Infinitely many solutions and asymptotics for resonant oscillatory problems

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Abstract

For a class of oscillatory resonant problems, involving Dirichlet problems for semilinear PDE's on balls and rectangles in R^n, we show the existence of infinitely many solutions, and study the global solution set. The first harmonic of the right hand side is not required to be zero, or small. We also derive asymptotic formulas in terms of the first harmonic of solutions, and illustrate their accuracy by numerical computations. The numerical method is explained in detail.

Key words: Global solution curves, asymptotic distribution of solutions.

AMS subject classification: 35J25, 35J61, 65N25.

1 Introduction

We study multiplicity of solutions for semilinear equations with linear part at resonance, a direction of research initiated by the classical paper of E.M. Landesman and A.C. Lazer [14]. Several classes of oscillatory resonant problems on balls and rectangles in R^n are considered. Our focus is on the existence of infinitely many solutions, which we establish by studying global solution curves. Our results are supported by asymptotic analysis and numerical computations, and they extend related research in [1],[2],[16],[17].

Next we describe one of our main results, and the approach used. Let B be the unit ball in R^2, $x^2+y^2<1$. For a class of oscillatory resonant problems, with $h(u)=\sqrt{u}\sin\left[\ln\left(u^{\frac{3}{2}}+1\right)\right]$ and $r=\sqrt{x^2+y^2}$,

$$\Delta u+\lambda_1 u+h(u)=g(x,y)=\mu_1\varphi_1(r)+e(x,y) \text{ for } x\in B\,, \quad u=0 \text{ on } \partial B \tag{1.1}$$

we show the existence of infinitely many solutions for any $g(x,y)\in L^2(B)\cap C^\alpha(B)$, $\alpha>0$. Here $\Delta u=u_{xx}(x,y)+u_{yy}(x,y)$, while $(\lambda_1,\varphi_1(r))$ is the principal eigenpair of the Laplacian on B, with zero boundary conditions, $\mu_1\in R$, $e(x,y)\in\varphi_1^\perp$ in $L^2(B)$, and $e(x,y)\in L^2(B)\cap C^\alpha(B)$, for some $\alpha\in(0,1)$. The function $e(x,y)$ is not assumed to be radially symmetric. Decompose solutions of (1.1) as $u(x,y)=\xi_1\varphi_1(r)+U(x,y)$, with $\xi_1\in R$ and $U(x,y)\in\varphi_1^\perp$ in $L^2(B)$ (similar decomposition is used throughout the paper). We prove that the solution set of (1.1) is exhausted by a continuous solution curve $(u(x,y),\mu_1)(\xi_1)$ parameterized by $\xi_1\in R$, and a section of this curve $\mu_1=\mu_1(\xi_1)$ oscillates toward $\pm\infty$ as $\xi_1\to\infty$, see Figure 1 below. We find this result to be rather surprising. A more typical behavior is that $\mu_1(\xi_1)\to 0$ as $\xi_1\to\infty$. The choice of nonlinearity $h(u)$ was dictated by rather restrictive conditions that we needed to impose to obtain a continuous curve of solutions. For more natural nonlinearities we can only assert the existence of a continuum of solutions (as in the case of (1.3) below), and in some cases we have to rely on the numerically produced solutions curves (e.g., for (1.2) below), for which we can still derive accurate asymptotic formulas. In dimension $n=1$ a similar result was proved in [12] for $h(u)=u^p\sin u$, $\frac{1}{2}<p<1$. We remark that solution curves of the form $(u(x,y),\mu_1)(\xi_1)$ appeared previously in R. Schaaf and K. Schmitt [17].

For a model resonant problem

$$\Delta u+\lambda_1 u+u\sin u=\mu_1\varphi_1(r)+e(x,y) \text{ for } x\in B\,, \quad u=0 \text{ on } \partial B \tag{1.2}$$

we provide asymptotical and computational evidence, as well as heuristic justification of the following conjecture: there exist two numbers $0<a<A$ so that the problem (1.2) has infinitely many solutions for $\mu_1\in(-a,a)$, there are at most finitely many solutions for μ_1 outside of $(-a,a)$, and no solutions exist for $|\mu_1|>A$. A similar situation occurs for rectangles in two dimensions. The existence of infinitely many solutions for an interval of μ_1's (bounded or unbounded) is a new, and actually a rare phenomenon (as evidenced by the results of this paper, including numerical computations).

For the problem (1.2) on balls and rectangles in dimensions higher than two, and for "most" other nonlinear terms, infinitely many solutions occur only at $\mu_1 = 0$. The restriction to the case $\mu_1 = 0$ is common in the literature, see e.g., [1], [4], [5], [16], [17]. The nonlinear term in (1.2) occurred previously in [8], in connection with the oscillatory bifurcation from infinity.

The problem (1.2) can be seen as a limiting case of another model problem

$$\tag{1.3} \begin{gathered} \Delta u + \lambda_1 u + u^p \sin u = \mu_1 \varphi_1(r) + e(x,y) \text{ for } x \in B\,, \\ u = 0 \text{ on } \partial B\,, \end{gathered}$$

with $p \in (0,1)$, to which the well known results of D. Costa et al [1] apply at $\mu_1 = 0$ (the results of [1] hold for more general domains), see also [16]. It follows from [1] that for $\mu_1 = 0$ the problem (1.3) has infinitely many solutions, and moreover $\dfrac{u}{\max_B u} \to \varphi_1(r)$ for large solutions. We derive a rather precise asymptotic formula for $\mu_1 = \mu_1(\xi_1)$ in case $|\xi_1|$ is large, and this formula tends to be accurate for small $|\xi_1|$ as well. The nonlinear term $u \sin u$ in (1.2) has linear growth at infinity, while (1.3) requires sublinear growth and is related to the result in [1].

In addition to balls in R^2, we obtain asymptotic formulas and perform computations of solution curves for rectangular domains, and for radial solutions on balls in any dimension.

In the last section we use a similar continuation approach to obtain solutions of nonlinear algebraic systems. A "toy example" at the end of the paper shows how this computation can be done.

2 Global solution set for a ball in R^2

Let $J_0(z)$ be the Bessel function of order zero, with $J_0(0) = 1$, and denote by $\nu_1 > 0$ its first root, $\nu_1 \approx 2.405$. The principal eigenpair of the Laplacian on the unit ball $B \in R^2$ is $\lambda_1 = \nu_1^2 \approx 5.78$, $\varphi_1(r) = c_0 J_0(\nu_1 r)$ with $r = \sqrt{x^2 + y^2}$, with c_0 chosen so that

$$\int_B \varphi_1^2(r)\, dxdy = 2\pi c_0^2 \int_0^1 J_0^2(\nu_1 r)\, r\, dr = 1\,,$$

which is

$$\tag{2.1} c_0 = \frac{1}{\sqrt{2\pi \int_0^1 J_0^2(\nu_1 r)\, r\, dr}} \approx 1.09\,.$$

Observe that $\varphi_1(0) = c_0$. We shall also need the second eigenvalue λ_2. Recall (see e.g. [15]) that the eigenvalues of the Laplacian on B with zero boundary condition are $\lambda_{n,m} = \alpha_{n,m}^2$ ($n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots$) with the corresponding eigenfunctions $J_n(\alpha_{n,m} r)(\alpha \cos n\theta + \beta \sin n\theta)$, where $\alpha_{n,m}$ is the m-th root of $J_n(x)$, the n-th Bessel function (α and β are arbitrary constants). One calculates $\lambda_2 = \alpha_{1,1}^2 \approx 14.62$, with $\alpha_{1,1} \approx 3.83$, and $\varphi_2 = J_1(\alpha_{1,1} r)(\alpha \cos\theta + \beta \sin\theta)$. The principal eigenvalue is simple, while all other eigenvalues have multiplicity two, because any two Bessel functions with indices different by an integer do not have any roots in common, see G.N. Watson [18].

Let us now recall the following result from [10] and [12]. It deals with PDE's on a general domain $\Omega \subset R^n$

$$\Delta u + h(u) = \mu_1 \varphi_1(r) + e(x) \text{ for } x \in \Omega, \;\; u = 0 \text{ on } \partial\Omega \,. \tag{2.2}$$

Here $x \in R^n$, $r = |x|$ and $(\lambda_1, \varphi_1(r))$ is the principal eigenpair of the Laplacian on Ω, with zero boundary conditions, $\mu_1 \in R$, $e(x) \in \varphi_1^\perp$ in $L^2(\Omega)$, and $e(x) \in C^\alpha(\Omega) \cap L^2(\Omega)$, for some $\alpha \in (0, 1)$.

Theorem 2.1 *For the problem (2.2) assume that $h(u) \in C^2(R)$, $f(x) \in L^2(\Omega)$, and*

$$h'(u) < \lambda_2 - \lambda_1, \;\; \textit{for all } u \in R\,, \tag{2.3}$$

$$|h(u)| < \gamma |u| + c, \;\; \textit{with } 0 < \gamma < \lambda_2 - \lambda_1,\; c \geq 0, \textit{ and } u \in R\,. \tag{2.4}$$

Then the solution set of (2.2) consists of a continuous curve $(u(x), \mu_1)(\xi_1)$ parameterized by $\xi_1 \in R$.

The following lemma extends a similar result in [12]. Recall that solutions of (2.2) are decomposed as $u(x) = \xi_1 \varphi_1(r) + U(x)$.

Lemma 2.1 *In the conditions of Theorem 2.1 assume that $\lim_{|u| \to \infty} \frac{h(uz)}{u} = 0$, uniformly in $z \in R$. Then as $\xi_1 \to \pm\infty$, the solutions of (2.2) satisfy $\frac{u(x)}{\xi_1} \to \varphi_1(x)$ in $H^1(\Omega)$ (and also in $L^2(\Omega)$). Moreover, if $h(\xi_1 u) = O\left(|\xi_1|^p\right)$ as $|\xi_1| \to \infty$ uniformly in $u \in R$, then $||U(x)||_{H^1(\Omega)} = O\left(|\xi_1|^{\frac{p}{2}}\right)$ as $|\xi_1| \to \infty$.*

Proof: By Theorem 2.1 we have a solution curve $(u(x), \mu_1)(\xi_1)$. From (1.1)

$$\Delta U + \lambda_1 U + h(\xi_1 \varphi_1 + U) = \mu_1 \varphi_1 + e\,.$$

Letting here $U = \xi_1 V$, obtain

$$\Delta V + \lambda_1 V = -\frac{h(\xi_1(\varphi_1 + V))}{\xi_1} + \frac{\mu_1}{\xi_1}\varphi_1 + \frac{e}{\xi_1}.$$

Multiplying by V and integrating, we conclude that $\int_\Omega |\nabla V|^2 \, dx = O\left(|\xi_1|^{p-1}\right)$, as $\xi_1 \to \pm\infty$, and the lemma follows (since $V \perp \varphi_1$ in $L^2(\Omega)$, obtain $-\int_\Omega \Delta V V \, dx = \int_\Omega |\nabla V|^2 \, dx \geq \lambda_2 \int_\Omega V^2 \, dx$). $\diamondsuit$

Next we present one of our main results.

Theorem 2.2 *There exist $h(u) \in C^2(R)$ for which the problem (1.1) has infinitely many positive solutions or any $g(x,y) \in L^2(B) \cap C^\alpha(B)$. Moreover, all solutions of (1.1) lie on a continuous solution curve $(u(x,y), \mu_1)(\xi_1)$, and $\mu_1(\xi_1)$ oscillates toward $\pm\infty$, as $\xi_1 \to +\infty$.*

Proof: We exhibit $h(u)$ satisfying the conditions of Theorem 2.1, for which $\mu_1(\xi_1)$ oscillates toward $\pm\infty$, as $\xi_1 \to +\infty$. Take $h(u) = \sqrt{u} \sin\left[\ln\left(u^{\frac{3}{2}} + 1\right)\right]$ for $u \geq 0$ (observe that $h(u) \in C^2[0,\infty)$ and $h'(u) < 1$ for $u \in [0,\infty)$), then extend $h(u)$ arbitrarily to $(-\infty, 0)$ so that $h(u) \in C^2(R)$ and the conditions (2.3) and (2.4) hold. Calculate (for $u \geq 0$) $h'(u) = \frac{\sin\left[\ln\left(u^{\frac{3}{2}}+1\right)\right]}{2\sqrt{u}} + \frac{3u\cos\left[\ln\left(u^{\frac{3}{2}}+1\right)\right]}{2\left(u^{\frac{3}{2}}+1\right)}$, and one of the anti-derivatives of $h(u)$:

$$H(u) = \frac{1}{3}\left(u^{\frac{3}{2}} + 1\right)\left(\sin\left[\ln\left(u^{\frac{3}{2}} + 1\right)\right] - \cos\left[\ln\left(u^{\frac{3}{2}} + 1\right)\right]\right)$$

$$= \frac{\sqrt{2}}{3}\left(u^{\frac{3}{2}} + 1\right)\sin\left[\ln\left(u^{\frac{3}{2}} + 1\right) - \frac{\pi}{4}\right].$$

Multiply the equation in (1.1) by φ_1 and integrate over B:

$$\mu_1 = \int_B h\left(\xi_1 \varphi_1 + U(x,y)\right)\varphi_1 \, dxdy. \tag{2.5}$$

Since $||U(x,y)||_{L^2(B)} = O(\xi_1^{\frac{1}{4}})$ by Lemma 2.1, and $||h'(\xi_1\varphi_1 + U)||_{L^2(B)} = O(\frac{1}{\xi_1^{\frac{1}{2}}})$ as $\xi_1 \to \infty$, we have by the mean-value theorem

$$||h\left(\xi_1\varphi_1 + U\right) - h\left(\xi_1\varphi_1\right)||_{L^2(B)} = o(1).$$

Then (2.5) becomes

$$\mu_1(\xi_1) = 2\pi \int_0^1 h(\xi_1\varphi_1)\,\varphi_1 r\,dr + o(1)\,. \tag{2.6}$$

Denoting $f(r) = \frac{r\varphi_1(r)}{\varphi_1'(r)}$, and integrating by parts (using $\varphi_1(0) = c_0$)

$$\int_0^1 h(\xi_1\varphi_1)\,\varphi_1 r\,dr = \frac{1}{\xi_1}\int_0^1 f(r)\frac{d}{dr}H(\xi_1\varphi_1)\,dr \tag{2.7}$$

$$= -\frac{1}{\xi_1}\int_0^1 f'(r)H(\xi_1\varphi_1)\,dr - \frac{1}{\xi_1}f(0)H(c_0\xi_1)\,.$$

The second term on the right oscillates toward $\pm\infty$ as $\xi_1 \to \infty$ with the amplitude approaching $\frac{\sqrt{2}}{3}c_0^{\frac{3}{2}}|f(0)|\xi_1^{\frac{1}{2}}$. We show next that it dominates the first term on the right in (2.7). A computation shows that $f'(r) > 0$ for all $r \in (0,1)$, and hence the first term is estimated in absolute value by a quantity approaching $\frac{\sqrt{2}}{3}c_0^{\frac{3}{2}}\xi_1^{\frac{1}{2}}\int_0^1 f'(r)J_0^{\frac{3}{2}}(\nu_1 r)\,dr$. Calculate $f(0) = \frac{\varphi_1(0)}{\varphi_1''(0)} = -\frac{2}{\nu_1^2}$, so that $|f(0)| = \frac{2}{\nu_1^2} \approx 0.34$. (From $\varphi_1'' + \frac{1}{r}\varphi_1' + \nu_1^2\varphi_1 = 0$, it follows that $2\varphi_1''(0) = -\nu_1^2\varphi_1(0)$.) Another computation shows that $\int_0^1 f'(r)J_0^{\frac{3}{2}}(\nu_1 r)\,dr \approx 0.1$, so that the second term dominates in (2.7). Since $g(x,y) \in C^\alpha(B)$ the convergence $\frac{u(x,y)}{\xi_1} \to \varphi_1(r)$ is in $C^2(B)$ by the elliptic regularity. It follows that $u(x,y) > 0$ for large ξ_1, and hence it satisfies (1.1) with the original $h(u)$ (before the extension). We conclude that for any $g(x,y) \in L^2(B) \cap C^\alpha(B)$ the problem (1.1) has infinitely many positive solutions. $\diamondsuit$

In Figure 1 we present an approximation of the solution curve $\mu_1 = \mu_1(\xi_1)$ for the problem (1.1), with $h(u) = \sqrt{u}\sin\left[\ln\left(u^{\frac{3}{2}}+1\right)\right]$, computed using the formula (2.6). We were able to perform computations on very large intervals (along both ξ_1 and μ_1 axes), and to make the resulting picture manageable a logarithmic scale is used for both ξ_1 and μ_1. (*Log* denotes the natural logarithm in *Mathematica*. Note that the curve in this presentation is not continuous. It only appears so, since all values with $|\mu| < 1$ have been omitted.) The result is the solution curve oscillating toward $\pm\infty$. In Figure 2 we show the same curve in the original (ξ_1, μ_1) coordinates. It is not apparent from that picture that $\mu_1(\xi_1) \to \pm\infty$ as $\xi_1 \to \infty$, since $\mu_1(\xi_1)$ keeps the same sign on large, and ever increasing intervals.

There are other $h(u)$ which can be handled similarly (including *Mathematica* being able to calculate the integral $H(u)$ in elementary functions). We mention $h(u) = u\sin\left[\ln\left(u^2+1\right)\right]$ and $h(u) = \sin\left[\ln\left(u+1\right)\right]$.

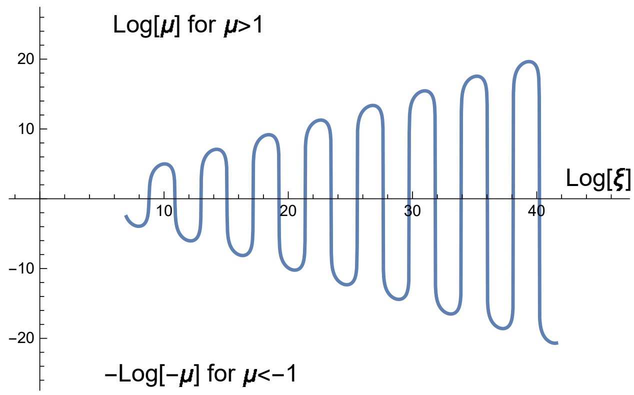

Figure 1: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (1.1), oscillating to $\pm\infty$. Values with $|\mu_1| < 1$ are not shown.

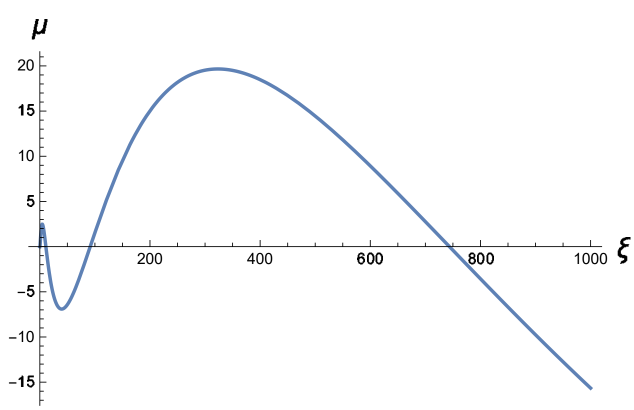

Figure 2: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (1.1) for $\xi_1 \in (0, 1000)$

Next we turn to more "natural" nonlinearities $h(u)$.

Theorem 2.3 *If $0 < p < 1$ there is a continuum of solutions of (1.3) $(u(x), \mu_1)(\xi_1)$ parameterized by the first harmonic $\xi_1 \in (-\infty, \infty)$. Along this continuum, $\lim_{\xi_1 \to \pm\infty} \mu_1(\xi_1) = 0$. Moreover, the asymptotic formula (2.10) below holds.*

Proof: The existence of a continuum of solutions of (1.3) $(u(x), \mu_1)(\xi_1)$ parameterized by the first harmonic $\xi_1 \in (-\infty, \infty)$ follows by the result of R. Schaaf and K. Schmitt [17], which was based on E.N. Dancer [3], see also P. Korman [12]. By the Lemma 2.1 and elliptic regularity it follows that $\frac{u}{\xi_1} \to \varphi_1$ in $C^2(B)$, as $|\xi_1| \to \infty$.

We now derive an asymptotic formula for $\mu_1(\xi_1)$. Multiply the equation (1.3) by φ_1, and integrate over B, then use Lemma 2.1 and elliptic regularity

$$\begin{aligned}(2.8)\qquad \mu_1 &= \textstyle\int_B u^p \sin u \, \varphi_1 \, dx \, dy = \int_0^{2\pi} \int_0^1 u^p \sin u \, \varphi_1 r \, dr \, d\theta \\ &= 2\pi \xi_1^p \textstyle\int_0^1 \varphi_1^{p+1} \sin(\xi_1 \varphi_1) \, r \, dr + o(\xi_1^p) \, .\end{aligned}$$

Integration by parts gives

$$(2.9)\qquad \int_0^1 \varphi_1^{p+1} \sin(\xi_1 \varphi_1) \, r \, dr = \int_0^1 \frac{\varphi_1^{p+1} r}{\xi_1 \varphi_1'} \frac{d}{dr} \left[-\cos(\xi_1 \varphi_1) \right] \, dr$$

$$= -\frac{1}{\xi_1} g(r) \cos(\xi_1 \varphi_1) \, \Big|_0^1 + \frac{1}{\xi_1} \int_0^1 g'(r) \cos(\xi_1 \varphi_1) \, dr \, ,$$

where $g(r) = \frac{\varphi_1^{p+1} r}{\varphi_1'}$. Observe that $g(1) = 0$, while

$$g(0) = c_0^{p+1} \lim_{r \to 0} \frac{r}{\varphi_1'(r)} = \frac{c_0^{p+1}}{\varphi_1''(0)} = -\frac{2 c_0^p}{\nu_1^2} \, .$$

(From $\varphi_1'' + \frac{1}{r} \varphi_1' + \nu_1^2 \varphi_1 = 0$, it follows that $2\varphi_1''(0) = -c_0 \nu_1^2$.) Hence

$$-g(r) \cos(\xi_1 \varphi_1) \, \Big|_0^1 = -\frac{2 c_0^p \cos c_0 \xi_1}{\nu_1^2} \, .$$

One checks that $g'(r) \in C[0,1]$ is a bounded function. It follows that the oscillating integral $\int_0^1 g'(r) \cos(\xi_1 \varphi_1) \, dr$ is $o(1)$. From (2.8)

$$(2.10)\qquad \mu_1 = -\frac{4\pi \xi_1^{p-1} c_0^p \cos c_0 \xi_1}{\nu_1^2} + o\left(\xi_1^{p-1}\right) \, .$$

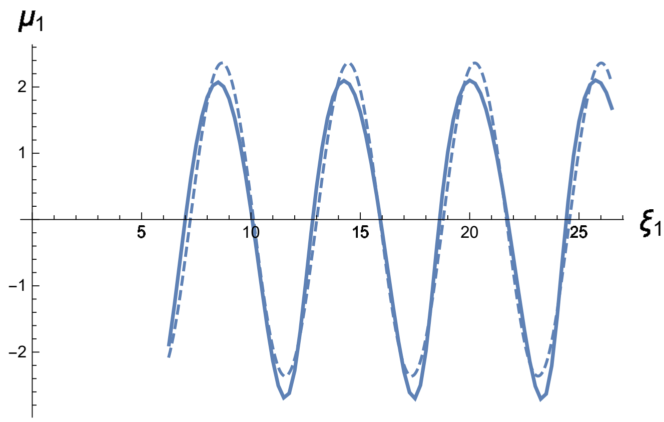

Figure 3: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (2.11), compared with (2.10)

It follows that $\mu_1(\xi_1) \to 0$ as $\xi_1 \to \infty$, concluding the proof. $\diamondsuit$

Remark In limiting case $p = 1$ the formula (2.10) (while not rigorously justified) indicates that $\mu_1(\xi_1)$ is asymptotic to a multiple of $\cos c_0\xi_1$, suggesting that there is a $\mu_0 > 0$ so that for $|\mu| < \mu_0$ the problem (1.3) has infinitely many solutions. The same conclusion is supported by our numerical computations, including the following example.

Example We computed the solution curve $\mu_1 = \mu_1(\xi_1)$ for the following example, with the linear part at resonance,

$$\begin{aligned} &\Delta u + \lambda_1 u + u \sin u = \mu_1 \varphi_1(r) + xy \ \text{ for } (x,y) \in B\,, \\ &u = 0 \ \text{ on } \partial B\,. \end{aligned} \tag{2.11}$$

Observe that $\int_B xy\,\varphi_1(r)\,dxdy = 0$. The solution curve $\mu_1 = \mu_1(\xi_1)$ (solid line) is presented in Figure 3. Notice an excellent agreement with the asymptotic formula (2.10) (dashed line).

Example We computed the solution curve $\mu_1 = \mu_1(\xi_1)$ for an example of (1.3), with $p = \frac{1}{2}$,

$$\begin{aligned} &\Delta u + \lambda_1 u + u^{\frac{1}{2}} \sin u = \mu_1 \varphi_1(r) + x^2 y - 3xy^4 \ \text{ for } x \in B\,, \\ &u = 0 \ \text{ on } \partial B\,. \end{aligned} \tag{2.12}$$

Observe that $\int_B (x^2y - 3xy^4)\ \varphi_1(r)\,dxdy = 0$. The solution curve $\mu_1 = \mu_1(\xi_1)$ (solid line) is presented in Figure 4. The solutions $u(x,y)$ *are not*

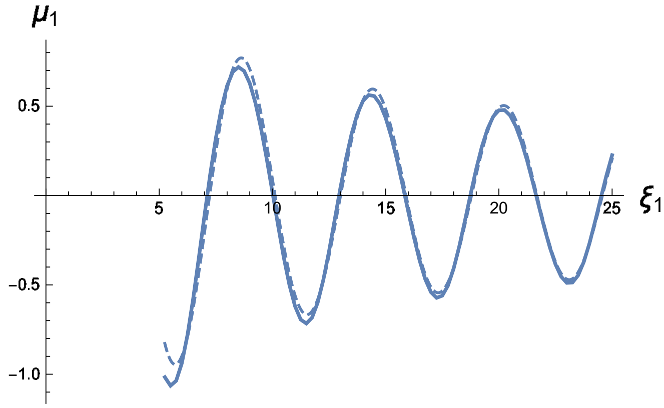

Figure 4: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (2.12), compared with (2.10)

radially symmetric, although *they get arbitrarily close to radially symmetric functions as* $|\xi_1| \to \infty$, according to our results. Again, we have an excellent agreement with the asymptotic formula (2.10) (dashed line).

3 Asymptotic formula in case of a rectangle

Let $R = \{0 < x < a\} \times \{0 < y < b\}$ be a rectangle in R^2. In this section we present computation of the solution curve $\mu_1 = \mu_1(\xi_1)$ on R for the nonlinearity considered above

$$(3.1) \qquad \begin{aligned} \Delta u + \lambda_1 u + u \sin u &= \mu_1 \varphi_1(x,y) + e(x,y) \ \text{ for } (x,y) \in R\,, \\ u &= 0 \ \text{ on } \partial R\,, \end{aligned}$$

and derive an asymptotic formula for $\mu_1(\xi_1)$. Here the principal eigenfunction $\varphi_1(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{\pi}{a} x \sin \frac{\pi}{b} y$ satisfies $\int_R \varphi_1^2 \, dxdy = 1$, and the corresponding principal eigenvalue of $-\Delta$ is $\lambda_1 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$. It is assumed that $\iint_R e(x,y)\varphi_1(x,y)\, dxdy = 0$. We decompose the solution of (3.1) as $u(x,y) = \xi_1 \varphi_1(x,y) + U(x,y)$, with $\iint_R U(x,y)\varphi_1(x,y)\, dxdy = 0$, and $\xi_1 = \iint_R u(x,y)\varphi_1(x,y)\, dxdy$.

Unlike the case of a ball, we shall use the stationary phase method to derive the following asymptotic formula for ξ_1 large:

$$(3.2) \qquad \mu_1(\xi_1) \sim \frac{4\sqrt{ab}}{\pi} \sin\left(\frac{2}{\sqrt{ab}}\,\xi_1 - \frac{\pi}{2}\right)\,,$$

after we recall the following known lemmas.

Lemma 3.1 *Assume that $f(x) \in C^2(x_0 - a_0, x_0 + a_0)$ for some constants $\alpha\,, a_0 > 0$. Then as $\mu \to \infty$*

$$\int_{x_0-a_0}^{x_0+a_0} f(x) e^{-i\alpha\mu(x-x_0)^2}\, dx = e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi}{\alpha\mu}} f(x_0) + O\left(\frac{1}{\mu}\right) .$$

This lemma follows from the part (i) of the following more general lemma, which we will also need, see e.g., p. 83 in [9] for the proof.

Lemma 3.2 *(i) Assume that $f(x)$ and $g(x)$ are of class $C^2[a,b]$ and $g(x)$ has a unique critical point x_0 on $[a,b]$, and moreover $x_0 \in (a,b)$ and $g''(x_0) \neq 0$ (so that x_0 gives a global max or global min on $[a,b]$). Then as $\mu \to \infty$ the following asymptotic formula holds*

$$\int_a^b f(x) e^{i\mu g(x)}\, dx = e^{i\left[\mu g(x_0) \pm \frac{\pi}{4}\right]} \sqrt{\frac{2\pi}{\mu |g''(x_0)|}}\, f(x_0) + O\left(\frac{1}{\mu}\right) ,$$

where one takes "plus" if $g''(x_0) > 0$ and "minus" if $g''(x_0) < 0$.

(ii) Assume that the functions $f(x)$ and $g(x) > 0$ are of class $C^2[0,1]$, and satisfy

$$g'(x) < 0 \quad \text{for all} \quad x \in (0,1], \quad \text{and} \quad g'(0) = 0,\ g''(0) < 0\,.$$

Then, as $\mu \to \infty$,

$$\int_0^1 f(x) e^{i\mu g(x)} dx = e^{i(\mu g(0) - \frac{\pi}{4})} \sqrt{\frac{\pi}{2\mu |g''(0)|}} f(0) + O\left(\frac{1}{\mu}\right) .$$

Turning to the derivation of (3.2), multiply (3.1) by φ_1 and integrate over R. Then use that $u \sim \xi_1 \varphi_1$ for ξ_1 large ($\sin u \sim \sin \xi_1 \varphi_1$ by the mean-value theorem)

$$\mu_1 = \iint_R u \sin u\, \varphi_1\, dxdy \sim \xi_1 \iint_R \varphi_1^2 \sin \xi_1 \varphi_1\, dxdy = \xi_1 \mathrm{Im} \iint_R \varphi_1^2 e^{i\xi_1\varphi_1}\, dxdy$$

$$= \frac{4\xi_1}{ab} \mathrm{Im} \iint_R \sin^2 \frac{\pi}{a} x \sin^2 \frac{\pi}{b} y\, e^{i\frac{2}{\sqrt{ab}}\xi_1 \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y}\, dxdy\,.$$

For ξ_1 large there are fast oscillations around zero, except near the stationary point $(x_0, y_0) = (\frac{a}{2}, \frac{b}{2})$. The approximation of the integral near the stationary point provides the dominant contribution to this integral. Using Taylor's formula near (x_0, y_0)

$$\sin \frac{\pi}{a} x \sin \frac{\pi}{b} y \approx 1 - \frac{\pi^2}{2a^2}(x - x_0)^2 - \frac{\pi^2}{2b^2}(y - y_0)^2 \,,$$

because the second mixed partials vanish at (x_0, y_0). Then

$$\mu_1 \sim \frac{4\xi_1}{ab} \mathrm{Im} \left(e^{i\frac{2}{\sqrt{ab}}\xi_1} \int_0^a \sin^2 \frac{\pi}{a} x \, e^{-i\xi_1 \alpha_1 (x-x_0)^2} \, dx \int_0^b \sin^2 \frac{\pi}{b} y \, e^{-i\xi_1 \alpha_2 (y-y_0)^2} \, dy \right),$$

where $\alpha_1 = \frac{\pi^2}{a^2\sqrt{ab}}$ and $\alpha_2 = \frac{\pi^2}{b^2\sqrt{ab}}$. Using Lemma 3.1, for large ξ_1,

$$\mu_1 \sim \frac{4\xi_1}{ab} \mathrm{Im} \left(e^{i\frac{2}{\sqrt{ab}}\xi_1} e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi}{\alpha_1 \xi_1}} \, e^{-i\frac{\pi}{4}} \sqrt{\frac{\pi}{\alpha_2 \xi_1}} \right) = \frac{4\sqrt{ab}}{\pi} \sin \left(\frac{2}{\sqrt{ab}} \xi_1 - \frac{\pi}{2} \right) .$$

This formula, as well as our numerical calculations, suggests that there exist two constants $0 < a < A$ so that the problem (3.1) has infinitely many solutions for $\mu_1 \in (-a, a)$, there are at most finitely many solutions for μ_1 outside of $(-a, a)$, and no solutions exist for $|\mu_1| > A$.

Example On the rectangle $R_1 = \{0 < x < 1\} \times \{0 < y < 2\}$ we computed the solution curve $\mu_1 = \mu_1(\xi_1)$ for the problem

$$\text{(3.3)} \quad \Delta u + \lambda_1 u + u \sin u = \mu_1 \varphi_1(x, y) + (x - \tfrac{1}{2})(y - 1) \ \text{ for } (x, y) \in R_1 \,, \qquad u = 0 \ \text{ on } \partial R_1 \,,$$

with $\lambda_1 = \frac{5\pi^2}{4}$ and $\varphi_1(x, y) = \sqrt{2} \sin \pi x \sin \frac{\pi}{2} y$. Observe that $\iint_R (x - \frac{1}{2})(y - 1)\varphi_1(x, y) \, dxdy = 0$. The solution curve $\mu_1 = \mu_1(\xi_1)$ (solid line) is presented in Figure 5. Once again, we have an excellent agreement with the asymptotic formula (3.2) (dashed line).

As in the case of balls, oscillations of $\mu_1(\xi_1)$ are decaying in the dimensions $n > 2$, as will follow from the asymptotic formula that we present next. Consider the n-dimensional rectangle $R = (0, a_1) \times (0, a_2) \times \cdots \times (0, a_n)$, and the problem (3.1) on R. The principal eigenfunction of the Laplacian on R with Dirichlet boundary conditions, and satisfying $\int_R \varphi_1^2 \, dx = 1$, is

$$\varphi_1 = \frac{2^{\frac{n}{2}}}{\sqrt{a_1 a_2 \cdots a_n}} \sin \frac{\pi}{a_1} x_1 \, \sin \frac{\pi}{a_2} x_2 \cdots \sin \frac{\pi}{a_n} x_n \,.$$

Similarly to the above derivation, one shows that

$$\mu_1(\xi_1) \sim \frac{2^{\frac{n}{2}(3-\frac{n}{2})} \, (a_1 a_2 \cdots a_n)^{\frac{n}{4}}}{\pi^{\frac{n}{2}}} \xi_1^{1-\frac{n}{2}} \sin \left(\frac{2^{\frac{n}{2}}}{\sqrt{a_1 a_2 \cdots a_n}} \xi_1 - n\frac{\pi}{4} \right) \to 0 \,,$$

as $\xi_1 \to \infty$, for $n > 2$.

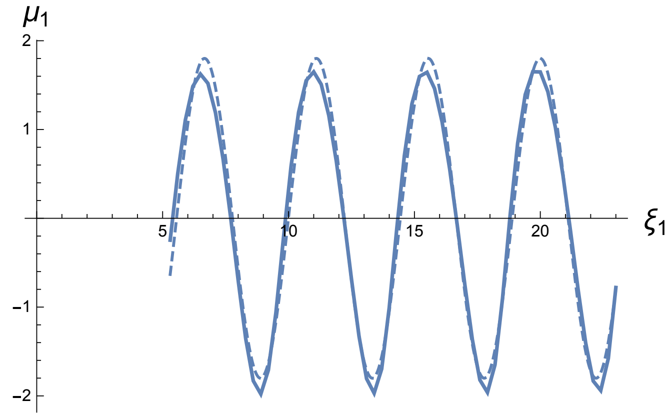

Figure 5: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (3.3), compared with (3.2)

4 Numerical computations

We now describe the *Mathematica* program that was used to produce the solution curves $\mu_1 = \mu_1(\xi_1)$, presented above. To avoid ambiguity of notation, in this section we shall write ξ instead of ξ_1 and μ instead of μ_1, so that $u = \xi\varphi_1 + U$, and $\mu = \mu(\xi)$.

Our program handles semilinear equations on rather general domains solving

$$\begin{aligned} &\Delta u + f(u) = \mu\varphi_1(x,y) + e(x,y) \ \text{ for } (x,y) \in \Omega\,, \\ &u = 0 \ \text{ on } \partial\Omega \end{aligned} \tag{4.1}$$

on bounded domains $\Omega \in R^2$, including rectangles and ellipses in R^2, and radially symmetric solutions on balls in R^n. Here $\Delta u = u_{xx}(x,y) + u_{yy}(x,y)$, while (λ_1, φ_1) is the principal eigenpair of the Laplacian on Ω, with zero boundary conditions and $\int_\Omega \varphi_1^2\,dxdy = 1$, $\mu \in R$, $e(x,y) \in \varphi_1^\perp$ in $L^2(\Omega)$. Choosing a step size h and an initial value $\xi = \xi_0$, let $\xi_n = \xi_0 + nh$. We are looking for $\mu = \mu_n$ for which the problem (4.1) has a solution $u(x) \equiv u_n(x,y)$, with

$$\int_\Omega u(x,y)\varphi_1(x,y)\,dxdy = \xi_n\,. \tag{4.2}$$

We utilize *Mathematica*'s ability (the NDSolve command) to solve linear Dirichlet problems of the type

$$\Delta u + a(x,y)u = b(x,y) \ \text{ for } (x,y) \in \Omega\,, \quad u = 0 \ \text{ on } \partial\Omega$$

on some domains $\Omega \in R^2$, including ellipses around the origin, and rectangles.

Assuming that $(\mu_n, u_n(x,y))(\xi_n)$ is already computed, we use Newton's method to calculate $(\mu_{n+1}, u_{n+1}(x,y))(\xi_{n+1})$. We calculate $(\mu_{n+1}, u_{n+1}(x,y))$ using a sequence of iterates $(\mu^k, u^k(x,y))$ beginning with $(\mu^0, u^0(x,y)) = (\mu_n, u_n(x,y))$. Assuming that $(\mu^k, u^k(x,y))$ is already computed, we approximate $f(u) \approx f(u^k) + f'(u^k)(u - u^k)$, and solve the linear problem

$$\Delta w + f'(u^k)w = \mu_1\varphi_1(x,y) + f'(u^k)u^k - f(u^k) + e(x,y) \text{ for } (x,y) \in \Omega\,,$$
$$w = 0 \text{ on } \partial\Omega$$

for μ and w by the algorithm that is described next. One can decompose the solution in the form $w(x,y) = \mu w_1(x,y) + w_2(x,y)$, where w_1 and w_2 are solutions of

$$\Delta w_1 + f'(u^k)w_1 = \varphi_1(x,y)\,, \text{ for } (x,y) \in \Omega\,, \quad w_1 = 0 \text{ on } \partial\Omega\,,$$

$$\Delta w_2 + f'(u^k)w_2 = f'(u^k)u^k - f(u^k) + e(x,y)\,, \text{ for } (x,y) \in \Omega\,, \tag{4.3}$$
$$w_2 = 0 \text{ on } \partial\Omega\,.$$

After calculating w_1 and w_2, we look for μ such that $\int_\Omega w(x,y)\varphi_1(x,y)\,dxdy = \xi_{n+1}$, and declare that value of μ to be our new iterate μ^{k+1}, so that

$$\mu^{k+1} = \frac{\xi_{n+1} - \int_\Omega w_2(x,y)\varphi_1(x,y)\,dxdy}{\int_\Omega w_1(x,y)\varphi_1(x,y)\,dxdy}\,. \tag{4.4}$$

The corresponding $w(x,y)$ is our next iterate $u^{k+1}(x,y) = \mu^{k+1}w_1(x,y) + w_2(x,y)$. The iterations are stopped once the relative error $\frac{\mu^{k+1}-\mu^k}{\mu^k}$ is small enough.

Controlling the accuracy of iterates is the major improvement of the present algorithm, compared with the one we used in [13].

Remark There is a better way of choosing the initial iterate $u^0(x,y)$ at ξ_{n+1} (corresponding to the "predictor" of the predictor-corrector method). Write $u = u(x,y,\xi)$. Approximate $u(x,y,\xi_{n+1}) \approx u(x,y,\xi_n) + u_\xi h = u_n(x,y) + u_\xi h$. Differentiate the equation 4.1) in ξ, and compare the result with the first formula in (4.3) to get: $u_\xi = \mu'(\xi_n)w_1 \approx \frac{\mu_{n+1}-\mu_n}{h}w_1$. So that we take $u^0(x,y) = u_n(x,y) + (\mu_{n+1} - \mu_n)\,w_1$, using the last function w_1 computed at ξ_n. Our experiments show considerably faster convergence.

5 Asymptotics and numerics for radial solutions

Let B denote the unit ball around the origin in R^n, $x \in R^n$ and $r = |x|$. We present computations of the solution curve $\mu_1 = \mu_1(\xi_1)$ for the radial solutions $u = u(r)$ of the model problem

$$\begin{aligned} (5.1) \quad & u''(r) + \tfrac{n-1}{r}u'(r) + \lambda_1 u + \sin u = \mu_1\varphi_1(r) + e(r)\,, \quad \text{for } 0 < r < 1 \\ & u'(0) = u(1) = 0\,, \end{aligned}$$

and derive an asymptotic formula for $\mu_1(\xi_1)$. By the Theorem 2.1 there exists a continuous solution curve $(u(r), \mu_1)(\xi_1)$ that exhausts the solution set of (5.1), and moreover $\frac{u(r)}{\xi_1} \to \varphi_1$ in $C^2(B)$ as $\xi_1 \to \infty$. (The same result holds if one replaces $e(r)$ by $e(x) \in \varphi_1^\perp$.) Restricting to radial solutions we can perform numerical computations for $n \geq 3$.) Recall that the principal eigenfunction of the Laplacian on B is $\varphi_1(r) = c_0 r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\nu_1 r)$, where ν_1 denotes the first root of the Bessel function $J_{\frac{n-2}{2}}(r)$, and c_0 is chosen so that

$$\int_B \varphi_1^2(r)\,dx = \omega_n c_0^2 \int_0^1 J^2_{\frac{n-2}{2}}(\nu_1 r)\,r\,dr = 1\,,$$

which is

$$c_0 = \frac{1}{\sqrt{\omega_n \int_0^1 J^2_{\frac{n-2}{2}}(\nu_1 r)\,r\,dr}}\,.$$

Here $\omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ gives the area of the unit ball in R^n, where Γ denotes the gamma function. The corresponding principal eigenvalue is $\lambda_1 = \nu_1^2$. It is assumed that $\iint_B e(r)\varphi_1(r)\,dx = \omega_n \int_0^1 e(r)\varphi_1(r) r^{n-1}\,dr = 0$. We decompose the solution of (3.1) as $u(r) = \xi_1\varphi_1(r) + U(r)$, with $\iint_B U(r)\varphi_1(r)\,dx = 0$, and $\xi_1 = \iint_B u(r)\varphi_1(r)\,dx$.

We now derive an asymptotic formula for the function $\mu_1(\xi_1)$, by using that $u(x) \sim \xi_1\varphi_1$ for large ξ_1. Multiplying the PDE version of the equation (5.1) by φ_1 and integrating over B

$$\begin{aligned} \mu_1 = \omega_n \textstyle\int_0^1 (\sin u(r))\,\varphi_1(r)\,r^{n-1}\,dr &\sim \omega_n \textstyle\int_0^1 (\sin \xi_1\varphi_1)\,\varphi_1(r)\,r^{n-1}\,dr \\ &= \tfrac{\omega_n}{\xi_1} \textstyle\int_0^1 \tfrac{\varphi_1(r)\,r^{n-1}}{\varphi_1'(r)} \tfrac{d}{dr}\left(-\cos \xi_1\varphi_1\right)\,dr\,. \end{aligned}$$

Integrating by parts, and denoting $f_1(r) \equiv \frac{\varphi_1(r)\,r^{n-1}}{\varphi_1'(r)}$, obtain ($\varphi_1(1) = 0$)

$$(5.2) \qquad \mu_1 \sim \frac{\omega_n}{\xi_1}\left(f_1(0)\cos \xi_1\varphi_1(0) + \int_0^1 f_1'(r)\cos \xi_1\varphi_1(r)\,dr\right)\,.$$

The next steps depend on the dimension n.

Assume that $n = 2$. Then $\omega_2 = 2\pi$, $\varphi_1(r) = c_0 J_0(\nu_1 r)$, where ν_1 is the first root of the Bessel's function $J_0(x)$, c_0 is given by (2.1), $f_1(r) = \frac{\varphi_1(r)\, r}{\varphi_1'(r)}$, and $f_1(0) = \frac{\varphi_1(0)}{\varphi_1''(0)} = -\frac{2}{\nu_1^2}$. We conclude that for large ξ_1 (using that $\varphi_1(0) = c_0$, and that the oscillatory integral in (5.2) is $o(1)$)

$$\mu_1 \sim -\frac{4\pi}{\xi_1 \nu_1^2} \cos c_0 \xi_1 \,, \tag{5.3}$$

which is consistent with (2.10) at $p = 0$.

Assume now that $n = 3$. Then $\omega_3 = 4\pi$. Since $J_{\frac{1}{2}}(x) = \frac{\sin x}{\sqrt{x}}$, we have $\nu_1 = \pi^2$, $\varphi_1(r) = \frac{1}{\sqrt{2\pi}} \frac{\sin \pi r}{r}$, $\varphi_1(0) = \sqrt{\frac{\pi}{2}}$. Also, $f_1(r) = \frac{\varphi_1 r^2}{\varphi_1'}$. It follows that $f_1(0) = 0$, so that the first term in (5.2) vanishes. Then by Lemma 3.2 (ii),

$$\begin{aligned}
\mu_1 &\sim \tfrac{4\pi}{\xi_1} \mathrm{Re} \int_0^1 f_1' e^{i\xi_1 \varphi_1}\, dr = \tfrac{4\pi}{\xi_1} \mathrm{Re}\, e^{i\left(\xi_1 \varphi_1(0) - \frac{\pi}{4}\right)} \sqrt{\tfrac{\pi}{2\xi_1 |\varphi_1''(0)|}} f_1'(0) \\
&= \tfrac{4\pi}{\xi_1} \cos\left(\xi_1 \sqrt{\tfrac{\pi}{2}} - \tfrac{\pi}{4}\right) \sqrt{\tfrac{\pi}{2\xi_1 |\varphi_1''(0)|}} f_1'(0) \,.
\end{aligned}$$

As above, calculate $3\varphi_1''(0) = -\pi^2 \varphi_1(0)$, so that $|\varphi_1''(0)| = \frac{\pi^{\frac{5}{2}}}{3\sqrt{2}}$. A short calculation shows that

$$f_1'(0) = \frac{\varphi_1(0)}{\varphi_1''(0)} = -\frac{3}{\pi^2} \,.$$

We conclude that for large ξ_1

$$\mu_1 \sim -\frac{12\sqrt{3\sqrt{2}}}{\sqrt{2}\xi_1^{\frac{3}{2}} \pi^{\frac{7}{4}}} \cos\left(\xi_1 \sqrt{\frac{\pi}{2}} - \frac{\pi}{4}\right) \,. \tag{5.4}$$

For numerical computations of radial solutions we used the algorithm described in a previous section. Computations can be performed accurately in any dimension, because the linearization of (5.1) involves a boundary value problem for an ODE, readily handled by *Mathematica*. The formula (4.4) takes the form

$$\mu^{k+1} = \frac{\xi_{n+1} - \omega_n \int_0^1 w_2(r) \varphi_1(r)\, r^{n-1}\, dr}{\omega_n \int_0^1 w_1(r) \varphi_1(r)\, r^{n-1}\, dr} \,.$$

Example For $n = 3$ we solved the problem (5.1) with $e(r) = \frac{\cos \pi r}{r}$ (observe that $\int_B e(r) \varphi_1(r)\, dx = 0$), see Figure 6. The solution of (5.1) (solid line) is in a good agreement with the asymptotic formula (5.4) (dashed line).

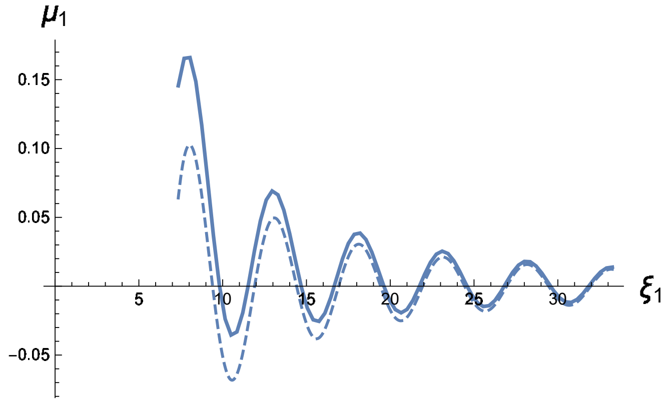

Figure 6: The solution curve $\mu_1 = \mu_1(\xi_1)$ of (5.1), $n = 3$, compared with the asymptotic formula (5.4)

6 Using μ-curves for algebraic systems

The Calculus prescription for finding the extrema of a function $F(x, y)$ involves solving the system of equations

$$F_x(x, y) = 0$$
$$F_y(x, y) = 0\,.$$

However, there are no general methods for solving nonlinear systems! In fact, for complicated $F(x, y)$ it may be easier to look for the extrema by direct minimization (maximization).

More generally, let us consider the system

$$f(x, y) = 0 \qquad (6.1)$$
$$g(x, y) = 0\,.$$

We shall show that under some conditions it is possible to compute *all solutions of (6.1)*, in any computer window, if one solution $x = \alpha$, $y = \beta$ is known.

Theorem 6.1 *Assume that the differentiable functions $f(x, y)$ and $g(x, y)$ satisfy*

(i) $f(\alpha, \beta) = g(\alpha, \beta) = 0$, *at some point* (α, β),

(ii) $g_y(x, y) \neq 0$ *for all* (x, y) ,

(iii) There exists a constant $N > 0$, so that for each x

$$g(x, y) \neq 0 \quad \textit{for } |y| > N.$$

Then all solutions of (6.1) can be computed by a curve following algorithm, beginning with $x = \alpha$, $y = \beta$.

Proof: Embed the system (6.1) into

$$\begin{aligned} f(x, y) &= \mu \\ g(x, y) &= 0\,, \end{aligned} \tag{6.2}$$

where μ is an extra variable. We shall regard x as a parameter, and show that all solutions of (6.2) lie on a single continuous curve $(y, \mu)\,(x)$. This curve can be broken into two sections, the curves $\mu = \mu(x)$ and $y = y(x)$. If at some x_0, one has $\mu(x_0) = 0$, then $(x_0, y(x_0))$ gives a solution of (6.1). By the implicit function theorem we can solve for y from the second equation $y = \varphi(x)$, and then obtain $\mu = f(x, \varphi(x))$ from the first equation. Hence the solution $x = \alpha$, $y = \beta$, $\mu = 0$ of (6.2) can be continued locally. Along this solution curve, it follows by the condition (iii) that $|y| < N$ for any fixed x, so that $|\mu|$ is also bounded, and hence one can continue the curve for all $x \in (-\infty, \infty)$. All solutions of (6.2) lie on this curve, in view of the condition (ii). $\diamondsuit$

Numerical algorithm. Choose a step size h, and let $x_n = nh$. Then y_n is obtained by solving

$$g(x_n, y) = 0\,,$$

followed by $\mu_n = f(x_n, y_n)$. If at some step N, $\mu_N \approx 0$ (within the desired tolerance), then (x_N, y_N) approximates a solution of (6.1).

Example To find the solutions of the following system (to which the Theorem 6.1 applies)

$$\begin{aligned} x \sin(x + y) + y &= 0 \\ \cos x + y + x^2 y^3 + y^7 &= 0\,, \end{aligned} \tag{6.3}$$

we computed the solution curve $\mu = \mu(x)$ for the problem

$$\begin{aligned} x \sin(x + y) + y &= \mu \\ \cos x + y + x^2 y^3 + y^7 &= 0\,, \end{aligned} \tag{6.4}$$

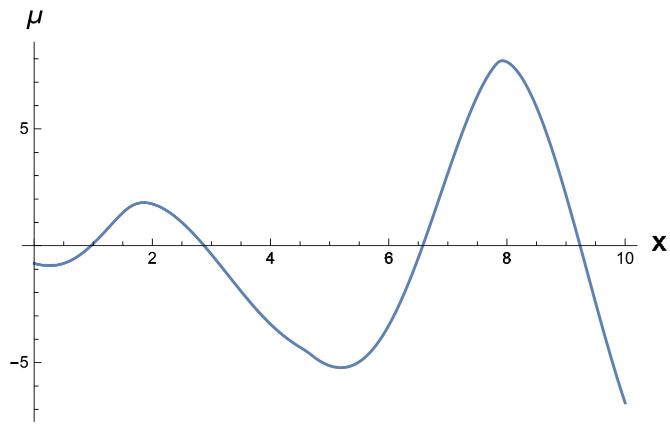

Figure 7: The solution curve $\mu = \mu(x)$ of (6.4)

using *Mathematica*. This curve is presented in Figure 7. It shows that for $x \in (0, 10)$ there are *exactly four solutions* of the system (6.3), corresponding to the points with $\mu = 0$. Up to two decimals these solutions (x, y) are $(0.96, -0.46)$, $(2.88, 0.40)$, $(6.58, -0.25)$ and $(9.24, 0.21)$. Our computations on larger intervals suggest that the system (6.3) has infinitely many solutions.

Remark The Jacobian matrix for the system (6.2) is

$$J = J_{(y,\mu)} = \begin{bmatrix} f_y & -1 \\ g_y & 0 \end{bmatrix}.$$

The condition (ii) assures that J is non-singular. Let us suppose now that the condition (ii) is not satisfied. Then the solution curve may reach a point $(\bar{x}, \bar{y})$ at which $g_y(\bar{x}, \bar{y}) = 0$, and then J is singular. There are many ways to modify the solution curve for $x > \bar{x}$ to assure that we can march forward in x. For example, if $f_y(\bar{x}, \bar{y}) \neq 0$, one can replace (6.2) by

$$\begin{aligned} f(x, y) - \mu &= 0 \\ g(x, y) - \mu &= 0\,, \end{aligned} \tag{6.5}$$

with non-singular Jacobian at $(\bar{x}, \bar{y})$, $J_{(y,\mu)} = \begin{bmatrix} f_y(\bar{x}, \bar{y}) & -1 \\ 0 & -1 \end{bmatrix}$, and now follow the solution curve of (6.5) (until another possible singular point). Of course, we are not going to get all possible solutions of (6.1) this way, in general (for example, in case there are two solutions of (6.1) with the same x). We recommend performing another calculation using now y as a parameter, to catch other solutions.

We remark that homotopy methods for solving nonlinear systems go back at least as far as D.F. Davidenko [3]. In the conditions of the Theorem 6.1 it is possible to compute *all solutions*, for any interval along the x-axis.

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