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COMPUTATION OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

GILBERT CHOUDURY
Department of Mathematical Sciences
University of Cincinnati, Cincinnati, OH 45221-0025
PHILIP KORMAN
Institute for Dynamics and Department of Mathematical Sciences
University of Cincinnati, Cincinnati, OH 45221-0025

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Abstract—We develop an explicit method allowing efficient computation of solutions of nonlinear boundary value problems with nonlinear boundary conditions. We apply our results to the nonlinear beam equation, and to second order problems in the case when the growth of nonlinearities in the first derivative of the solution is not restricted, for which there are very few theoretical results.

1. INTRODUCTION

We develop an explicit monotone scheme for solving boundary value problems for nonlinear ordinary differential equations with, possibly, nonlinear boundary conditions. We replace the discretized boundary value problems with corresponding parabolic problems, and show that solutions of the parabolic problems approach steady states, provided the initial data is either a super- or subsolution and the time step is sufficiently small. The algorithm is easily implemented and good accuracy is usually achieved after a moderate number of time steps.

This algorithm was previously implemented by Huy, McKenna and Walter [1] for semilinear elliptic systems and, independently, by Korman [2] for more general quasilinear nonlinearities. It was then extended by Choudury and Korman [3] to wide classes of fully nonlinear problems. For semilinear monotone elliptic systems, similar theoretical results were established by Sattinger [4].

In Section 2, we extend this algorithm to fully nonlinear boundary conditions of first order. The novelty here is in proving monotone convergence while working with fictitious mesh points, introduced to approximate the boundary conditions. We consider semilinear ordinary differential equations for simplicity, although our results can be easily applied in the PDE case.

In Section 3, we consider quasilinear nonlinearities of the type $f(x,u,u')$ without restricting the growth of f in u'. In [2], it was shown that if f is subquadratic in u', then a result similar to the Theorem 2.1 holds. Such conditions (of Nagumo's type) are usually placed in the PDE literature, see e.g., [5], since otherwise one cannot expect gradient estimates, which are usually used to prove existence. We present a heuristic argument showing that, in the general case, one can expect the iterates to be monotone on the average when one starts with a super- or subsolution. This conclusion was confirmed by our numerical experiments.

In Section 4, we consider fourth order problems, describing static displacements of a uniform beam by a nonlinear force. Using ideas from the previous two sections, we show that one can again expect monotone convergence, on the average, for the corresponding evolution equation.

Once a solution is computed we can represent it by its Fourier series. This is discussed in Section 5. We draw some interesting conclusions, in particular, that problems of the type (3.1) can be reasonably accurately solved using only the first harmonic.

2. NONLINEAR BOUNDARY CONDITIONS FOR SEMILINEAR EQUATIONS

We are interested in the numerical solution of the following two-point boundary value problem

$$\begin{aligned} u'' + f(x,u) &= 0, \qquad \text{for } a < x < b, \\ \alpha(u(a), u'(a)) &= 0, \quad \beta(u(b), u'(b)) = 0, \end{aligned} \tag{2.1}$$

with given functions f, α and β. We divide the interval [a,b] into N equal parts of length $h = (b-a)/N$ each, denote $x_0 = a$, $x_k = x_0 + kh$ for $k = 1, 2, \dots, N, u_k = u(x_k)$, and then replace the equation in (2.1) by its finite difference version

$$F[u_k] \equiv \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + f(x_k, u_k) = 0, \qquad k = 1, \dots, N-1. \tag{2.2}$$

To approximate the boundary conditions in (2.1), one could approximate $u'(a)$ by $(u_1 - u_0)/h$ and $u'(b)$ by $(u_N - u_{N-1})/h$, however, our numerical experiments suggest that a much better accuracy is achieved by the following procedure (which is common for Neumann boundary conditions, see e.g., [6]). We introduce fictitious mesh points x_{-1} and x_{N+1}, and replace $u'(a)$ by $(u_1 - u_{-1})/2h$ and $u'(b)$ by $(u_{N+1} - u_{N-1})/2h$. We then write (2.2) for $k = 0$ and $k = N$, which allows us to express u_{-1} through u_0 and u_1, and u_{N+1} through u_{N-1} and u_N. Our boundary conditions then take the form

$$\begin{aligned} \alpha[u_k] &\equiv \alpha\left(u_0, \frac{u_1 - u_0}{h} + \frac{h}{2} f(a, u_0)\right) = 0, \\ \beta[u_k] &\equiv \beta\left(u_N, \frac{u_N - u_{N-1}}{h} - \frac{h}{2} f(b, u_N)\right) = 0. \end{aligned} \tag{2.3}$$

Our goal is to solve (2.2) together with (2.3).

We set up the following iteration scheme ($p = 0, 1, \dots$)

$$\frac{u_k^{p+1} - u_k^p}{\tau} = \frac{u_{k+1}^p - 2u_k^p + u_{k-1}^p}{h^2} + f(x_k, u_k^p), \qquad k = 1, \dots, N-1, \tag{2.4}$$

$$u_0^{p+1} = u_0^p + \tau_1 \alpha\left(u_0^p, \frac{u_1^p - u_0^p}{h} + \frac{h}{2} f(a, u_0^p)\right), \tag{2.5}$$

$$u_N^{p+1} = u_N^p + \tau_2 \beta\left(u_N^p, \frac{u_N^p - u_{N-1}^p}{h} - \frac{h}{2} f(b, u_N^p)\right), \tag{2.6}$$

with suitably chosen initial mesh functions u_k^0, $k = 0, \dots, N$, and positive parameters τ, τ_1 and τ_2.

DEFINITION. *A mesh function $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_N) \equiv \varphi_k$ is called a supersolution of the system (2.2), (2.3), provided that*

$$F[\varphi_k] \le 0, \quad \alpha[\varphi_k] \le 0, \quad \beta[\varphi_k] \le 0; \qquad k = 1, \dots, N-1. \tag{2.7}$$

A mesh function $\psi = (\psi_0, \psi_1, \dots, \psi_n)$ is called a subsolution of (2.2), (2.3), if it satisfies the opposite inequalities.

We remark that φ and ψ can often be obtained as discretizations of the super- and subsolutions of the continuous problem (2.1).

The following result establishes existence of solutions for (2.2), (2.3) and provides an explicit computational algorithm.

THEOREM 2.1. *Assume that the problem (2.2), (2.3) possesses a supersolution φ_k and a subsolution ψ_k, with $\psi_k \le \varphi_k$, for all k. Assume that $f(x,u)$ is continuous in both variables and Lipschitz continuous in u uniformly in $a \le x \le b$, and $\min_k \psi_k \le u \le \max_k \varphi_k$. Assume that $\alpha(u,u')$ and $\beta(u,u')$ are continuous in both variables and continuously differentiable in u', and $\alpha_{u'} \ge 0$ and $\beta_{u'} \le 0$ when $\psi_0 \le u \le \varphi_0$ and $\psi_N \le u \le \varphi_N$, respectively, and $u' \in R$. Then, the problem (2.2), (2.3) has a solution u_k with $\psi_k \le u_k \le \varphi_k$ for all k. Moreover, if one starts the* *scheme (2.4)–(2.6) with $u_k^0 = \varphi_k$, then for τ, τ_1 and τ_2 sufficiently small (precise conditions may be found in the proof) the sequence u_k^p is monotone decreasing in p for each k. Similarly, if one starts with $u_k^0 = \psi_k$, then the corresponding sequence, call it v_k^p, is monotone increasing in p, and*

$$\psi_k \le v_k^1 \le v_k^2 \le \dots \le u_k \le \dots \le u_k^2 \le u_k^1 \le \varphi_k, \qquad \text{for all } k. \tag{2.8}$$

The sequences $\{u_k^p\}$ and $\{v_k^p\}$ converge, correspondingly, to the maximal and minimal solutions of (2.2), (2.3).

PROOF. Let $u_k^1 = \varphi_k$ and denote $w_k^p = u_k^{p+1} - u_k^p$. From the definition of the supersolution it follows that $w_k^0 \le 0$, for all k. We prove by induction that $w_k^p \le 0$ for all k and $p \ge 1$. Assume that $w_k^{p-1} \le 0$, for all $k = 0, 1, \dots, N$. Using the mean-value theorem, we obtain (with L the Lipschitz constant of $f(x,u)$)

$$\begin{aligned}
w_k^p &\le \left(1 - \frac{2\tau}{h^2} - \tau L\right) w_k^{p-1} + \frac{\tau}{h^2}\left(w_{k+1}^{p-1} + w_{k-1}^{p-1}\right),\\
w_0^p &= w_0^{p-1} + \tau_1\, \alpha_u\, w_0^{p-1} + \tau_1\, \alpha_{u'}\, \frac{w_1^{p-1} - w_0^{p-1}}{h} + \frac{\tau_1 h}{2}\, \alpha_{u'} \left(f(a, u_0^p) - f(a, u_0^{p-1})\right)\\
&\le \left(1 + \tau_1\, \alpha_u - \frac{\tau_1}{h}\, \alpha_{u'} - \frac{\tau_1 h L}{2}\, \alpha_{u'}\right) w_0^{p-1} + \frac{\tau_1}{h}\, \alpha_{u'}\, w_1^{p-1},\\
w_N^p &\le \left(1 + \tau_2\, \beta_u + \frac{\tau_2}{h}\, \beta_{u'} + \frac{\tau_2 h L}{2}\, \beta_{u'}\right) w_N^{p-1} - \frac{\tau_2}{h}\, \beta_{u'}\, w_{N-1}^{p-1},
\end{aligned}$$

with α_u, $\alpha_{u'}$ and β_u, $\beta_{u'}$, evaluated at some intermediate points. Fixing h, and choosing τ, τ_1 and τ_2 so that all three expressions in brackets in front of w_k^{p-1} are positive, we conclude that $w_k^p \le 0$, for $k = 0, 1, \dots, N$. This implies that the sequence $\{u_k^p\}$ is decreasing in p. A similar argument shows that the sequence $\{v_k^p\}$ is increasing in p. Applying the same argument once more, we see that $v_k^p \le u_k^p$ for all k and p, justifying (2.8). Hence, the sequences $\{u_k^p\}$ and $\{v_k^p\}$ converge, and it is clear from (2.4)–(2.6) that their limits are solutions of (2.2), (2.3). If u_k is any solution of (2.2), (2.3), such that $\psi_k \le u_k \le \varphi_k$ for all k, then the same comparison argument as above implies that $u_k \le u_k^p$ for all p and k and, hence, the sequence $\{u_k^p\}$ converges to the maximal solution.

EXAMPLE. We have solved

$$\begin{aligned}
-u'' &= \sqrt{u} - \sqrt{2-x}, \qquad 0 < x < 2,\\
u'(0) - u^3(0) + 9 &= 0, \qquad u(2) = 0,
\end{aligned}$$

starting with a supersolution $\varphi = 4 - x^2$, taking $h = 0.1$ and $\tau = 0.0015$. By $p = 6000$, the iterations had six decimal digits stabilized, giving the exact values of the solution $u = 2 - x$ at the mesh points.

3. DIRICHLET PROBLEM FOR QUASILINEAR EQUATIONS

We discuss numerical solutions of the problem

$$-u'' = f(x, u, u') \qquad \text{for } a < x < b, \quad u(a) = u(b) = 0. \tag{3.1}$$

We assume the function f is continuous in all variables and continuously differentiable in u and u', but do not assume any structure conditions, in particular, we do not place Nagumo type restrictions on the growth of f in u'. Proceeding as in the previous section we obtain the finite difference approximation of (3.1),

$$-\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = f\left(x_n, u_n, \frac{u_{n+1} - u_{n-1}}{2h}\right), \qquad n = 1, \dots, N-1, \quad u_0 = u_N = 0. \tag{3.2}$$

To solve (3.2), we set up the explicit scheme ($p = 0, 1, \dots$)

$$\begin{aligned} &\frac{u_n^{p+1} - u_n^p}{\tau} = \frac{u_{n+1}^p - 2u_n^p + u_{n-1}^p}{h^2} + f\left(x_n, u_n^p, \frac{u_{n+1}^p - u_{n-1}^p}{2h}\right), \qquad n = 1, \dots, N-1, \\ &\quad u_0^{p+1} = (1-\tau_1)\, u_0^p, \quad u_N^{p+1} = (1-\tau_2)\, u_N^p, \end{aligned} \tag{3.3}$$

with a suitably chosen initial guess u_n^0, and positive τ, τ_1 and τ_2.

Denoting as before, $w_n^p = u_n^{p+1} - u_n^p$, we express, from (3.3),

$$w_n^p = \left(1 - \frac{2\tau}{h^2} + \tau f_u\right) w_n^{p-1} + \frac{\tau}{h^2}\left(w_{n+1}^{p-1} + w_{n-1}^{p-1}\right) + \tau f_{u'} \frac{w_{n+1}^{p-1} - w_{n-1}^{p-1}}{2h}. \tag{3.4}$$

Denoting $W^p = \sum_{n=1}^{N-1} w_n^p$, we obtain, from (3.4),

$$W^p = W^{p-1} + \tau \sum_{n=1}^{N-1} f_u\, w_n^{p-1} - \frac{\tau}{h^2}\left(\tau_1 u_0^{p-1} + \tau_2 u_N^{p-1}\right) + \frac{\tau}{2h} \sum_{n=1}^{N-1} f_{u'} \left(w_{n+1}^{p-1} - w_{n-1}^{p-1}\right).$$

Assume now that f_u and $f_{u'}$ do not change much from one mesh point to another. Then the last term on the right telescopes up to the boundary terms, which we ignore since they are quadratic in τ, τ_1 and τ_2. Then,

$$W^p \simeq (1 + \tau f_u^*)\, W^{p-1}, \tag{3.5}$$

where f_u^* is some fixed value of f_u.

Let φ_n be a supersolution of (3.2) defined by

$$\begin{aligned} &-\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{h^2} \ge f\left(x_n, \varphi_n, \frac{\varphi_{n+1} - \varphi_{n-1}}{2h}\right), \qquad n = 1, \dots, N-1, \\ &\qquad\qquad \varphi_0 \ge 0, \quad \varphi_N \ge 0. \end{aligned} \tag{3.6}$$

If we now start the scheme (3.3) with $u_n^0 = \varphi_n$, then $w_n^1 \le 0$, for all $n = 0, 1, \dots, N$, and hence, $W^1 \le 0$. For sufficiently small τ we can expect, from (3.5), that $W^p \le 0$ for all p, i.e., the iterations, on the average, go down. If we start with a subsolution ψ (defined by reversing the inequality signs in (3.6)) with $\psi \le \varphi$, then iterations are expected to go up on the average, and stay, on the average, below the iterations starting with φ. It is then natural to expect convergence.

EXAMPLE. The problem

$$-u'' = e^{u'} + 6x - 4 - e^{4x - 3x^2}, \qquad 0 < x < 2, \quad u(0) = u(2) = 0,$$

has an explicit solution $u = x^2(2-x)$. We have solved it starting with a supersolution $\varphi = -4x^2 + 16$, taking $h = 0.1$, $\tau = 0.0025$, $\tau_1 = 2\tau$, $\tau_2 = 1$. We obtained stabilization of six decimal digits by $p = 4000$, with values close to the exact solution (we used overrelaxation with $\omega = 1.8$).

We now discuss the role of τ_1 and τ_2 in the scheme (3.3). They are placed there to ensure stability. In the example above we were able to set $\tau_2 = 1$, since the initial guess φ vanishes at $x = 2$. If one tries to do the same at the left end, $x = 0$, one gets large values for $(u_2 - u_0)/2h \simeq u'(x_1)$, and an immediate overflow when computing $e^{u'}$.

4. DISPLACEMENTS OF A NONLINEAR ELASTIC BEAM

We consider a model problem

$$\begin{aligned} &u'''' = f(x, u, u'), \quad \text{for } a < x < b, \\ &u(a) = \alpha, \quad u'(a) = \beta, \quad u(b) = \gamma, \quad u'(b) = \delta, \end{aligned} \tag{4.1}$$

with a given function f and given constants α, β, γ and δ, describing vertical deflections of a static beam subject to a nonlinear force f. The finite difference approximation of the equation (4.1) is

$$\frac{u_{n+2}-4u_{n+1}+6u_n-4u_{n-1}+u_{n-2}}{h^2}=f\left(x_n,u_n,\frac{u_{n+1}-u_{n-1}}{2h}\right),\quad n=2,3,\dots,N-2. \tag{4.2}$$

As in the previous sections, for $n=2,3,\dots,N-2$, we set up an iteration scheme ($p=0,1,\dots$)

$$u_n^{p+1}=u_n^p-\tau\,\Delta_h^2\,u_n^p+\tau\,f\left(x_n,u_n^p,\frac{u_{n+1}^p-u_{n-1}^p}{2h}\right), \tag*{(4.3)$_n$}$$

where $\Delta_h^2\,u_n^p$ denotes the left hand side of (4.2) (notice that $u_t+u_{xxxx}=f$ is the evolution equation corresponding to (4.1)).

We use the formula (4.3) at x_1 also. The right hand side of $(4.3)_1$ will then involve u_{-1}^p. We approximate the second boundary condition by $(u_1^p-u_{-1}^p)/2h=\beta$. We express $u_{-1}^p=u_1^p-2h\beta$ and use this in $(4.3)_1$. We apply the same procedure at x_{N+1}. This leads us to the iteration formulas

$$u_1^{p+1}=u_1^p-\frac{\tau}{h^4}\left(-2h\beta-4\alpha+7u_1^p-4u_2^p+u_3^p\right)+\tau\,f\left(x_1,u_1^p,\frac{u_2^p-u_0^p}{2h}\right), \tag{4.4}$$

$$\begin{aligned}u_{N-1}^{p+1}=&u_{N-1}^p-\frac{\tau}{h^4}\left(2h\delta-4\gamma+7u_{N-1}^p-4u_{N-2}^p+u_{N-3}^p\right)\\&+\tau f\left(x_{N-1},u_{N-1}^p,\frac{u_N^p-u_{N-2}^p}{2h}\right),\end{aligned} \tag{4.5}$$

$$u_0^{p+1}=\alpha+(1-\tau_1)\,(u_0^p-\alpha), \tag{4.6}$$

$$u_N^{p+1}=\gamma+(1-\tau_2)\,(u_N^p-\gamma). \tag{4.7}$$

A function $\varphi(x)$ is called a supersolution of (4.1) if

$$\begin{gathered}\varphi''''\geq f(x,\varphi,\varphi'),\qquad \text{for } a<x<b,\\ \varphi(a)\geq\alpha,\quad \varphi'(a)\geq\beta,\quad \varphi(b)\geq\gamma,\quad \varphi'(b)\leq\delta.\end{gathered} \tag{4.8}$$

A subsolution ψ is defined by reversing the signs of inequalities in (4.8). Assume that $\psi(x)\leq\varphi(x)$, for all x in $[a,b]$.

By an analysis similar to that of the previous section, it follows for the scheme (4.3)–(4.7), that if one sets $u_n^0=\varphi(x_n), n=0,1,\dots,N$, then $\sum_{n=0}^N u_n^p$ is expected to decrease in p and converge to the solution of (4.1). This was confirmed by our numerical experiments, however, in contrast to the previous sections, the convergence was rather slow.

EXAMPLE. We solved the problem

$$\begin{gathered}u''''=u^2+1,\qquad 0<x<2,\\ u(0)=u'(0)=u(2)=u'(2)=0,\end{gathered} \tag{4.9}$$

taking $h=0.1$, $\tau=0.00001$ (taking $\tau=0.000015$ leads to overflow) and $\tau_1=\tau_2=1$. Starting with a subsolution $\psi=0$, we obtained stabilization of six decimal digits by $p=60000$ (the computation took under 9 minutes on a SUN 386). We verified the result by a program based on the monotone iteration scheme, developed in [7].

REMARKS. The scheme (4.3)–(4.7) is quite flexible, in particular, we could include lower order terms on the left in the equation (4.1). We believe it can also be extended to handle functions of the type $f=f(x,u,u',u'',u''')$. However, for an important special class of problems of the type $(a(x)u'')''=f(x,u)$ with $a(x)>0$, we suggest using the monotone iteration scheme of [7], applying the Green's function method (see [8]) to solve the linear problems, particularly if one wants computations with small h.

5. A POSTERIORI FOURIER ANALYSIS

Let us consider the problem (3.1) with $a = 0$, $b = 1$. Any solution can be written in the form

$$u(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x, \qquad c_n = 2\int_0^1 u(x) \sin n\pi x \, dx.$$

Once the solution of (3.1) is computed at the grid points x_n, we can evaluate the integrals in c_n by (say) the trapezoid rule, using the same grid points.

EXAMPLE 1. Take $-u'' = 10u^2 + 1$, $u(0) = u(1) = 0$. The problem was solved with $h = 0.1$, $\tau = 0.0025$, $\psi = 0$. The first 20 Fourier coefficients of the solution are given in the Table 1. Since the equation above is invariant under the change of variables $x \to 1-x$, all even numbered Fourier coefficients of the solution are equal to zero. We find it remarkable that c_2, c_4, c_6, c_8 and c_{10} are so close to zero, given the approximation errors. Coefficients from c_{11} on should be disregarded as numerical noise (similar noise was obtained when running $-u'' = 1$, $u(0) = u(1) = 0$). Also notice that, with reasonable accuracy, $u(x) \simeq c_1 \sin \pi x$. Similar results were obtained for other nonlinearities of the type $f = f(u)$.

EXAMPLE 2. Take $-u'' = (u')^3 + 1$, $u(0) = u(1) = 0$. The problem was solved with h, τ and ψ as above, and Fourier coefficients are given in Table 2. This time, even numbered coefficients are not expected to be zero, but c_1 still predominates, although to a smaller degree. The last property was common to all nonlinearities $f = f(x, u, u')$ that we tried.

Table 1.

n	c
1	0.14769862
2	-0.00000002
3	0.00441024
4	-0.00000001
5	0.00097014
6	-0.00000001
7	0.00031481
8	-0.00000001
9	0.00007992
10	-0.00000001
11	-0.00686936
12	0.01246702
13	-0.01972931
14	0.01168754
15	-0.03117548
16	0.03991352
17	-0.03433395
18	-0.03873912
19	-0.02372190
20	-0.10781452

Table 2.

n	c
1	0.12900708
2	0.00074879
3	0.00476743
4	0.00020581
5	0.00100431
6	0.00006308
7	0.00032285
8	0.00002099
9	0.00008173
10	-0.00000001
11	-0.00604312
12	0.01088602
13	-0.01728411
14	0.01015632
15	-0.02728043
16	0.03440854
17	-0.03056843
18	-0.03465806
19	-0.02108706
20	-0.09382150

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