Bounded Solution to a Differential Equation — **June 2000**

1577. *Proposed by Philip Korman, University of Cincinnati, Cincinnati, OH.*

Consider the differential equation $x''(t) + a(t)x^3(t) = 0$ on $0 \le t < \infty$, where $a(t)$ is continuously differentiable and $a(t) \ge \kappa > 0$.

(a) If $a'(t)$ has only finitely many changes of sign, prove that any solution $x(t)$ is bounded.

(b) If one does not assume that $a'(t)$ has only finitely many sign changes, is $x(t)$ necessarily bounded?

Solution by the proposer.

(a) Define the "energy" function E by

$$E(t) = \frac{1}{2}\left(x'(t)\right)^2 + a(t)\frac{\left(x(t)\right)^4}{4}. \tag{1}$$

Using the differential equation we find

$$E'(t) = a'(t)\frac{\left(x(t)\right)^4}{4}. \tag{2}$$

If $a'(t) \le 0$ on an interval $[t_1, t_2]$, then $E'(t) \le 0$ on this interval. It follows that $E(t) \le E(t_1)$ for $t_1 \le t \le t_2$. If $a'(t) \ge 0$ on $[t_1, t_2]$, then from (1) and (2) we conclude that $E'(t) \le a'(t)\dfrac{E(t)}{a(t)}$ on this interval. Integrating this expression we find that

$$E(t) \le \frac{E(t_1)}{a(t_1)}a(t) \le \frac{a(t_2)}{a(t_1)}E(t_1), \qquad t_1 \le t \le t_2. \tag{3}$$

In particular, $E(t)$ can increase by at most a factor of $a(t_2)/a(t_1)$ on $[t_1, t_2]$. From (1) and (3) we also conclude that

$$\frac{\left(x(t)\right)^4}{4} \le \frac{E(t_1)}{a(t_1)}, \qquad t_1 \le t \le t_2. \tag{4}$$

Now assume that $a(t)$ changes sign at points $c_1, c_2, \ldots, c_n$. Because $E(t)$ is non-negative and non-increasing on any interval on which $a'(t) \le 0$, and increases by at most a factor $a(c_{k+1})/a(c_k)$ on any bounded interval $[c_k, c_{c+1}]$ on which $a'(t) \ge 0$, it follows that $E(t)$ remains bounded on $[0, c_n]$. If $a'(t) \le 0$ on $[c_n, \infty)$, then by (2), $E(t)$ is non-increasing on this interval, so remains bounded. This implies that $x(t)$ is bounded on $[0, \infty)$. If $a'(t) \ge 0$ on (c_n, ∞), then by (4), $x^4(t) \le 4E(c_n)/a(c_n)$ for $t \ge c_n$, again showing that $x(t)$ is bounded.

(b) Please supply a solution.

Solution to part (b).

The answer is "no", one can construct $a(t)$ which will "pump up" the energy function $E(t)$, and consequently $x(t)$ will become unbounded. We outline the construction. We construct $a(t)$ depending on the solution itself. Let us start with the initial conditions $x(0) = 1$ and $x'(0) = 1$, and $a(t) = t$. By (2) the energy is increasing. Slightly before the time $t = 2$ we smoothly change $a(t)$ to a constant function $a(t) = 2$, and keep it constant for a while, which keeps the energy unchanged. It is well known that for constant $a(t)$ solutions of our equation move on closed curves around the origin in (x, x') plane. Hence at some time $t_1 > 2$ we will have $x(t_1)$ small. Near t_1 we quickly but smoothly decrease $a(t)$ to $a(t) = 1$. This will result in a loss of energy, which by (2) is very small. We now keep $a(t) = 1$, until a time $t_2 > t_1$, at which $x(t)$ is the largest possible at this energy level (when $x'(t_2) = 0$). At this time we quickly and smoothly increase $a(t)$ to $a(t) = 2$. This will increase the energy considerably. We continue this process, which will increase the energy without bound, and since $a(t) \leq 2$, this will imply that $x(t)$ will become unbounded (at times t when $x'(t) = 0$).

The above procedure can be informally described as "buy high and sell low".