



A computer assisted study of uniqueness of ground state solutions

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ABSTRACT

For the problem (here $u = u(x)$)

$$\Delta u - u^p + \alpha u^q + \beta u^r = 0, \quad x \in \mathbb{R}^n, \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

with constants $1 \leq p < q < r < \frac{n+2}{n-2}$, and $\alpha, \beta > 0$, uniqueness of radial solution (called *ground state solution*) is not known. We present a procedure, which opens the way to produce computer assisted proofs of uniqueness for specific p, q, r , and n .

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1. Introduction

We study positive solutions of semilinear problem (here $x \in \mathbb{R}^n$, $u = u(x)$)

$$\Delta u + \lambda f(u) = 0, \tag{1.1}$$

depending on a positive parameter λ , with $f(u) \in C^1[0, \infty)$. If one considers this equation on a unit ball B around the origin in \mathbb{R}^n , with Dirichlet boundary condition $u = 0$ on ∂B , then by the classical theorem of Gidas et al. [1], any positive classical solution is radially symmetric, i.e., $u = u(r)$ with $r = |x|$, and moreover $u'(r) < 0$. I.e. $u(r)$ satisfies

$$\begin{aligned} u'' + \frac{n-1}{r} u' + \lambda f(u) &= 0, & u(r) &> 0, & u'(r) < 0, & \text{for } r \in (0, 1), \\ u'(0) &= u(1) = 0. \end{aligned} \tag{1.2}$$

This remarkable theorem does not require any assumptions on $f(u)$, except for differentiability. If one considers Neumann problem on a unit ball, then there is no analog of Gidas et al.'s theorem, but we may still consider an important class of radial and decreasing solutions:

$$\begin{aligned} u'' + \frac{n-1}{r} u' + \lambda f(u) &= 0, & u(r) &> 0, & u'(r) < 0, & \text{for } r \in (0, 1), \\ u'(0) &= u'(1) = 0. \end{aligned} \tag{1.3}$$

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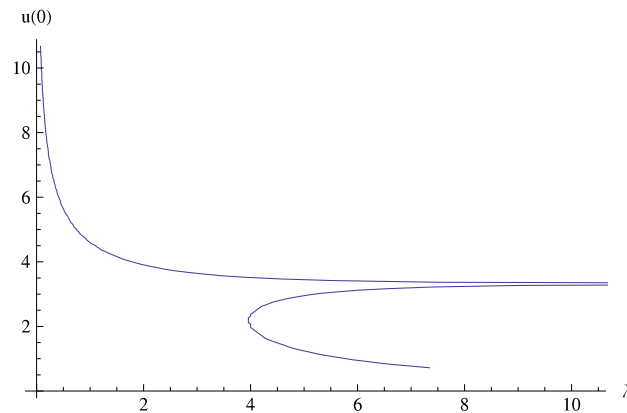


Fig. 1. The Neumann and the Dirichlet curves.

We can also consider Eq. (1.1) on all of \mathbb{R}^n , and look for “ground-state” solutions, satisfying $u(x) \rightarrow 0$ for $|x| \rightarrow \infty$ (assuming that $f(0) = 0$). There are a number of analogs of Gidas et al.’s theorem in this case, under various conditions on $f(u)$, see e.g. [2], while for some $f(u)$ such result is not available. But in any case, we can still consider radial and decreasing solutions:

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad u(r) > 0, \quad u'(r) < 0, \quad \text{for } r \in (0, \infty), \quad (1.4)$$

$$u'(0) = u(\infty) = 0.$$

There is no parameter λ here, since it can be removed by scaling r . Now suppose that (α and β are positive constants)

$$f(u) = -u^p + \alpha u^q + \beta u^r, \quad \text{with constants } 1 \leq p < q < r < 2^*, \quad (1.5)$$

where $2^* = \frac{n+2}{n-2}$ for $n \geq 3$, and $2^* = \infty$ if $n = 1, 2$ (i.e., for some $b > 0$, $f(u) < 0$ on $(0, b)$, and $f(u) > 0$ on (b, ∞)), and let us ask: what is the structure of the set of solutions for the problems (1.2)–(1.4)? It appears that the answer is not known for all three of these problems. For the Neumann problem exact multiplicity has not been studied much. For the other two problems, uniqueness and exact multiplicity of solutions have been studied extensively, but this class of equations has not been covered (for both the Dirichlet and the ground state problems one usually needs that the function $K(u) \equiv \frac{uf'(u)}{f(u)}$ is decreasing, see [3–7]), which does not hold here.

In Fig. 1 we present numerically computed solution curves for the Dirichlet and Neumann problems, in case $f(u) = -u + u^3 + 2u^4$, and $n = 3$. It is known that $u(0)$, the maximum value of solution, uniquely identifies the solution pair $(\lambda, u(r))$ for both the Dirichlet and the Neumann problems, see [8], so we plot $u(0)$ versus λ . Here is what the picture tells us. The lower curve is the solution curve of the Neumann problem (1.3). We shall refer to it as *Neumann curve*. It bifurcates from the constant solution $u = b \simeq 0.657$ to the left, it makes exactly one turn to the right, and then continues for all λ , with $u(0)$ tending to a limit, $u(0) \rightarrow \bar{u}$. The upper curve is the *Dirichlet curve*, the solution curve of (1.2). It bifurcates from infinity at $\lambda = 0$, and continues without turns for all $\lambda > 0$, with $u(0) \rightarrow \bar{u}$. The initial value $u(0) = \bar{u}$ gives us the unique ground state. We saw a similar picture for all other $f(u)$ of class (1.5) that we have tried. The question is: can we justify that what we see is actually true, by using some additional numerical computations? Observe that $f'(u) < 0$ for small u , while $u(r)$ is small for large r . We show that if solution of the variational equation is negative and decreasing at one point in the region where $f'(u(r)) < 0$, then uniqueness of ground state follows. This condition is easy to verify numerically. So that one can “compute uniqueness” for any specific equation of our class.

Our analysis and numerical computations provide strong evidence of uniqueness of ground state solutions for the above example, as well as all other examples that we tried. We believe that a computer assisted proof of uniqueness can be produced this way, but we did not carry this out. Our computations suggest that the ground state is unique for all $1 \leq p < q < r < 2^*$, and $n \geq 1$, but of course our approach can be applied only on a case by case basis.

We mention that a number of authors have been using computer assisted proofs recently, see [9–14,4,15]. For theoretical studies of ground state solutions please consult [16–20].

2. Shooting and scaling

Let us consider “shooting”, i.e., we solve the initial value problem

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0 \quad (2.1)$$

for various values of $\alpha > 0$. We shall consider only those α 's for which $f(\alpha) > 0$, and consequently $u''(0) = -\frac{1}{n}f(\alpha) < 0$, by the L'Hospital rule. It follows that for small $r > 0$, the function $u(r)$ is positive and decreasing:

$$u(r) > 0 \quad \text{and} \quad u'(r) < 0. \quad (2.2)$$

We continue the solution $u(r)$ for increasing r , while (2.2) holds. I.e. we stop as soon as (2.2) is violated. This may happen in two ways: solution becomes zero, or it develops zero slope. If $u(R) = 0$, then we make a change of variables $r = R\xi$, and see that the function $u(\xi)$ satisfies the Dirichlet problem (1.2), with $\lambda = R^2$. Clearly, $u|_{\xi=0} = \alpha$. We then say that the initial value α belongs to *Dirichlet range*. Similarly, if $u'(R) = 0$, then using the same change of variables $r = R\xi$, we see that $u(\xi)$ satisfies the Neumann problem (1.3), with $\lambda = R^2$. Then α belongs to *Neumann range*. Conversely, all values of $\alpha = u(0)$ assumed by any solution of the Dirichlet problem (1.2) belong to the Dirichlet range (with similar correspondence of Neumann curve and Neumann range). So, $\alpha = u(0)$ belongs to Dirichlet (Neumann) range if and only if for some λ there is a solution of (1.2) and (1.3), with $u(0) = \alpha$.

We assume that $f(u)$ satisfies

$$\begin{aligned} f(u) &\in C^1[0, \infty), \quad f(0) = 0, \quad \text{and for some } b > 0 \text{ we have:} \\ f(u) &< 0 \quad \text{on } (0, b), \quad \text{and} \quad f(u) > 0 \quad \text{for } b < u < \infty, \quad f'(0) < 0, \quad f'(b) > 0. \end{aligned} \quad (2.3)$$

We shall also assume that $f(u)$ is sub-critical, and asymptotic to a power, i.e., for some $1 < p < \frac{n+2}{n-2}$, and $a > 0$, we have

$$\lim_{u \rightarrow \infty} \frac{f(u)}{u^p} = a. \quad (2.4)$$

The following lemma is known (see e.g., [21]), so we just sketch its proof.

Lemma 2.1. Assume that $f(u)$ satisfies the conditions (2.3) and (2.4). Then all α 's sufficiently large belong to the Dirichlet range, i.e., the solution of (2.1) is decreasing, and becomes zero at some $r = r_0$.

Proof. Write $f(u) = g(u) + au^p$, with $\lim_{u \rightarrow \infty} \frac{g(u)}{u^p} = 0$. In (2.1) we set $u = \alpha z$, $r = \frac{1}{\alpha^\beta} \xi$. Letting $2\beta = p - 1$, and using primes for the derivatives of $z(\xi)$, we get

$$z'' + \frac{n-1}{\xi} z' + \frac{g(\alpha z)}{\alpha^p} + az^p = 0, \quad z(0) = 1, \quad z'(0) = 0. \quad (2.5)$$

For the problem

$$z'' + \frac{n-1}{\xi} z' + az^p = 0, \quad z(0) = 1, \quad z'(0) = 0$$

it is well known that solution is decreasing, becoming zero at some $r = R$, and then it continues to decrease, taking negative values. By continuity, for α large, the solution of (2.5) is decreasing, becoming zero near $r = R$, since the term $\frac{g(\alpha z)}{\alpha^p}$ is uniformly small for $r \in (0, R)$, for α large. \square

Remark. The result is not true for super-critical $f(u)$. Indeed, consider

$$u'' + \frac{n-1}{r} u' - u^{\frac{n+2}{n-2}} + u^p = 0, \quad u(0) = \alpha, \quad u'(0) = 0 \quad (2.6)$$

with $p > \frac{n+2}{n-2}$. Our nonlinearity $f(u) = -u^{\frac{n+2}{n-2}} + u^p$ satisfies the condition (2.3). We claim that for any $\alpha > 0$, $u(r) > 0$ for all r , which means that the Dirichlet range is empty. We recall the well-known Pohozaev's identity, which is easy to check, using Eq. (2.1). It says that with $H(r) \equiv \frac{1}{2}[ru'^2 + (n-2)uu'] + rF(u)$, we have

$$H' + \frac{n-1}{r} H = nF - \frac{n-2}{2} u f(u) = u^{p+1} \left(\frac{n}{p+1} - \frac{n-2}{2} \right) < 0,$$

or

$$[r^{n-1} H(r)]' < 0. \quad (2.7)$$

Assume on the contrary that $u(r)$ vanishes at some $r = \xi$. Integrating (2.7) over $(0, \xi)$, we get

$$0 \leq \frac{1}{2} \xi^n u'^2(\xi) < 0,$$

a contradiction.

We shall need the linearized problem for the Neumann problem (1.3):

$$w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0, \quad \text{for } r \in (0, 1), \quad w'(0) = w'(1) = 0. \quad (2.8)$$

We define a number θ by the relation $\int_0^\theta f(u) du = 0$. We now recall the following results from [22].

Theorem 2.1. Consider the Neumann problem (1.3), with $f(u)$ satisfying (2.3). There is a curve of solutions, bifurcating from the trivial solution $u = b$ at $\lambda = \lambda_1/f'(b)$. This curve continues globally, with the maximum value $u(0)$ strictly increasing along this solution curve, and with λ eventually tending to infinity. Moreover, $\lim_{\lambda \rightarrow \infty} u(0, \lambda) > \theta$. Any other solution curve of (1.3) has at least one turn to the right, with both branches extending to infinity along the λ axis (i.e., λ tends to infinity on both branches).

If, in addition, the linearized problem (2.8) admits only the trivial solution, then the solution curve bifurcating from $u = b$ does not turn, and there are no other solution curves of (1.3).

Theorem 2.2. With $f(u)$ given by (1.5), the solution set for the Dirichlet problem (1.2) consist of a single curve, extending to infinity. Assume that the solution set for the Neumann problem (1.3) also consists of a single curve. Then the set of $\alpha = u(0)$ giving ground state solutions (i.e., solutions of (1.4)) is either one point or a bounded interval.

3. Numerical computations

We assume throughout this section that $f(u)$ is given by (1.5), so that Theorems 2.1 and 2.2 apply. If there is an interval of α 's leading to ground states $u(r, \alpha)$ (i.e., solutions of (1.4)), then $w \equiv u_\alpha(r, \alpha)$ satisfies

$$\begin{aligned} w''(r) + \frac{n-1}{r} w'(r) + f'(u(r))w(r) &= 0, \quad 0 < r < \infty \\ w'(0) &= 0, \quad w(0) = 1, \quad w(\infty) = 0. \end{aligned} \quad (3.1)$$

Similarly, if there is a turn on a curve of solutions of Neumann problem (1.3), then the corresponding linearized problem (1.4) has a non-trivial solution, i.e., after rescaling

$$\begin{aligned} w''(r) + \frac{n-1}{r} w'(r) + f'(u)w(r) &= 0, \quad 0 < r < \lambda^2 \\ w'(0) &= 0, \quad w(0) = 1, \quad w'(\lambda^2) = 0, \end{aligned} \quad (3.2)$$

for some $\lambda > 0$. Both possibilities can be usually ruled out by the same set of computations, justifying uniqueness of ground state solution. Ruling out the possibility of turns on Neumann curves implies that the Neumann curve bifurcating from $u = b$ is unique by Theorem 2.1, and hence by Theorem 2.2 the set of α 's giving rise to ground states is either a point or an interval. And it is a point, once we rule out the case of an interval. To rule out an interval of ground states, we need to show that the problem (3.1) has no solution. Our crucial observation is the following: not only we do not need to know $w(r)$ all the way to infinity, we do not need to compute $w(r)$ too far. Namely, as soon as $u(r)$ enters the region where $f'(u(r)) < 0$, if at some point r_0 we have $w(r_0) < 0$ and $w'(r_0) < 0$, then we have $w(r) < 0$ and $w'(r) < 0$ for all $r > r_0$, since $w(r)$ cannot have points of local negative minimums, as is clear from Eq. (3.1).

Proposition 1. Assume that at some point $r_0 > 0$ we have $w(r_0) < 0$ and $w'(r_0) < 0$, and moreover $f'(u) < 0$ for $u < u(r_0)$. Then

$$w(r) < 0 \quad \text{and} \quad w'(r) < 0, \quad \text{for all } r > r_0.$$

Proof. Assuming otherwise, we could find a point $r_1 > r_0$ (which is either a point of local minimum or an inflection point), at which $w(r_1) < 0$, $w'(r_1) = 0$, and $w''(r_1) \geq 0$. Then at $r = r_1$ the left hand side of Eq. (3.1) is positive, a contradiction. \square

Example. $f(u) = -u + u^3 + 2u^4$, $n = 3$. We wish to prove uniqueness of ground state, i.e., of solution to (1.4). One computes the values of $b \simeq 0.657$, and $\theta \simeq 0.904$. We consider “shooting” for the problem

$$u'' + \frac{2}{r}u' - u + u^3 + 2u^4 = 0, \quad u(0) = \alpha, \quad u'(0) = 0. \quad (3.3)$$

According to Theorem 2.2, the Dirichlet range is (d, ∞) , for some $d > 0$, see also Fig. 1, where the Dirichlet curve is the one above. Our numerical computations show that $d < 3.4$. Theorem 2.1 shows that the Neumann curve bifurcating from $u \equiv b \simeq 0.657$ extends above $u(0) = \theta \simeq 0.904$. Hence the interval (b, θ) is definitely in the Neumann range (i.e., decreasing positive solutions of (3.3) develop zero slope). It follows that intervals of ground states, or Neumann branches with a turn, are only possible for $\alpha \in (\theta, 3.4)$. This is a finite interval, for which one can perform and validate computations (of the type we give below) to show that the Neumann range is actually (b, d) , and then $\alpha = d$ gives rise to the unique ground state solution. Actually, we can narrow this interval further. It is clear from our computations that the maximum value on the Neumann curve goes much higher than $\theta \simeq 0.904$, see the lower curve in Fig. 1. This leaves us with the interval $(3.3, 3.4)$, where intervals of ground states, or Neumann branches with a turn, might happen. (Computations show that the Neumann range goes above 3.3 and the Dirichlet range goes below 3.4, but we leave some “safety”, so that the numbers 3.3 and 3.4 could be validated, using estimates and continuity arguments.)

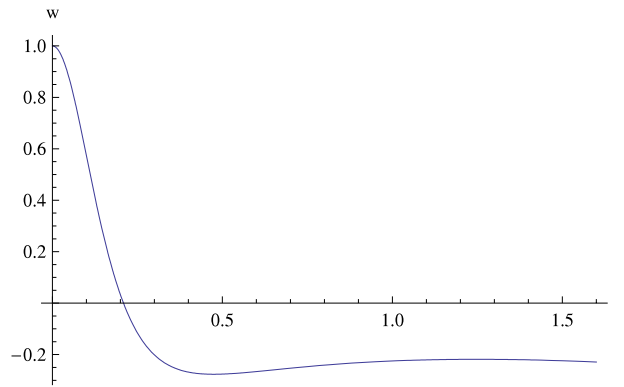


Fig. 2. Solution of the variational problem (3.4).

We took $\alpha = 3.35$ in the middle of the range, and computed the solution $u(r)$ of (3.3), and then solved the corresponding variational equation

$$w''(r) + \frac{2}{r} w'(r) + (-1 + 3u^2(r) + 8u^3(r))w(r) = 0, \quad w(0) = 1, \quad w'(0) = 0 \quad (3.4)$$

on the interval $r \in [0, 1.6]$. The result is plotted in Fig. 2.

We see that $w(r)$ is decreasing for small r , then nearly flat for a while, and then at, say $r = 1.6$, it is negative and decreasing, with $w(1.6) \simeq -0.229$, and $w'(1.6) \simeq -0.0587$. We claim that $w(r)$ continues to decrease for all $r > 1.6$, which implies that $w(r)$ cannot be solution of the problem (3.1). One computes $u(1.6) \simeq 0.089$. Since $u(r)$ is decreasing, it follows that

$$-1 + 3u^2(r) + 8u^3(r) < 0 \quad \text{for all } r > 1.6.$$

Now, $w(r)$ is negative and decreasing at $r = 1.6$, and it cannot turn around to go to zero as $r \rightarrow \infty$, since at the turning point, where $w'(r) = 0$ the left hand side of the equation in (3.4) would be positive, a contradiction. Our computations showed similar results for other α 's in the critical range (3.3, 3.4), and that was to be expected, since small changes in α produce small changes in both $u(r)$ and $w(r)$. These computations rule out the possibility of an interval of ground states.

Fig. 2 does not rule out yet the possibility of a Neumann curve with a turn. Indeed, $w'(r)$ vanishes on the interval (1, 1.5), but not for $r > 1.5$. So, if the Neumann problem

$$u'' + \frac{n-1}{r} u' + f(u) = 0, \quad \text{for } 0 < r < R, \quad u'(0) = u'(R) = 0 \quad (3.5)$$

has a singular solution, then $R \in (1, 1.5)$. However, our computations show that for $\alpha \in (3.3, 3.4)$ solution of (3.3) is strictly decreasing for $r \in (0, 1.5)$, and so the problem (3.5) has no solution for $R \in (1, 1.5)$. Hence, the problem (3.2) has no non-trivial solutions.

Mathematica's computations are very accurate, moreover using continuity arguments, with some standard estimates, it is straightforward, albeit tedious, to complete a computer assisted proof of uniqueness of the ground state, i.e., solution of (1.4). We used *Mathematica*'s **NDSolve** command to solve the problems (3.3) and (3.4). To avoid a singularity at $r = 0$, we took a small $h = 0.0001$, and approximated $u(h) \simeq \alpha + \frac{1}{2}u''(0)h^2 = \alpha - \frac{f(\alpha)}{2n}h^2$, and $u'(h) \simeq u''(0)h = -\frac{f(\alpha)}{n}h$, and then used *Mathematica* to compute $u(r)$ for $r > h$ (and we had solved the problem (3.4) similarly). This approximation is reasonably accurate, provided α is not too large, as was the case in our computations.

Finally, we mention the example of $f(u) = -u + u^3$, $n = 3$, considered in Korman [22]. For this $f(u)$, the function $\frac{uf'(u)}{f(u)}$ is decreasing, and so uniqueness of the ground state is known, see Kwong and Zhang [23], or Ouyang and Shi [7]. In this case, solution of the variational equation $w(r)$ was strictly decreasing, making the argument easier. Looking at Fig. 2, one understands why uniqueness of ground state is tough (and in fact not known) for the class of functions (1.5). One needs to prove that $w(\infty) \neq 0$, but for a while $w(r)$ is negative, small in absolute value, and increasing.

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