CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 7, Number 3, Fall 1999

MULTIPLICITY AND MORSE INDICES OF SIGN-CHANGING SOLUTIONS FOR SEMILINEAR EQUATIONS

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ABSTRACT. We study Morse indices of sign-changing solutions for Dirichlet two-point problems. If solution has k interior roots, we show that its Morse index is equal to either k or $k+1$, with a precise answer possible in some important cases. As an application we establish exact multiplicity of sign-changing solutions for two classes of problems, which arise in population modeling.

1. Introduction. We consider both positive and sign-changing solutions of the two-point problem

$$u'' + f(u) = 0 \quad \text{for } x \in (-1,1), \quad u(-1) = u(1) = 0, \tag{1.1}$$

with nonlinearity $f(u)$ of class C^1. Problems of this kind arise in many applications, and there is vast literature devoted to them, see, e.g., [7] and the references given there. Most of the previous work is concerned with positive (or negative) solutions. There is a good reason for this emphasis: positive (or negative) solutions are the only ones with a chance of being stable. Recall that solution of (1.1) is called stable if the corresponding linearized equation (see (2.4) below) has only positive eigenvalues. In other words, the Morse index of the positive solution, defined as the number of negative eigenvalues of the linearized equation, is zero. We shall show that more is true: if the solution has k interior roots, then its Morse index is either k or $k+1$. In particular, the Morse index of positive solution is either zero or one. We use the information on Morse index to establish exact multiplicity of sign changing solutions for several classes of problems, obtaining a complete description of the solution set. In addition, the Morse index provides us with the dimension of the unstable manifold for the corresponding parabolic equation.

Accepted for publication on August 5, 1999.
AMS *Mathematics Subject Classification.* 34B15.
Key words and phrases. Morse indices, multiplicity of sign-changing solutions.

Our results on the problem (1.1) depend heavily on the autonomous nature of the equation. We show that our results on positive solutions extend to a class of problems with nonlinearity depending on x, which we have studied extensively previously, see, e.g., [**8**],

$$u'' + f(x,u) = 0 \quad \text{for } x \in (-1,1), \quad u(-1) = u(1) = 0, \tag{1.2}$$

with f even in x and $xf_x < 0$ for $x > 0$. We show that the Morse index of any positive solution of (1.2) is either zero or one, and give some natural conditions under which it is equal to zero or to one.

2. Morse index of positive and sign-changing solutions. We study Morse index of positive solutions for the two-point boundary value problem

$$u'' + f(x,u) = 0 \quad \text{for } x \in (-1,1), \quad u(-1) = u(1) = 0. \tag{2.1}$$

We assume that the function $f \in C^1$ satisfies

$$f(-x,u) = f(x,u) \quad \text{for all} \quad u > 0 \quad \text{and} \quad x \in (-1,1), \tag{2.2}$$

$$xf_x(x,u) \le 0 \quad \text{for all} \quad u > 0 \quad \text{and} \quad x \in (-1,1). \tag{2.3}$$

By the theorem of Gidas, Ni and Nirenberg [4], it follows that any positive solution of (2.1) is an even function, with $u'(x) < 0$ for $0 < x < 1$ (see also Korman [**5**] for a different proof and more general results in one dimension). Corresponding to any positive solution of (2.1) we consider an eigenvalue problem

$$w'' + f_u(x,u)w + \lambda w = 0 \quad \text{for } x \in (-1,1), \quad w(-1) = w(1) = 0. \tag{2.4}$$

By the Morse index of u we understand the number of negative eigenvalues of (2.4).

Theorem 2.1. *Under the conditions* (2.2) *and* (2.3), *the Morse index of any positive solution of* (2.1) *is either zero or one.*

Proof. We recall that (2.4) has an increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots$, with one-dimensional eigenspaces, and that kth eigenfunction has $k-1$ interior roots, see, e.g., [2]. Since the first eigenfunction, call it $\phi(x)$, can be assumed to be positive, it follows that it has to be an even function (since otherwise $\phi(-x)$ is also a first eigenfunction, which is impossible since all eigenspaces are one-dimensional). Arguing similarly, the second eigenfunction, call it $\psi(x)$, has one interior zero, which has to be at $x=0$ (otherwise consider $\psi(-x)$ and obtain a contradiction as before). If we assume the Morse index of $u(x)$ to be greater than or equal to two, then $\lambda_2<0$ and $\psi(x)$ satisfy

$$\psi''+f_u(x,u)\psi+\lambda_2\psi=0 \quad \text{for } x\in(-1,1), \quad \psi(-1)=\psi(1)=0. \tag{2.5}$$

By the above $\psi(x)$ has its only interior root at $x=0$, and so we may assume that $\psi(x)$ is positive on $(-1,0)$ and negative on $(0,1)$, with $\psi'(-1)>0$ and $\psi'(1)>0$. We now differentiate our equation (2.1)

$$u'''+f_u(x,u)u'+f_x(x,u)=0. \tag{2.6}$$

Multiply the equation (2.5) by u', and subtract from it (2.6) multiplied by ψ. Then we integrate the result over $(-1,1)$, obtaining

$$u'(1)\psi'(1)-u'(-1)\psi'(-1)+\lambda_2\int_{-1}^{1}\psi u'\,dx-\int_{-1}^{1}f_x\psi\,dx=0. \tag{2.7}$$

Since all terms on the left are nonpositive, and the third one is negative, we obtain a contradiction.

More detailed conclusion is possible provided that

$$f(\pm 1,0)\geq 0. \tag{2.8}$$

Theorem 2.2. *Assume conditions of the previous theorem and* (2.8) *hold. Then the Morse index of any positive solution of* (2.1) *is either zero or one, and if it is one, then the second eigenvalue is not zero.*

Proof. Condition (2.8) and the Hopf's boundary lemma imply that $u'(-1)>0$ and $u'(1)<0$. Then the first two terms in (2.7) are negative, and we obtain the same contradiction as before if we assume that $\lambda_2=0$.

Remark. If $f(\pm 1, 0) < 0$, then the second eigenvalue may be zero. Indeed, let us consider a simple example

$$u'' + u^2 - \lambda = 0 \quad \text{for } x \in (-1,1), \quad u(-1) = u(1) = 0. \tag{2.9}$$

Then for small values of the parameter λ there is a curve of positive solutions, and the solutions on this curve stop being positive for large λ, see, e.g., Korman [6]. Since for positive solutions $u'(x,\lambda) < 0$ for all $x > 0$, the only way this may happen is for $u'(1,\bar{\lambda}) = 0$ at some $\lambda = \bar{\lambda}$, and then solution becomes negative near the endpoints for $\lambda > \bar{\lambda}$. Differentiating the equation (2.9), we see that zero is an eigenvalue of the linearized equation, and it has to be the second one since the corresponding eigenfunction, $u'(x,\bar{\lambda})$, changes sign exactly once.

Theorem 2.3. *In addition to the conditions* (2.2) *and* (2.3), *assume that* $(f(x,u)/u)$ *is an increasing function of* u *for all* $u > 0$ *and* $x \in (-1,1)$. *Then the Morse index of any positive solution of* (2.1) *is one. If, on the other hand,* $(f(x,u)/u)$ *is a decreasing function of* u *for all* x, *then the Morse index of any positive solution of* (2.1) *is zero.*

Proof. Let μ be the principal eigenvalue of the linearized equation and $\phi(x)$ the corresponding eigenfunction, i.e.,

$$\phi'' + f_u(x,u)\phi + \mu\phi = 0 \quad \text{for } x \in (-1,1), \quad \phi(-1) = \phi(1) = 0. \tag{2.10}$$

Recall that $\phi(x) > 0$ is an even function. We now multiply (2.1) by $\phi(x)$ and subtract from that the equation (2.10) multiplied by $u(x)$, and then integrate the result over $(0,1)$. We obtain

$$\int_0^1 u\phi\left(\frac{f(x,u)}{u} - f_u(x,u)\right) dx - \mu \int_0^1 u\phi \, dx = 0. \tag{2.11}$$

Assume first that $(f(x,u)/u)$ is increasing in u. This implies that $(f(x,u)/u) - f_u(x,u) < 0$ for all $x \in (0,1)$ and $u > 0$. If, contrary to what we want to prove, the Morse index of u is zero, then $\mu \geq 0$. It follows that the first term in (2.11) is negative, while the second one is nonpositive, a contradiction.

Similarly, if we assume that $(f(x,u)/u)$ is decreasing in u but the Morse index of u is one, then $\mu < 0$, and we obtain a similar contradiction in (2.11).

Next we give an application of the above theorem for the autonomous problem (1.1).

Theorem 2.4. *Assume that* $f(u) \in C^2(\overline{R}_+)$ *satisfies* $f(0) \leq 0$, *and the condition*

$$f''(u) > 0 \quad \text{for all } u > 0. \tag{2.12}$$

Then the problem (1.1) *admits at most one positive solution and its Morse index is one. If, on the other hand,* $f(u) \in C^2(\overline{R}_+)$ *satisfies* $f(0) \geq 0$ *and the condition*

$$f''(u) < 0 \quad \text{for all } u > 0, \tag{2.13}$$

then the problem (1.1) *admits at most one positive solution, and its Morse index is zero.*

Proof. We prove the first part, the other one being similar. Positive solutions of (1.1) are even functions, decreasing on $(0,1)$, see, e.g., [**4**]. Moreover, different positive solutions of (1.1) are strictly ordered on $(-1,1)$, see, e.g., [**5**] where more general results are available. Differentiating the equation (1.1),

$$u_x'' + f'(u)u_x = 0. \tag{2.14}$$

If $v(x) > u(x)$ is another solution of (1.1), then

$$v_x'' + f'(u)v_x + [f'(v) - f'(u)]v_x = 0. \tag{2.15}$$

Denoting $p(x) \equiv u''v' - u'v''$, we conclude from (2.14) and (2.15)

$$p'(x) = [f'(v) - f'(u)]u_x v_x > 0 \quad \text{on } (0,1). \tag{2.16}$$

Clearly, $p(0) = 0$, while

$$p(1) = -f(0)(v'(1) - u'(1)) \leq 0,$$

since $v'(1) < u'(1)$. This contradicts (2.16), proving the uniqueness. Turning to the Morse index, we notice that $(f(u)/u)' = (g(u)/u^2)$, where $g(u) = f'(u)u - f(u)$. Since by our conditions $g(0) \geq 0$ and

$g'(u) > 0$ for all $u > 0$, it follows that the function $(f(u)/u)$ is increasing, and hence the previous theorem applies.

More detailed results on Morse index are possible in case $f = f(u)$. But first we need a lemma, which rephrases the Sturm's comparison theorem.

Lemma 2.1. *Consider solutions of two linear equations on some interval $I = (\alpha, \beta)$*

$$v'' + a(x)v = 0, \tag{2.17}$$

and

$$\phi'' + a(x)\phi + b(x)\phi = 0, \tag{2.18}$$

with continuous functions $a(x)$ and $b(x)$. If $b(x) \geq 0$ on I, then ϕ oscillates faster than v, i.e., if ϕ keeps the same sign on some subinterval J of I (i.e., either $\phi > 0$ on J or the opposite inequality holds), then v cannot have two roots on J. On the other hand, if $b(x) \leq 0$ on I, then ϕ cannot have two roots on any subinterval $\bar{J}$ of I, where v keeps the same sign.

We now consider sign-changing solutions of the problem

$$u'' + f(u) = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0. \tag{2.19}$$

We shall need the corresponding eigenvalue problem

$$w'' + f'(u)w + \lambda w = 0 \quad \text{for } x \in (0,1), \quad w(0) = w(1) = 0. \tag{2.20}$$

We denote by m the Morse index of the solution $u(x)$, i.e., m is the number of negative eigenvalues of (2.20). Let $k \geq 0$ denote the number of interior roots of $u(x)$ (i.e., roots inside $(0,1)$).

Theorem 2.5. *Let $u(x)$ be any solution of (2.19) with $u'(0) \neq 0$. Assume that $f(u) \in C^1(R)$. Then either $m = k$ or else $m = k+1$.*

Proof. Assume that eigenvalues of (2.20) are

$$\lambda_1 < \cdots \lambda_m < 0 \leq \lambda_{m+1} < \cdots,$$

and $\phi_i(x)$ are the corresponding eigenfunctions. Given any function $v(x)$, we shall denote by n_v the number of roots of $v(x)$ inside the interval $(0,1)$. Observe that it is well known that $n_{\phi_m} = m-1$. Since $u(x)$ vanishes at the endpoints, $n_{u_x} = k+1$. Also observe that if all roots of any function $v(x)$ are simple, then the number of subregions of $(0,1)$, where $v(x)$ keeps the same sign is equal to $n_v + 1$.

Differentiate the equation (2.19)

$$u_x'' + f'(u)u_x = 0. \tag{2.21}$$

Since $\lambda_{m+1} \geq 0$, applying Lemma 2.1 to the equations (2.20) and (2.21), we see that the number of roots of u_x is not greater than the number of regions, where ϕ_{m+1} keeps the same sign. In other words,

$$k+1 = n_{u_x} \leq n_{\phi_{m+1}} + 1 = m+1,$$

so that $k \leq m$. On the other hand, since $\lambda_m < 0$, ϕ_m can have at most one root in any region where u_x keeps the same sign, except two corner regions where ϕ_m cannot have any interior roots at all. Indeed, since $\phi_m(0) = 0$, while $u_x(0) \neq 0$, it follows by Lemma 2.1 that $\phi_m(x)$ can have no roots on $(0, x_1)$ where x_1 is the smallest root of u_x. A similar argument applies near the $x=1$ corner. It follows that

$$m-1 = n_{\phi_m} \leq n_{u_x} + 1 - 2 = k,$$

i.e., $m \leq k+1$. The proof follows.

3. Curves of sign-changing solutions. We consider sign-changing solutions of the two-point problem

$$u'' + f(u) = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0. \tag{3.1}$$

We call solution $u(x)$ of (3.1) nonsingular if the corresponding linearized problem

$$w'' + f'(u)w = 0 \quad \text{for } x \in (0,1), \quad w(0) = w(1) = 0 \tag{3.2}$$

has only the trivial solution $w(x) \equiv 0$. We shall also need the corresponding eigenvalue problem

$$\phi'' + f'(u)\phi + \mu\phi = 0 \quad \text{for } x \in (0,1), \quad \phi(0) = \phi(1) = 0. \tag{3.3}$$

Theorem 3.1. *Assume that $f \in C^1(R)$ and either*

$$\frac{f(u)}{u} - f'(u) > 0 \quad \text{for almost all real } u, \tag{3.4}$$

or

$$\frac{f(u)}{u} - f'(u) < 0 \quad \text{for almost all real } u. \tag{3.5}$$

Then any solution of the problem (3.1) *satisfying $u'(0) \neq 0$ is nonsingular.*

Proof. Assume on the contrary that the problem (3.2) has a nontrivial solution $w(x)$.

Step 1. We show that the number of interior roots of u and w differs by one. Assume for definiteness that condition (3.4) holds. Then $f'(u) < (f(u)/u)$ for almost all u and hence, by Sturm's comparison theorem, the function $u(x)$ has a root between any two roots of $w(x)$. Since both functions vanish at the endpoints, $x = 0$ and $x = 1$, it follows that w has one fewer interior root than u.

Step 2. We show that u and w have the same number of interior roots. This will result in a contradiction, proving the theorem. The functions w and u_x satisfy the same equation and hence their roots are interlaced. Since $w(x)$ vanishes at the endpoints and u_x does not, it follows that $n_{u_x} = n_w + 1$ (as before n_v denotes the number of interior roots of v). Since $n_{u_x} = n_u + 1$, the claim follows.

Remark. Notice that condition (3.4) means that the function $(f(u)/u)$ is increasing for $u < 0$ and decreasing for $u > 0$.

Theorem 3.2. *Let $u(x)$ be the solution of* (3.1) *with k interior roots, and assume that the condition* (3.4) *holds. Then the Morse index of $u(x)$ is k. If, on the other hand, the condition* (3.5) *holds, then the Morse index of $u(x)$ is $k+1$.*

Proof. Assume (3.4) holds. By Theorem 2.5 the Morse index $m = m(u)$ is either k or $k+1$. Assume that, on the contrary, $m = k+1$.

Consider $k+1$ eigenpair $\mu = \mu_{k+1} < 0$ and $\phi = \phi_{k+1}$ with k interior roots on $(0,1)$. Since ϕ and u have the same number of interior roots and both vanish at the endpoints, one of three arrangements of interior roots must occur: the smallest root of ϕ is to the left of the smallest root of u, the largest root of ϕ is to the right of the largest root of u or there are two consecutive roots of ϕ with no root of u in between. In any case we can find two consecutive roots $0 \le \alpha < \beta \le 1$ of $\phi(x)$ so that either $u > 0$ on (α, β) or $u < 0$ on (α, β). Assume for definiteness that $u > 0$ on (α, β) (the other case is similar), and we may also assume that $\phi > 0$ on (α, β). Multiplying the equation (3.1) by ϕ and the eigenvalue problem (3.3) by u, subtracting and integrating, we obtain

$$- u(\beta)\phi'(\beta) + u(\alpha)\phi'(\alpha) + \int_\alpha^\beta \left[\frac{f(u)}{u} - f'(u)\right] u\phi \, dx \\ - \mu \int_\alpha^\beta u\phi \, dx = 0.$$

Since the first two terms on the left are nonnegative, and the integrals are positive, we have a contradiction.

Assume now that condition (3.5) holds but $m(u) = k$. We again consider $k+1$ eigenpair $\mu = \mu_{k+1} \ge 0$ and $\phi = \phi_{k+1}$ with k interior roots on $(0,1)$. Arguing as above, and reversing the roles of $u(x)$ and $\phi(x)$, we can find two consecutive roots $0 \le \alpha < \beta \le 1$ of $u(x)$ so that either $\phi > 0$ on (α, β) or $\phi < 0$ on (α, β). Assume for definiteness this time that $u < 0$ on (α, β) (the other case is similar), and we may also assume that $\phi > 0$ on (α, β). Proceeding as before, we get

$$u'(\beta)\phi(\beta) - u'(\alpha)\phi(\alpha) + \int_\alpha^\beta \left[\frac{f(u)}{u} - f'(u)\right] u\phi \, dx \\ - \mu \int_\alpha^\beta u\phi \, dx = 0.$$

Since all terms on the left are nonnegative and the third one is positive, we again have a contradiction.

As an application we now give a complete description of the solution set for a class of problems

$$u'' + \lambda u - u|u|^p = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0, \tag{3.6}$$

depending on a real parameter λ. Here $p > 0$ is a constant. It is well known that near $\lambda_k = k^2\pi^2$ there are curves of solutions bifurcating off the trivial solution $u \equiv 0$, see [**3**]. We show next that these curves do not turn and that they exhaust the solution set.

Theorem 3.3. *The set of nontrivial solutions of* (3.6) *consists of infinitely many solution curves, bifurcating off the trivial solution* $u \equiv 0$ *at* $\lambda = \lambda_k$, $k = 1, 2, \dots$. *These curves consist of nonsingular solutions and continue without turns for all* $\lambda > \lambda_k$. *All solutions on the kth curve have exactly* $k - 1$ *interior roots, and their Morse index is equal to* k.

Proof. Existence of nontrivial solution curves bifurcating to the right (i.e., for increasing λ) off $u = 0$ at λ_k is well known, see [**3**]. By the Hopf's boundary lemma condition $u'(0) \neq 0$ is satisfied for any nontrivial solution, and hence by Theorem 3.1 (condition (3.4) holds here) there are no singular solutions, and so all solution curves continue without any turns. Since, by maximum principle $\max_{[0,1]} |u(x)|$ is bounded for any fixed λ, it follows that the kth curve continues for all $\lambda > \lambda_k$.

We claim next that the problem (3.6) has no solutions except the ones lying on the curves through λ_k. Assuming existence of such a solution, we continue it for decreasing λ and discover that it has no place to go. Indeed, by the above, this solution curve cannot turn, it cannot approach the trivial solution at λ_k by uniqueness of bifurcating curve, and finally the solution curve cannot continue until $\lambda = 0$ since at $\lambda = 0$ there are no nontrivial solutions (multiply (3.6) by u and integrate over $(0, 1)$).

The curve bifurcating off λ_k is asymptotic to $\sin k\pi x$ for λ near λ_k (see [**3**]), and hence solutions on this curve have $k - 1$ interior roots. Since, as noted above, the condition $u'(0) \neq 0$ is always satisfied, it follows that the number of interior roots stays equal to $k - 1$ throughout the curve.

The remaining conclusions follow by Theorems 3.1 and 3.2.

Remark. The exact multiplicity part of the theorem can also be established by a direct integration, see, e.g., [**1**]. Our approach, in addition to the extra information it gives, appears to be more flexible. Consider, for example, the problem

$$(3.7) \qquad u'' + \lambda u - g(u) = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0,$$

with a function $g(u) \in C^1(R)$ satisfying $g(0) = g'(0) = 0$, $\lim_{|u|\to\infty}(g(u)/u) = \infty$, and the condition (3.4). Then all conclusions of Theorem 3.3 hold, and the proof is exactly the same. Similar results for problem (3.7) were obtained recently by Shi and Wang [9] (their results also apply to radial solutions on balls). Our Theorems 3.1 and 3.2 cover general situations where solution branches are not necessarily bifurcating off the trivial solution.

Similar reasoning applies to the problem

$$(3.8) \qquad u'' + \lambda u + u|u|^p = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0.$$

Theorem 3.4. *The set of nontrivial solutions of* (3.8) *consists of infinitely many solution curves, bifurcating off the trivial solution* $u \equiv 0$ *at* $\lambda = \lambda_k$, $k = 1, 2, \ldots$. *These curves consist of nonsingular solutions and continue without turns for all* $\lambda < \lambda_k$. *All solutions on the kth curve have exactly* $k-1$ *interior roots, and their Morse index is equal to* $k+1$.

Example. Consider the problem

$$(3.9) \qquad u'' + u|u|^p = 0 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0,$$

with a constant $p > 0$. Our results, combined with simple scaling arguments, imply that the problem (3.9) has a unique positive solution $u_0(x)$, and for each integer $k \geq 1$ it has exactly two solutions $\pm u_k(x)$ with k interior zeros. The roots of $u_k(x)$ subdivide the interval $(0,1)$ into $k+1$ equal parts, and solution on each subinterval is a constant multiple of a stretched version of the positive solution $u_0(x)$. The Morse index of $u_k(x)$ is equal to $k+1$ for all $k \geq 0$.

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