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MULTIPLICITY OF POSITIVE SOLUTIONS FOR SEMILINEAR EQUATIONS ON CIRCULAR DOMAINS

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Abstract

We study uniqueness and exact multiplicity of positive solutions for several classes of Dirichlet problems on both ball and annular domain. Crucial to our work is establishing positivity of nontrivial solutions for the corresponding linearized problem.

Key words: Uniqueness and multiplicity of solutions, positivity for linearized equation.

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1 Introduction

We study uniqueness and exact multiplicity of positive solutions for both balls and annular domains. We begin by describing our results for the problem

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < R,\ u = 0 \quad \text{for } |x| = R, \tag{1.1}$$

i.e, a Dirichlet problem on a ball in R^n, depending on a positive parameter λ. In view of the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [5] any positive solution of (1.1) is radial, i.e. $u = u(r)$, where $r = |x|$, and the problem (1.1) takes the form

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0 \quad r \in (0, R),\ \ u'(0) = u(R) = 0. \tag{1.2}$$

Exact multiplicity for (1.1) is a notoriously difficult question, and few results our available (even for radial solutions). One could compare this problem to finding the exact number of critical points for some implicitly defined function $y = y(x)$. Unless one has some strong information about $y(x)$, it is hard to exclude the possibility of "wiggles" for some range of x.

Recently together with Y. Li and T. Ouyang (see [9] and the references there to our earlier papers) we developed a bifurcation theory approach to the exact multiplicity question. Instead of studying the solution at some particular λ, we take a more global approach, and study the solution curves. We then follow these curves to some corner in (λ, u) "plane", where some extra information is available (usually it is either where $\lambda = 0$, or where $u = \infty$). This way we can often prove uniqueness of the solution curve. If, moreover, we can prove that this solution curve does not turn, we obtain a uniqueness result. If solution curve admits exactly one turn, we get an exact multiplicity result. To continue solutions we use a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz, which is recalled below. The crucial ingredient of our approach is to prove positivity of any nontrivial solution of the corresponding linearized problem

$$(1.3) \quad w'' + \frac{n-1}{r}w' + \lambda f'(u)w = 0 \quad r \in (0, R), \;\; w'(0) = w(R) = 0.$$

This information is used to compute the direction of bifurcation, and to prove monotonicity of some solution branches.

We begin by establishing positivity of any nontrivial solution of (1.2), and the corresponding uniqueness result for sublinear nonlinearities (i.e., $f'(u) \leq f(u)u$). We then discuss concave nonlinearities for both the case $f(0) \geq 0$, and for the harder, so called *non-positone* case $f(0) < 0$. The difficulty in non-positone case is that as we continue along solution branches, solution may cease being positive (which is impossible in the other case, in view of strong maximum principle). We show that there are only two types of possible solution curves, and show that a complete classification of bifurcation diagrams is possible. Here again positivity for the linearized equation is important. After this work was completed, we became aware that A. Castro, S. Gadam and R. Shivaji has derived similar results for concave nonlinearities with $f(0) < 0$, see [2] and several earliear papers cited there. We believe our approach is simpler, and it gives some extra information (e.g. on monotonicity properties of the branches).

In another direction, on an annulus $\Omega = \{x | A < |x| < B\}$ in R^n we consider the problem

$$(1.4) \qquad \Delta U + \lambda |x|^{2p} f(U) = 0 \quad \text{in } \Omega, \;\; U = 0 \text{ on } \partial\Omega.$$

For the values of the parameter $-n+1 < p < 1$ we prove positivity for the corresponding linearized equation, which implies various uniqueness and exact multiplicity results. Uniqueness for $p = 0$ is known only for the "thin" annulus, see W.-M. Ni and R. Nussbaum [11], and also P. Korman [7].

Next we state a bifurcation theorem of Crandall-Rabinowitz [3].

Theorem 1.1 *[3] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R}\times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) = span\,\{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda}+\tau(s), \overline{x}+sx_0+z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R}\times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2 Multiplicity of positive solutions on a ball

In this section we consider positive solutions of the problem (depending on a positive parameter λ)

$$(2.1) \quad u''(r) + \frac{n-1}{r} u'(r) + \lambda f(u) = 0 \quad r \in (0,1), \;\; u'(0) = u(1) = 0.$$

In view of [5] this problem is equivalent to the Dirichlet problem (1.1) on a ball, the radius of which, without loss of generality, is assumed to be one. We present next two conditions under which any nontrivial solution of the linearized equation

$$(2.2) \qquad w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0 \;\; r \in (0,1), \;\; w'(0) = w(1) = 0,$$

is of one sign.

Lemma 2.1 *Assume that the function $f(u) \in C^2(\bar{R}_+)$ satisfies*

$$(2.3) \qquad f'(u)u \le f(u) \quad \textit{for all } u > 0.$$

Then the problem (2.2) admits no nontrivial solutions.

Proof. We claim that any nontrivial solution of (2.2) would have to be of one sign. It is well-known that $w'(0) \neq 0$ for any nontrivial solution of 2.2, see e.g., [12]. Hence we may (and will) assume that $w'(0) > 0$. Set $v(r) = ru_r + \mu u$, with a constant μ to be specified. One easily checks that v satisfies the equation

$$(2.4)\ v'' + \frac{n-1}{r}v' + \lambda f'(u)v = \lambda\left[\mu(f'(u)u - f(u)) - 2f(u)\right] \equiv g_\mu(u).$$

Assuming the claim to be false, let $r_1 \in (0,1)$ be the smallest zero point of $w(r)$. Choosing μ sufficiently large we may, in view of the assumption (2.3), obtain

$$(2.5) \qquad v(r) > 0 \ \text{ and } \ g_\mu(u(r)) < 0 \ \text{ on } \ (0, r_1].$$

We now multiply the equation (2.4) by w, and subtract from it the equation (2.2) multiplied by v, obtaining

$$(2.6) \qquad \frac{d}{dr}\left[r^{n-1}(v'w - vw')\right] = r^{n-1}g_\mu(u(r))w(r).$$

Integrate this over $(0, r_1)$

$$(2.7) \qquad -r_1^{n-1}v(r_1)w'(r_1) = \int_0^{r_1} r^{n-1}g_\mu(u(r))w(r)\,dr.$$

Using (2.5) we see that the left hand side of (2.7) is positive, while the right hand side is negative, a contradiction.

It follows that any nontrivial solution of (2.2) can be assumed to be positive. Comparing the equations (2.1) and (2.2) via the Sturm comparison theorem, and using condition (2.3), we conclude that this is impossible, proving the Lemma.

Lemma 2.2 *Assume that the function $f(u) \in C^2(\bar{R}_+)$ satisfies*

$$(2.8) \qquad f''(u) < 0 \quad \text{for all } u > 0.$$

Then any nontrivial solution of (2.2) is of one sign.

Proof. Let as before $r_1 \in (0,1)$ denote the smallest zero point of $w(r)$. By Lemma 2.3 in Korman [6]

$$(2.9) \qquad f(u(r_1)) \leq 0.$$

If $f(u)$ is a positive function, we are done, otherwise (since f is concave) we may assume that $f < 0$ on $(0, r_1)$. We now select the test function

$v(r) = ru_r$, i.e. we take $\mu = 0$. Integrating the relation (2.6) over $(r_1, 1)$, we get

$$r_1^{n-1} v(r_1) w'(r_1) = \int_{r_1}^{1} r^{n-1} g_0(u(r)) w(r)\, dr. \tag{2.10}$$

Since $g_0 = -2f(u) > 0$, the integral on the right is positive, while the left hand side of (2.10) is negative, a contradiction.

The following theorem is implicitly proved in [6]. We extract its proof for completeness.

Theorem 2.1 *Assume that the problem (2.2) admits only the trivial solution. Then any branch $u(r, \lambda)$ of positive solutions of (2.1) can loose its positivity only for increasing λ.*

Proof. If $f(0) \geq 0$ then it is follows by the Hopf's boundary lemma that positivity cannot be lost at all (see e.g. [5]), so we assume that

$$f(0) < 0. \tag{2.11}$$

Since any positive solution of (2.1) is a decreasing function of r, the only way the positivity can get lost is that for some λ_1 we have $u'(1, \lambda_1) = 0$ and then for $\lambda < \lambda_1$ the solution $u(r, \lambda)$ becomes negative near $r = 1$. Then $u_\lambda(r, \lambda_1)$ would have to be positive near $r = 1$. (By the definition of λ_1 it is clear that $u_\lambda(r, \lambda_1)$ cannot be negative on an interval containing $r = 1$. If $u_\lambda(r, \lambda_1)$ failed to be positive in some interval containing $r = 1$, we could find a sequence $r_n \to 1$, such that $u_\lambda(r_n, \lambda_1) = 0$. Let $\mu_n \to 1$ be points of positive local maximums of $u_\lambda(r, \lambda_1)$, i.e. $u_\lambda(\mu_n, \lambda_1) > 0$ and $u'_\lambda(\mu_n, \lambda_1) = 0$. We now evaluate the equation satisfied by u_λ

$$u''_\lambda(r) + \frac{n-1}{r} u'_\lambda(r) + \lambda f'(u) u_\lambda(r) = -f(u) \ \text{ for } \ r \in (0, 1) \tag{2.12}$$

at $r = \mu_n$. The first term on the left is negative, the second one is zero, and the third term is positive and tending to zero. The right hand side tends to $-f(0) > 0$, a contradiction.) It is now easy to see that $z(r) \equiv ru_r(r, \lambda_1) - 2\lambda_1 u_\lambda(r, \lambda_1)$ is a nontrivial solution of the problem (2.2) at $\lambda = \lambda_1$, contradicting the assumption (notice that $z(1) = 0$, and $z(r)$ is negative near $r = 1$).

Theorem 2.2 *Assume that any nontrivial solution of the problem (2.2) is of one sign. Let $f(0) < 0$, and assume that we have a branch of positive solutions of (2.1). Then such a branch can loose its positivity only for increasing λ.*

Proof. Assuming that $u'(1) = 0$, we shall show that the linearized problem (2.2) cannot have a nontrivial solution, and then the proof follows by the previous theorem. So assume on the contrary that (2.2) has a solution $w(r) > 0$. Differentiate the equation (2.1)

$$u_r'' + \frac{n-1}{r}u_r' + \lambda f'(u)u_r = \frac{n-1}{r^2}u'. \tag{2.13}$$

From the equations (2.2) and (2.13) we obtain

$$\left[r^{n-1}\left(u''w - u'w'\right)\right] |_0^1 = \int_0^1 (n-1)r^{n-3}u'w\, dr.$$

The integral on the right is negative, while the quantity on the left is equal to zero, a contradiction.

Corollary. Assume that we have a branch of positive solutions of (2.1) with $u_\lambda(0,\lambda)$ decreasing and

$$u'(1,\lambda) < 0 \text{ at some } \lambda = \lambda_2 > 0. \tag{2.14}$$

Then the same inequality holds for all $\lambda < \lambda_2$.

The following lemma was proved in [6]. We present a much simpler proof.

Lemma 2.3 *Assume that the function $f(u) \in C^2(\bar{R}_+)$, and the problem (2.2) has a nontrivial solution w at some λ. Then*

$$\int_0^1 f(u)wr^{n-1}\, dr = \frac{1}{2\lambda}u'(1)w'(1). \tag{2.15}$$

Proof. The function $v = ru_r - u_r(1)$ satisfies

$$\begin{aligned} \Delta v + \lambda f'(u)v &= -2\lambda f(u) - \lambda f'(u)u'(1) \text{ for } |x| < 1, \\ v &= 0 \text{ for } |x| = 1. \end{aligned} \tag{2.16}$$

By the Fredholm alternative the right hand side of (2.16) is orthogonal to w, i.e.

$$\int_0^1 f(u)wr^{n-1}\, dr = -\frac{1}{2}u'(1)\int_0^1 f'(u)wr^{n-1}\, dr. \tag{2.17}$$

Integrating the equation (2.2), we get

$$\lambda \int_0^1 f'(u)wr^{n-1}\, dr = -w'(1).$$

Using this in (2.17), we conclude the lemma.

The following lemma is known, see e.g. E.N. Dancer [4]. We present its proof for completeness.

Lemma 2.4 *Positive solutions of the problem (2.1) are globally parameterized by their maximum values* $u(0,\lambda)$. *I.e., for every* $p > 0$ *there is at most one* $\lambda > 0$, *for which* $u(0,\lambda) = p$.

Proof. If $u(r,\lambda)$ is a solution of (2.1) with $u(0,\lambda) = p$, then $v \equiv u(\frac{1}{\sqrt{\lambda}}r)$ solves

$$v'' + \frac{n-1}{r}v' + f(v) = 0, \;\; v(0) = p, \, v'(0) = 0. \tag{2.18}$$

If $u(0,\mu) = p$ for some $\mu \neq \lambda$, then $u(\frac{1}{\sqrt{\mu}}r)$ is another solution of the same problem. This is a contradiction in view of the uniqueness of solutions for initial value problems of the type (2.18), see [12].

We are now ready to present our first uniqueness result.

Theorem 2.3 *Assume that the function* $f(u) \in C^2(\bar{R}_+)$ *satisfies the condition (2.3). Then for any* $\lambda > 0$ *the problem (2.1) admits at most one positive solution.*

Proof. By (2.3) it follows that $f(0) \geq 0$. We distinguish two cases.

(i) Assume first that

$$f(0) = 0. \tag{2.19}$$

We assume that $f'(0) > 0$, since in case $f'(0) < 0$ we see by (2.3) that $f(u) < 0$ for all $u > 0$, and then by maximum principle the problem (2.1) has no positive solutions, while the case $f'(0) = 0$ will be excluded later. It follows that $f(u) > 0$ near $u = 0$, and then we can integrate (2.3), obtaining

$$f(u) \leq \frac{f(u_1)}{u_1}u, \tag{2.20}$$

for any $u_1 > 0$ small. Letting $u_1 \to 0$, we obtain in view of (2.19)

$$f(u) \leq f'(0)u. \tag{2.21}$$

We now see that in case $f'(0) = 0$ there is nothing to prove, since $f(u)$ is identically zero. Also notice that (2.21) holds for all $u > 0$, since if $f(u)$ becomes negative after some $\bar{u}$, by (2.3) it will stay negative for all $u > \bar{u}$. We may assume that

$$f'(0) > \lambda_1, \tag{2.22}$$

where λ_1 is the principal eigenvalue of $-\Delta$ on the unit ball, since otherwise the problem (2.1) has no positive solution (just multiply it by u and integrate). We now write

$$f(u) = f'(0)u - ug(u),$$

with $g(u) \geq 0$, and observe that $g(0) = 0$, and by (2.3)

$$g'(u) > 0 \text{ for all } u > 0. \tag{2.23}$$

Observe that for small enough $\epsilon > 0$, the function $\epsilon\phi_1(x)$, with $\phi_1(x) > 0$ a principal eigenfunction of the Laplacian on the unit ball, is a subsolution of (2.1). It follows that the problem (2.1) has a minimal solution $p(x) > 0$, i.e. any other solution $q(x)$ satisfies $q(x) > p(x)$ for all x with $|x| < 1$. From the equations for $p(x)$ and $q(x)$ we now obtain

$$\int_{|x|<1} pq\,[g(p) - g(q)]\,dx = 0,$$

which is a contradiction in view of (2.23).

(ii) We assume that

$$f(0) > 0. \tag{2.24}$$

We begin with the solution $(\lambda = 0, u = 0)$. At this point the implicit function theorem applies, and by the maximum principle we conclude that $u_\lambda > 0$, hence we have a curve of positive solutions for small $\lambda > 0$. By Lemma 2.1 we can continue this curve for increasing λ by using the implicit function theorem. By the Lemma 2.4 $u(0,\lambda)$ will increase on this curve. We distinguish two subcases.

(a) $f(u) > 0$ for all $u \geq 0$.

We claim that $u(0,\lambda) \to \infty$ as λ tends either to a critical $\bar{\lambda} > 0$ or to infinity. Indeed, assuming that $u(0,\lambda)$ tends to a finite $\bar{u}$, we conclude from (2.1) that the derivative of the function $r^{n-1}u'$ has to become large on $(0,1)$, and hence $u(r)$ has to become large, contradicting to the fact that the total variation of $u(r)$ is $2\bar{u}$. By the Lemma 2.4 the problem (2.1) has exactly one solution curve (since for any $p > 0$ there is at most one $\lambda > 0$ with $u(0,\lambda) = p$). It follows that the problem (2.1) has at most one positive solution for any $\lambda > 0$.

(b) $f(u_1) = 0$ for some $u_1 > 0$.

Notice that by (2.3) $f(u) < 0$ for all $u > u_1$, and by the maximum any positive solution of (2.1) is below u_1. We argue as before, this time obtaining a unique solution curve with $u(0, \lambda) \to u_1$ as $\lambda \to \infty$.

The following result will provide us with complete understanding of the problem (2.1) in case of concave nonlinearities.

Theorem 2.4 *Assume that the function $f(u) \in C^2(\bar{R}_+)$ satisfies the condition (2.8). If $f(0) \geq 0$ then the problem (2.1) has at most one positive solution. If $f(0) < 0$ then the problem (2.1) admits at most one solution curve, which admits at most one turn (to the right). If such a turn occurs, then any two solutions on the curve, corresponding to the same λ are strictly ordered. For λ sufficiently large the problem (2.1) has at most one positive solution.*

Proof. We distinguish between two cases.
Case 1. $f(0) \geq 0$. Setting $p(u) = uf'(u) - f(u)$, we see that $p(0) \leq 0$ and $p'(u) = uf''(u) < 0$. It follows that the condition (2.3) holds, and the previous theorem applies.
Case 2. $f(0) < 0$. If $f(u)$ is negative for all $u > 0$ then the problem (2.1) has no positive solutions. We therefore assume that $f(u_1) = 0$ at some $u_1 > 0$. We distinguish two subcases.

(i) There exists $u_2 > u_1$ such that $f(u_2) = 0$, and $f(u) < 0$ for $u > u_2$. By the maximum principle any positive solution of (2.1) satisfies

$$u(r) < u_2 \quad \text{for all } r \in [0, 1).$$

Let (λ_0, u_0) be any solution of (2.1), with $u_0'(1) < 0$. If the corresponding linearized equation has only the trivial solution, then this solution continues by the implicit function theorem. Otherwise, as was shown in [6], the Crandall-Rabinowitz Theorem 1.1 applies. (The Lemma 2.3 verifies that $F_\lambda \notin R(F_u)$, the crucial condition of that theorem.) We claim that in such a case a turn to the right in the (λ, u) "plane" occurs. Indeed by a standard calculation the function $\tau(s)$ defined in the Theorem 1.1 satisfies

$$\tau''(0) = -\lambda_0 \frac{\int_0^1 f''(u_0) w^3 r^{n-1}\, dr}{\int_0^1 f(u_0) w r^{n-1}\, dr}, \tag{2.25}$$

see e.g. [9]. By the condition (2.8) and Lemma 2.2, the numerator in (2.25) is negative. By the Lemma 2.3 the denominator is positive. It follows that $\tau''(0) > 0$, proving the claim. Let us consider a non-singular point (λ_0, u_0).

By the Lemma 2.4 $u(0,\lambda)$ is either strictly increasing or decreasing near (λ_0, u_0). Assume that $u(0,\lambda)$ is decreasing (the other case will be clear from the proof). We now continue this curve for decreasing λ. By the Theorem 2.2 solutions on this curve stay positive, and moreover the condition $u'(1,\lambda) < 0$ is preserved. Solutions on this curve are bounded (by u_2), with increasing maximums, and so this curve cannot enter the trivial solution at $\lambda = 0$, which is the only solution at $\lambda = 0$. Hence at some critical $\bar{\lambda}$ this curve must turn. Since only turns to the right are possible, and it will follow from the Lemma 2.5 below that condition $u'(1) < 0$ is preserved after the turn, this curve will continue after the turn for all $\lambda > \bar{\lambda}$ without any more turns. We now state the above mentioned lemma, but postpone its proof.

Lemma 2.5 *After the turn at $\lambda = \bar{\lambda}$, we have*

$$u_\lambda(r,\lambda) > 0 \quad \textit{for all } r \in [0,1) \textit{ and all } \lambda > \bar{\lambda}. \tag{2.26}$$

Arguing as in Theorem 2.3 we conclude that after the turn

$$u(r,\lambda) \to u_2 \text{ for all } r \in [0,1) \text{ as } \lambda \to \infty. \tag{2.27}$$

We now return to the solution (λ_0, u_0), and continue the curve for increasing λ. If at some λ we reach a critical point, then as above a turn to the right must occur, which is impossible since we arrived at that point from the left. We claim that the lower branch cannot be continued for all λ, but the branch looses positivity at some $\lambda_1 > \lambda_0$. Indeed, as we noticed in the proof of the previous theorem, if solutions on the lower branch stay positive for all $\lambda > \lambda_0$, then $f(u(r))$ must become small for λ large on any subinterval of $(0,1)$. This implies that $u(r)$ is close to either u_1 or u_2. Since $u(0)$ is decreasing on the lower branch, it follows that $u(r)$ cannot get close to u_2 for any r. Hence it suffices to exclude the possibility that $u(r,\lambda) \to u_1$ on some interval $(r_1, 1) \subset (0,1)$. From the equation (2.1) we see that the function $u(r)$ is convex for $u < u_1$. The only way $u(r,\lambda)$ can tend to u_1 is that if $\xi = \xi(\lambda)$ is defined by $u(\xi) = u_1$, then $\xi \to 1$ as $\lambda \to \infty$. Clearly, $|u'(\xi)|$ is large (in fact, it is of order $const \cdot \sqrt{\lambda}$, see e.g. [9]). Since $u(x)$ is convex at ξ the quantity $|u'(x)|$ is still increasing as we decrease x near ξ. In order for $|u'(x)|$ to decrease for decreasing x (as is necessary for a function close to a constant over $(r_1, 1)$), the function $\lambda f(u(r))$ has to get large enough (i.e. $u(r)$ has to get sufficiently larger than u_1) to "kill" the $\frac{n-1}{r}u'$ term, so that $u(r)$ can get concave, as seen from (2.1). But once $\lambda f(u(r))$ gets large, it stays large over $(r_1, 1)$ (since $u'(r) < 0$ for all r by [5]). This implies that the derivative of $r^{n-1}u'$ is large over $(r_1, 1)$, which leads to a contradiction as in the previous theorem.

We have completed the description of the solution curve, the curve 1 in Figure 1. Lemma 2.4 and (2.27) imply uniqueness of the solution curve, completing the proof for this case.

(ii) Assume that $f(u) > 0$ for all $u > u_1$. Since f is a concave function, this implies that it tends to either positive constant or infinity as $\lambda \to \infty$. The proof proceeds exactly as before, only now when we continue the curve of solutions leftward, there is a possiblity that it goes to infinity at a finite λ, producing a curve of type 2 in Figure 1. By Lemma 2.4 and Theorem 2.2, if a curve of type 2 is present no other positive solutions are possible. (If there is another curve of solutions, continue is leftward. It cannot loose positivity by Theorem 2.2, and it cannot be another curve of type 1 or 2 by Lemma 2.4).

Proof of Lemma 2.5 We follow closely the argument in [9]. For λ close to $\bar{\lambda}$ the Lemma follows from Lemma 2.2 and Theorem 1.1, since u_λ is then positive. Our goal is to show that u_λ remains positive for all $\lambda > \bar{\lambda}$. For that we shall show that u_λ cannot develop a zero in the interior of the interval $(0, 1)$, or zero slope at $r = 1$. But first we establish a preliminary inequality. We show next that for any solution of (2.1)

$$\int_0^1 f(u)u'r^{n-1}\,dr < 0. \tag{2.28}$$

Indeed, write (2.1) in the form

$$(r^{n-1}u')' + \lambda r^{n-1} f(u) = 0. \tag{2.29}$$

Multiply (2.29) by $r^{n-1}u'$ and integrate over $(0, 1)$

$$\int_0^1 f(u)u'r^{2(n-1)}dr = -\frac{1}{2\lambda}u'^2(1) < 0. \tag{2.30}$$

The function $f(u(r))u'(r)$ is negative near $r = 0$ and positive near $r = 1$, and it changes sign once, say at $r = r_0$. It follows that

$$\begin{aligned}\int_0^1 f(u)u'r^{2(n-1)}dr &> r_0^{n-1}\int_0^{r_0} f(u)u'r^{n-1}\,dr + r_0^{n-1}\int_{r_0}^1 f(u)u'r^{n-1}\,dr \\ &= r_0^{n-1}\int_0^1 f(u)u'r^{n-1}\,dr.\end{aligned}$$

In view of (2.30), the inequality (2.28) follows. If r_1 is any point in $(0,1)$, then we also have

$$\int_0^{r_1} f(u)u'r^{n-1}\,dr < \int_0^1 f(u)u'r^{n-1}\,dr < 0. \tag{2.31}$$

(We may assume that $r_1 \in (r_0, 1)$, since otherwise this inequality is trivial.) Let now λ_1 be the supremum of $\lambda > \bar{\lambda}$ where the inequality $u_\lambda(r,\lambda) > 0$ (for all $r \in (0,1)$) holds. Several cases are possible.

Case (i) $u_\lambda(r, \lambda_1) \geq 0$ for all $r \in (0,1)$, and $u_\lambda(r_1, \lambda_1) = 0$ for some $r_1 \in (0,1)$. Notice that u_λ satisfies

$$\begin{gathered} u_\lambda'' + \tfrac{n-1}{r}u_\lambda' + \lambda f'(u)u_\lambda + f(u) = 0 \quad r \in (0,1), \\ u_\lambda'(0) = u_\lambda(1) = 0. \end{gathered} \tag{2.32}$$

Since r_1 is a point of minimum for $u_\lambda(r,\lambda_1)$, it follows that $u_\lambda'(r_1,\lambda_1) = 0$. From the equation for u_r (obtained by differentiating of (2.1)) and (2.32) we obtain as before

$$(u_\lambda' u' - u_\lambda u'')' + \frac{n-1}{r}(u_\lambda' u' - u_\lambda u'') + \frac{n-1}{r^2}u_\lambda u' + f(u)u' = 0. \tag{2.33}$$

Letting $p = u_\lambda' u' - u_\lambda u''$, we express from (2.33)

$$(r^{n-1}p)' = -(n-1)r^{n-3}u_\lambda u' - f(u)u'r^{n-1}. \tag{2.34}$$

Integrate (2.34) over $(0, r_1)$, and use (2.31)

$$r_1^{n-1}p(r_1) \geq -\int_0^{r_1} f(u)u'r^{n-1}\,dr > 0. \tag{2.35}$$

But $p(r_1) = 0$, a contradiction.

Case (ii) $u_\lambda'(1, \lambda_1) = 0$. Integrating (2.34) over $(0,1)$ we obtain the same contradiction.

Case (iii) $u_\lambda(0, \lambda_1) = 0$. The nonnegative function u_λ takes its minimum at $r = 0$. Again from (2.32) we see that $f(u(0,\lambda_1)) \leq 0$, a contradiction.

Remark. Our proof shows that there are only two types of solution curves possible. They are sketched in Figure 1.

Remark. The class of concave $f(u)$ with $f(0) < 0$, $f(u_1) = 0$ and $f(u)$ is positive for some range of u is exhausted by the following four cases.

Case 1. There is a $u_2 > u_1$, such that $f(u_2) = 0$. If $\int_{u_1}^{u_2} f(t)\,dt \le 0$, then (2.1) has no positive solution for any $\lambda > 0$, see e.g. [9]. If, on the other hand, $\int_{u_1}^{u_2} f(t)\,dt > 0$, then existence of positive solutions at some λ is known (see e.g. [9]), and we conclude that the solution set consists of a curve of type 1, with upper branch tending to u_2 as $\lambda \to \infty$, and the lower branch loosing positivity at some finite λ.
Case 2. There is an $\alpha > 0$ so that $\lim_{u\to\infty} f(u) = \alpha$. Again solution set consists of one curve of type 1, this time solutions on the upper branch tend to infinity as $\lambda \to \infty$.
Cases 3 and 4. $f(u)$ is asymptotic to a straight line when $u \to \infty$, i.e. we can write

$$f(u) = au + b + g(u) \quad \text{where } g(u) \to 0 \text{ as } u \to \infty,$$

with constants $a > 0$ and b ($g(u) < 0$ since $f(u)$ is concave). According to A. Ambrosetti et al [1] there is a solution curve bifurcating from infinity at $\lambda = \frac{\lambda_1}{a}$. Moreover, bifurcation is to the left if $b > 0$, and to the right if $b < 0$. It follows that in case $b > 0$ the solution set of (2.1) consists of one curve of type 1, with the upper branch going to infinity at some finite λ_1, and the lower branch loosing positivity at some finite λ_2 (λ_1 and λ_2 can be in any order). In case $b < 0$ the solution set consists of one curve of type 2.

3 Exact multiplicity of positive solutions on annular domains

We begin with a class of Dirichlet problems involving ordinary differential equations, for which we establish positivity of nontrivial solutions of the corresponding linearized equation. This result will be of use to us later, and it is also of independent interest. We consider boundary value problems ($u = u(s)$)

$$u'' + f(s,u) = 0 \quad \text{for } a < s < b,\ u(a) = u(b) = 0. \tag{3.1}$$

The corresponding linearized equation is

$$w'' + f_u(s,u)w = 0 \quad \text{for } a < s < b,\ w(a) = w(b) = 0. \tag{3.2}$$

Theorem 3.1 *Let $u(s)$ be a positive solution of (3.1). Assume that on $[a,b] \times R_+$ the function $f(s,u)$ is of class C^2, and it satisfies*

$$f_s < 0, \tag{3.3}$$

$$-2f - sf_s < 0. \tag{3.4}$$

Then any nontrivial solution of (3.2) is of one sign, i.e. we may assume that $w > 0$ on (a, b).

Proof. We assume that (3.3) and (3.4) hold, with the other case being similar. These conditions imply that $u(s)$ is a concave positive function, and hence it has a unique point of maximum, which is a point of global maximum, at some $s_0 \in (a, b)$. Assume that contrary to what we want to prove $w(s_1) = 0$ at some $s_1 \in (a, b)$. We distinguish between two cases.

(i) Assume that $s_1 \in (s_0, b)$. We may assume that $w > 0$ on (s_0, b). Differentiate the equation (3.2),

$$u_s'' + f_u u_s + f_s = 0. \tag{3.5}$$

We now multiply the equation (3.5) by w, and subtract from it the equation (3.1) multiplied by u_s. We then integrate over (s_1, b). Obtain

$$-u'(b)w'(b) + u'(s_1)w'(s_1) + \int_{s_1}^{b} f_s w \, ds = 0. \tag{3.6}$$

Since all three terms on the left in (3.6) are negative, we have a contradiction.

(ii) Assume that $s_1 \in (a, s_0)$. Again we may assume that $w > 0$ on (a, s_0). Observe that the function $v = su_s$ satisfies

$$v'' + f_u v = -2f - sf_s \equiv g(s), \tag{3.7}$$

and the function $g(s)$ is negative by the assumption (3.4). We now multiply the equation (3.7) by w, the equation (3.2) by v, subtract and then integrate over (a, s_1), obtaining

$$-v(s_1)w'(s_1) + v(a)w'(a) = \int_a^{s_1} g(s)w(s) \, ds. \tag{3.8}$$

Since the function $v(s)$ is positive on (a, s_0), it follows that both terms on the left are positive, while the integral on the right is negative. This contradiction completes the proof.

On an annulus $\Omega = \{x | A < |x| < B\}$ in R^n, $n \geq 3$, we now consider the problem (p is a real parameter)

$$\Delta U + \lambda |x|^{2p} f(U) = 0 \quad \text{in } \Omega, \; U = 0 \text{ on } \partial\Omega. \tag{3.9}$$

We study positive radially symmetric solutions of (3.9), depending on a positive parameter λ. Writing $U = U(r)$ with $r = |x|$, we are then led to consider the problem

$$(3.10) \qquad \begin{aligned} U'' + \tfrac{n-1}{r}U' + \lambda r^{2p} f(U) = 0, \quad \text{for } A < r < B, \\ U(A) = U(B) = 0. \end{aligned}$$

We make a standard change of variables, letting $s = r^{2-n}$ and $u(s) = U(r)$, transforming (3.10) into the problem

$$(3.11) \qquad u'' + \lambda\alpha(s) f(u) = 0, \ \text{ for } a < s < b, \ u(a) = u(b) = 0,$$

where $\alpha(s) = (n-2)^{-2}s^{-2k}$ with $k = 1 + \frac{p+1}{n-2}$, $a = B^{2-n}$ and $b = A^{2-n}$. We shall also need the corresponding linearized equation

$$(3.12) \qquad \begin{aligned} w'' + \lambda\alpha(s) f'(u) w = 0, \ \text{ for } a < s < b, \\ w(a) = w(b) = 0. \end{aligned}$$

We wish to show that any nontrivial solution of (3.12) is of one sign. If $p = 0$, our Theorem 3.1 just fails. We therefore ask the parameter p to "help". The following lemma provides a key to obtaining various uniqueness and exact multiplicity results. Its proof follows from Theorem 3.1.

Lemma 3.1 *Assume that $f(u) \in C^2[0,\infty)$ satisfies $f(u) > 0$ for almost all $u > 0$. Assume that the parameter p satisfies $-n+1 < p < -1$. Let $u(s)$ be any positive solution of (3.9). Then if the corresponding linearized equation (3.12) admits a nontrivial solution, we may assume that $w > 0$.*

Remark. In [7] we proved a similar result for $p = 0$, but the annulus was assumed to be "thin".

In case $f(u)$ is a power of u, we can allow positive p, as the following result shows.

Theorem 3.2 *Assume that in (3.9) we have $f(u) = u^q$, with $q > 1$ real, and*

$$(3.13) \qquad 2p > q(n-2) - n.$$

Then for any positive solution $u(s)$ of (3.9), the corresponding linearized equation has only the trivial solution.

Proof. It suffices to prove that any nontrivial solution of the linearized equation (3.12) is positive. Indeed, writing our nonlinearity as $g \equiv \lambda\alpha(s)u^q$, we notice that $g_u > \dfrac{g}{u}$. Then in view of Sturm's comparison theorem it is impossible for both the equations (3.11) and (3.12) to have positive solutions. To show that $w > 0$ we proceed similarly to the Theorem 3.1. We shall use the test function $v = su_s + \mu u$, with a constant μ to be specified. Compute

$$
\begin{aligned}
v'' + g_u v &= \mu(g_u u - g) - 2g - sg_s \\
&= g\left[\mu(q-1) - 2 + 2k\right] \equiv g_\mu(s).
\end{aligned} \tag{3.14}
$$

We now select

$$\mu = -\frac{2(k-1)}{q-1} < 0.$$

Then $g_\mu = 0$.

We claim next that our test function $v = su_s - \frac{2(k-1)}{p-1}u$ changes sign exactly once on (a, b). As before we denote by s_0 the unique point of global maximum of $u(s)$, and as before we have $u_s > 0$ for all $s \in (a, s_0)$. Observe that $v(a) > 0$,

$$v' = su_{ss} + (1-\mu)u_s < (1 - \frac{2(k-1)}{q-1})u_s < 0,$$

in view of (3.13), and $v(s_0) < 0$. It follows that v changes sign exactly once on (a, s_0). Since v is clearly negative on (s_0, b), we conclude existence of $\bar{s} \in (a, s_0)$ such that $v > 0$ for $s \in (a, \bar{s})$, and $v < 0$ for $s \in (\bar{s}, b)$.

Assume that $w(s_1) = 0$ for some $s_1 \in (a, b)$. We distinguish two cases.

(i) Assume that $s_1 \in (a, \bar{s})$, and $w > 0$ on $(a, \bar{s})$. From the equations (3.12) and (3.14) we obtain integrating over (a, s_1) as before

$$v(s_1)w'(s_1) - v(a)w'(a) = 0.$$

Since both terms on the left are negative, we have a contradiction.

(ii) Assume that $s_1 \in (\bar{s}, b)$, and $w > 0$ on $(\bar{s}, b)$. From the equations (3.12) and (3.14) we obtain integrating over (s_1, b),

$$v(b)w'(b) - v(s_1)w'(s_1) = 0.$$

Both terms on the left are now positive, a contradiction.

Given positivity of any nontrivial solution of the linearized equation, one can obtain a number of uniqueness and exact multiplicity results, similarly to [7]. We give an example of such a result next. Its proof is identical to that of Theorem 2.3 in Korman [7]. It applies in particular for the nonlinearity $f(u) = e^u$, which occurs in combustion theory.

Theorem 3.3 *Assume that in addition to the conditions of Lemma 3.1 , the function $f(u)$ is superlinear, i.e.* $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$ *and it satisfies*

$$f(u) \geq c_1 u^q + c_2 \text{ for all } u \geq 0,$$

with the constants $c_1, c_2 > 0$, $q > 1$, and

$$f''(u) > 0 \text{ for all } u > 0.$$

Then there is a critical $\lambda_0 > 0$, such that the problem (3.9) has exactly two positive solutions for $0 < \lambda < \lambda_0$, it has exactly one positive solution at $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve, which for $\lambda \in (0, \lambda_0)$ has two branches $u^-(r, \lambda)$ and $u^+(r, \lambda)$, with $u^-(r, \lambda) < u^-(r, \lambda)$ for all $r = |x| \in (A, B)$. The lower branch $u^-(r, \lambda)$ is strictly monotone increasing in λ, and $\lim_{\lambda\to 0^+} u(r, \lambda) = 0$ for all $r \in (A, B)$. For the upper branch $\lim_{\lambda\to 0^+} \max_r u(r, \lambda) = \infty$.

Based on the Theorem 3.2 we can give an uniqueness result for the problem (3.9) in case $f(u) = u^q$, which is similar to the Theorem 2.4 in [7]. We omit the details.

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Type 1

Type 2

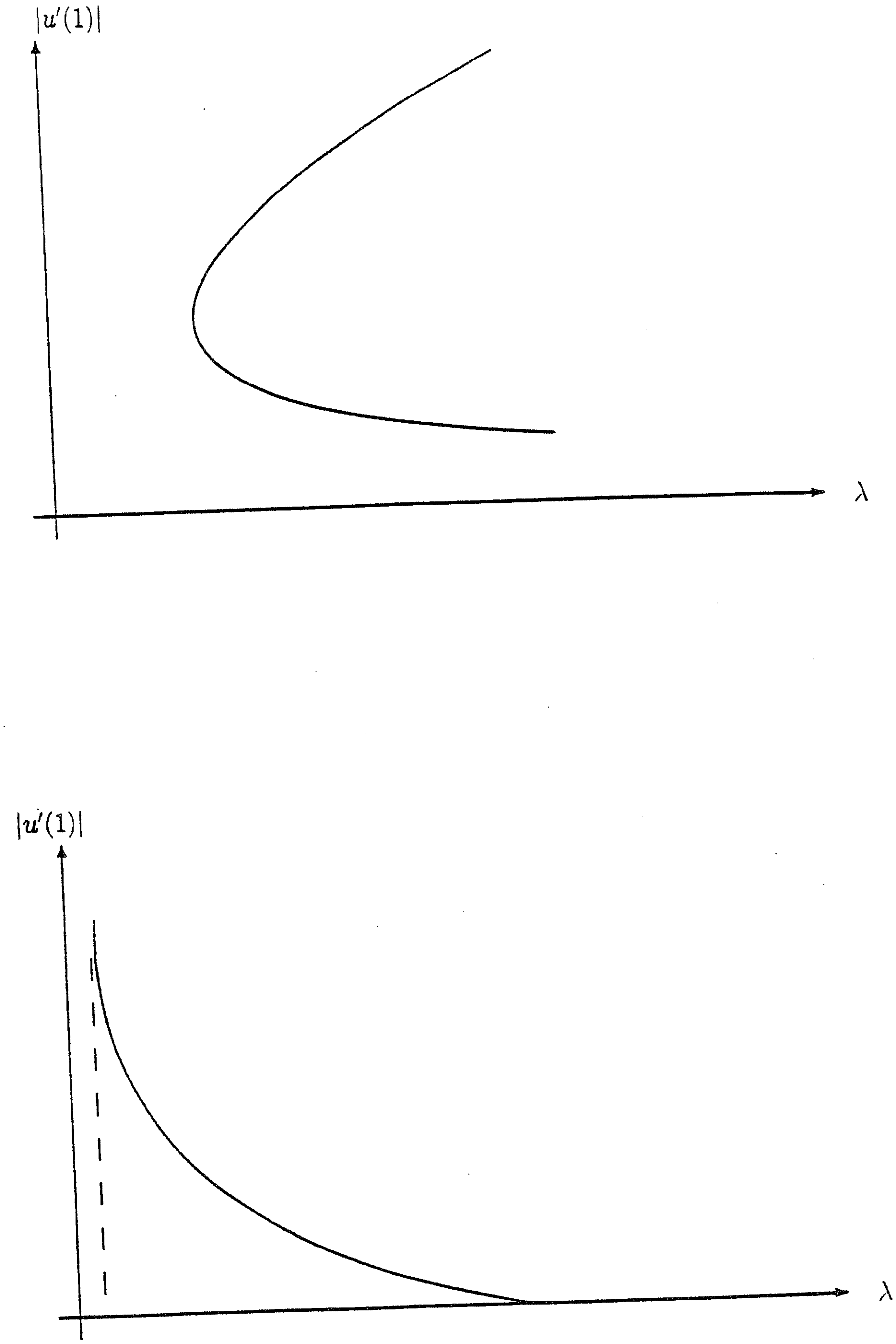

Figure 1. Two types of solution curves