does not exist, i.e. the solutions do not intersect. Indeed, by Lemma 2.3 and the definition of $\bar{u}$, $E_2(\bar{u}) = E_1(\bar{u}) > 0$, while

$$r_1^n(\bar{u})Q_2(\bar{u}) - r_2^n(\bar{u})Q_1(\bar{u}) = r_1^{n-1}(\bar{u})r_2^{n-1}(\bar{u})\frac{r_1(\bar{u}) - r_2(\bar{u})}{r_1'(\bar{u})} < 0,$$

i.e. $\Psi(\bar{u}) = E_1(\bar{u})[r_1^n(\bar{u})Q_2(\bar{u}) - r_2^n(\bar{u})Q_1(\bar{u})] < 0$.

On the other hand, let us define $\phi(u) = P_1(u)Q_2(\bar{u}) - P_2(u)Q_1(\bar{u})$. Since $P_1(\alpha_1) = 0$ and $P_2(\alpha_1) > 0$ (by Lemma 2.5), we see that $\phi(\alpha_1) = -P_2(\alpha_1)Q_1(\bar{u}) > 0$. By (2.13) and Lemmas 2.2 and 2.4, we see that on the interval $(\bar{u}, \alpha_1)$

$$\begin{aligned} \phi'(u) &= n\Phi(u)\big[Q_1(u)Q_2(\bar{u}) - Q_2(u)Q_1(\bar{u})\big] \\ &= n\Phi(u)Q_2(u)Q_2(\bar{u})\big[S(u) - S(\bar{u})\big] < 0. \end{aligned} \tag{2.14}$$

It follows that $\phi(\bar{u}) > 0$, but $\phi(\bar{u}) = \Psi(\bar{u}) < 0$, a contradiction. □

The condition $\Phi(u) > 0$ may be written as

$$\frac{f'(u)F(u)}{f^2(u)} < \frac{n+2}{2n}. \tag{2.15}$$

It holds for many positive functions $f(u)$.

Example 1. $f(u) = u^p + u^q$, with $1 \leqslant p < q < \frac{n+2}{n-2}$. We show that (2.15) holds, provided that $q - p < 1$. Compute

$$f'(u)F(u) = \frac{p}{p+1}u^{2p} + \left(\frac{q}{p+1} + \frac{p}{q+1}\right)u^{p+q} + \frac{q}{q+1}u^{2q}.$$

Clearly, $\frac{p}{p+1} < \frac{q}{q+1}$, and we also have

$$\frac{q}{p+1} + \frac{p}{q+1} < 2\frac{q}{q+1},$$

since $(q-p)^2 < q - p$. It follows that $f'(u)F(u) < \frac{q}{q+1}f^2(u)$, i.e.

$$\frac{f'(u)F(u)}{f^2(u)} < \frac{q}{q+1} < \frac{n+2}{2n},$$

since $q < \frac{n+2}{n-2}$. Of course, for $n \geqslant 6$ the condition $q - p < 1$ holds automatically.

Example 2. $f(u) = u^p + a$, with $p > 1$. Calculate

$$f'(u)F(u) - \frac{n+2}{2n} f^2(u) = \left(\frac{p}{p+1} - \frac{n+2}{2n}\right) u^{2p} + a\left(p - \frac{n+2}{n}\right) u^p - \frac{n+2}{2n} a^2.$$

On the right we have a quadratic polynomial in u^p, whose leading coefficient is negative, provided that $p < \frac{n+2}{n-2}$. This quadratic is negative for all u, and any $a > 0$, if

$$\left(p - \frac{n+2}{n}\right)^2 < -4\left(\frac{p}{p+1} - \frac{n+2}{2n}\right)\frac{n+2}{2n}. \tag{2.16}$$

The difference between the right-hand side and the left-hand side of (2.16) is $\frac{p^2(4+n-np)}{n(p+1)}$. It follows that (2.16) is satisfied, and hence condition (2.15) holds for

$$1 < p < \frac{n+4}{n}. \tag{2.17}$$

(Observe that $\frac{n+4}{n} < \frac{n+2}{n-2}$ for all $n \geqslant 1$.)

Recall that we have denoted $E(r) = \frac{1}{2} u'^2(r) + F(u(r))$, and that $E'(r) = -\frac{n-1}{r} u'^2(r)$. Compute

$$\left(r^k E(r)\right)' = r^{k-1}\left[\frac{k}{2} u'^2 + kF(u)\right] - (n-1) r^{k-1} u'^2.$$

If we choose here $k = 2(n-1)$ we obtain the following identity of L.A. Peletier and J. Serrin [16]

$$\left(r^{2(n-1)} E(r)\right)' = 2(n-1) r^{2n-3} F\big(u(r)\big). \tag{2.18}$$

The following result is due to L.A. Peletier and J. Serrin [15], who considered ground state solutions. However, their argument works equally well for Dirichlet problems, as was observed previously by M.K. Kwong and L. Zhang [11].

Theorem 2.2. *(See [15].) Assume that $F(u) < 0$ for $0 < u < \beta$. Let r_0 be a point of intersection of two positive solutions of* (2.1) *$u_1(r)$ and $u_2(r)$, with $u_0 = u_1(r_0) = u_2(r_0)$. Then*

$$u_0 > \beta. \tag{2.19}$$

Proof. Assume that $u_1'(r_0) > u_2'(r_0)$. Recall that given the intersection point (r_0, u_0), we can find $\bar{u}$, satisfying (2.4). Let $\bar{r}_1$ and $\bar{r}_2$ be defined by $u_1(\bar{r}_1) = u_2(\bar{r}_2) = \bar{u}$. We now integrate the identity (2.18) for the first solution over the interval $(r_0, \bar{r}_1)$ (denoting $E_1 = \frac{1}{2} u_1'^2(r) + F(u_1(r))$), and switch to the u variable, $u = u_1(r)$, with $dr = r_1'(u)\,du$

$$\bar{r}_1^{2(n-1)} E_1(\bar{r}_1) - r_0^{2(n-1)} E_1(r_0) = -2(n-1) \int_{\bar{u}}^{u_0} r_1^{2n-3}(u) F(u) r_1'(u)\,du.$$

Similarly, we integrate the identity (2.18) for the second solution over the interval $(r_0, \bar{r}_2)$, and then subtract from the first formula the second one (with $E_2(r) = E(u_2(r))$, and $k = 2(n-1)$, and observing that $E_1(\bar{r}_1) = E_2(\bar{r}_2)$)

$$\begin{aligned}&\left(\bar{r}_1^k - \bar{r}_2^k\right)E_1(\bar{r}_1) + r_0^k\left(E_2(r_0) - E_1(r_0)\right)\\ &\quad = k\int_{\bar{u}}^{u_0}\left(r_2^{k-1}r_2'(u) - r_1^{k-1}r_1'(u)\right)F(u)\,du.\end{aligned} \tag{2.20}$$

The first term on the left is positive by Lemma 2.3, and the second one is positive, since u_2 is steeper than u_1 at r_0. By (2.4) we have

$$r_1^{k-1}r_1'(u) < r_2^{k-1}r_2'(u) < 0 \quad \text{for } u \in (\bar{u}, u_0).$$

Hence, if we assume that $u_0 \leqslant \beta$, then $F(u) < 0$, and hence the right-hand side of (2.20) is negative, giving a contradiction. □

Corollary. *If, moreover $F(\beta) = 0$, and $f(u) > 0$, $\Phi(u) > 0$ for $u > \beta$, then any two positive solutions of* (2.1) *cannot intersect more than once.*

3. Uniqueness and exact multiplicity results

We study uniqueness and exact multiplicity of positive solutions for the problem (here λ is a positive parameter, and $f(u) \in C^2(\bar{R}_+)$)

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < 1, \qquad u = 0 \quad \text{when } |x| = 1, \tag{3.1}$$

by using the methods of bifurcation theory, as developed in P. Korman, Y. Li and T. Ouyang [8], and further extended by T. Ouyang and J. Shi [13,14]. By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5] positive solutions of (3.1) are radially symmetric, i.e. they satisfy $(r = |x|)$

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \quad r \in (0,1), \qquad u'(0) = 0, \qquad u(1) = 0. \tag{3.2}$$

Moreover, $u'(r) < 0$ for all $r \in (0,1)$, i.e. $u(0)$ is the maximum value of the solution. It turns out that the value of $u(0)$ uniquely identifies both λ and $u(r)$, as follows easily by scaling λ out of (3.2), and using uniqueness result for initial value problems of type (3.2) from L.A. Peletier and J. Serrin [15], see also E.N. Dancer [3]. Hence the solution set of (3.2) can be faithfully depicted by planar curves in $(\lambda, u(0))$ plane. It is customary to refer to these curves as *solution curves*.

By a result of C.S. Lin and W.-M. Ni [12], any non-trivial solution of the corresponding linearized problem is also radially symmetric, and hence the linearized problem takes the form

$$\begin{aligned}&w''(r) + \frac{n-1}{r}w'(r) + \lambda f'(u)w(r) = 0, \quad r \in (0,1),\\ &w'(0) = 0, \qquad w(1) = 0.\end{aligned} \tag{3.3}$$

We call solution $(\lambda_0, u_0(r))$ of (3.2) *non-singular* if the corresponding linearized problem (3.3) admits only the trivial solution $w(r) = 0$, and we call the solution $(\lambda_0, u_0(r))$ *singular*, in case (3.3) admits non-trivial solutions. At a non-singular solution, the implicit function theorem applies, and the solution can be continued to nearby λ, i.e. for $|\lambda - \lambda_0|$ small. At a singular solution it turns out that the bifurcation theorem of M.G. Crandall and P.H. Rabinowitz [2] applies, see [6,8,13]. The Crandall–Rabinowitz bifurcation theorem implies that either the solution curve continues through the singular solution (with λ being either increasing or decreasing), or that a simple turn occurs. To study the direction of the turn, one usually needs to show that any non-trivial solution of the linearized problem (3.3) is of one sign. The following lemma provides a crucial link to the results of the preceding section.

Lemma 3.1. *Let* $(\lambda_0, u_0(r))$ *be a singular solution of* (3.2)*, at which a turn occurs. Assume that any two positive solutions of* (3.2) *do not intersect* (*for all* λ)*. Then any non-trivial solution of the linearized problem* (3.3) *is of one sign, i.e. we may assume that*

$$w(r) > 0 \quad \text{for all } r \in [0, 1). \tag{3.4}$$

Proof. The Crandall–Rabinowitz theorem [2] provides a detailed description of the solution curve in a neighborhood of a turning point. Namely, in a neighborhood of $(\lambda_0, u_0(r))$ we have

$$\lambda = \lambda_0 + \tau(s) \quad \text{and} \quad u = u_0 + sw + \phi(s),$$

where the smooth functions $\tau(s)$ and $\phi(s)$ satisfy $\tau(0) = \tau'(0) = 0$ and $\phi(0) = \phi'(0) = 0$, and s is some parameter, with $|s| < \epsilon$ (usually, $s = u(0) - u_0(0)$). Since a turn occurs, $\tau(s)$ is of one sign, say $\tau(s) > 0$. Hence for every fixed $\lambda > \lambda_0$, and λ close to λ_0, we can find two solutions $u_1 = u_0 + s_1 w + o(s_1)$ and $u_2 = u_0 - s_2 w + o(s_2)$, with some s_1, s_2 positive and small. If we assume that w changes sign, then u_1 and u_2 would have to intersect for small s_1 and s_2 (since $u_1 - u_2 \simeq (s_1 + s_2)w$ for $|s|$ small), contrary to our assumption. □

We consider positive solutions of ($x \in R^n$)

$$\Delta u + \lambda\left(u^p + u^q\right) = 0 \quad \text{for } |x| < 1, \qquad u = 0 \quad \text{when } |x| = 1, \tag{3.5}$$

where $1 < p < q < s_n$, with $s_n = \frac{n+2}{n-2}$ for $n > 2$, and $s_n = \infty$ for $n = 2$, and λ is a positive parameter. We shall assume $n \geqslant 2$, since for $n = 1$ uniqueness is easy to prove for all $1 < p < q < \infty$. The following result is due essentially to L. Erbe and M. Tang [4], although we add some information on the solution curves. (The main point is that our proof is much shorter.)

Theorem 3.1. *If* $n \geqslant 6$*, or if* $2 \leqslant n \leqslant 5$ *and* $q - p < 1$*, the problem* (3.5) *has a unique positive solution for all* $\lambda > 0$*. Moreover, all positive solutions lie on a unique smooth solution curve, which tends to infinity as* $\lambda \to 0$*, and to zero when* $\lambda \to \infty$*.*

Proof. As is well known, there is a curve of positive solutions bifurcating from infinity at $\lambda = 0$. We now continue this curve for increasing λ. At non-singular solutions we apply the implicit function theorem. Now, suppose a singular solution $(\lambda_0, u_0(r))$ is reached. If no turn occurs at this solution, we continue the solution curve forward in λ. Now suppose a turn occurs. By Example 1, solutions of (3.5) cannot intersect, and then by Lemma 3.1, $w(r) > 0$. Since

$f(u) = u^p + u^q$ is convex, it follows that at *any* turning point a turn to the left must occur in $(\lambda, u(0))$ plane, see [8,13]. But this means that no turn can occur at $(\lambda_0, u_0(r))$, since no more turns would be possible, while if λ is always decreasing, the solution curve has no place to go (multiplying (3.5) by u and integrating, we see that the solution curve cannot approach the point $(\lambda = 0, u = 0)$). Hence, there are no turns on the solution curve, implying both existence and uniqueness of solutions. Along the curve, $u(0, \lambda)$ is monotone decreasing, see [8,13], and if its limit was not zero, as $\lambda \to \infty$, we would have a contradiction by the Sturm comparison theorem (solution would have to become sign-changing for large λ). $\square$

The same approach provides a partial result in the case when $0 < p < 1 < q < \frac{n+2}{n-2}$, provided that $q - p < 1$ (i.e. it implies the conclusions of Theorem 3.2 below under the additional assumption that $q - p < 1$). The nonlinearity $f(u) = u^p + u^q$ is then *concave-convex*, i.e. $f(u)$ is concave on $(0, u_0)$ and convex on (u_0, ∞), for some $u_0 > 1$. The optimal result given by P. Korman [7], based on Adimurthi, F. Pacella and S.L. Yadava [1], and independently by M. Tang [20], is the following.

Theorem 3.2. *Consider the problem* (3.5), *with* $0 < p < 1 < q < \frac{n+2}{n-2}$. *There is a critical* $\lambda_0 > 0$, *such that for* $\lambda > \lambda_0$ *there are no positive solutions, for* $\lambda = \lambda_0$ *there is exactly one positive solution, and for* $\lambda < \lambda_0$ *there are exactly two positive solutions. Moreover, all positive solutions lie on a unique solution curve, which has two branches for* $\lambda < \lambda_0$, *denoted by* $0 < u^-(r, \lambda) < u^+(r, \lambda)$, *with* $u^-(r, \lambda)$ *strictly monotone increasing in* λ, *and* $\lim_{\lambda \to 0} u^+(0, \lambda) = \infty$.

We now turn to our main exact multiplicity result. It covers many $f(u)$, but does not include the preceding result (except when $q - p < 1$). It basically says that the solution curve in $(\lambda, u(0))$ plane is parabola-like, with a turn to the left. Recall that we assume $f(u) \in C^2(\bar{R}_+)$.

Theorem 3.3. *For the problem* (3.1) *we assume that* $f(u) > 0$ *and* $\Phi(u) > 0$ *for* $u > 0$, *and either* $f(0) > 0$ *and* $f(u)$ *is convex, or* $f(0) \geqslant 0$ *and* $f(u)$ *is concave-convex. Assume finally there is* $\beta \geqslant 0$ *so that* $uf'(u) - f(u) > 0$ *on* (β, ∞). *Then for the problem* (3.1) *there exist three critical numbers* $\lambda_0 \geqslant 0$, $\lambda_\infty \geqslant 0$ *and* $\lambda^* > 0$, *with* $\max(\lambda_0, \lambda_\infty) < \lambda^*$, *such that there are no positive solutions for* $0 \leqslant \lambda \leqslant \min(\lambda_0, \lambda_\infty)$ *and for* $\lambda > \lambda^*$, *there is exactly one positive solution for* $\min(\lambda_0, \lambda_\infty) < \lambda \leqslant \max(\lambda_0, \lambda_\infty)$ *or* $\lambda = \lambda^*$, *and for* $\max(\lambda_0, \lambda_\infty) < \lambda < \lambda^*$ *there are exactly two positive solutions. Moreover, all positive solutions lie on a unique smooth solution curve, which has two branches for* $\lambda < \lambda^*$, *denoted by* $0 < u^-(r, \lambda) < u^+(r, \lambda)$, *with* $u^-(r, \lambda)$ *strictly monotone increasing in* λ *and* $\lim_{\lambda \to \lambda_0} u^-(0, \lambda) = 0$, *while* $\lim_{\lambda \to \lambda_\infty} u^+(0, \lambda) = \infty$. *If, moreover,* $\lim_{u \to \infty} \frac{f(u)}{u} = \infty$, *then* $\lambda_\infty = 0$. *If, moreover,* $\lim_{u \to 0} \frac{f(u)}{u} = \infty$, *then* $\lambda_0 = 0$.

Proof. We begin the solution curve from the zero solution. If $f(0) > 0$ then solution curve starts from $(\lambda = 0, u = 0)$ by the implicit function theorem. If $f(0) = 0$ and $\lim_{u \to 0} \frac{f(u)}{u} = \infty$, then solution curve again starts from $(0, 0)$, this time existence is proved by minimization of the corresponding functional. If $f(0) = 0$ and $\lim_{u \to 0} \frac{f(u)}{u} = f'(0) > 0$, then we have standard bifurcation from zero at $(\lambda_1 / f'(0), 0)$, where λ_1 is the principal eigenvalue of the Laplacian on the unit ball. (And the final possibility, $f(0) = f'(0) = 0$, is inconsistent with concavity and positivity of $f(u)$ for small u.) We now continue the solution curve for increasing λ, by applying the implicit function theorem. We cannot continue this curve for all λ, since our assumptions imply that $f(u) > au$ for all $u > 0$ and some $a > 0$, which implies that the problem (3.1) has no solutions for large λ (just multiply by the principal eigenfunction and integrate). Let λ^* be the supremum of λ's for which we can continue the curve. The solution curve cannot go to infinity at $\lambda = \lambda^*$, since our assumption $uf'(u) - f(u) > 0$ on (β, ∞) implies that any bifurcation from infinity is supercritical (i.e. forward in λ), see Corollary 3.8 in T. Ouyang and J. Shi [14]. Hence we have a turning point at $\lambda = \lambda^*$.

In view of Theorem 2.1 and Lemma 3.1, any non-trivial solution of the linearized equation is of one sign. As in [8,14] this allows one to show that the solution curve can make only turns to the left (and hence there is only one turning point, the one at $\lambda = \lambda^*$). It is here that we use positivity of solutions of the linearized problem, and our convexity assumptions on $f(u)$. If we assume further that $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, then the solution curve can go to infinity only at $\lambda = 0$, by the Sturm comparison theorem. The rest of proof is just standard bifurcation analysis, see the proof of Theorem 6.21 in [14]. $\square$

Example. $f(u) = u^p + a$. In view of Example 2, our result applies if $1 < p < \frac{n+4}{n}$, for any $n \geqslant 1$, and any constant $a > 0$. Here $\lambda_0 = \lambda_\infty = 0$. Theorem 6.21 in T. Ouyang and J. Shi [14] also applies (and gives the same conclusions) if $n \geqslant 4$ and $1 < p < \frac{n}{n-2}$. Our result is better for all $n \neq 4$ (and is the same for $n = 4$).

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