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ON COMPUTATION OF SOLUTIONS OF ELLIPTIC SYSTEMS

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1. INTRODUCTION

We apply the monotone iteration techniques to solve boundary value problems for nonlinear elliptic equations and systems. We replace the discretized elliptic equations by the corresponding parabolic ones, and show that solutions of the parabolic problems tend to those of the elliptic ones, assuming that the time step τ and the spatial step h are sufficiently small, $\tau = \tau(h)$, and that the initial guess is either a super- or subsolution. This allows us to solve wide classes of elliptic problems without ever having to solve linear systems. The algorithm is easily implemented, and good accuracy is usually achieved at very moderate final times (see sec. 5).

In section 4 we take up systems of equations with singular right-hand sides, for which there are no apparent supersolutions. We present a technique, which allows one to conclude existence of solutions by a computation.

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2. SEMILINEAR EQUATIONS AND SYSTEMS

We are interested in numerical solution of the boundary value problems of the type

(2.1) $\quad -\Delta u = f(x,u) \quad x \in \Omega,\ u = g(x)$ for $x \in \partial\Omega \quad (\Omega \subset R^d)$.

Since our focus is on computations we restrict ourselves to $d = 1$ or 2, although most of our results carry over to arbitrary number of dimensions. We replace (1.1) by the corresponding finite-difference problem (say $d = 2$)

(2.1)′ $\quad -\Delta_h u_n = f(x_n, u_n) \quad x \in \Omega_h,\ u_n = g(x_n)$ for $x_n \in \partial\Omega_h$.

Here we replace the domain Ω by uniform square mesh Ω_h of step size h, denoting $n = (n_1, n_2)$, $x_n = (n_1 h, n_2 h)$ and $u_{n_1,n_2} = u(x_n)$. The Laplacian is replaced by the central difference quotient

$$\Delta_h u_n = \frac{1}{h^2}[u_{n_1+1,n_2} + u_{n_1,n_2+1} - 4u_{n_1,n_2} + u_{n_1-1,n_2} + u_{n_1,n_2-1}],$$

for $d = 2$. We obtain the solution of (2.1)′ as a steady-state of the corresponding parabolic problem ($u_n^k = u_{n_1,n_2}^k$)

$$(2.2) \qquad \frac{u_n^{k+1} - u_n^k}{\tau} = \Delta_h u_n^k + f(x_n, u_n^k) \text{ in } \Omega_h,\ u_n^k = g(x_n) \text{ on } \partial\Omega_h,$$

with properly chosen u_n^0 and the time step size τ.

Next we define the concept of super- and subsolutions for a grid function ϕ_n on the mesh Ω_h.

<u>Definition.</u> A grid function ϕ_n is called a supersolution of (2.1)′ iff

(2.3) $\quad -\Delta_h \phi_n \geq f(x_n, \phi_n) \quad x_n \in \Omega_h,\quad \phi_n \geq g(x_n)$ on $\partial\Omega_h$.

A grid function ψ_n is called a subsolution if it satisfies the opposite inequalities.

Theorem 1. Assume that the problem (2.1)′ possesses a supersolution ϕ_n and a subsolution ψ_n with $\psi_n \le \phi_n$ for all $n = (n_1, n_2)$. Assume also that $f(x,u)$ is continuous on $\Omega \times R$, and $g(x)$ is continuous in a neighborhood of $\partial\Omega$. Then the problem (2.1)′ has a solution u_n with $\psi_n \le u_n \le \phi_n$ for all n. Moreover, if one starts the scheme (2.2) with $u_n^0 = \phi_n$, then for τ sufficiently small the sequence u_n^k is monotone decreasing in k for all n. Similarly, if one starts with $u_n^0 = \psi_n$ then the corresponding sequence, call it v_n^k, is monotone increasing in k and

$$(2.4)\quad \psi_n \le v_n^1 \le v_n^2 \le \dots \le u_n \le \dots \le u_n^2 \le u_n^1 \le \phi_n \quad \text{for all } n.$$

If solution of (2.1)′ is not unique, we obtain existence of a minimal and maximal solutions.

Proof. Letting $u_n^0 = \phi_n$, rewrite (2.2) for d = 2 as

$$(2.5)\quad u_n^{k+1} = u_n^k\left(1-\frac{4\tau}{h^2}\right) + \tau f(x_n, u_n^k) + \frac{\tau}{h^2}[u_{n_1+1,n_2}^k + u_{n_1,n_2+1}^k + u_{n_1-1,n_2}^k + u_{n_1,n_2-1}^k].$$

Denote $w_n^k = u_n^k - u_n^{k+1}$. It follows from (2.2) that $w_n^0 \ge 0$ for all n. We prove by induction that $w_n^k \ge 0$ for all $k \ge 1$. Assume that τ is chosen so small that

$$(2.6)\quad K \equiv 1 - \frac{4\tau}{h^2} - \tau \max_{\Omega \times [a,b]} |f_u(x,u)| > 0,$$

where $a = \min_n \psi_n$, $b = \max_n \phi_n$. Then writing (2.5) for k-1, subtracting and using the mean-value theorem, we estimate

$$w_n^k \ge K w_n^{k-1} + \frac{\tau}{h^2}[w_{n_1+1,n_2}^{k-1} + w_{n_1,n_2+1}^{k-1} + w_{n_1-1,n_2}^{k-1} + w_{n_1,n_2-1}^{k-1}] \ge 0.$$

Hence, $\{u_n^k\}$ is a decreasing sequence, and similarly $\{v_n^k\}$ is an increasing one in k. An argument similar to the above shows that $u_n^k \ge v_n^k$ for all n and k, justifying (2.4). Call $u_n = \lim_{k\to\infty} u_n^k$, $v_n =$

$\lim_{k\to\infty} v_n^k$. Clearly u_n and v_n are the maximal and minimal solutions of (2.1)′. (If w_n is a solution of (2.1)′ with $\psi_n \le w_n \le \phi_n$, then the same argument applied once more implies that $\psi_n^k \le w_n \le \phi_n^k$ for all k and n.)

Next we present a simple application of the theorem 1 to a class of nonlinear problems. First, we recall some simple facts on the eigenvalues of the discrete Laplacian Δ_h.

Lemma 2.1. Consider the eigenvalue problem

(2.7) $\Delta_h u_n + \lambda u_n = 0, \quad \text{in } \Omega_h, \quad u_n = 0 \text{ on } \partial\Omega_h.$

Then all eigenvalues are positive, the smallest one (call it λ_1) is simple and its eigenvector can be chosen to have positive entries (call it ψ_n^1).

Proof (sketch). One easily shows that the maximum principle holds for Δ_h. This implies that the Dirichlet problem for Δ_h is uniquely solvable. Hence the inverse matrix Δ_h^{-1} exists, and all its entries are positive. The rest follows by the Perron-Frobenius theorem.

Proposition 2.1 Consider the problem

(2.8) $\Delta_h u_n + a u_n - f(x_n, u_n) = 0, \text{ in } \Omega_h, \quad u_n = 0 \text{ on } \partial\Omega_h.$

Assume that $a > \lambda_1$ and the function $f(x,u)$ is such that $aM - f(x,M) < 0$ for some $M > 0$ and all x, and $f(x,u) = o(|u|)$ as $u \to$ uniformly in x, $x \in \bar{\Omega}$. Then the problem (2.8) has a positive solution, which can be computed using the scheme (2.2).

Proof. One easily sees that $\phi_n = M$ and $\psi_n = \varepsilon\psi_n^1$ are super- and subsolutions, provided ε is small.

Next we turn to the system case. To simplify the presentation we restrict ourselves to the case of two equations with two

unknowns. Similar approach works for systems of arbitrary size, which are essentially quasimonotone increasing (see [4] for the definition and complete characterization of such systems).

We are interested in solving a discrete version of an elliptic system

$$
\begin{aligned}
&\Delta_h u_n + f(x_n, u_n, v_n) = 0, \quad x \in \Omega_h, \\
(2.9) \qquad &\Delta_h v_n + g(x_n, u_n, v_n) = 0, \quad x \in \Omega_h, \\
&u_n = g_1(x), v_n = g_2(x) \quad \text{for } x \in \partial\Omega_h.
\end{aligned}
$$

Similarly to the case of one equation we shall use the following scheme with suitably chosen u_n^0 and v_n^0,

$$
\begin{aligned}
&\frac{u_n^{k+1} - u_n^k}{\tau} = \Delta_h u_n^k + f(x_n, u_n^k, v_n^k) \text{ in } \Omega_h, \ u_n^k = g_1(x_n) \text{ on } \partial\Omega_h, \\
(2.10) \qquad & \\
&\frac{v_n^{k+1} - v_n^k}{\tau} = \Delta_h v_n^k + g(x_n, u_n^k, v_n^k) \text{ in } \Omega_h, \ v_n^k = g_2(x_n) \text{ on } \partial\Omega_h.
\end{aligned}
$$

<u>Theorem 2.</u> Assume that the problem (2.9) has a pair of coupled super- and subsolutions (ϕ_n^1, ϕ_n^2) and (ψ_n^1, ψ_n^2) respectively, i.e.,

$$
\begin{aligned}
&-\Delta\phi_n^1 \ge f(x_n, \phi_n^1, v) \quad x_n \in \Omega_h, \quad \phi_n^1 \ge g_1(x_n) \quad \text{on } \partial\Omega_h, \\
(2.11) \qquad & \\
&-\Delta\phi_n^2 \ge g(x_n, u, \phi_n^2) \quad \text{in } \Omega_h, \quad \phi_n^2 \ge g_2(x_n) \text{ on } \partial\Omega_h,
\end{aligned}
$$

for all $(u,v) \in V \equiv \{(u,v) \mid \psi_n^1 \le u \le \phi_n^1, \ \psi_n^2 \le v \le \phi_n^2\}$, and the reverse inequalities holding for subsolutions. Assume that either of the two conditions hold in $V \times \Omega_h$:

(i) $f_u \ge 0, \ g_v \ge 0,$

(ii) $f_u \le 0, \ g_v \le 0.$

Assume finally that f, g, g_1, g_2 are continuous functions on $\Omega_h \times V$.

Then the problem (2.9) has a solution $(u_n, v_n) \in V$. Moreover, for τ sufficiently small one can construct monotone sequences converging to a solution, by taking in case (i) $u_n^0 = \psi_n^1$, $v_n^0 = \psi_n^2$ ($u_n^0 = \phi_n^1$, $v_n^0 = \phi_n^2$) to obtain two increasing (decreasing) in k sequences $\{u_n^k\}$ and $\{v_n^k\}$. In case (ii) we can take $u_n^0 = \psi_n^1$, $v_n^0 = \phi_n^2$ ($u_n^0 = \phi_n^1$, $v_n^0 = \psi_n^2$) to obtain an increasing in k sequence for u and a decreasing one for v (decreasing for u and increasing for v).

We omit the straightforward proof of this theorem, since it is similar to that of the theorem 1 (see also [7] for the continuous version of this result). In our numerical experiments we found that convergence usually occurs for more general initial guesses.

3. QUASILINEAR EQUATIONS

We are interested in approximate solution of the Dirichlet problem

$$(3.1) \qquad -\Delta u = f(x, u, u_{x_1}, \ldots, u_{x_d}) \quad \text{for } x \in \Omega, \quad u = g(x) \text{ on } \partial\Omega.$$

We approximate this problem by (taking d = 2 for simplicity)

$$(3.2) \qquad -\Delta_h u_n = f\left(x_n, u_n, \frac{u_{n+e_1} - u_n}{h}, \frac{u_{n+e_2} - u_n}{h}\right) \quad \text{for } x_n \in \Omega_h,$$

and $u_n = g(x_n)$ on $\partial\Omega_h$. Here $e_1 = (1,0)$, $e_2 = (0.1)$.

To solve (3.2) we construct the iterations, $k = 1, 2, \ldots$

$$(3.3) \qquad \frac{u_n^{k+1} - u_n^k}{\tau} = \Delta_h u_n^k + f(x_n, u_n^k, D_h^1 u_n^k, D_h^2 u_n^k) \quad \text{for } x_n \in \Omega_h,$$

and $u_n^k = g(x_n)$ for all k and $x_n \in \partial\Omega_n$. Here we denote

$D_h^i u = \frac{1}{h}(u_{n+e_i} - u_n)$.

<u>Theorem 3.</u> Assume that the function $f(x, u, p_1, p_2)$ is continuous in all arguments and for $x \in \Omega$ and $|u| \le K$ satisfies

$$|f_{p_i}| \le c(1 + |p_1| + |p_2|)^\alpha, \quad i = 1,2 \quad , \alpha < 1, \quad c = c(K).$$

Assume that the problem (3.2) possesses a supersolution $\phi_n = \phi$ defined by

$$(3.5) \qquad -\Delta_h\phi \geq f(x_n,\phi,\phi_{x_1},\phi_{x_2}) \quad x_n \in \Omega_h, \qquad \phi \geq g(x_n) \text{ on } \partial\Omega_h,$$

and a supersolution ψ_n, defined by reversing the above inequalities. Moreover, $\phi_n \geq \psi_n$ for all n. Then for h sufficiently small the problem (3.2) has a solution, which can be approximated using (3.3). Namely, letting $u_n^0 = \phi_n$ ($u_n^0 = \psi_n$) and choosing h and $\tau = \tau(h)$ sufficiently small, we get a decreasing (increasing) in k sequence u_n^k converging to a solution of (3.2).

Proof. Denote by u_n^k the iterates obtained by (3.3) starting with $u_n^0 = \phi_n$, and by v_n^k the ones starting with $v_n^0 = \psi_n$. We claim that

$$(3.6) \qquad \psi_n \leq v_n^1 \leq \ldots \leq v_n^k \leq \ldots \leq u_n^k \leq \ldots \leq u_n^1 \leq \phi_n,$$

from which the proof of the theorem follows as in the theorem 1. The proof of (3.6) is by induction in k. By definition of super- and subsolutions it follows that $\psi_n \leq v_n^1$, $u_n^1 \leq \phi_n$ for all n. Assume (3.6) to be true up to k. From (3.3) express

$$(3.7) \qquad u_n^{k+1} = u_n^k(1-\frac{4\tau}{h^2}) + \frac{\tau}{h^2}(u_{n+e_1}^k + u_{n+e_2}^k + u_{n-e_1}^k + u_{n-e_2}^k)$$
$$\tau f(x_n, u_n^k, \frac{1}{h}(u_{n+e_1}^k - u_n^k), \frac{1}{h}(u_{n+e_2}^k - u_n^k)).$$

Define $w_n^k = u_n^{k+1} - u_n^k$. Using the mean-value theorem we express from (3.7)

$$(3.8) \qquad w_n^k = w_n^{k-1}(1- \frac{4\tau}{h^2} + \tau f_u - \frac{\tau}{h} f_{p_1} - \frac{\tau}{h} f_{p_2}) + \frac{\tau}{h^2}(w_{n-e_1}^{k-1} + w_{n-e_2}^{k-1})$$
$$+ \frac{\tau}{h^2}(1 + hf_{p_1})w_{n+e_1}^{k-1} + \frac{\tau}{h^2}(1 + hf_{p_2})w_{n+e_2}^{k-1},$$

with f_u, f_{p_1}, f_{p_2} evaluated at an intermediate point. By the inductive assumption $w_n^{k-1} \leq 0$. By (3.4) we estimate

$$|hf_{p_i}| \le ch(1 + (\frac{\max_n |\phi_n - \psi_n|}{h})^\alpha) < 1, \ i = 1,2,$$

by chosing h sufficiently small. Next we choose $\tau = \tau(h)$ so that

$$1 - \frac{4\tau}{h^2} + \tau f_u - \frac{\tau}{h} f_{p_1} - \frac{\tau}{h} f_{p_2} > 0.$$

It follows from (3.8) that $w_n^{k+1} \le 0$. A similar argument shows that $u_n^k \ge v_n^k$ for all k and n, completing the proof of (3.6), and the theorem follows.

We remark that for the continuous case a similar result appears to be known only in the one-dimensional case [2].

4. ELLIPTIC SYSTEMS WITH SINGULAR RIGHT-HAND SIDES

We begin with a single equation

$$(4.1) \qquad -\Delta u = \frac{1}{1-u} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

which has received considerable recent attention in connection with the phenomenon of "quenching" for the corresponding parabolic equation, see e.g., [2,6]. Clearly, $u \equiv 0$ is a subsolution of (4.1). However, it is not clear which function can serve as a supersolution, which does not allow us to proceed as in the theorem 1. Instead we proceed as follows. Starting with $u \equiv 0$ we obtain an increasing sequence of Picard iterates. If we can show that they are bounded from above by a constant less than one, then the sequence must converge to a solution of (4.1). Similarly we approach the system (4.5) below. We formulate our results for differential equations, however it is clear that similar results hold for the corresponding difference equations.

Define U(x) to be solution of

$$(4.2) \qquad -\Delta U = 1 \quad \text{in } \Omega, \quad U = 0 \text{ on } \partial\Omega,$$

and $M_0 = \max_\Omega U$. Starting with $u_0 \equiv 0$ in Ω, let $u_n(x)$, $n \geq 1$, be solutions of

$$(4.3) \qquad -\Delta u_n = \frac{1}{1-u_{n-1}} \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Denoting $M_n \geq \max_\Omega u_n(x)$, one shows inductively that we may take

$$(4.4) \qquad M_n = \frac{1}{1-M_{n-1}} M_0, \quad n = 1,2,\ldots ,$$

assuming that $M_n < 1$ for all n.

The recursive relation (4.4) is easily analysed. Indeed, M_n form an increasing sequence so long as $M_n < 1$. Hence, if this sequence is to remain bounded by 1, it has to converge. If M is the limit, then $M = \frac{1}{2} - \sqrt{1/4 - M_0}$, i.e. we need that $M_0 \leq \frac{1}{4}$. So that using standard arguments, concerning passage to the limit, we conclude that (4.1) has a positive solution bounded by M, provided the domain Ω is such that $\max_\Omega U(x) \leq \frac{1}{4}$. The last condition can be verified computationally.

Next we consider a system case

$$(4.5) \qquad \begin{aligned} -\Delta u &= \frac{1}{1-a_1u-b_1v} \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \\ -\Delta v &= \frac{1}{1-a_2u-b_2v} \quad \text{in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \end{aligned}$$

Defining $u_0(x) = v_0(x) = 0$, we construct approximations

$$(4.6) \qquad \begin{aligned} -\Delta u_n &= \frac{1}{1 - a_1u_{n-1} - b_1v_{n-1}} \quad \text{in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega, \\ -\Delta v_n &= \frac{1}{1 - a_2u_{n-1} - b_2v_{n-1}} \quad \text{in } \Omega, \quad v_n = 0 \text{ on } \partial\Omega. \end{aligned}$$

Clearly, $u_n(x) \geq u_{n-1}(x)$, $v_n(x) \geq v_{n-1}(x)$ for all $n \geq 1$ and $x \in \Omega$.

Theorem 4. In the (u,v) plane consider the region D defined by

$$u > 0,\ v > 0,\quad 1 - a_1u - b_1v > 0,\ 1 - a_2u - b_2v > 0,$$

and the curves Γ_1, Γ_2 defined by (M_0 as defined above)

$$\Gamma_1 : \quad a_1u^2 + b_1uv - u = -M_0$$

$$\Gamma_2 : \quad a_2uv + b_2v^2 - v = -M_0.$$

If the curves Γ_1 and Γ_2 intersect in the region D, then the iterations constructed in (4.6) converge to a positive solution of (4.5).

Proof. Define $U_n \geq \max_\Omega u_n(x)$, $V_n \geq \max_\Omega v_n(x)$, then by induction we may take

$$(4.7)\qquad \begin{aligned} U_n &= \frac{1}{1 - a_1U_{n-1} - b_1V_{n-1}} M_0, \quad U_0 = 0, \\ V_n &= \frac{1}{1 - a_2U_{n-1} - b_2V_{n-1}} M_0, \quad V_0 = 0. \end{aligned}$$

These recursive relations define a map $T : (U_{n-1},V_{n-1}) \to (U_n,V_n)$, and we are interested in its iterates, starting with $(0,0)$.

By induction one sees that both sequences U_n and V_n are increasing in n, while $(U_n,V_n) \in D$. Also, one checks that the map T is monotone in D, i.e., if $\bar{U}_{n-1} \geq U_{n-1}$, $\bar{V}_{n-1} \geq V_{n-1}$, and $\bar{U}_n = T(\bar{U}_{n-1},\bar{V}_{n-1})$, $\bar{V}_n = T(\bar{U}_{n-1},\bar{V}_{n-1})$ then $\bar{U}_n \geq U_n$, $\bar{V}_n \geq V_n$. Notice next that the curves Γ_1 and Γ_2 are unimodal and on Γ_1, $v \to -\infty$ as $u \to 0$, while on Γ_2, $u \to -\infty$ as $v \to 0$. Let $(\bar{U},\bar{V})$ be the point of intersection of Γ_1,Γ_2 which is closer to $(0,0)$. Notice that $(\bar{U},\bar{V})$ is a fixed point of T. By monotonicity of T,

$$(4.8)\qquad U_n \leq \bar{U}, \quad V_n \leq \bar{V} \quad \text{for all } n.$$

Since U_n,V_n are increasing sequences they must converge. Since they have to stay in the region defined by (4.8), it follows that

$(U_n, V_n) \to (U, V)$. By standard arguments $u_n(x)$, $v_n(x)$ then converge to a classical solution of (4.5).

5. NUMERICAL EXPERIMENTS

A common feature of our numerical experiments was stabilization of solutions in rather small time $t = \tau k$. For example, by the time $t = 6$ we had obtained the solution of

$$-u'' = u(5 - u), \quad u(0) = u(2) = 0,$$

with six stabilized digits. We took $h = 0.1$, $\tau = 0.001$, and started with a supersolution $\phi = 6$. The result was verified by a program based on the usual monotone iteration scheme. For a similar two-dimensional problem

$$u_{xx} + u_{yy} = u(12-u) \quad \text{for } 0 < x < 2,\ 0 < y < 2,$$

$u = 0$ on the boundary of the rectangle, with $h = 0.1$, $\tau = 0.001$ and $\phi = 13$, we obtained stabilization of two decimal digits by $t = 1$. We have studied a competing species model with constant a, b, c, d, e, f (see e.g. [3])

$$u'' + u(a - bu - cv) = 0, \quad u(0) = u(2) = 0 \quad (0 < x < 2)$$

$$v'' + v(d - eu - fv) = 0, \quad v(0) = v(2) = 0.$$

Without loss of generality we have always assumed that $b = f = 1$. We found that it is usually not necessary to start computations with super-subsolutions, as described in the theorem 2. For example for $a = 8$, $d = 9$, $c = 0.5$, $e = 0.9$, we obtained stabilization of three digits by time $t = 6$, starting with $u = v = 6$. (Monotonicity was, of course, lost.) A very interesting result was obtained by letting $a = 5.000001$, $d = 5$, $c = e = 10$. At the time $t = 2.5$ the solutions u and v were identical to three

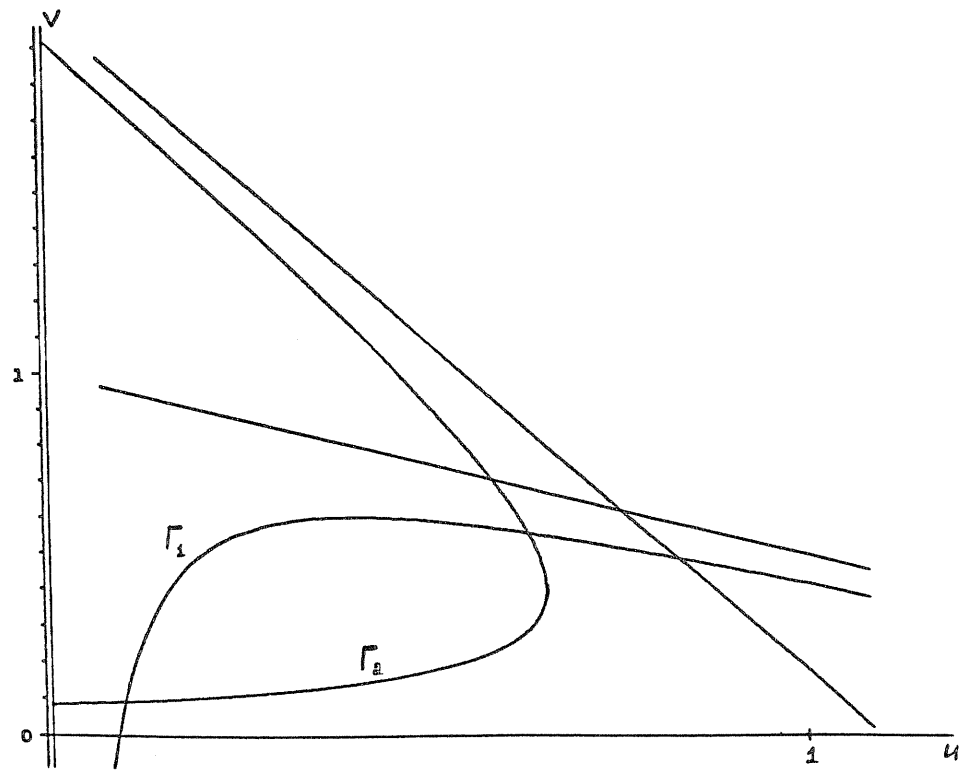

Figure 1

decimal places. By the time t = 5, the dominant species u drove the species v to extinction (zero to 36 decimal places).

For the problem

$$u'' = \sqrt{u} \quad \text{for } 0 < x < 10, \qquad u(0) = u(10) = 1,$$

we were able to compute the "dead core", which appears to extend over the interval [3.5,6.5]. For the quasilinear problem (see [1])

$$-u'' = 1 + 0.49u'^2 \quad \text{for } 0 < x < 1, \qquad u(0) = u(1) = 0,$$

we obtained monotone convergence starting with a subsolution $\psi = \frac{1}{2}x(1-x)$. We took $h = 0.05$, $\tau = 0.0003$. Iterations converged to the exact solution, which is easily obtained.

For the problem (4.5) we took $a_1 = 0.5$, $b_1 = 1$, $a_2 = 0.9$, $b_2 = 0.5$, $M_0 = 0.08$. The picture we obtained on a PC screen can be seen in Fig. 1. It suggests very strongly existence of a positive solution for (4.5). (By a careful justification of the fact that Γ_1 and Γ_2 intersect, it is possible to give a computer assisted proof.) Incidentally, in case Γ_1 and Γ_2 do not intersect, iterations of the map T produce very interesting patterns on the screen, suggesting possibility of strange attractors (computations of D. Schmidt).

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