

Infinitely many solutions for three classes of self-similar equations, with the p -Laplace operator: Gelfand, Joseph-Lundgren and MEMS problems

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Abstract

We study the global solution curves, and prove the existence of infinitely many positive solutions for three classes of self-similar equations, with p -Laplace operator. In case $p = 2$, these are well-known problems involving the Gelfand equation, the equation modeling electrostatic micro-electromechanical systems (MEMS), and a polynomial nonlinearity. We extend the classical results of D.D. Joseph and T.S. Lundgren [11] to the case $p \neq 2$, and we generalize the main result of Z. Guo and J. Wei [9] on the equation modeling MEMS.

Key words: Parameterization of the global solution curves, infinitely many solutions.

AMS subject classification: 35J60, 35B40.

1 Introduction

We consider radial solutions on a ball in R^n for three special classes of equations, involving the p -Laplace operator, the ones self-similar under scaling. We now explain our approach for one of the classes, involving the p -Laplace version of the equation which arises in modeling of electrostatic micro-electromechanical systems (MEMS), see [16], [8], [9] (with $p > 1$, $\alpha > 0$, $q > 0$, $u = u(x)$, $x \in R^n$, $n \geq 1$)

$$(1.1) \quad \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + \lambda \frac{|x|^\alpha}{(1-u)^q} = 0, \text{ for } |x| < 1 \quad u = 0, \text{ when } |x| = 1.$$

Here λ is a positive parameter. We are looking for solutions satisfying $0 < u < 1$. Radial solutions of this equation satisfy

$$(1.2) \quad \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda \frac{r^\alpha}{(1-u)^q} = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1,$$

with $\varphi(v) = v|v|^{p-2}$. It is easy to see that $u'(r) < 0$ for all $0 < r < 1$, which implies that the value of $u(0)$ gives the maximum value (or the L^∞ norm) of our solution. Moreover, $u(0)$ is a *global parameter*, i.e., it uniquely identifies the solution pair $(\lambda, u(r))$, see e.g., P. Korman [13]. It follows that a two-dimensional curve in the $(\lambda, u(0))$ plane completely describes the solution set of (1.2). The self-similarity of this equation allows one to parameterize the global solution curve, using the solution of a single initial value problem:

$$(1.3) \quad \varphi(w')' + \frac{n-1}{t} \varphi(w') = \frac{t^\alpha}{w^q}, \quad w(0) = 1, \quad w'(0) = 0.$$

Its solution $w(t)$ is a positive and increasing function, which can be easily computed numerically. Following J.A. Pelesko [16], we show that the global solution curve of (1.2) is given by

$$(\lambda, u(0)) = \left(\frac{t^{\alpha+p}}{w^{p+q-1}(t)}, 1 - \frac{1}{w(t)} \right),$$

parameterized by $t \in (0, \infty)$. In particular, $\lambda = \lambda(t) = \frac{t^{\alpha+p}}{w^{p+q-1}(t)}$, and

$$\lambda'(t) = t^{\alpha+p-1} w^{-p-q} [(\alpha+p)w - t(p+q-1)w'],$$

so that the solution curve travels to the right (left) in the $(\lambda, u(0))$ plane if $(\alpha+p)w - t(p+q-1)w' > 0$ (< 0). This makes us interested in the roots of the function $(\alpha+p)w - t(p+q-1)w'$. If we set this function to zero

$$(\alpha+p)w - t(p+q-1)w' = 0,$$

then the general solution of this equation is

$$w(t) = ct^\beta, \quad \beta = \frac{\alpha+p}{p+q-1}.$$

Quite remarkably, if we choose the constant $c = c_0 = \left[\frac{1}{\beta^{p-1}[(p-1)(\beta-1)+n-1]} \right]^{\frac{1}{p+q-1}}$ then

$$w_0(t) = c_0 t^\beta$$

also solves the equation in (1.3), along with $w(t)$. We show that $w(t)$ tends to $w_0(t)$ as $t \rightarrow \infty$, and the solution curve of (1.2) makes infinitely many turns if and only if $w(t)$ and $w_0(t)$ intersect infinitely many times. We give a sharp condition for that to happen, thus generalizing the main result in Z. Guo and J. Wei [9] to the case of $p \neq 2$ (with a simpler proof). In [12] we called $w(t)$ *the generating solution*, and $w_0(t)$ *the guiding solution*.

We apply a similar approach to a class of equations with polynomial $f(r, u)$ generalizing the well-known results of D.D. Joseph and T.S. Lundgren [11], and to the p -Laplace version of the generalized Gelfand equation, where we easily recover the corresponding result of J. Jacobsen and K. Schmitt [10].

For each of the three classes of equations we show that along the solution curves (as $u(0) \rightarrow \infty$), the solutions tend to a singular solution (for which $u(r) \rightarrow \infty$, or $u'(r) \rightarrow \infty$, as $r \rightarrow 0$). Moreover, one can calculate the singular solutions explicitly, which is truly a remarkable feature of self-similar equations. Singular solutions were studied previously by many authors, including C. Budd and J. Norbury [3], F. Merle and L. A. Peletier [15], and I. Flores [6].

2 Parameterization of the solution curves

We begin with the p -Laplace version of the generalized Gelfand equation

$$(2.1) \quad \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda r^\alpha e^u = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = 0, \quad u(1) = 0,$$

where $\varphi(v) = v|v|^{p-2}$, $p > 1$. Observe that $\varphi(sv) = s^{p-1}\varphi(v)$ for any constant $s > 0$. Assume that $u(0) = a > 0$. We set $u = w + a$, $t = br$. The constants a and b are assumed to satisfy

$$\lambda = b^{\alpha+p} e^{-a}.$$

Then (2.1) becomes

$$(2.2) \quad \varphi(w')' + \frac{n-1}{t} \varphi(w') + t^\alpha e^w = 0, \quad w(0) = 0, \quad w'(0) = 0.$$

The solution of this problem $w(t)$, which is a negative and decreasing function, is defined for all $t > 0$, and it may be easily computed numerically. (Write this equation as $[t^{n-1}\varphi(w')] = -t^{n+\alpha-1}e^w < 0$, and conclude that $t^{n-1}\varphi(w') < 0$, and then $w'(t) < 0$ for all t .) We have

$$0 = u(1) = a + w(b),$$

so that $a = -w(b)$, and then $\lambda = b^{\alpha+p}e^{w(b)}$. The solution curve for (2.1) is

$$(\lambda, u(0)) = \left(b^{\alpha+p}e^{w(b)}, -w(b) \right),$$

parameterized by $b \in (0, \infty)$. The solution of (2.1) at b is $u(r) = w(br) - w(b)$. It will be convenient to write the solution curve as

$$(2.3) \quad (\lambda, u(0)) = \left(t^{\alpha+p}e^{w(t)}, -w(t) \right),$$

parameterized by $t \in (0, \infty)$, and $w(t)$ is the solution of (2.2). The solution of (2.1) at the parameter value t is $u(r) = w(tr) - w(t)$.

We consider next the problem

$$(2.4) \quad \varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda \frac{r^\alpha}{(1-u)^q} = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1,$$

which arises in modeling of electrostatic micro-electromechanical systems (MEMS), see [16], [8], [9]. Here λ is a positive parameter, $q > 0$ and $\alpha > 0$ are constants, and as before $\varphi(v) = v|v|^{p-2}$, $p > 1$. Any solution $u(r)$ of (2.4) is a positive and decreasing function (by the maximum principle), so that $u(0)$ gives its maximum value. Our goal is to compute the solution curve $(\lambda, u(0))$. Let $1 - u = v$. Then $v(r)$ satisfies

$$(2.5) \quad \varphi(v')' + \frac{n-1}{r}\varphi(v') = \lambda \frac{r^\alpha}{v^q} \quad \text{for } 0 < r < 1, \quad v'(0) = 0, \quad v(1) = 1.$$

Assume that $v(0) = a$. We scale $v(r) = aw(r)$, and $t = br$. The constants a and b are assumed to satisfy

$$(2.6) \quad \lambda = a^{p+q-1}b^{\alpha+p}.$$

Then (2.5) becomes

$$(2.7) \quad \varphi(w')' + \frac{n-1}{t}\varphi(w') = \frac{t^\alpha}{w^q}, \quad w(0) = 1, \quad w'(0) = 0.$$

The solution of this problem is a positive increasing function, which is defined for all $t > 0$. We have

$$1 = v(1) = aw(b),$$

and so $a = \frac{1}{w(b)}$, and then $\lambda = \frac{b^{\alpha+p}}{w^{p+q-1}(b)}$. The solution curve $(\lambda, u(0))$ is $\left(\frac{b^{\alpha+p}}{w^{p+q-1}(b)}, 1 - \frac{1}{w(b)}\right)$, parameterized by $b \in (0, \infty)$. It will be convenient to write the solution curve in the form

$$(2.8) \quad (\lambda, u(0)) = \left(\frac{t^{\alpha+p}}{w^{p+q-1}(t)}, 1 - \frac{1}{w(t)} \right),$$

parameterized by $t \in (0, \infty)$. In case $p = 2$, this parameterization was first derived by J.A. Pelesko [16], and was then used in [8]. The solution of (2.4) at t is $u(r) = 1 - \frac{w(tr)}{w(t)}$.

Finally, we consider the problem (with the constants $p > 1, q > 1, \alpha \geq 0$)

$$(2.9) \quad \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda r^\alpha (1+u)^q = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0,$$

which was analyzed in case $p = 2$ and $\alpha = 0$ by D.D. Joseph and T.S. Lundgren [11]. If we set $1+u = v$, then $v(r)$ satisfies

$$(2.10) \quad \varphi(v')' + \frac{n-1}{r} \varphi(v') + \lambda r^\alpha v^q = 0, \quad v'(0) = 0, \quad v(1) = 1.$$

Assuming that $v(0) = a$, we scale $v(r) = aw(r)$, and $t = br$. The constants a and b are assumed to satisfy

$$(2.11) \quad \lambda = \frac{b^{p+\alpha}}{a^{q-p+1}}.$$

Then (2.10) becomes

$$(2.12) \quad \varphi(w')' + \frac{n-1}{t} \varphi(w') + t^\alpha w^q = 0, \quad w(0) = 1, \quad w'(0) = 0.$$

The solution of (2.12) satisfies $w'(t) < 0$, so long as $w(t) > 0$ (the function $t^{n-1} \varphi(w'(t))$ is zero at $t = 0$, and its derivative is negative). It follows that either there is a t_0 , so that $w(t_0) = 0$ and $w(t) > 0$ on $(0, t_0)$, or $w(t) > 0$ on $(0, \infty)$ and $\lim_{t \rightarrow \infty} w(t) = a \geq 0$. It is easy to see that $a = 0$ in the second case. Indeed, assuming that $a > 0$, we have $[t^{n-1} \varphi(w')] \leq -a^q t^{n+\alpha-1}$, and integrating we conclude that $w(t) \leq 1 - ct^\gamma$, with some $c > 0$, and $\gamma = \frac{\alpha+p}{p-1} > 0$, contradicting that $w(t) > 0$ on $(0, \infty)$.

Lemma 2.1 *Assume that*

$$(2.13) \quad q > \frac{np - n + p + p\alpha}{n - p}.$$

Then $w(t) > 0$, and $w'(t) < 0$ on $(0, \infty)$, with $\lim_{t \rightarrow \infty} w(t) = 0$.

Proof: In view of the above remarks, we need to exclude the possibility that $w(t_0) = 0$ and $w(t) > 0$ on $(0, t_0)$. Recall that for the equation

$$\varphi(w')' + \frac{n-1}{t}\varphi(w') + f(t, w) = 0,$$

the Pohozaev function

$$P(t) = t^n [(p-1)\varphi(w')w' + pF(t, w)] + (n-p)t^{n-1}\varphi(w')w$$

is easily seen to satisfy

$$P'(t) = t^{n-1} [npF(t, w) - (n-p)wf(t, w) + ptF_t(t, w)],$$

where $F(t, w) = \int_0^w f(t, z) dz$, see e.g., [13], p. 136. Here

$$P'(t) = t^{n-1+\alpha} \left[\frac{np}{q+1} - (n-p) + \frac{p\alpha}{q+1} \right] w^{q+1} < 0.$$

Since $P(0) = 0$, and $P(t_0) > 0$, we have a contradiction. \diamond

As before, we have

$$1 = v(1) = aw(b),$$

and so $a = \frac{1}{w(b)}$, and then $\lambda = b^{p+\alpha}w^{q-p+1}(b)$. Under the condition (2.13), the solution curve $(\lambda, u(0))$ is $\left(b^{p+\alpha}w^{q-p+1}(b), \frac{1}{w(b)} - 1\right)$, parameterized by $b \in (0, \infty)$. The solution at b is $u(r) = \frac{w(br)}{w(b)} - 1$. It will be convenient to write the solution curve in the form

$$(2.14) \quad (\lambda, u(0)) = \left(t^{p+\alpha}w^{q-p+1}(t), \frac{1}{w(t)} - 1\right),$$

parameterized by $t \in (0, \infty)$. The solution of (2.9) at t is $u(r) = \frac{w(tr)}{w(t)} - 1$.

3 The equation modeling MEMS

We consider the problem (2.4), whose solution curve is given by (2.8), where $w(t)$ is the solution of (2.7). We have $\lambda(t) = \frac{t^{\alpha+p}}{w^{p+q-1}(t)}$, where $w(t)$ is the solution of (2.7), and so

$$\lambda'(t) = t^{\alpha+p-1}w^{-p-q}[(\alpha+p)w - t(p+q-1)w'].$$

We are interested in the roots of the function $(\alpha+p)w - t(p+q-1)w'$. If we set this function to zero

$$(\alpha+p)w - t(p+q-1)w' = 0,$$

then the general solution of this equation is

$$w(t) = ct^\beta, \quad \beta = \frac{\alpha + p}{p + q - 1}.$$

Quite remarkably, if we choose the constant $c = c_0 = \left[\frac{1}{\beta^{p-1}[(p-1)(\beta-1)+n-1]} \right]^{\frac{1}{p+q-1}}$, under the condition that

$$(3.1) \quad (p-1)(\beta-1) + n - 1 > 0,$$

then

$$w_0(t) = c_0 t^\beta$$

also solves the equation in (2.7), along with $w(t)$. We shall show that $w(t)$, the solution of the initial value problem (2.7), tends to $w_0(t)$ as $t \rightarrow \infty$, and the issue turns out to be whether $w(t)$ and $w_0(t)$ cross infinitely many times as $t \rightarrow \infty$.

Lemma 3.1 *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.4) makes infinitely many turns.*

Proof: Assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote the points of intersection. At $\{t_n\}$'s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial value problems). Since $(\alpha + p)w_0(t_n) - t_n(p + q - 1)w'_0(t_n) = 0$, it follows that $(\alpha + p)w(t_n) - t_n(p + q - 1)w'(t_n) < 0$ (> 0) if $w(t)$ intersects $w_0(t)$ from below (above) at t_n . Hence, on any interval (t_n, t_{n+1}) there is a point t_0 , where $(\alpha + p)w(t_0) - t_0(p + q - 1)w'(t_0) = 0$, i.e., $\lambda'(t_0) = 0$, and t_0 gives a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}) . \diamond

We shall need the following well-known Sturm-Picone's comparison theorem, see e.g., p. 5 in [14].

Lemma 3.2 *Let $u(t)$ and $v(t)$ be respectively classical solutions of*

$$(3.2) \quad (a(t)u')' + b(t)u = 0,$$

$$(3.3) \quad (a_1(t)v')' + b_1(t)v = 0.$$

Assume that the given differentiable functions $a(t)$, $a_1(t)$, and continuous functions $b(t)$ and $b_1(t)$, satisfy

$$(3.4) \quad b_1(t) \geq b(t), \quad \text{and} \quad 0 < a_1(t) \leq a(t) \quad \text{for } t \geq t_0 > 0.$$

In case $a_1(t) = a(t)$ and $b_1(t) = b(t)$ for all t , assume additionally that $u(t)$ and $v(t)$ are not constant multiples of one another. Then, for $t \geq t_0$, $v(t)$ has a root between any two consecutive roots of $u(t)$.

Lemma 3.3 *Consider the equation*

$$(3.5) \quad (a_0(t) (1 + f(t)) v')' + \frac{n-1}{t} a_0(t) (1 + f(t)) v' + b_0(t) (1 + g(t)) v = 0,$$

with given differentiable functions $a_0(t) > 0$ and $f(t)$, and continuous functions $b_0(t) > 0$ and $g(t)$. Assume that $\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} g(t) = 0$, and there is an $\epsilon > 0$ such that any solution of

$$(3.6) \quad (a_0(t) (1 + \epsilon) v')' + \frac{n-1}{t} a_0(t) (1 + \epsilon) v' + b_0(t) (1 - \epsilon) v = 0$$

has infinitely many roots. Then any solution of (3.5) has infinitely many roots.

Proof: We rewrite (3.5) in the form (3.2), with $a(t) = t^{n-1} a_0(t) (1 + f(t))$, and $b(t) = t^{n-1} b_0(t) (1 + g(t))$, and we rewrite (3.6) in the form (3.3), with $a_1(t) = t^{n-1} a_0(t) (1 + \epsilon)$, and $b_1(t) = t^{n-1} b_0(t) (1 - \epsilon)$. For large t , the inequalities in (3.4) hold, and the Lemma 3.2 applies. \diamond

The linearized equation for (2.7) is

$$(\varphi'(w') z')' + \frac{n-1}{t} \varphi'(w') z' = -q t^\alpha w^{-q-1} z.$$

At the solution $w = w_0(t)$, this becomes

$$(3.7) \quad (a_0(t) z')' + \frac{n-1}{t} a_0(t) z' + b_0(t) z = 0,$$

with $a_0(t) = \varphi'(w'_0) = (p-1) c_0^{p-2} \beta^{p-2} t^{(p-2)(\beta-1)}$, and $b_0(t) = q t^\alpha w_0^{-q-1} = q c_0^{-q-1} t^{\alpha-\beta(q+1)}$. One simplifies (3.7) to read

$$z'' + \frac{[(p-2)(\beta-1) + n-1]}{t} z' + \frac{q\beta[(p-1)(\beta-1) + n-1]}{(p-1)t^2} z = 0,$$

which is an Euler equation! The roots of its characteristic equation,

$$r(r-1) + [(p-2)(\beta-1) + n-1] r + \frac{q\beta[(p-1)(\beta-1) + n-1]}{(p-1)} = 0,$$

are complex valued, provided that

$$[(p-2)(\beta-1) + n-2]^2 < \frac{4q\beta[(p-1)(\beta-1) + n-1]}{p-1}.$$

We write this inequality in the form

$$(3.8) \quad A\beta^2 + B\beta - C > 0,$$

with $A = 4(p-1)q - (p-1)(p-2)^2$, $B = 4q(n-p) - 2(p-1)(p-2)(n-p)$, and $C = (p-1)(n-p)^2$. We shall have $A > 0$, provided that

$$(3.9) \quad 4q - (p-2)^2 > 0.$$

For (3.8) to hold, we need $\beta = \frac{\alpha+p}{p+q-1}$ to be greater than the larger root of this quadratic, i.e., $\beta > \frac{-B+\sqrt{B^2+4AC}}{2A}$ (assuming (3.9)), which gives

$$(3.10) \quad \frac{\alpha+p}{p+q-1} > \frac{(p-n)(2q-p^2+3p-2) + 2|n-p|\sqrt{q(p+q-1)}}{(p-1)[4q-(p-2)^2]}.$$

Theorem 3.1 *Assume that $q > 0$, $p > 1$, with*

$$(3.11) \quad (p-1)(\beta-1) + n-1 > \beta,$$

and the conditions (3.9) and (3.10) hold. Then the solution curve of

$$(3.12) \quad \varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda \frac{r^\alpha}{(1-u)^q} = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1$$

makes infinitely many turns. Moreover, along this curve (as $u(0) \rightarrow \infty$), $\lambda \rightarrow \lambda_0 = \frac{1}{c_0^{q-1}} = \beta^{p-1}[(p-1)(\beta-1) + n-1]$, and $u(r)$ tends to $1-r^\beta$ for $r \neq 0$, which is a solution of the equation in (3.12).

Proof: In view of Lemma 3.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(3.13) \quad (a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0,$$

where

$$(3.14) \quad a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w_0'(t)) \, ds,$$

$$(3.15) \quad b(t) = q t^\alpha \int_0^1 \frac{1}{[sw(t) + (1-s)w_0(t)]^{q+1}} \, ds.$$

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞) , and tend to a constant as $t \rightarrow \infty$. Assuming the contrary, write

$$a(t) = (p-1) (w'_0)^{p-2} \int_0^1 \left| s \frac{w'(t)}{w'_0(t)} + (1-s) \right|^{p-2} ds = a_0(t) (1 + o(1)) ,$$

$$b(t) = q t^\alpha \frac{1}{w_0^{q+1}} \int_0^1 \frac{1}{\left[s \frac{w(t)}{w_0(t)} + (1-s) \right]^{q+1}} ds = b_0(t) (1 + o(1)) .$$

as $t \rightarrow \infty$. (Observe that $\frac{w(t)}{w_0(t)} \rightarrow 1$, since $P(t)$ tends to a constant, and $\frac{w'(t)}{w'_0(t)} \rightarrow 1$, by L'Hospital's rule, as $t \rightarrow \infty$.) Since Euler's equation (3.7) has infinitely many roots on (t_0, ∞) , we conclude by Lemma 3.3 that $P(t)$ must vanish on that interval too, a contradiction.

Next we show that if $P(t_0) = 0$, then $P(t)$ remains bounded for all $t > t_0$. Assume that $P'(t_0) < 0$, and the case when $P'(t_0) > 0$ is similar. Then $P(t) < 0$ for $t > t_0$, with $t - t_0$ small. From (3.13), $t^{n-1}a(t)P'(t)$ is increasing for $t > t_0$, so that

$$P'(t) > -\frac{a_0}{a(t)t^{n-1}}, \quad \text{for } t > t_0 \quad (\text{with } a_0 = -t_0^{n-1}a(t_0)P'(t_0) > 0) .$$

Since solutions of the linear equation (3.13) cannot go to infinity over a bounded interval, we may assume that t_0 is large, and then by the above $a(t) \sim a_0(t) \sim a_1 t^{(p-2)(\beta-1)}$ for $t > t_0$, and some $a_1 > 0$. It follows that for some $a_2 > 0$

$$(3.16) \quad P'(t) > -\frac{a_2}{t^{n-1+(p-2)(\beta-1)}} = -\frac{a_2}{t^{1+\epsilon}}, \quad \text{for } t > t_0 ,$$

with $\epsilon = n - 2 + (p-2)(\beta-1) > 0$, in view of (3.11). Integrating over (t_0, t) , and using that $n \geq 3$, we conclude the boundness of $P(t)$, so long as $P(t) < 0$. If another root of $P(t)$ is encountered, we repeat the argument. Hence, $P(t)$ remains bounded for all $t > t_0$.

From the equation (3.13), we see that $P(t)$ cannot have points of positive minimum or points of negative maximum. We claim that if $P(t)$ has one root, it has infinitely many roots. Indeed, assume that $P(t_1) = 0$, and say $P'(t_1) > 0$. For $t > t_1$, $P(t)$ remains bounded, but cannot tend to a constant. Hence, $P(t)$ will have to turn back and become decreasing, but it cannot have a positive local minimum, or tend to a constant. Hence, $P(t_2) = 0$ at some $t_2 > t_1$, and so on.

We have $P(0) = 1$, so that $(t^{n-1}a(t)P'(t))' < 0$ for small $t > 0$. The function $q(t) \equiv t^{n-1}a(t)P'(t)$ satisfies $q(0) = 0$ and $q'(t) < 0$, and so $q(t) < 0$. It follows that $P'(t) < 0$ for small $t > 0$. Since $P(t)$ cannot turn around, or tend to a constant, we conclude the existence of the first root t_1 of $P(t)$, implying the existence of infinitely many roots.

We show next that $w(t) \rightarrow w_0(t)$ as $t \rightarrow \infty$. Let t_k and t_{k+1} be two consecutive roots of $P(t)$, and $P'(t_k) < 0$, so that $P(t) < 0$ on (t_k, t_{k+1}) . Let τ_k be the unique point of minimum of $P(t)$ on (t_k, t_{k+1}) . For negative $P(t)$ we have the inequality (3.16), with t_k in place of t_0 . Integrating this inequality over (t_k, τ_k) , we get

$$P(\tau_k) > \bar{c} \left(\tau_k^{-\epsilon} - t_k^{-\epsilon} \right) \quad (\text{with some } \bar{c} > 0),$$

which implies that $|P(\tau_k)| \rightarrow 0$, as $k \rightarrow \infty$. The case when $P'(t_k) > 0$ is similar, so that $w(t) \rightarrow w_0(t)$ along the solution curve. Since $u(r) = 1 - \frac{w(tr)}{w(t)}$, it follows that along the solution curve $u(r)$ tends to $1 - \frac{w_0(tr)}{w_0(t)} = 1 - r^\beta$, while $\lambda(t)$ tends to $\frac{1}{c_0^{q-1}}$. \diamond

Observe that in case $\beta \in (0, 1)$, the limiting solution $1 - r^\beta$ is *singular*, because $u'(0)$ is not defined. Notice also that the condition (3.11) implies (3.1). Finally, observe that in case $\beta \in (0, 1)$ the condition (3.11) implies that $n \geq 2$. Indeed, we can rewrite (3.11) as $n > 2\beta + p(1 - \beta)$, which is a point between $p > 1$, and 2.

One special case when this theorem applies is the following. Assume that $n \geq p$, so that (3.10) becomes

$$\frac{\alpha + p}{p + q - 1} > (n - p) \frac{2\sqrt{q(p + q - 1)} + p^2 - 3p + 2 - 2q}{(p - 1)[4q - (p - 2)^2]}.$$

Then (3.10) holds, provided that

$$(3.17) \quad \begin{aligned} & 2\sqrt{q(p + q - 1)} + p^2 - 3p + 2 - 2q > 0, \\ & 4q > (p - 2)^2, \\ & p \leq n < p + \frac{(\alpha + p)(p - 1)[4q - (p - 2)^2]}{(p + q - 1)(2\sqrt{q(p + q - 1)} + p^2 - 3p + 2 - 2q)}. \end{aligned}$$

Observe that the third inequality ($n \geq p$) implies that the condition (3.1) holds, and the second inequality is just (3.9). Hence, the three inequalities in (3.17) imply the theorem. In case $p = 2$, the first and the second inequalities hold automatically, while the third one gives the condition in Z. Guo and J. Wei [9].

4 The generalized Joseph-Lundgren problem

We now study the problem (2.9). Its solution curve is represented by (2.14), under the condition (2.13), where $w(t)$ is the solution of (2.12). In particular, $\lambda(t) = t^{p+\alpha} w^{q-p+1}(t)$, and we wish to know how many times this function changes the direction of monotonicity for $t \in (0, \infty)$. (Here $w(t)$ is the solution of (2.12), the generating solution.) Compute

$$\lambda'(t) = t^{p+\alpha-1} w^{q-p}(t) [(p+\alpha)w(t) + (q-p+1)tw'(t)] ,$$

so that we are interested in the roots of the function $(p+\alpha)w + (q-p+1)tw'$. If we set this function to zero

$$(p+\alpha)w + (q-p+1)tw' = 0 ,$$

then the general solution of this equation is $w(t) = at^{-\beta}$, with $\beta = \frac{p+\alpha}{q-p+1}$. If we choose the constant a as

$$a = a_0 = \left[(n-p)\beta^{p-1} - (p-1)\beta^p \right]^{\frac{1}{q-p+1}}$$

then $w_0(t) = a_0 t^{-\beta}$ is a solution of (2.12), the guiding solution (we have $(n-p)\beta^{p-1} - (p-1)\beta^p > 0$, under the condition (2.13), if $n > p$).

Lemma 4.1 *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.9) makes infinitely many turns.*

Proof: Indeed, assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote their points of intersection. At $\{t_n\}$'s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial value problems). Since $(p+\alpha)w_0(t_n) + (q-p+1)t_n w_0'(t_n) = 0$, it follows that $(p+\alpha)w(t_n) + (q-p+1)t_n w'(t_n) > 0$ (< 0) if $w(t)$ intersects $w_0(t)$ from below (above) at t_n . Hence, on any interval (t_n, t_{n+1}) there is a point t_0 , where $(p+\alpha)w(t_0) + (q-p+1)t_0 w'(t_0) = 0$, i.e., $\lambda'(t_0) = 0$, and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}) . \diamond

The linearized equation for (2.12) is

$$(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' + qt^\alpha w^{q-1}z = 0 .$$

At the solution $w = w_0(t)$, this becomes

$$(4.1) \quad (a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0 ,$$

with $a_0(t) = \varphi'(w'_0)$, and $b_0(t) = qt^\alpha w_0^{q-1}$. One simplifies (4.1) to Euler's equation

$$(4.2) \quad z'' + \frac{[-(\beta+1)(p-2) + n-1]}{t} z' + \frac{qa_0^{q-p+1}}{(p-1)\beta^{p-2}t^2} z = 0.$$

Let us consider first the case when $p = 2$ and $\alpha = 0$, and $n > 2$. Then $\beta = \frac{2}{q-1}$, $a_0 = [\beta(n-\beta-2)]^{\frac{1}{q-1}}$, and the equation (4.2) becomes

$$t^2 z'' + (n-1)tz' + q\beta(n-\beta-2)z = 0.$$

Its characteristic equation

$$r(r-1) + (n-1)r + q\beta(n-\beta-2) = 0$$

has the roots

$$r = \frac{-(n-2) \pm \sqrt{(n-2)^2 - 4q\beta(n-\beta-2)}}{2}.$$

These roots are complex if

$$(n-2)^2 - 4q\beta(n-2) + 4q\beta^2 < 0.$$

On the left we have a quadratic in $n-2$, with two positive roots. The largest value of $n-2$, for which this inequality holds, corresponds to the larger root of this quadratic, i.e.,

$$(4.3) \quad n-2 < \frac{4q}{q-1} + 4\sqrt{\frac{q}{q-1}}.$$

We shall show that infinitely many solutions occur if (4.3) holds, and

$$(4.4) \quad q > \frac{n+2}{n-2}.$$

(The last condition ensures that the generating solution $w(t)$ is defined for all $t > 0$, by Lemma 2.1.) In terms of n , the conditions (4.3) and (4.4) imply

$$(4.5) \quad \frac{2+2q}{q-1} < n < 2 + \frac{4q}{q-1} + 4\sqrt{\frac{q}{q-1}},$$

which is the condition from [11] (it implies that $n > 2$). Thus we shall recover the following classical theorem of D.D. Joseph and T.S. Lundgren [11].

Theorem 4.1 *Assume that the conditions (4.3) and (4.4) hold (or (4.5) holds). Then the solution curve of (2.9) makes infinitely many turns. Moreover, along this curve (as $u(0) \rightarrow \infty$), $\lambda \rightarrow \lambda_0 = a_0^{q-1}$, and $u(r)$ tends to $r^{-\beta} - 1$ for $r \neq 0$, which is a singular solution of the equation in (2.9).*

We shall give a proof of more general result below.

For general p and α , the characteristic equation for (4.2) is

$$(4.6) \quad r(r-1) + Ar + B = 0,$$

with $A = -\beta(p-2) + n - p + 1$, and $B = \frac{q(n-p)}{p-1}\beta - q\beta^2$. The roots of (4.6)

$$r = \frac{-(A-1) \pm \sqrt{(A-1)^2 - 4B}}{2}$$

are complex, provided that

$$(A-1)^2 - 4B < 0,$$

which simplifies to

$$(4.7) \quad (n-p)^2 - \theta(n-p) + \gamma < 0,$$

with

$$(4.8) \quad \theta = 2\beta(p-2) + \frac{4q\beta}{p-1}, \quad \gamma = (p-2)^2\beta^2 + 4q\beta^2.$$

On the left in (4.7) we have a quadratic in $n-p$, with two positive roots. The largest value of $n-p$, for which the inequality (4.7) holds, corresponds to the larger root of this quadratic, i.e.,

$$(4.9) \quad n-p < \frac{\theta + \sqrt{\theta^2 - 4\gamma}}{2}.$$

We shall show that infinitely many solutions occur if the conditions (2.13) and (4.9) hold. In terms of n , the conditions (2.13) and (4.9) imply that

$$(4.10) \quad \frac{pq + p + p\alpha}{q - p + 1} < n < p + \frac{\theta + \sqrt{\theta^2 - 4\gamma}}{2}.$$

The first inequality in (4.10) implies that

$$(4.11) \quad (\beta + 1)(p-2) < n-2,$$

which in turn gives that $n > p$.

The critical exponent in (4.9) was computed earlier in X. Cabré and M. Sanchón [4] in the context of semi-stable and extremal solutions of p -Laplace equations. That paper considered equations on general domains, and more general $f(u)$, see also [2] and [5].

Theorem 4.2 Assume that $\lim_{t \rightarrow \infty} \frac{w(t)}{w_0(t)} = 1$ (in case $p = 2$, this follows by Lemma 2.2 in [2]). Assume also that the conditions (2.13) and (4.9) hold (or (4.10) holds). Then the solution curve of

$$(4.12) \quad \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda r^\alpha (1+u)^q = 0 \quad \text{for } 0 < r < 1,$$

$$u'(0) = u(1) = 0$$

makes infinitely many turns. Moreover, along this curve (as $u(0) \rightarrow \infty$), $\lambda \rightarrow \lambda_0 = a_0^{q-1}$, and $u(r)$ tends to $r^{-\beta} - 1$ for $r \neq 0$, which is a singular solution of the equation in (4.12).

Proof: In view of Lemma 4.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times, and they tend to each other as $t \rightarrow \infty$. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(4.13) \quad (a(t)P')' + \frac{n-1}{t} a(t)P' + b(t)P = 0,$$

where

$$(4.14) \quad a(t) = \int_0^1 \varphi' (s w'(t) + (1-s) w_0'(t)) \, ds,$$

$$(4.15) \quad b(t) = q t^\alpha \int_0^1 [s w(t) + (1-s) w_0(t)]^{q-1} \, ds.$$

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞) . Assuming the contrary, write $(a_0(t)$ and $b_0(t)$ were defined in (4.1))

$$a(t) = (p-1) (-w_0')^{p-2} \int_0^1 \left| s \frac{w'(t)}{w_0'(t)} + (1-s) \right|^{p-2} \, ds = a_0(t) (1 + o(1)),$$

$$b(t) = q t^\alpha w_0^{q-1} \int_0^1 \left[s \frac{w(t)}{w_0(t)} + (1-s) \right]^{q-1} \, ds = b_0(t) (1 + o(1)).$$

as $t \rightarrow \infty$. We have $\frac{w(t)}{w_0(t)} \rightarrow 1$, and then $\frac{w'(t)}{w_0'(t)} \rightarrow 1$, by L'Hospital's rule, as $t \rightarrow \infty$. Since Euler's equation (3.7) has infinitely many solutions on (t_k, ∞) , we conclude by Lemma 3.3 that $P(t)$ must vanish on that interval too, a contradiction. It follows that $P(t)$ has infinitely many roots, which implies that $w(t)$ and $w_0(t)$ have infinitely many points of intersection, and hence the solution curve makes infinitely many turns.

Since $u(r) = \frac{w(tr)}{w(t)} - 1$, it follows that along the solution curve $u(r)$ tends to $\frac{w_0(tr)}{w_0(t)} - 1 = r^{-\beta} - 1$ for $r \neq 0$. \diamond

5 The generalized Gelfand problem

We now use the representation (2.3) for the solution curve of (2.1). In particular, $\lambda(t) = t^{\alpha+p}e^{w(t)}$, where $w(t)$ is the solution of (2.2), and the issue is how many times this function changes its direction of monotonicity for $t \in (0, \infty)$. Compute

$$\lambda'(t) = te^w (\alpha + p + tw') ,$$

so that we are interested in the roots of the function $\alpha + p + tw'$. If we set this function to zero

$$\alpha + p + tw' = 0 ,$$

then the solution of this equation is of course $w(t) = a - (\alpha + p) \ln t$. Quite surprisingly, if we choose the constant $a = a_0 = \ln [(n-p)(\alpha+p)^{p-1}]$, assuming that $n > p$, then

$$w_0(t) = \ln [(n-p)(\alpha+p)^{p-1}] - (\alpha+p) \ln t$$

is a solution of the equation in (2.2)! We shall show that $w(t)$ (the solution of the initial value problem (2.2)) tends to $w_0(t)$ as $t \rightarrow \infty$, and give a condition for $w(t)$ and $w_0(t)$ to cross infinitely many times as $t \rightarrow \infty$.

Lemma 5.1 *Assume that $w(t)$ and $w_0(t)$ intersect infinitely many times. Then the solution curve of (2.1) makes infinitely many turns.*

Proof: Indeed, assuming that $w(t)$ and $w_0(t)$ intersect infinitely many times, let $\{t_n\}$ denote the points of intersection. At $\{t_n\}$'s, $w(t)$ and $w_0(t)$ have different slopes (by uniqueness for initial value problems). Since $\alpha + p + t_n w'_0(t_n) = 0$, it follows that $\alpha + p + t_n w'(t_n) > 0$ (< 0) if $w(t)$ intersects $w_0(t)$ from below (above) at t_n . Hence, on any interval (t_n, t_{n+1}) there is a point t_0 , where $\alpha + p + t_0 w'(t_0) = 0$, i.e., $\lambda'(t_0) = 0$, and t_0 is a critical point. Since $\lambda'(t_n)$ and $\lambda'(t_{n+1})$ have different signs, the solution curve changes its direction over (t_n, t_{n+1}) . \diamond

The linearized equation for (2.2) is

$$(\varphi'(w')z')' + \frac{n-1}{t}\varphi'(w')z' + t^\alpha e^w z = 0 .$$

At the solution $w = w_0(t)$, this becomes

$$(5.1) \quad (a_0(t)z')' + \frac{n-1}{t}a_0(t)z' + b_0(t)z = 0 ,$$

with $a_0(t) = \varphi'(w'_0) = \frac{(p-1)(p+\alpha)^{p-2}}{t^{p-2}}$, and $b_0(t) = t^\alpha e^{w_0} = \frac{(n-p)(p+\alpha)^{p-1}}{t^p}$. Simplifying (5.1) gives

$$(p-1)t^2 z'' + (p-1)(n-p+1)tz' + (n-p)(p+\alpha)z = 0,$$

which is Euler's equation! Its characteristic equation

$$(p-1)r(r-1) + (p-1)(n-p+1)r + (n-p)(p+\alpha) = 0$$

has the roots

$$r = \frac{-(p-1)(n-p) \pm \sqrt{((p-1)(n-p)[p-1)(n-p) - 4(p+\alpha)]}}{2(p-1)}.$$

The roots are complex if $n-p > 0$, and the quantity in the square brackets is negative (the opposite inequalities lead to a vacuous condition), i.e., when

$$(5.2) \quad p < n < \frac{p^2 + 3p + 4\alpha}{p-1}.$$

We now easily recover the following result of J. Jacobsen and K. Schmitt [10], which was a generalization of the famous theorem of D.D. Joseph and T.S. Lundgren [11].

Theorem 5.1 *Assume that the condition (5.2) holds. Then the solution curve of*

$$(5.3) \quad \varphi(u')' + \frac{n-1}{r}\varphi(u') + \lambda r^\alpha e^u = 0 \text{ for } 0 < r < 1, \quad u'(0) = 0, \quad u(1) = 0$$

makes infinitely many turns. Moreover, along this curve (as $u(0) \rightarrow \infty$), $\lambda \rightarrow e^{a_0} = (n-p)(p+\alpha)^{p-1}$, and $u(r)$ tends to $-(p+\alpha)\ln r$ for $r \neq 0$, which is a singular solution of the equation in (5.3).

Proof: We follow the proof of the Theorem 3.1. In view of Lemma 5.1, we need to show that $w(t)$ and $w_0(t)$ intersect infinitely many times. Let $P(t) = w(t) - w_0(t)$. Then $P(t)$ satisfies

$$(5.4) \quad (a(t)P')' + \frac{n-1}{t}a(t)P' + b(t)P = 0,$$

where

$$(5.5) \quad a(t) = \int_0^1 \varphi'(sw'(t) + (1-s)w'_0(t)) \, ds,$$

$$(5.6) \quad b(t) = t^\alpha \int_0^1 e^{sw(t) + (1-s)w_0(t)} \, ds.$$

Compared with the proof of the Theorem 3.1, we have a complication here: in case $P(t)$ tends to a constant p_0 as $t \rightarrow \infty$, we cannot conclude that $b(t) = b_0(t)(1 + o(1))$, unless $p_0 = 0$.

We claim that it is impossible for $P(t)$ to keep the same sign over some infinite interval (t_0, ∞) , and tend to a constant $p_0 \neq 0$ as $t \rightarrow \infty$. Assume, on the contrary, that $P(t) > 0$ on (t_0, ∞) , and $\lim_{t \rightarrow \infty} P(t) = p_0 > 0$. We may assume that

$$(5.7) \quad P(t) > \frac{1}{2}p_0 > 0 \quad \text{on } (t_1, \infty), \text{ with some } t_1 > t_0.$$

Write (5.4) as

$$(5.8) \quad \left(t^{n-1} a(t) P' \right)' = -t^{n-1} b(t) P.$$

As before,

$$(5.9) \quad a(t) = a_0(t) (1 + f(t)), \quad \text{with } f(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Writing $b(t) = t^\alpha e^{w_0(t)} \int_0^1 e^{sP(t)} ds$, we see that

$$(5.10) \quad b(t) = b_0(t) (p_1 + g(t)),$$

with $p_1 = \int_0^1 e^{sp_0} ds > 1$, and $g(t) \rightarrow 0$ as $t \rightarrow \infty$. By (5.8), (5.7), and (5.10)

$$\left(t^{n-1} a(t) P' \right)' < -c_1 t^{n-p-1} \quad \text{on } (t_1, \infty),$$

for some constant $c_1 > 0$. Integrating this inequality over (t_1, t) , we get

$$(5.11) \quad t^{n-1} a(t) P' < c_2 - c_3 t^{n-p} \quad \text{on } (t_1, \infty),$$

for some constants $c_2 > 0$, and $c_3 > 0$ (using that $n > p$). By (5.9)

$$a(t) > c_4 t^{-p+2} \quad \text{on } (t_2, \infty),$$

for some constants $c_4 > 0$, and $t_2 > t_1$. Using this in (5.11), we have

$$P' < \frac{c_2}{c_4} t^{-n+p-1} - \frac{c_3}{c_4} t^{-1} \quad \text{on } (t_2, \infty).$$

Integrating this over (t_2, t) , and using that $n > p$

$$P(t) < c_5 + \frac{c_2}{c_4(-n+p)} t^{-n+p} - \frac{c_3}{c_4} \ln t < c_5 - \frac{c_3}{c_4} \ln t,$$

for some constant $c_5 > 0$. Hence, $P(t)$ has to vanish at some $t > t_2$, contradicting the assumption that $P(t) > 0$ on (t_0, ∞) . This proves that $p_0 = 0$. We conclude that $p_1 = 1$ in (5.10), and the rest of the proof is similar to that of Theorem 3.1. \diamond

If $p = 2$ and $\alpha = 0$, the condition (5.2) becomes $2 < n < 10$, the classical condition of D.D. Joseph and T.S. Lundgren [11].

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