On computation of solutions of fully nonlinear elliptic problems

Gilbert Choudury and Philip Korman *

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, United States

Abstract

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We present an explicit monotone scheme for solving boundary value problems for fully nonlinear elliptic equations. We replace the discretized elliptic problems by their parabolic versions, with initial data being either super- or subsolution. For sufficiently fine meshes we obtain monotone iterations, for which convergence can often be proved.

Keywords: Fully nonlinear elliptic problems; monotone iterations.

1. Introduction

We develop an explicit algorithm for solving boundary value problems for fully nonlinear elliptic problems. We replace the discretized elliptic problems by the corresponding parabolic ones, and show that solutions of the parabolic problems tend to those of the elliptic ones, assuming that the spatial step h and the time step τ are small enough, $\tau = \tau(h)$, and that the initial data is either a super- or subsolution. While we usually need to choose the time step τ rather small, we typically get good accuracy after a moderate number of time steps.

This algorithm was previously used [5] for quasilinear Dirichlet problems (including systems) of the type $\Delta u = f(x, u, Du)$, and in [2], in case f is independent of Du. While in the one-dimensional case the extension is rather straightforward, difficulties involving mixed

derivatives arise in higher dimensions. In Section 3 we use a nine-point approximation of the Laplacian to overcome this problem.

Our results can be easily extended to systems (see [2,5,7]). Preliminary computations indicated that a similar approach may work for the obstacle problems (see also [6]).

2. One-dimensional problem

We are interested in the numerical solution of the fully nonlinear boundary value problem with general separated boundary conditions

$$f[u] \equiv f(u'', u', u, x) = 0, \quad \text{for } a < x < b,$$
$$B_1 u \equiv \alpha u(a) - \beta u'(a) = A, \qquad B_2 u \equiv \gamma u(b) + \delta u'(b) = B, \tag{2.1}$$

with nonnegative constants α, β, γ, δ and real A, B. We divide the interval $[a, b]$ into N equal parts of length $h = (b - a)/N$ each, denote $x_0 = a$, $x_k = x_0 + kh$ for $k = 1, 2, \ldots, N$, $u_k = u(x_k)$, and replace (2.1) by its finite-difference version

$$f[u_k] \equiv f\left(\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}, \frac{u_{k+1} - u_{k-1}}{2h}, u_k, x_k \right) = 0, \quad k = 1, \ldots, N-1,$$

$$B_1 u_0 \equiv \alpha u_0 - \beta \frac{u_1 - u_0}{h} = A, \qquad B_2 u_N \equiv \gamma u_N + \delta \frac{u_N - u_{N-1}}{h} = B. \tag{2.2}$$

The following result establishes existence of solution for (2.2) and provides an explicit algorithm for its computation.

We shall obtain the solution(s) of (2.2) as the steady state of the corresponding "parabolic" problem ($p = 0, 1, \ldots$)

$$\frac{u_k^{p+1} - u_k^p}{\tau} = f[u_k^p], \quad k = 1, \ldots, N-1, \qquad B_1 u_k^p = A, \qquad B_2 u_k^p = B, \tag{2.3}$$

with a suitably chosen u_k^0 and "time" step τ.

Definition 2.1. A grid function ϕ_k is called a *supersolution* of (2.2) provided

$$f[\phi_k] \leq 0, \quad k = 1, \ldots, N-1, \qquad B_1 \phi_0 \geq A, \qquad B_2 \phi_N \geq B. \tag{2.4}$$

A grid function ψ_k is called a *subsolution* if it satisfies the opposite inequalities.

Theorem 2.2. *Assume that the problem* (2.2) *possesses a supersolution* ϕ_k *and a subsolution* ψ_k, *with* $\psi_k \leq \phi_k$ *for all* k. *Assume that for* $h \leq h_0$ *sufficiently small,*

$$f_{u''}[u_k] > \tfrac{1}{2} h \, | f_{u'}[u_k] |, \quad \text{for } k = 1, \ldots, N-1, \tag{2.5}$$

and all u_k *such that* $\psi_k \leq u_k \leq \phi_k$ *for all* k. *Assume finally that* f *is continuously differentiable in* u, u' *and* u''. *Then for* $h \leq h_0$ *the problem* (2.2) *has a solution* u_k, *with* $\psi_k \leq u_k \leq \phi_k$ *for all* k.

Moreover, if one starts the scheme (2.3) *with* $u_k^0 = \phi_k$, *then for* $h \leqslant h_0$ *and* $\tau = \tau(h)$ *sufficiently small, the sequence* u_k^p *is monotone decreasing in* p *for all* k. *Similarly, if one starts with* $v_k^0 = \psi_k$, *then the corresponding sequence is monotone increasing in* k, *and*

$$\psi_k \leqslant v_k^1 \leqslant v_k^2 \leqslant \cdots \leqslant u_k \leqslant \cdots \leqslant u_k^2 \leqslant u_k^1 \leqslant \phi_k, \quad \text{for all } k. \tag{2.6}$$

The sequences $\{u_k^p\}$ *and* $\{v_k^p\}$ *converge correspondingly to the maximal and minimal solutions of* (2.2).

Proof. Letting $u_k^0 = \phi_k$, rewrite (2.3) as

$$u_k^{p+1} = u_k^p + \tau f[u_k^p], \quad \text{for } k = 1, \ldots, N-1,$$
$$u_0^{p+1} = \frac{1}{\alpha h + \beta}(Ah + \beta u_1^p), \qquad u_N^{p+1} = \frac{1}{\delta h + \gamma}(Bh + \gamma u_{N-1}^p). \tag{2.7}$$

Denote $w_k^p = u_k^{p+1} - u_k^p$. From the definition of the supersolution it follows that $w_k^0 \leqslant 0$ for all k. We prove by induction that $w_k^p \leqslant 0$ for all k and $p \geqslant 1$. Using the mean-value theorem we obtain from (2.7) (assuming $w_k^{p-1} \leqslant 0$),

$$w_k^p = w_k^{p-1}\left(1 - \frac{2\tau}{h^2}f_{u''} + \tau f_u\right) + \tau w_{k+1}^{p-1}\left(\frac{1}{h^2}f_{u''} + \frac{1}{2h}f_{u'}\right) + \tau w_{k-1}^{p-1}\left(\frac{1}{h^2}f_{u''} - \frac{1}{2h}f_{u'}\right), \tag{2.8}$$

with $f_{u''}$, $f_{u'}$ and f_u evaluated at $[\theta u_k^{p-1} + (1-\theta)u_k^p]$ for some $0 < \theta < 1$. Hence, if we choose h small enough to satisfy (2.5), and $\tau = \tau(h)$ so small that

$$1 - \frac{2\tau}{h^2}f_{u''}[u_k] + \tau f_u[u_k] > 0, \quad \text{for all } \psi_k \leqslant u_k \leqslant \phi_k, \ k = 1, \ldots, N-1, \tag{2.9}$$

it will follow from (2.8) that $w_k^p \leqslant 0$ for $k = 1, \ldots, N-1$. Using the second line of (2.7), we then conclude $w_k^p \leqslant 0$ for all $k = 0, 1, \ldots, N$.

A similar argument shows that v_k^p form an increasing in p sequence, and that (2.6) holds, which finishes the proof. $\square$

Remark 2.3. The following two natural conditions (see also [8]) are sufficient for (2.5) to hold:

$$f_{u''}[u_k] \geqslant c_1 > 0 \quad \text{(ellipticity)}, \tag{2.10}$$

$$|f_{u'}[u_k]| \leqslant c_2\left(1 + \frac{1}{h^\alpha}\right), \quad \text{with } 0 < \alpha < 1 \text{ and } c_1, c_2 \text{ independent of } h, \tag{2.11}$$

for $\psi_k \leqslant u_k \leqslant \phi_k$, $k = 1, \ldots, N-1$, and any $h > 0$ (which is implied by the condition $|f_{u'}(x, u, u', u'')| \leqslant c(1 + |u'|^\alpha)$, $c = c(u)$, since we can then set

$$c_2 = \max_k \max_{\psi_k \leqslant u_k \leqslant \phi_k} c(u_k) \cdot \max_k |\phi_k - \psi_k|).$$

Remark 2.4. Existence of super- and subsolutions implies satisfaction of the compatibility conditions, which are required for the Neumann problem. Consider for example the linear problem

$$-u''(x) = f(x), \quad \text{for } 0 < x < 1, \qquad u'(0) = u'(1) = 0. \tag{2.12}$$

Writing down the definitions of super- and subsolutions, and integrating in both cases, we conclude that $\int_0^1 f(x)\,dx = 0$, i.e., the problem above is solvable.

Next we show that for the Dirichlet problem the solution of (2.2) converges to that of (2.1). We shall need an a priori estimate given in Lemma 2.5, which is a slight modification of [1, p.147]. We shall sketch its proof, but first we state the following standard maximum principle, see, e.g., [3, p.439].

Lemma 2.5. *Let A_i, B_i, C_i be given numbers, $1 \leqslant i \leqslant N - 1$, such that*

$$A_i, B_i, C_i > 0, \quad A_i - 2B_i + C_i \leqslant 0, \quad 1 \leqslant i \leqslant N - 1.$$

Let the grid function w_i satisfy

$$A_i w_{i+1} - 2B_i w_i + C_i w_i + C_i w_{i-1} \geqslant 0, \quad 1 \leqslant i \leqslant N - 1.$$

Then for all $1 \leqslant i \leqslant N - 1$,

$$w_i \leqslant \max(w_0, w_N).$$

For the grid function u_k, $k = 0, \ldots, N$, we define $\|u\| = \|u_k\| = \max_{0 \leqslant k \leqslant N} |u_k|$.

Lemma 2.6. *On the grid x_k defined above consider the problem*

$$a_k(h)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + b_k(h)\frac{u_{k+1} - u_{k-1}}{2h} + c_k(h)u_k = f_k, \quad k = 1, \ldots, N-1, \tag{2.13}$$

$$u_0 = 0, \qquad u_N = 0.$$

Assume that the functions $a_k(h)$, $b_k(h)$ and $c_k(h)$ are continuous and $a_k(h) \geqslant \delta > 0$ and $c_k(h) \leqslant 0$ for all $k = 1, \ldots, N - 1$, and $0 \leqslant h \leqslant h_1$. Then there exist positive constants h_0 and c_0, such that for $h \leqslant h_0$,

$$\|u\| \leqslant c_0 \|f\|. \tag{2.14}$$

Proof. Introduce $w(x) = (1 + x - a)^\beta$, where β is a positive integer to be chosen later. Let $w_k = w(x_k)$. Since $w(x)$ and all its derivatives are nonnegative for $x \geqslant a$, it follows that (with $-1 < \theta < 1$)

$$\frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} = w''(x_k + \theta_1 h) \geqslant w''(x_k - h),$$

$$\frac{w_{k+1} - w_{k-1}}{2h} = w'(x_k + \theta_2 h) \leqslant w'(x_k + h).$$

If $R > 0$ is sufficiently large, then

$$L_h w_k \equiv a_k(h)\frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} + b_k(h)\frac{w_{k+1} - w_{k-1}}{2h} + c_k(h)w_k$$

$$\geqslant \delta w''(x_k - h) - R w'(x_k + h) - R w(x_k)$$

$$= (1 + x_k - a)^{\beta - 2}\beta(\beta - 1)$$

$$\times \left[\delta\left(\frac{1 + x_k - a - h}{1 + x_k - a}\right)^{\beta - 2} - R\frac{1 + x_k - a}{\beta - 1}\left(\frac{1 + x_k - a + h}{1 + x_k - a}\right)^{\beta - 1} \right.$$

$$\left. - R\frac{(1 + x_k - a)^2}{\beta(\beta - 1)} \right]$$

$$\geqslant \beta(\beta - 1)\left[\delta(1 - h)^{\beta - 2} - \frac{R(1 + b - a)}{\beta - 1}(1 + h)^{\beta - 1} - R\frac{(1 + b - a)^2}{\beta(\beta - 1)} \right] \geqslant S,$$

where S is a positive constant, provided we choose h sufficiently small and β sufficiently large. Then with a sufficiently large constant $P > 0$,

$$L_h\left(P w_k \pm \frac{u_k}{\|f\|}\right) \geqslant PS - 1 \geqslant 0.$$

Letting $z_k = P w_k \pm u_k / \|f\|$, we rewrite the last inequality as

$$A_k z_{k+1} - 2B_k z_k + C_k z_{k-1} \geqslant 0, \tag{2.15}$$

with $A_k = a_k(h) + \frac{1}{2}hb_k(h)$, $B_k = a_k(h) - \frac{1}{2}h^2c_k(h)$, $C_k = a_k(h) - \frac{1}{2}hb_k(h)$. Decreasing h if necessary, we see that Lemma 2.5 applies to (2.15), and hence for all $k = 1, \ldots, N - 1$,

$$\pm\frac{u_k}{\|f\|} \leqslant z_k \leqslant P \max(w_0, w_N),$$

and the lemma follows. $\quad\square$

Theorem 2.7. *Let* $u(x) \in C^4[a, b]$ *be solution of* (2.1) *with* $u(a) = u(b) = 0$. *Assume that for* $x \in [a, b]$, $\psi \leqslant u \leqslant \phi$, *and all values of* u' *and* u'', *the following conditions hold:*
   (i) $f_{u''} > 0$,
   (ii) $f_u \leqslant 0$,
   (iii) f *is continuous and continuously differentiable in* u'', u' *and* u, *with derivatives* f_u, $f_{u'}$, $f_{u''}$ *uniformly bounded.*
   Denote by u_k *the solution of* (2.2) *with* $u_0 = u_N = 0$, *and by* $[u]_k$ *the solution of* (2.1) *with* $u(a) = u(b) = 0$ *evaluated at the grid points* x_k. *Then one can find positive constants* c *and* h_0 *such that*

$$\|u_k - [u]_k\| \leqslant ch^2, \quad \text{for } h \leqslant h_0.$$

Proof. We follow the standard scheme: stability plus approximation implies convergence. By our assumptions (using the Taylor's formula),

$$f[[u]_k] = f(u''(x_k) + O(h^2), u'(x_k) + O(h^2), u_k, x_k)$$
$$= f(u''(x_k), u'(x_k), u_k, x_k) + O(h^2) = O(h^2), \qquad (2.16)$$

since $u(x)$ solves (2.1). Subtracting (2.16) from (2.2), using the mean-value theorem, and denoting $w_k = u_k - [u]_k$, we obtain

$$f_{u''}\frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} + f_{u'}\frac{w_{k+1} - w_{k-1}}{2h} + f_u w_k = O(h^2), \qquad (2.17)$$

where $f_{u''}[\cdot]$, $f_{u'}[\cdot]$, $f_u[\cdot]$ are evaluated at $\theta[u]_k + (1 - \theta)u_k$ with $0 < \theta < 1$. Applying Lemma 2.6 to (2.17), we conclude the proof. □

3. Two and more dimensions

We present our results for the two-dimensional case, although they easily generalize to arbitrary number of dimensions. We start by considering the Dirichlet problem

$$f[u] \equiv f(u_{xx}, u_{yy}, u_x, u_y, u, x, y) = 0, \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$u(x, y) = g(x, y), \qquad \text{on } \partial\Omega, \qquad (3.1)$$

where the equation does not explicitly depend on u_{xy}. We replace the domain Ω by the uniform square mesh Ω_h of step size h, denoting $k = (k_1, k_2)$, $x_k = (k_1 h, k_2 h)$ and $u_k = u(x_k)$ with positive integers k_1, k_2, (assuming Ω to lie in the first quadrant). We replace (3.1) by its finite-difference version

$$f[u_k] \equiv f(\delta_x^2 u, \delta_y^2 u, \delta_x u, \delta_y u, u_k, x_k) = 0, \quad \text{in } \Omega_h,$$
$$u_k = g(x_k), \qquad \text{on } \partial\Omega_h, \qquad (3.2)$$

where

$$\delta_x u = \frac{u_{k+e_1} - u_{k-e_1}}{2h}, \quad \delta_x^2 u = \frac{u_{k+e_1} - 2u_k + u_{k-e_1}}{h^2}, \qquad e_1 = (1, 0),$$

$\delta_y u$ and $\delta_y^2 u$ are defined similarly, and the boundary values are defined on $\partial\Omega_h$ in a standard way. We obtain the solution of (3.2) using the scheme ($p = 0, 1, \dots$)

$$\frac{u_k^{p+1} - u_k^p}{\tau} = f[u_k^p], \quad \text{in } \Omega_h, \quad u_k^p = g(x_k), \quad \text{on } \partial\Omega_h, \qquad (3.3)$$

with properly chosen u_k^0 and the step size τ.

Definition 3.1. A grid function ϕ_k is called a *supersolution* of (3.2) if

$$f[\phi_k] \leqslant 0, \quad \text{in } \Omega_h, \quad \phi_k \geqslant g(x_k), \quad \text{on } \partial\Omega_h. \qquad (3.4)$$

A *subsolution* ψ_k is defined by reversing the above inequalities.

The following result is similar to Theorem 2.2.

Theorem 3.2. *Assume that the problem (3.2) has a supersolution ϕ_k and a subsolution ψ_k, with $\psi_k \leq \phi_k$ for all k. Assume that for $h \leq h_0$ sufficiently small, and all u_k such that $\psi_k \leq u_k \leq \phi_k$ in Ω_h,*

$$f_{u_{xx}}[u_k] > \tfrac{1}{2}h \left| f_{u_x}[u_k] \right|, \quad f_{u_{yy}}[u_k] > \tfrac{1}{2}h \left| f_{u_y}[u_k] \right|, \quad in \ \Omega_h. \tag{3.5}$$

Assume finally that f is continuously differentiable in u, u_x, u_y, u_{xx}, u_{yy} for $\psi \leq u \leq \phi$ and all values of other arguments. Then conclusions are word for word the same as in Theorem 2.2.

Proof. Starting with $u_k^0 = \phi_k$, define the iterates u_k^p by $u_k^{p+1} = u_k^p + \tau f[u_k^p]$, and similarly starting with $v_k^0 = \psi_k$ define the iterates v_k^p. We claim that the inequalities (2.6) hold. Indeed, with $w_k^p = u_k^{p+1} - u_k^p$, we obtain, using the mean-value theorem,

$$w_k^p = w_k^{p-1}\left(1 - \frac{2\tau}{h^2}f_{u_{xx}} - \frac{2\tau}{h^2}f_{u_{yy}} + \tau f_u\right)$$

$$+ \frac{\tau}{h^2}w_{k+e_1}^{p-1}\left(f_{u_{xx}} + \tfrac{1}{2}hf_{u_x}\right) + \frac{\tau}{h^2}w_{k-e_1}^{p-1}\left(f_{u_{xx}} - \tfrac{1}{2}hf_{u_x}\right)$$

$$+ \frac{\tau}{h^2}w_{k+e_2}^{p-1}\left(f_{u_{yy}} + \tfrac{1}{2}hf_{u_y}\right) + \frac{\tau}{h^2}w_{k-e_2}^{p-1}\left(f_{u_{yy}} - \tfrac{1}{2}hf_{u_y}\right).$$

In view of the conditions (3.5) we can make the last four brackets on the right positive by choosing h sufficiently small, and then we can make the first bracket positive by choosing $\tau = \tau(h)$ small enough. The inequalities (2.6) then follow by induction, and they imply the theorem. $\square$

We can allow the u_{xy} term in (3.1), provided the equation has a special form:

$$f[u] \equiv f\left(\Delta u, u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y\right) = 0, \quad in \ \Omega,$$
$$u = g(x, y), \quad \text{on } \partial\Omega. \tag{3.6}$$

To preserve the monotonicity properties we use the nine-point approximation of the Laplacian

$$\Delta_h u_k = \frac{1}{6h^2}\left[4u_{k+e_1} + 4u_{k-e_1} + 4u_{k+e_2} + 4u_{k-e_2} + u_{k+e_1+e_2}\right.$$

$$\left. + u_{k-e_1+e_2} + u_{k-e_1-e_2} + u_{k+e_1-e_2} - 20u_k\right].$$

(The usual five-point approximation does not allow one to control the terms like $u_{k+e_1+e_2}$ introduced by u_{xy}.) We then replace (3.6) by

$$f[u_k] \equiv f\left(\Delta_h u_k, \delta_x^2 u_k, \delta_{xy}^2 u_k, \delta_y^2 u_k, \delta_x u_k, \delta_y u_k, u_k, x_k\right) = 0, \quad in \ \Omega_h,$$
$$u_k = g(x_h), \quad \text{on } \partial\Omega_h, \tag{3.7}$$

where

$$\delta_{xy}^2 u_k = \frac{1}{4h^2}\left[u_{k+e_1+e_2} - u_{k-e_1+e_2} - u_{k+e_1-e_2} + u_{k-e_1-e_2}\right].$$

To solve (3.7) we again use the scheme (3.3) (a numerical example is given at the end of Section 4).

Theorem 3.3. *Assume that the problem* (3.7) *has a supersolution* ϕ_k *and a subsolution* ψ_k, *with* $\psi_k \leqslant \phi_k$ *for all k. Assume that for $h \leqslant h_0$ sufficiently small and all u_k such that $\psi_k \leqslant u_k \leqslant \phi_k$ in* Ω_h, *the following conditions hold:*

$$\tfrac{2}{3} f_{\Delta u}[u_k] + f_{u_{xx}}[u_k] > \tfrac{1}{2} h \left| f_{u_x}[u_k] \right|, \qquad \tfrac{2}{3} f_{\Delta u}[u_k] + f_{u_{yy}}[u_k] > \tfrac{1}{2} h \left| f_{u_y}[u_k] \right|,$$

$$f_{\Delta u}[u_k] > \tfrac{3}{2} \left| f_{u_{xy}}[u_k] \right|. \tag{3.8}$$

Assume finally that f is continuously differentiable in Δu, u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u for $\psi \leqslant u \leqslant \phi$ and all values of other arguments. Then conclusions are exactly the same as in Theorem 2.2.

Proof. Letting $u_k^0 = \phi_k$, rewrite (3.3) (for (3.6)) as

$$u_k^{p+1} = u_k^p + \tau f \left(\frac{1}{6h^2} \left(4u_{k+e_1}^p + \cdots + u_{k+e_1+e_2}^p + \cdots - 20 u_k^p \right), \right.$$

$$\frac{1}{4h^2} \left(u_{k+e_1+e_2}^p \pm \cdots \right), \ \frac{1}{h^2} \left(u_{k+e_1}^p - 2u_k^p + u_{k-e_1}^p \right),$$

$$\frac{1}{h^2} \left(u_{k+e_2}^p - 2u_k^p + u_{k-e_2}^p \right), \ \frac{1}{2h} \left(u_{k+e_1}^p - u_{k-e_1}^p \right),$$

$$\left. \frac{1}{2h} \left(u_{k+e_2}^p - u_{k-e_2}^p \right), \ u_k^p, \ x_k \right),$$

where $+ \cdots$ and $\pm \cdots$ denote similar terms. Denoting $w_k^p = u_k^{p+1} - u_k^p$, and choosing $h \leqslant h_0$, and τ so small that

$$1 - \frac{10\tau}{3h^2} f_{\Delta u} - \frac{2\tau}{h^2} f_{u_{xx}} - \frac{2\tau}{h^2} f_{u_{yy}} + \tau f_u > 0, \quad \text{for all } \psi_k \leqslant u_k \leqslant \phi_k,$$

we conclude by induction as in Theorem 2.2 that

$$w_k^p = w_k^{p-1} \left(1 - \frac{10\tau}{h^2} f_{\Delta u} - \frac{2\tau}{h^2} f_{u_{xx}} - \frac{2\tau}{h^2} f_{u_{yy}} + \tau f_u \right)$$

$$+ w_{k+e_1}^{p-1} \left(\frac{2\tau}{3h^2} f_{\Delta u} + \frac{\tau}{h^2} f_{u_{xx}} + \frac{\tau}{2h} f_{u_x} \right)$$

$$+ \cdots + w_{k+e_1+e_2}^{p-1} \left(\frac{1}{6h^2} f_{\Delta u} + \frac{1}{4h^2} f_{u_{xy}} \right) + \cdots \leqslant 0,$$

for all k and p, which implies that the inequalities (2.6) and the proof follows (here again $+ \cdots$ denotes similar terms). $\square$

4. Numerical experiments

It was an easy matter to implement the explicit scheme (3.3) in both one- and two-dimensional cases. While the time step τ has to be taken rather small to ensure convergence, we had usually obtained stabilization of solution after rather moderate number of time steps. To speed

up the convergence an overrelaxation was tried, i.e., after computing u_k^{p+1} by (3.3), we would set $u_k^{p+1} := u_k^p + \omega(u_k^{p+1} - u_k^p)$, with typically $\omega = 1.8$ (by analogy with linear problems) and other values of ω. This did not seem to intervene with the monotonicity of the convergence (typically we compared the results after every ten iterations). However, the overrelaxation tended to have a destabilizing effect, making it necessary to choose considerably smaller time steps, and so it was not effective. In most cases we used either super- or subsolution, as it seems to be a luxury to have both for fully nonlinear problems. Control examples may also be hard to come by for such problems, however for ODEs [4] is an excellent source.

The problem (see [4, problem 6.137])

$$2uu'' + u'^2 + 1 = 0, \quad \text{for } 0 < x < 2, \qquad u(0) = u(2) = 0, \tag{4.1}$$

was solved using $\psi \equiv 0$, $h = 0.1$ and $\tau = 0.004$. By the time $t = 1$ (i.e., after 1000 iterations), our iterations had six decimal digits stabilized, converging to the exact solution, which is a cycloid $x = \pi^{-1}(t - \sin t)$, $u = \pi^{-1}(1 - \cos t)$ with $0 \leqslant t \leqslant 2\pi$. (We kept six digits since the discretized version of (4.1) is also of interest. Of course, because of the approximation error, some of the digits have no significance for the original equation (4.1).) For example, at $x = 1$ the exact value of the solution is $2/\pi \simeq 0.6366$. We obtained $u(1) \simeq 0.6261$. When we repeated the computation with $h = 0.05$ and $\tau = 0.001$, we obtained a more accurate value $u(1) \simeq 0.6316$, with similar improvements at the other mesh points. The accuracy was decreasing towards the boundary. For example, $u(0.1)$ was computed with relative error $\simeq 10\%$ when $h = 0.1$, and the relative error $\simeq 5\%$ when $h = 0.05$. The problem (4.1) also has a positive subsolution $\psi_1 = \epsilon \sin \frac{1}{2}\pi x$, provided ϵ is small enough. To verify the condition (2.5) compute

$$f_{u''}[u_k] \pm \tfrac{1}{2}hf_{u'}[u_k] = 2u_k \pm hu_k' > u_k \pm hu_k' \simeq u_{k \pm 1} \geqslant 0, \quad \text{for } u \geqslant \psi_1.$$

Computed values at different times t are shown in Fig. 1. It is clear from the figure that by $t = 0.4$ at least two decimal digits are stabilized.
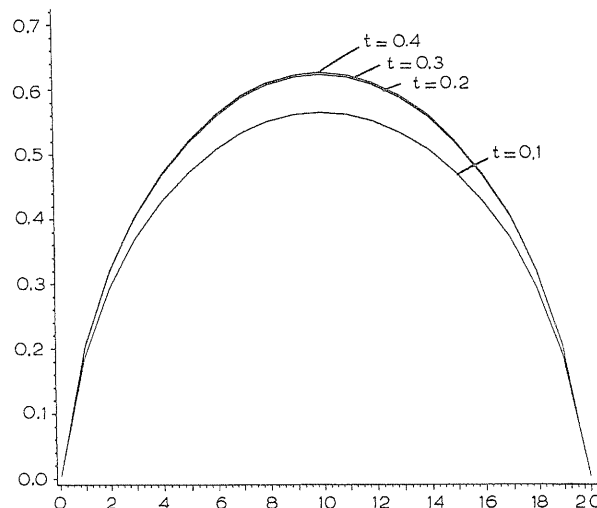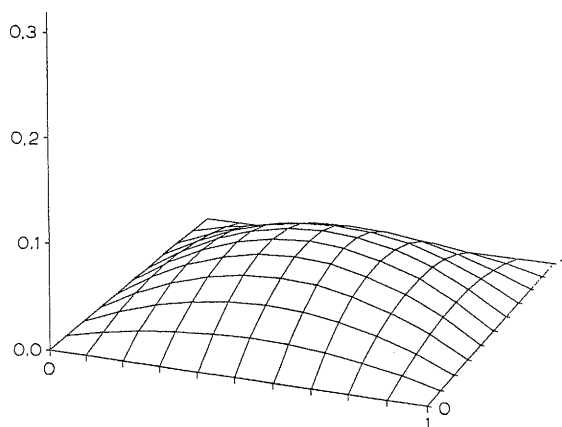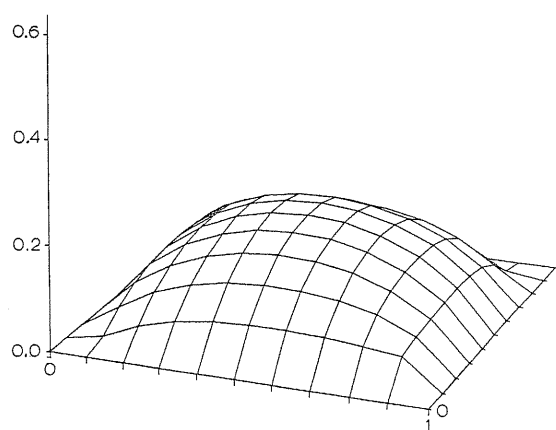

Fig. 1.

Fig. 2.

Fig. 3.

For the problem

$$u''^3 + u'' + (u' + 2)x - 2u + 10 = 0, \quad \text{for } 0 < x < 2, \qquad u(0) = u(2) = 0,$$

we took $h = 0.1$, $\tau = 0.0002$, $\psi = 0$, and obtained the exact values of the solution, which is $u = x(2 - x)$, at all mesh points by $t = 0.3$. When we tried $h = 0.1$ and $\tau = 0.001$ as in the previous example, the scheme was unstable and the program was aborted. The critical τ is here $\simeq 0.0003$. One sees that dependence of τ on h varies from one equation to another. We have no theoretical explanation on how τ should be selected, except that it usually should be considerably lower than $\tau = \frac{1}{2}h^2$ from the linear theory.

For the two-dimensional problem on $[0, 1] \times [0, 1]$,

$$f[u] = x^3(1 - x)^3 u_{xx}^3 + y^3(1 - y)^3 u_{yy}^3 + u_{xx} + u_{yy} + 2y(1 - y) + 2x(1 - x) + 16u^3$$

$$= 0, \tag{4.2}$$

$$u(x, 0) = u(x, 1) = 0, \quad \text{for } 0 < x < 1, \qquad u(0, y) = u(1, y) = 0, \quad \text{for } 0 < y < 1, \tag{4.3}$$

for which the solution is $u = x(1 - x)y(1 - y)$, we started with a subsolution $\psi = 0$, and by taking $h = 0.1$, $\tau = 0.001$, we obtained three stabilized digits by $t = 0.4$ (see Fig. 2).

Finally, we present numerical results for the following equation on $\Omega = (0, 1) \times (0, 1)$:

$$(1 + \sqrt{u})\Delta u + (1 - 10u)u_{xy}^3 - 4u(3 - u) + y(4x^2 - 1) + 5 = 0, \quad u = 0, \qquad \text{on } \partial\Omega,$$

which is of the type (3.6) and for which the solution is not known. One can easily see that conditions (3.8) are satisfied at $u_k \equiv 0$, which makes it reasonable to try the scheme (3.3), and that $\psi \equiv 0$ and $\phi \equiv 1$ are a sub- and supersolution, respectively.

The graph in Fig. 3 represents values at $t = 0.5$, computed with $h = 0.1$, $\tau = 0.002$, and $\psi \equiv 0$ as a starting point (using the nine-point approximation of Δu). In this example, four decimal digits are stabilized by $t = 0.5$.

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