Infinitely many solutions and Morse index for non-autonomous elliptic problems

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Abstract

Our first result is a change of variables which transforms radial k-Hessian equations into radial p-Laplace equations. Then we generalize the classical results of D.D. Joseph and T.S. Lundgren [9] by using the method we developed in [12] and [13]. We provide a considerably simpler approach, which yields additional information on the Morse index of solutions.

Key words: Global solution curves, Infinitely many solutions, Morse index.

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1 Introduction

J. Jacobsen and K. Schmitt [8] considered a class of radial quasilinear elliptic equations on the unit ball

$$\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)' + r^{\gamma}f(u) = 0\,, \quad u'(0) = u(1) = 0\,, \tag{1.1}$$

with parameters α, β and γ. This class of equations includes: the Laplacian, in case $\alpha = n-1$, $\beta = 0$, $\gamma = n-1$, p-Laplacian ($p > 1$), for $\alpha = n-1$, $\beta = p-2$, $\gamma = n-1$, and k-Hessian in case $\alpha = n-k$, $\beta = k-1$, $\gamma = n-1$, where n is the dimension of the space, and $1 \le k \le n$ is an integer. Recall that the k-Hessian is defined as the sum of all principal $k \times k$ minors of the Hessian matrix D^2u. So that 1-Hessian is Δu, and n-Hessian is $\det D^2u$, the Monge-Ampere operator. The paper of J. Sánchez and V. Vergara [20] has a detailed exposition of k-Hessian equations, and the previous contributions include L. Caffarelli et al [2], N.S. Trudinger and X.-J. Wang [21], J. Jacobsen [7], and J. Jacobsen and K. Schmitt [8].

Our first result is a change of variables which transforms the problem (1.2) into a Dirichlet problem for a p-Laplacian, with $p = \beta + 2$, in $\frac{\theta}{q+1} + 1$ dimensions, on another ball around the origin, of radius $(q+1)^{\frac{1}{q+1}}$, where θ and q are defined below. In particular, one can reduce any k-Hessian equation to a p-Laplacian, and a class of non-autonomous p-Laplace equations to autonomous p-Laplace equations. This extends our previous result in [14].

Then we present exhaustive results on multiplicity of positive solutions, and global solution curves for ($p > 1$, $\alpha \geq 0$)

$$u'' + \frac{n-1}{r}u' + \lambda r^{\alpha}(1+u)^p = 0\,, \quad 0 < r < 1, \; u'(0) = 0, \quad u(1) = 0\,, \tag{1.2}$$

generalizing the classical results of D.D. Joseph and T.S. Lundgren [9], who considered the case $\alpha = 0$. The approach in [9] was rather involved, and it relied on the old results of E. Hopf [6], and S. Chandrasekhar [3] (see p. 261 in [9]). We use a considerably simpler method, involving generating and guiding solutions, that we developed in [12] and [13]. In the present paper we further simplify and improve this method, particularly in the case when the solution curve is monotone, see Theorem 3.2 below. Our approach uses the self-similar nature of the problem (1.2), which allows parameterization of the entire solution curve, see [12] for the discussion. Singular solutions, arising as limits of the regular solutions, are easy to understand for self-similar problems, while for other equations with polynomial nonlinearities this issue is rather involved, see e.g., F. Merle and L.A. Peletier [16].

In the last section we use the generating and guiding solutions to show that all turning points of the problem are non-degenerate, and that the Morse index of solutions increases by one at each turning point, when one follows the solution curve of (1.2) in the direction of increasing $u(0)$. For the Gelfand problem, where $f(u) = e^u$, such a result was first proved in K. Nagasaki and T. Suzuki [17], and then in P. Korman [13], by using the generating and guiding solutions.

2 Reduction of k-Hessian equation to p-Laplacian

Similarly to J. Jacobsen and K. Schmitt [8], we consider the problem

$$\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)' + r^{\gamma}f(u) = 0\,, \quad u'(0) = 0\,, \tag{2.1}$$

with parameters α, β and γ. We make a change of variables

$$t = \frac{r^{q+1}}{q+1}\,, \tag{2.2}$$

with q to be chosen shortly. Clearly, $u_r = u_t r^q$, and we have

$$\left(r^{\alpha+q(\beta+1)}|u_t|^\beta u_t\right)' + r^{\gamma-q} f(u) = 0\,.$$

We now choose q to equalize the powers of r:

$$\alpha + q(\beta+1) = \gamma - q \equiv \theta\,,$$

$$q = \frac{\gamma-\alpha}{\beta+2}\,. \tag{2.3}$$

The common power θ is then

$$\theta = \frac{(\beta+1)\gamma+\alpha}{\beta+2}\,, \tag{2.4}$$

and we have

$$\left(r^\theta |u_t|^\beta u_t\right)' + r^\theta f(u) = 0\,. \tag{2.5}$$

Theorem 2.1. *Assume that $f(u(0)) > 0$, $\beta+1 > 0$, and*

$$\gamma > \alpha - \beta - 2\,. \tag{2.6}$$

The change of variables (2.2), with q given by (2.3), transforms the problem (2.1) into

$$\left(t^{\frac{\theta}{q+1}}|u'(t)|^\beta u'(t)\right)' + t^{\frac{\theta}{q+1}} f(u) = 0\,,\ \ u'(0) = 0\,, \tag{2.7}$$

i.e., into $(\beta+2)$-Laplacian in $\frac{\theta}{q+1}+1$ dimensions.

Proof: Replacing in (2.5), $r = (q+1)^{\frac{1}{q+1}} t^{\frac{1}{q+1}}$, and using primes to denote the derivatives in t, we obtain the equation in (2.7).

Since $u'(r) < 0$ for small $r > 0$, and $\beta+1 > 0$, we express from (2.1)

$$\left(-u'(r)\right)^{\beta+1} = \frac{1}{r^\alpha}\int_0^r z^\gamma f(u(z))\,dz\,.$$

Using again that $\beta+1>0$, we have (observe that $r\to 0$ as $t\to 0$, since $q+1>0$, by (2.6))

$$-\frac{du}{dt}(0)=\lim_{r\to 0}\frac{-u'(r)}{r^q}=\lim_{r\to 0}\left[\frac{\left(-u'(r)\right)^{\beta+1}}{r^{q(\beta+1)}}\right]^{\frac{1}{\beta+1}}\,,$$

and

$$\lim_{r\to 0}\frac{\left(-u'(r)\right)^{\beta+1}}{r^{q(\beta+1)}}=\lim_{r\to 0}\frac{1}{r^{\alpha+q(\beta+1)}}\int_0^r z^\gamma f(u(z))\,dz=0\,,$$

since $\gamma>\alpha+q(\beta+1)-1$, by (2.6). $\diamondsuit$

In particular the k-Hessian equation is transformed into (2.7), with $\beta=k-1$, and $\frac{\theta}{q+1}=\frac{k(n-2)+n}{2k}$, i.e., into a p-Laplace equation, with $p=\beta+2=k+1$, in $\frac{\theta}{q+1}+1=\frac{kn+n}{2k}$ dimensions.

Radial solutions of (here $s>-1$ is a real parameter)

$$\Delta u+r^s f(u)=0\,,\;\; u'(0)=0 \tag{2.8}$$

in n dimensions satisfy

$$\left(r^{n-1}u'\right)'+r^{n+s-1}f(u)=0\,,\;\; u'(0)=0\,. \tag{2.9}$$

The change of variables (2.2) becomes $t=\frac{r^{s/2+1}}{s/2+1}$, and it transforms (2.9) into

$$\left(t^{m-1}u'\right)'+t^{m-1}f(u)=0\,,\;\; \frac{du}{dt}(0)=0\,. \tag{2.10}$$

with $m=\frac{n+s}{s/2+1}$. This corresponds to $\Delta u+f(u)=0$ in m dimensions. So that the non-autonomous term r^s in (2.8) got removed (with the dimension changing to m). We first proved this result in [14].

3 A generalization of a result of Joseph and Lundgren

By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5], all positive solutions of the Dirichlet problem (here $u=u(x)$, $x\in R^n$)

$$\Delta u+\lambda\,(1+u)^p=0\;\text{ for }|x|<1\,,\;\; u=0\;\text{ when }|x|=1$$

are radially symmetric, i.e., $u = u(r)$, $r = |x|$, and they satisfy

$$u''+\frac{n-1}{r}u'+\lambda\,(1+u)^p = 0 \text{ for } 0 < r < 1,\; u'(0) = 0,\;\; u(1) = 0\,. \tag{3.1}$$

Here λ is a positive parameter, $p > 1$ a constant. Similarly to J.A. Pelesko [19], we set $v = 1 + u$, followed by $v = aw$, and $t = br$, where $a = v(0) = 1 + u(0)$. The constants a and b are assumed to satisfy

$$\lambda = \frac{b^2}{a^{p-1}}\,. \tag{3.2}$$

Then (3.1) becomes

$$w'' + \frac{n-1}{t}w' + w^p = 0,\;\; w(0) = 1,\;\; w'(0) = 0\,. \tag{3.3}$$

The solution of this problem is easily seen to be a decreasing function, going to zero. In case of sub-critical p, $1 < p < \frac{n+2}{n-2}$, it is known that $w(t)$ vanishes at some $t_0 > 0$, since the Dirichlet problem for the equation in (3.3) has a (unique) solution on any ball (as follows by the mountain pass lemma). If $p \geq \frac{n+2}{n-2}$, then $w(t)$ has no roots on $(0, \infty)$, which follows by Pohozhaev's identity (also a well known fact). Once we compute $w(t)$ from (3.3), $u(r) = -1 + aw(br)$, and since $u(1) = 0$, we have

$$1 = aw(b)\,,$$

so that $a = \frac{1}{w(b)}$, and then $\lambda = b^2 w^{p-1}(b)$. The global solution curve is

$$(\lambda, u(0)) = \left(b^2 w^{p-1}(b)\,, -1 + \frac{1}{w(b)}\right)\,,$$

parameterized by $b \in (0, t_0)$ in the sub-critical case, and $b \in (0, \infty)$ for the super-critical and critical cases. The solution of (3.1) at the parameter value of b is

$$u(r) = aw(br) - 1 = \frac{w(br)}{w(b)} - 1$$

It will be convenient to use the letter t instead of b as the parameter. So that the global solution curve for $p \geq \frac{n+2}{n-2}$ is

$$(\lambda, u(0)) = \left(t^2 w^{p-1}(t)\,, -1 + \frac{1}{w(t)}\right)\,,\;\; t \in [0, \infty)\,, \tag{3.4}$$

and the solution of (3.1) at the parameter value of t is $u(r) = \frac{w(tr)}{w(t)} - 1$, where $w(t)$ is the solution of (3.3).

In particular, $\lambda = \lambda(t) = t^2 w^{p-1}(t)$, and

$$\lambda'(t) = t w^{p-2} \left[2w + (p-1) t w' \right] , \tag{3.5}$$

so that the solution curve travels to the right (left) in the $(\lambda, u(0))$ plane if $2w + (p-1)tw' > 0$ (< 0). The turning points correspond to the roots of the function $2w + (p-1)tw'$. If we set this function to zero

$$2w + (p-1)tw' = 0 \, ,$$

then the general solution of this equation is

$$w(t) = ct^{-\beta} \, , \quad \text{with } \beta = \tfrac{2}{p-1} \, .$$

If we choose

$$c = c_0 = [\beta(n-1) - \beta(\beta+1)]^{\frac{1}{p-1}} \, ,$$

then $w_0(t) = c_0 t^{-\beta}$ is also a solution of the equation in (3.3), and the issue turns out to be how many times $w(t)$ and $w_0(t)$ cross as $t \to \infty$, as the following lemma shows.

Lemma 3.1. *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (3.1) makes infinitely many turns.*

Proof: Let $\{t_n\}$ denote the points of intersection. At $\{t_n\}$'s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial value problems). Since $2w_0(t_n) + (p-1)t_n w_0'(t_n) = 0$, it follows that $2w(t_n) + (p-1)t_n w'(t_n) > 0$ (< 0) if $w(t)$ intersects $w_0(t)$ from below (above) at t_n. Hence, on any interval (t_n, t_{n+1}) there is a point t_0, where $2w(t_0) + (p-1)t_0 w'(t_0) = 0$, i.e., $\lambda'(t_0) = 0$, and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}). $\diamondsuit$

Using the terminology from [12], $w(t)$ is called the *generating solution*, while $w_0(t)$ is the *guiding solution*.

The linearized equation for (3.3) is

$$z'' + \frac{n-1}{t} z' + p w^{p-1} z = 0 \, .$$

At $w = w_0(t)$, the linearized equation becomes

$$z'' + \frac{n-1}{t} z' + p\gamma \frac{1}{t^2} z = 0 \, , \quad \text{with} \quad \gamma = \beta(n-1) - \beta(\beta+1) \, . \tag{3.6}$$

This is Euler's equation. The roots of its characteristic equation are

$$r = \frac{-(n-2) \pm \sqrt{(n-2)^2 - 4p\gamma}}{2}\,.$$

If $(n-2)^2 - 4p\gamma < 0$, the roots are complex, the fundamental solution set of (3.6) consists of $t^{-\frac{n-2}{2}} \cos(\omega \ln t)$ and $t^{-\frac{n-2}{2}} \sin(\omega \ln t)$, with $(n-2)^2 - 4p\gamma \equiv -4\omega^2$, and it is natural to expect that $w(t) - w_0(t)$ changes sign infinitely many times, and then the solution curve makes infinitely many turns. By the result of P. Korman [10], this may happen only if $p > \frac{n+2}{n-2}$.

Lemma 3.2. *Assume that $p > \frac{n+2}{n-2}$. Then $(n-2)^2 - 4p\gamma < 0$ if and only if*

$$\frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}} > n-2\,. \tag{3.7}$$

Proof: We have

$$(n-2)^2 - 4p\gamma = (n-2)^2 - 4p\beta(n-2) + 4p\beta^2 < 0\,,$$

provided that $n-2$ lies between the roots of this quadratic, i.e.,

$$\frac{4p}{p-1} - 4\sqrt{\frac{p}{p-1}} < n-2 < \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}\,.$$

The condition $p > \frac{n+2}{n-2}$ implies that $n > \frac{2p+2}{p-1}$. Then

$$n-2 > \frac{4}{p-1} > \frac{4p}{p-1} - 4\sqrt{\frac{p}{p-1}}\,,$$

completing the proof. ♢

Observe that the left hand side of (3.7) is a decreasing function, tending to 8 as $p \to \infty$. It follows that in dimensions $2 < n \leq 10$ the condition (3.7) holds for all $p > 1$.

The following result is known as *Bihari's inequality.*

Lemma 3.3. *Assume that the functions $a(t) \geq 0$, and $u(t) \geq 0$ are continuous for $t \geq t_0$, and we have*

$$u(t) \leq C + \int_{t_0}^{t} a(s) u^m(s)\, ds\,, \quad \textit{for } t \geq t_0\,, \tag{3.8}$$

with some constants $C > 0$ and $m > 1$. Assume also that

$$C^{1-m} - (m-1) \int_{t_0}^{t} a(s)\, ds > 0\,, \quad \textit{for } t \geq t_0\,.$$

Then

$$u(t) \leq \frac{1}{\left[C^{1-m} - (m-1) \int_{t_0}^{t} a(s)\, ds\right]^{\frac{1}{m-1}}}\,, \quad \textit{for } t \geq t_0\,.$$

Proof: Denote the right hand side of (3.8) by $w(t)$. Then $w(t_0) = C$, and

$$w' = a(t) u^m \leq a(t) w^m\,.$$

Divide by w^m, and integrate over (t_0, t). $\diamondsuit$

Lemma 3.4. *Assume that $p > \frac{n+2}{n-2}$, $n \geq 3$, and in case $n > 10$ assume additionally that (3.7) holds. The general solution of*

$$y'' + \frac{n-1}{t} y' + \frac{p\gamma}{t^2} y = f(t)$$

is

$$y(t) = C t^{-\frac{n-2}{2}} \sin\left(\omega \ln t + D\right) + \frac{1}{\omega} t^{-\frac{n-2}{2}} \int_{t_0}^{t} \sin\left(\omega \ln \frac{t}{s}\right) s^{\frac{n}{2}} f(s)\, ds\,,$$

with $(n-2)^2 - 4p\gamma \equiv -4\omega^2$, for any constant t_0 fixed, and arbitrary constants C and D.

Proof: By Lemma 3.2, the fundamental set of the corresponding homogeneous equation consists of $y_1(t) = t^{-\frac{n-2}{2}} \cos\left(\omega \ln t\right)$ and $y_2(t) = t^{-\frac{n-2}{2}} \sin\left(\omega \ln t\right)$. Compute their Wronskian $W(y_1, y_2) = \omega t^{1-n}$, and apply the method of variation of parameters. $\diamondsuit$

The following result is known, see e.g., J. Dávila et al [4]. Our proof is a little different from the usual one, in that we avoid the language of heteroclinic connections.

Lemma 3.5. *Assume that $p > \frac{n+2}{n-2}$, and let $w(r)$ and $w_0(r)$ be as above. Then $\lim_{r \to \infty} \frac{w(r)}{w_0(r)} = 1$.*

Proof: In the initial value problem determining $w(r)$:

$$y'' + \frac{n-1}{r} y' + y^p = 0\,, \quad y(0) = 1\,,\ y'(0) = 0\,, \quad r > 0 \tag{3.9}$$

we make a change of variables $y(r) = w_0(r)v(r)$, followed by $r = e^t$. Then $v(t)$ satisfies

$$v'' + av' - \gamma(v - v^p) = 0\,, \;\; v(-\infty) = v'(-\infty) = 0\,, \;\; t \in (-\infty, \infty)\,, \tag{3.10}$$

with $a = -2\beta + n - 2 > 0$, and as above $\gamma = \beta\,(n - \beta - 2) > 0$. (Since $v = \frac{1}{c_0} r^\beta y$, we have $\frac{dv}{dt}\,|_{t=-\infty} = \frac{dv}{dr}\frac{dr}{dt}\,|_{t=-\infty} = \frac{1}{c_0}\left[\beta r^{\beta-1} y(r) + r^\beta y'(r)\right] r\,|_{r=0} = 0$.) We need to show that $\lim_{t\to\infty} v(t) = 1$. Since the energy

$$E(t) = \frac{1}{2}{v'}^2 - \gamma\left(\frac{v^2}{2} - \frac{v^{p+1}}{p+1}\right)$$

is decreasing along the solution of (3.10), it follows that $v(t)$ is bounded, and it cannot tend to zero as $t \to \infty$. (Observe that $E(-\infty) = 0$, and hence $E(t) < 0$ for all t.) From the equation (3.10), $v(t) > 0$ can have local minimums only if $0 < v < 1$, and local maximums only if $v > 1$, and it can tend only to 1, as $t \to \infty$. Hence, either $v(t)$ tends to 1 monotonously, or it oscillates infinitely often around 1. We show next that in the latter case $v(t)$ also tends to 1.

Let $v - 1 = z$. By our assumption, $z(t)$ has infinitely many roots, $-1 < z < z_0$, for some $z_0 > 0$, and it satisfies

$$z'' + az' + \gamma f(z) = 0\,,$$

where $f(z)$ behaves like $z + z^2$ on $(-1, z_0)$, to which $f(z)$ is equal in case $p = 2$. The energy $\bar{E}(t) = \frac{1}{2}{z'}^2 + \gamma F(z)$, where $F(z) = \int_0^z f(t)\,dt$, is positive for $z > -1$, and decreasing, since

$$\bar{E}'(t) = -a{z'}^2\,. \tag{3.11}$$

Let $E_0 = \lim_{t\to\infty} \bar{E}(t) \geq 0$. Assume that $E_0 > 0$. If t_k are the roots of $z(t)$, then $|z'(t_k)|$ get arbitrarily close to $\sqrt{2E_0}$, for k large. From the equation (3.10), we get a bound on $|z''(t_k)|$. Hence, $|z'(t)|$ cannot change fast, so that we can find an interval of fixed length $(t_k, t_k + \theta)$ on which $|z'(t)| > \frac{1}{2}\sqrt{2E_0}$, independently of k. Integrating (3.11), we conclude that $\bar{E}(t)$ drops by at least $\frac{\theta}{2}\sqrt{2E_0}$ for each k, and eventually $E(t)$ drops below E_0, a contradiction. Hence, $E_0 = 0$.

Let τ_k be the point of maximum (minimum) of $z(t)$ on (t_k, t_{k+1}). Then $F(z(\tau_k)) = \bar{E}(\tau_k) \to 0$, proving that $z(t) \to 0$ as $t \to \infty$. ♢

Remarks

1. For $p = \frac{n+2}{n-2}$, the lemma does not hold (the proof breaks down since $a = 0$ in (3.10)). In that case, the generating solution, i.e., the solution of (3.3) is well-known to be $w(r) = \frac{1}{\left(1+\frac{1}{n(n-2)}r^2\right)^{\frac{n-2}{2}}}$. In standard terminology, $w(r)$ has *slow decay* for $p > \frac{n+2}{n-2}$, and *fast decay* at $p = \frac{n+2}{n-2}$.

2. The transformation $y = w_0 v$ is equivalent to the classical Fowler transformation $w = t^{-\frac{2}{p-1}} v$. So that the Fowler transformation can be seen as the extension of the elementary "reduction of order" method, when one of the solutions (here $w_0(t)$) is known.

Theorem 3.1. *Assume that $p > \frac{n+2}{n-2}$, $n \geq 3$, and in case $n > 10$ assume additionally that (3.7) holds. Then the solution curve of (3.1) makes infinitely many turns. Moreover, the problem (3.1) has infinitely many solutions at $\lambda = \gamma = \beta(n-1) - \beta(\beta+1)$, with $\beta = \frac{2}{p-1}$, and at most finitely many solutions at other λ's. As $u(0) \to \infty$, the solutions of (3.1) tend to $r^{-\beta} - 1$, for $r \neq 0$, which is a singular solution of (3.1).*

Proof: By Lemma 3.2, the roots of the characteristic equation of (3.6) are $-\frac{n-2}{2} \pm i\omega$, with $(n-2)^2 - 4p\gamma = -4\omega^2$. We will show that $y(t) \equiv w(t) - w_0(t)$ has infinitely many roots. This will follow from the following asymptotic formula

$$t^{\frac{n-2}{2}} y(t) = C \sin(\omega \ln t + D) + O(t^{-n/2+3-\beta(p-2)}), \quad \text{as } t \to \infty, \tag{3.12}$$

with some constants $C \neq 0$ and D, and with $-n/2 + 3 - \beta(p-2) < 0$.

The function $y(t)$ satisfies

$$y'' + \frac{n-1}{t} y' + p w_0^{p-1} y = f(t),$$

or

$$y'' + \frac{n-1}{t} y' + \frac{p\gamma}{t^2} y = f(t), \tag{3.13}$$

where $f(t) = -\left[(y(t) + w_0(t))^p - w_0^p(t) - p w_0^{p-1}(t) y(t)\right]$. Using Lemma 3.4, the general solution of (3.13) is

(3.14)

$$y(t) = C t^{-\frac{n-2}{2}} \sin(\omega \ln t + D) + \frac{1}{\omega} t^{-\frac{n-2}{2}} \int_{t_0}^{t} \sin\left(\omega \ln \frac{t}{s}\right) s^{\frac{n}{2}} f(s)\, ds.$$

We claim that for large t

$$|f(t)| \leq c_1 \omega t^{-\beta(p-2)} y^2 \,, \tag{3.15}$$

with some constant $c_1 > 0$. Indeed, by the two-term Taylor's formula with a remainder term

$$-f(t) = \frac{p(p-1)}{2} \left[\theta w(t) + (1-\theta) w_0(t)\right]^{p-2} y^2 \leq \hat{c}\, w_0^{p-2} y^2 \,,$$

with some $\theta \in (0,1)$ and $\hat{c} > 0$, giving (3.15). (We estimated $w(t)$ by a multiple of w_0 from above (below), in case $p \geq 2$ ($p < 2$), using Lemma 3.5.)

Setting $v(t) \equiv t^{\frac{n-2}{2}} |y(t)|$, we estimate from (3.14)

$$v(t) \leq |C| + c_1 \int_{t_0}^{t} s^{-n/2+2-\beta(p-2)} v^2(s)\, ds \,. \tag{3.16}$$

Since $p > \frac{n+2}{n-2}$, it follows that

$$-n/2 + 2 - \beta(p-2) < -1 \,.$$

By Bihari's inequality (Lemma 3.3)

$$v(t) \leq \frac{1}{|C|^{-1} - c_1 \int_{t_0}^{t} s^{-n/2+2-\beta(p-2)}\, ds} \leq \frac{1}{c_2} \,, \tag{3.17}$$

with t_0 chosen large enough, so that $|C|^{-1} - c_1 \int_{t_0}^{t} s^{-n/2+2-\beta(p-2)}\, ds > c_2 > 0$ for all $t > t_0$, and some constant c_2. By (3.14),

$$y(t) = C t^{-\frac{n-2}{2}} \sin\left(\omega \ln t + D\right) - \frac{1}{\omega} t^{-\frac{n-2}{2}} \int_t^{t_1} \sin\left(\omega \ln \frac{t}{s}\right) s^{\frac{n}{2}} f(s)\, ds$$

gives for $t < t_1$, with t_1 large and fixed, the unique solution of (3.13), which satisfies the appropriate initial conditions at t_1 (equal to the right hand side and its derivative evaluated at t_1). This solution is written using an integral of itself. Then

$$t^{\frac{n-2}{2}} y(t) = C \sin\left(\omega \ln t + D\right) - \frac{1}{\omega} \int_t^{t_1} \sin\left(\omega \ln \frac{t}{s}\right) s^{\frac{n}{2}} f(s)\, ds \,,$$

and using (3.17) we estimate, for t large,

$$\left| - \int_t^{t_1} \sin\left(\omega \ln \frac{t}{s}\right) s^{\frac{n}{2}} f(s)\, ds \right| \leq c_1 \int_t^{t_1} s^{-n/2+2-\beta(p-2)} v^2(s)\, ds \,.$$

$$\leq \frac{c_1}{c_2^2} \int_t^{t_1} s^{-n/2+2-\beta(p-2)}\, ds = O(t^{-n/2+3-\beta(p-2)}) = o(1)\,,$$

proving (3.12). Hence, $w(t)$ and $w_0(t)$ have infinitely many points of intersection, and by Lemma 3.1, the solution curve makes infinitely many turns.

Turning to the other claims, for large t, we have, by Lemma 3.5, $u(r) = \frac{w(tr)}{w(t)} - 1 \sim \frac{w_0(tr)}{w_0(t)} - 1 = r^{-\beta} - 1$, which is the singular solution of (3.1), and $\lambda = t^2 w^{p-1} \sim t^2 w_0^{p-1} = \gamma$, gives the vertical asymptote of the solution curve of (3.1). $\diamondsuit$

Remark In case $n \geq 11$, define p_0 as the solution of

$$f(p) \equiv \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}} = n-2\,.$$

The function $f(p)$ is decreasing, $f(\frac{n+2}{n-2}) > n-2$, so that $\frac{n+2}{n-2} < p_0$, and the theorem requires that

$$\frac{n+2}{n-2} < p < p_0\,.$$

To treat the case when the condition (3.7) fails, we need the following elementary lemmas.

Lemma 3.6. *Let the functions $y(r)$ and $z(r)$ of class $C^2[0,\infty)$ satisfy*

$$y'' + \frac{n-1}{r} y' + a(r) y = 0\,, \tag{3.18}$$

$$z'' + \frac{n-1}{r} z' + a(r) z > 0\,, \tag{3.19}$$

$$0 < y(r_0) = z(r_0)\,,\ \ y'(r_0) = z'(r_0) \ \text{ at some } r_0 > 0\,, \tag{3.20}$$

for some $a(r) \in C[0,\infty)$. Assume that $y(r) > 0$ on (r_0,∞). Then $z(r) > 0$ on (r_0,∞).

Proof: Assuming the contrary, let $r_1 > r_0$ be the first root of $z(r)$. From (3.18) and (3.19) we get

$$\left[r^{n-1}\left(z'y - zy'\right)\right]' > 0 \ \text{ on } (r_0, r_1)\,.$$

Integrating over (r_0, r_1), we get

$$r_1^{n-1} z'(r_1) y(r_1) > 0\,,$$

which is a contradiction. $\diamondsuit$

Lemma 3.7. *Consider the function $y(r) = \alpha r^{s_1} + \beta r^{s_2}$, with constants $s_1 < s_2 < 0$ and $\alpha, \beta \in \mathbb{R}$. Assume that at some $r_0 > 0$ we have $y(r_0) = A > 0$, $y'(r_0) = B < 0$, and*

$$B - A\frac{s_1}{r_0} > 0\,. \tag{3.21}$$

Then $y(r) > 0$ for $r > r_0$.

Proof: Write

$$y(r) = c_1 \left(\frac{r}{r_0}\right)^{s_1} + c_2 \left(\frac{r}{r_0}\right)^{s_2} = \left(\frac{r}{r_0}\right)^{s_1} \left[c_1 + c_2 \left(\frac{r}{r_0}\right)^{s_2 - s_1}\right],$$

with $c_1 = \alpha r_0^{s_1}$ and $c_2 = \beta r_0^{s_2}$. We have $s_2 - s_1 > 0$, and the proof will follow if $c_2 > 0$. (We have $y(r_0) > 0$, and if $y(r)$ has a root, it lies to the left of r_0.) Since

$$\begin{aligned} y(r_0) &= c_1 + c_2 = A \\ y'(r_0) &= \tfrac{s_1}{r_0} c_1 + \tfrac{s_2}{r_0} c_2 = B\,, \end{aligned}$$

it follows that

$$c_2 = \left(B - A\frac{s_1}{r_0}\right) \frac{r_0}{s_2 - s_1} > 0\,,$$

in view of (3.21). ♢

Theorem 3.2. *Assume that $p > \frac{n+2}{n-2}$, $n \geq 11$, and assume that (3.7) fails, i.e.,*

$$\frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}} \leq n - 2\,. \tag{3.22}$$

Then the solution curve of (3.1) is monotone increasing in the $(\lambda, u(0))$ plane, i.e., $\lambda'(t) > 0$ for all $t > 0$. Moreover, $\lim_{t\to\infty} \lambda(t) = \gamma = \beta(n-1) - \beta(\beta+1)$, with $\beta = \frac{2}{p-1}$. It follows that that the problem (3.1) has a unique positive solution for $\lambda < \gamma$, and no positive solution for $\lambda \geq \gamma$. As $u(0) \to \infty$, the solutions of (3.1) tend to $r^{-\beta} - 1$, for $r \neq 0$, which is a singular solution of (3.1).

Proof: Assume that the inequality in (3.22) is strict, the case of equality is similar. The roots of the characteristic equation of (3.6) are now real and negative

$$s_1 = \frac{-(n-2) - \sqrt{(n-2)^2 - 4p\gamma}}{2} < s_2 = \frac{-(n-2) + \sqrt{(n-2)^2 - 4p\gamma}}{2}\,. \tag{3.23}$$

We claim that the generating and guiding solutions satisfy

$$w(r) < w_0(r) \ \text{ for all } r > 0\,. \tag{3.24}$$

Let $z(r) = w_0(r) - w(r)$. We wish to show that $z(r) > 0$ for all $r > 0$. We have

$$z'' + \frac{n-1}{r} z' + p{w_0}^{p-1} z = w^p - w_0^p - p{w_0}^{p-1}(w - w_0) > 0\,.$$

Denote $A = A(r_0) = w_0(r_0) - w(r_0)$, and $B = B(r_0) = w_0'(r_0) - w'(r_0)$. Let $y(r)$ be the solution of

$$y'' + \frac{n-1}{r} y' + p{w_0}^{p-1} y = 0\,, \ \ y(r_0) = A, \ \ y'(r_0) = B\,,$$

so that $y(r) = \alpha r^{s_1} + \beta r^{s_2}$, with some constants α, β, and $s_1 < s_2 < 0$ as in (3.23). To apply Lemma 3.7, we shall find $r_0 > 0$ small, so that $A > 0$, $B < 0$, and

$$\frac{r_0 B(r_0)}{A(r_0)} > s_1\,. \tag{3.25}$$

Recalling that $w_0(r) = c_0 r^{-\beta}$, with $\beta = \frac{2}{p-1}$, and c_0 as above, we rewrite (3.25) as

$$\frac{-c_0\beta - r_0^{\beta+1} w'(r_0)}{c_0 - r_0^{\beta} w(r_0)} > s_1\,.$$

This will hold for sufficiently small r_0, provided that

$$\frac{-c_0\beta}{c_0} > s_1 = \frac{-(n-2) - \sqrt{(n-2)^2 - 4p\gamma}}{2}\,.$$

The last inequality follows from $\beta = \frac{2}{p-1} < \frac{n-2}{2}$, which is in turn equivalent to $p > \frac{n+2}{n-2}$. By Lemma 3.7, $y(r) > 0$ for $r > r_0$, and then by Lemma 3.6, $w(r) < w_0(r)$ for $r > r_0$, and since r_0 can be chosen arbitrarily small, (3.24) is justified.

By (3.5) we have $\lambda'(t) > 0$ for all $t > 0$, provided that

$$\frac{2}{p-1} + \frac{t w'(t)}{w(t)} > 0\,, \quad \text{for all } t > 0\,. \tag{3.26}$$

Recall that $w(t)$ and $w_0(t)$ are both solutions of $w'' + \frac{n-1}{t} w' + w^p = 0$. Writing this equation at $w(t)$ and at $w_0(t)$, we conclude, in view of (3.24),

$$\left[t^{n-1}\left(w_0' w - w_0 w'\right)\right]' = w_0 w \left(w^{p-1} - w_0^{p-1}\right) < 0\,, \quad \text{for all } t > 0\,.$$

The function $q(t) \equiv t^{n-1}\left(w_0'w - w_0 w'\right)$ satisfies $q(0) = 0$ (observe that $n - 1 > \beta + 1$), and $q'(t) < 0$. It follows that $q(t) < 0$, or that $\frac{w'(t)}{w(t)} > \frac{w_0'(t)}{w_0(t)}$ for all $t > 0$. Then

$$\frac{2}{p-1} + \frac{tw'(t)}{w(t)} > \frac{2}{p-1} + \frac{tw_0'(t)}{w_0(t)} = 0\,,$$

justifying (3.26), and so the solution curve is monotone.

Turning to the other claims, for large t, we have, by Lemma 3.5, $u(r) = \frac{w(tr)}{w(t)} - 1 \sim \frac{w_0(tr)}{w_0(t)} - 1 = r^{-\beta} - 1$, and $\lambda = t^2 w^{p-1} \sim t^2 w_0^{p-1} = \gamma$, while using (3.24), $\lambda(t) = t^2 w^{p-1} < t^2 w_0^{p-1} = \gamma$, for all $t > 0$. $\diamondsuit$

We now consider the problem (3.1) for the critical and sub-critical cases, $p \le \frac{n+2}{n-2}$. For $p = \frac{n+2}{n-2}$, the generating solution, i.e., the solution of (3.3) is well-known:

$$w(t) = \frac{1}{\left(1 + \frac{1}{n(n-2)}t^2\right)^{\frac{n-2}{2}}}\,,$$

see e.g., [14] for the references. In view of (3.4), the global solution curve of

$$u'' + \frac{n-1}{r}u' + \lambda\left(1+u\right)^{\frac{n+2}{n-2}} = 0 \ \text{ for } 0 < r < 1,\ u'(0) = 0,\ \ u(1) = 0$$

is given explicitly by

$$(\lambda, u(0)) = \left(\frac{t^2}{\left(1 + \frac{1}{n(n-2)}t^2\right)^2}\,,\left(1 + \frac{1}{n(n-2)}t^2\right)^{\frac{n-2}{2}} - 1\right)\,, \tag{3.27}$$

for $t \in [0,\infty)$ (observe that here $p - 1 = \frac{4}{n-2}$).

Theorem 3.3. *Consider the problem*

$$u'' + \frac{n-1}{r}u' + \lambda\left(1+u\right)^p = 0 \ \text{ for } 0 < r < 1,\ u'(0) = 0,\ \ u(1) = 0\,,$$

with $p \le \frac{n+2}{n-2}$. All positive solutions lie on a single smooth solution curve, which begins at $(\lambda = 0, u = 0)$, and makes exactly one turn to the left at some $\lambda_0 > 0$, tending to infinity as $\lambda \to 0$.

Proof: In case $p < \frac{n+2}{n-2}$, this is Theorem 2.20 in [11]. For $p = \frac{n+2}{n-2}$, the result follows from the representation (3.27), particularly from

$$\lambda(t) = \left[\frac{t}{1 + \frac{1}{n(n-2)}t^2}\right]^2\,,$$

which shows that $\lambda(t)$ has a unique point of maximum, and tends to zero as $t \to \infty$. One can explicitly compute $\lambda_0 = \frac{n(n-2)}{4}$. $\diamondsuit$

We now consider the problem

$$u'' + \frac{n-1}{r} u' + \lambda r^s (1+u)^p = 0, \quad 0 < r < 1, \;\; u'(0) = 0, \;\; u(1) = 0 \,. \tag{3.28}$$

We shall assume that the constant $p > 0$, while the case $p < 0$ we considered in [12] and [13], in the context of so called MEMS problems. Our result will allow $s < 0$, under some conditions, i.e., singular problems. The following is the main result of this section, which for $s = 0$ was proved in D.D. Joseph and T.S. Lundgren [9].

Theorem 3.4. *Assume that $n \geq 1$. Define $m = \frac{n+s}{s/2+1}$, $f(p) = \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$, and assume that $m \geq 1$.*

(i) If $0 < p < 1$, the problem (3.28) has a unique positive solution for any $0 < \lambda < \infty$.

(ii) If $1 < p \leq \frac{m+2}{m-2}$, then there is a critical $\lambda_0 > 0$ so that the problem (3.28) has exactly two positive solutions for $\lambda \in (0, \lambda_0)$, exactly one positive solution at $\lambda = \lambda_0$, and no solutions for $\lambda > \lambda_0$.

(iii) If $p > \frac{m+2}{m-2}$ and $f(p) > m-2$, then the solution curve of (3.28) makes infinitely many turns. Moreover, the problem (3.28) has infinitely many solutions at $\lambda = \bar{\gamma} \equiv (s/2+1)^{-2} \left[\beta(m-1) - \beta(\beta+1)\right]$, with $\beta = \frac{2}{p-1}$, and at most finitely many solutions at other λ's.

(iv) If $p > \frac{m+2}{m-2}$ and $f(p) \leq m-2$, then the solution curve of (3.28) is monotone increasing in the $(\lambda, u(0))$ plane. Moreover, $\lim_{t\to\infty} \lambda(t) = \bar{\gamma}$, $\lim_{t\to\infty} u(0) = \infty$. It follows that the problem (3.28) has a unique positive solution for $\lambda < \bar{\gamma}$, and no positive solution for $\lambda \geq \bar{\gamma}$.

Proof: By above, see (2.9) and (2.10), the change of variables $t = \dfrac{r^{s/2+1}}{s/2+1}$, followed by $\tau = (s/2+1)\, t$, and $\mu = (s/2+1)^2 \lambda$, transforms (3.28) into

$$u'' + \frac{m-1}{\tau} u' + \mu \,(1+u)^p = 0 \,, \quad 0 < \tau < 1, \;\; u'(0) = 0, \;\; u(1) = 0 \,,$$

with $m = \frac{n+s}{s/2+1}$. Then the result follows by the Theorems 3.1, 3.2 and 3.3, while the case $0 < p < 1$ is well-known. $\diamondsuit$

4 Morse index of solutions

We now use the generating and guiding solutions to show that all turning points of the problem (3.1) are non-degenerate, and that the Morse index of solutions increases by one at each turning point. We address the non-degeneracy first.

Theorem 4.1. *Let $u(t_n)$ be a singular solution of (3.1), i.e., $\lambda'(t_n) = 0$. Then $u(t_n)$ is non-degenerate, i.e., $\lambda''(t_n) \neq 0$.*

Proof: Recall that

$$\lambda'(t) = tw^{p-2}\left[2w + (p-1)tw'\right] \,,$$

Since $\lambda'(t_n) = 0$, we have

$$2w(t_n) + (p-1)t_n w'(t_n) = 0\,. \tag{4.1}$$

Then

$$\lambda''(t_n) = t_n w^{p-2}(t_n)\left[(p+1)w'(t_n) + (p-1)t_n w''(t_n)\right] \,, \tag{4.2}$$

and we need to show that

$$S \equiv (p+1)w'(t_n) + (p-1)t_n w''(t_n) \neq 0\,,$$

to conclude that $\lambda''(t_n) \neq 0$. Using the equation (3.3), and then (4.1), we express

$$S = \left[p+1-(n-1)(p-1)\right] w'(t_n) - (p-1)t_n w^p(t_n)$$

$$= w(t_n)\left[-\frac{2\left[p+1-(n-1)(p-1)\right]}{(p-1)t_n} - (p-1)t_n w^{p-1}(t_n)\right]. \tag{4.3}$$

For the guiding solution $w_0(t) = c_0 t^{-\beta}$, with $\beta = \frac{2}{p-1}$ and $c_0 = \left[\beta(n-1) - \beta(\beta+1)\right]^{\frac{1}{p-1}}$, we have by a direct computation

$$-\frac{2\left[p+1-(n-1)(p-1)\right]}{(p-1)t_n} - (p-1)t_n w_0^{p-1}(t_n) = 0\,.$$

Observing that the quantity in the square bracket in (4.3) is a decreasing function of $w(t_n)$, we conclude that $S \neq 0$, once we show that

$$w(t_n) \neq w_0(t_n)\,. \tag{4.4}$$

If, on the contrary, $w(t_n) = w_0(t_n)$, we conclude from (4.1) and the identity

$$2w_0(t_n) + (p-1)t_n w_0'(t_n) = 0\,,$$

that $w'(t_n) = w_0'(t_n)$, and then by the uniqueness for initial value problems $w(t) = w_0(t)$, a contradiction. $\diamondsuit$

By C.S. Lin and W.-M. Ni [15], any solution of the linearized problem for (3.1) is radially symmetric, and hence it satisfies

$$\omega'' + \frac{n-1}{r}\omega' + \lambda p(1+u)^{p-1}\omega = 0,\ 0 < r < 1,\ \ \omega'(0) = \omega(1) = 0\,. \tag{4.5}$$

We call $u(r)$ a *singular solution* of (3.1) if the problem (4.5) has a non-trivial solution. (Differentiating (3.1) in t, and setting $t = t_n$, it is easy to see that a solution is singular if and only if $\lambda'(t_n) = 0$.) The following lemma gives explicitly any non-trivial solution of (4.5).

Lemma 4.1. *Let $u(r)$ be a singular solution of (3.1). Then*

$$\omega(r) = ru'(r) + \frac{2}{p-1}u(r) + \frac{2}{p-1}\,.$$

gives a solution of (4.5).

Proof: The function $v(r) \equiv ru'(r) + \frac{2}{p-1}u(r) + \frac{2}{p-1}$ solves the equation in (4.5), and we have $v'(0) = \omega'(0) = 0$, $v(0) > 0$. By scaling of $\omega(r)$, we may assume that $\omega(0) = v(0)$, and then by the uniqueness result for this type of initial value problems (see [18]), it follows that $\omega(r) \equiv v(r)$. $\diamondsuit$

We now present the main result of this section for positive solutions of the problem

$$\Delta u + \lambda\,(1+u)^p = 0,\ \ \text{for } |x| < 1,\ \ u = 0,\ \ \text{when } |x| = 1\,. \tag{4.6}$$

Theorem 4.2. *Let $u(t_n) > 0$ be a singular solution of (4.6), which means that the corresponding linearized problem (4.5) has non-trivial solutions, or that $\lambda'(t_n) = 0$. Then $u(t_n)$ is non-degenerate, i.e., $\lambda''(t_n) \neq 0$. Moreover, when one follows the solution curve of (4.6) in the direction of increasing $u(0)$, the Morse index of solution increases by one at each turn.*

Proof: By [5] positive solutions of (4.6) are radially symmetric, and hence they satisfy (3.1). The Morse index of solution is the number of negative eigenvalues μ of

$$\Delta\omega + \lambda p\,(1+u)^{p-1}\,\omega + \mu\omega = 0,\ \ \text{for } |x| < 1,\ \ \omega = 0,\ \ \text{when } |x| = 1\,.$$

By [15] solutions of this problem are radially symmetric. At a singular solution $\mu = 0$, and then $\omega(r) = ru'(r) + \frac{2}{p-1}u(r) + \frac{2}{p-1}$ by Lemma 4.1. Assume that at a singular solution $u(t_n)$, $\mu(t_n) = 0$ is the k-th eigenvalue. Following [17], we will show that $\mu'(t_n) < 0$, which means that for $t < t_n$ ($t > t_n$) the k-th eigenvalue is positive (negative), i.e., the Morse index increases by one through $t = t_n$. We shall show that the sign of $\mu'(t_n)$ is the same as that of $-\left(\lambda''(t_n)\right)^2$, which is negative by the Theorem 4.1. Denoting $f(u) = (1+u)^p$, recall the following known formulas, which also hold for general $f(u)$ (here $u = u(t_n)$, ω is a solution of (4.5), and B is the unit ball around the origin in R^n):

$$\mu'(t_n)\int_B \omega^2\,dx = -\lambda(t_n)\int_B f''(u)\omega^3\,dx \quad \text{(p. 11 in [11])}\,,$$

$$-\lambda(t_n)\int_B f''(u)\omega^3\,dx = \lambda''(t_n)\int_B f(u)\omega\,dx \quad \text{(p. 3 in [11])}\,,$$

$$\int_B f(u)\omega\,dx = \frac{1}{2\lambda(t_n)}u'(1)\omega'(1) \quad \text{(p. 5 in [11])}\,.$$

Putting them together, we conclude

$$\mu'(t_n)\int_B \omega^2\,dx = \frac{\lambda''(t_n)}{2\lambda(t_n)}\,u'(1)\omega'(1)\,. \tag{4.7}$$

By Lemma 4.1,

$$\omega'(1) = \frac{p+1}{p-1}u'(1) + u''(1)\,.$$

Recall that $u(r) = \frac{w(rt_n)}{w(t_n)} - 1$, so that $u'(1) = \frac{w'(t_n)t_n}{w(t_n)}$ and $u''(1) = \frac{w''(t_n)t_n^2}{w(t_n)}$. Using these expressions and (4.2), we have

$$\omega'(1) = \frac{\lambda''(t_n)}{(p-1)t_n w^{p-1}(t_n)}\,,$$

and finally, from (4.7),

$$\mu'(t_n)\int_B \omega^2\,dx = \frac{\left(\lambda''(t_n)\right)^2 u'(1)}{2\lambda(t_n)(p-1)t_n w^{p-1}(t_n)} < 0\,,$$

because $u'(1) < 0$ and $\lambda''(t_n) \neq 0$. We conclude that $\mu'(t_n) < 0$, completing the proof. $\diamondsuit$

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