Solutions Manual
Lectures on Linear Algebra and its Applications
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Chapter 1

Section 1.1

1. d. Set $z = t$, an arbitrary number. From the second equation $y = t + 3$. Substitute these expressions into the first equation

$$x - (t + 3) + 2t = 0\,,$$

so that $x = -t + 3$.

1. e. From the last equation $u = 0$. Update the system:

$$\begin{aligned} x + y - z &= 2 \\ 3y - 3z &= 3\,. \end{aligned}$$

Set $z = t$. From the second equation $y = t + 1$. Then from the first equation $x = 1$.

2. f. From the second equation subtract the first one, and from the third equation subtract twice the first one:

$$\begin{aligned} x - y + 2z &= 0 \\ y - z &= 3 \\ y - z &= 3\,. \end{aligned}$$

Discard the third equation. Set $z = t$. From the second equation $y = t + 3$. Then from the first equation $x = -t + 3$.

3. The point $(1, 0, 2)$ lying on the plane $ax + by + cz = d$ implies that $a + 2c = d$. Similarly for the other two points, giving the following three equations for the unknowns a, b, c, d

$$\begin{aligned} a \quad + 2c &= d \\ b + 5c &= d \\ 2a + b + c &= d\,. \end{aligned}$$

From the second equation subtract twice the first one:

$$\begin{aligned} a \quad + 2c &= d \\ b + 5c &= d \\ b - 3c &= -d\,. \end{aligned}$$

From the third equation subtract the second one:

$$\begin{aligned} a \quad + 2c &= d \\ b + 5c &= d \\ -8c &= -2d\,. \end{aligned}$$

While the plane through three points is unique, the equation of the plane is not. One can multiply the equation by an arbitrary number p to obtain $pax + pby + pcz = pd$. By choosing p one can make the right side of this equation to be an arbitrary number. In other words, in the equation $ax + by + cz = d$, d can be taken to be an arbitrary number. In the last system we choose a convenient $d = 4$, and obtain by back substitution $c = 1$, $b = -1$ and $a = 2$. Obtain the plane $2x - y + z = 4$.

4. Multiply the first equation by a, and the second one by 2:

$$\begin{aligned} 2ax - 3ay &= -a \\ 2ax - 12y &= 10\,. \end{aligned}$$

From the second equation subtract the first one:

$$\begin{aligned} 2ax - 3ay &= -a \\ (3a - 12)y &= 10 + a\,. \end{aligned}$$

If $3a - 12 \neq 0$, or $a \neq 4$, by back substitution one produces a unique solution. In case $a = 4$, the second equation becomes

$$0 = 14\,,$$

and the system has no solutions.

For the system to have infinitely many solutions, the second equation would need to be

$$0 = 0\,,$$

which does not happen for any a.

5. Solve for y: $y = \frac{5x-1}{3} = 2x - \frac{x+1}{3}$. Since x and y are integers, $\frac{x+1}{3}$ is an integer too. Set $\frac{x+1}{3} = n$, an integer. Then $x = 3n-1$, leading to $y = 5n-2$, where n is an arbitrary integer.

Section 1.2

Let us consider one equation with two unknowns

$$x - y = 1\,.$$

It has infinitely many solutions: $x = 2$ and $y = 1$, $x = 3$ and $y = 2$, $x = \frac{3}{2}$ and $y = \frac{1}{2}$, and so on (and on). One way to represent all solutions is to let y be arbitrary and solve for x, $x = y + 1$. A slightly different way is to let $y = t$, an arbitrary number and solve for x, $x = t + 1$.

1(a). The pivots are circled:

$$\left[\begin{array}{cc|c} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{3} & 6 \end{array}\right].$$

Restore the system:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_2 &= 6\,. \end{aligned}$$

From the second equation $x_2 = 1$. Using that in the first equation gives

$$2x_1 - 2 = 0\,,$$

so that $x_1 = 1$.

1.(b). The pivot is circled:

$$\left[\begin{array}{cc|c} \textcircled{2} & -2 & 4 \\ 0 & 0 & 0 \end{array}\right].$$

Discard the second equation. Restore the first equation

$$2x_1 - 2x_2 = 4\,.$$

Set $x_2 = t$, an arbitrary number and solve for x_2: $x_1 = t + 2$.

1(e). The pivots are circled:

$$\left[\begin{array}{rrr:r} \textcircled{1} & -1 & 1 & 3 \\ 0 & \textcircled{1} & 2 & -1 \end{array}\right].$$

Restore the system:

$$\begin{aligned} x_1 - x_2 + x_3 \quad &= 3 \\ x_2 + 2x_3 &= -1\,. \end{aligned}$$

The variable x_3 is free. Set $x_3 = t$ and arbitrary number. Then $x_2 = -2t - 1$ and then $x_1 = -3t + 2$.

1(f). The pivots are circled:

$$\left[\begin{array}{rrr:r} \textcircled{2} & -1 & 0 & 2 \\ 0 & 0 & \textcircled{1} & -4 \end{array}\right].$$

Restore the system:

$$\begin{aligned} 2x_1 - x_2 \quad &= 2 \\ x_3 &= -4\,. \end{aligned}$$

Answer. $x_1 = \dfrac{1}{2}x_2 + 1$, $x_3 = -4$, x_2 is free.

2(d). Write down the augmented matrix, then apply $R_1 \leftrightarrow R_2$ (i.e., switch the first and second rows to avoid fractions) to get

$$\left[\begin{array}{rrr:r} 1 & 2 & 1 & -1 \\ 3 & -2 & -1 & 0 \\ 1 & -6 & -3 & 2 \end{array}\right].$$

Apply $R_2 - 3R_1$ and $R_3 - R_1$:

$$\left[\begin{array}{rrr:r} 1 & 2 & 1 & -1 \\ 3 & -2 & -1 & 0 \\ 1 & -6 & -3 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{rrr:r} 1 & 2 & 1 & -1 \\ 0 & -8 & -4 & 3 \\ 0 & -8 & -4 & 3 \end{array}\right].$$

Apply $R_3 - R_2$:

$$\left[\begin{array}{rrr:r} 1 & 2 & 1 & -1 \\ 0 & -8 & -4 & 3 \\ 0 & -8 & -4 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{rrr:r} \textcircled{1} & 2 & 1 & -1 \\ 0 & \textcircled{-8} & -4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Pivot variables are x_1 and x_2, while x_3 is free. The second equation becomes

$$-8x_2 - 4t = 3\,,$$

giving $x_2 = -\frac{1}{2}t - \frac{3}{8}$. Then from the first equation

$$x_1 = -2x_2 - x_3 - 1 = -2\left(-\frac{1}{2}t - \frac{3}{8}\right) - t - 1 = -\frac{1}{4}\,.$$

2(e). Apply $R_2 - 2R_1$, followed by $R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 2 & -1 & 1 & 1 & -3 \\ 0 & 1 & 1 & -1 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -5 \\ 0 & 1 & 1 & -1 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} \textcircled{1} & -1 & 0 & 1 & 1 \\ 0 & \textcircled{1} & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Pivot variables are x_1 and x_2, while x_3 and x_4 are free. Set $x_3 = t$, $x_4 = s$, and solve for $x_2 = -t + s - 5$, $x_1 = x_2 - x_4 + 1 = -t - 4$.

3(a). Apply $R_2 - 2R_1$ and $R_3 - R_1$, followed by $R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 2 & 3 & 1 & -4 \\ 1 & 5 & 1 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 7 & 1 & -8 \\ 0 & 7 & 1 & -7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 7 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The last equation is

$$0 = 1\,.$$

The system is inconsistent.

3(c). Apply $R_2 - 2R_1$ and $R_3 - R_1$:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 3 & 1 \\ 0 & 0 & \textcircled{3} & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right].$$

The second column has no pivot, but the third one does. Then $R_3 - R_2$ gives

$$\left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 3 & 1 \\ 0 & 0 & \textcircled{3} & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The third row is discarded. The pivot variables are x_1 and x_3, while x_2 and x_4 are free. Restore the system, take x_2 and x_4 to the right, then set $x_2 = s$, $x_4 = t$:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 3x_3 - 6x_4 &= 3\,, \end{aligned}$$

$$\begin{aligned} x_1 - x_3 &= 2x_2 - 3x_4 + 1 = 2s - 3t + 1 \\ 3x_3 &= 6x_4 + 3 = 6t + 3\,. \end{aligned}$$

Then $x_3 = 2t + 1$, and $x_1 = x_3 + 2x_2 - 3x_4 + 1 = -t + 2s + 2$.

d. Apply $R_2 - 2R_1$ and $R_3 - 3R_1$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & -1 & 1 \\ 3 & -3 & 2 & 0 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & -3 & 2 \end{array}\right].$$

The second column has no pivot, but the third one does. Then $R_3 - 2R_2$ gives

$$\left[\begin{array}{cccc|c} \textcircled{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -3 & 1 \\ 0 & 0 & 0 & \textcircled{3} & 0 \end{array}\right].$$

The last equation reads

$$3x_4 = 0\,,$$

so that $x_4 = 0$. Then the second equation gives $x_3 = 1$, and from the first equation $x_1 = x_2$.

5. In case $a = 1$, the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 1 & 1 \end{array}\right].$$

Apply $R_3 - R_1$ to get

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{array}\right].$$

Then $R_3 - R_2$ gives
$$\left[\begin{array}{ccc:c} \textcircled{1} & -1 & 2 & 3 \\ 0 & \textcircled{1} & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$
Pivot variables: x_1 and x_2. Free variable x_3. From the second equation $x_2 = x_3 - 2$, and from the first equation $x_1 = -x_3 + 1$.

In case $a \neq 0$, the same process leads to
$$\left[\begin{array}{ccc:c} \textcircled{1} & -1 & 2 & 3 \\ 0 & \textcircled{1} & -1 & -2 \\ 0 & 0 & a-1 & 0 \end{array}\right].$$
Since $a - 1 \neq 0$, the system is inconsistent.

6. Each pivot occupies its own row and its own column. Therefore the maximal possible number of pivots for a $m \times n$ matrix is equal to the smaller of the numbers m and n. So that for a 5×6 matrix, the maximal possible number of pivots is 5. For a 11×3 matrix, it is 3.

Section 1.3

1. Form a system of equations with the augmented matrix $[C_1\, C_2\, C_3\, |b]$:
$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 3 & 4 \end{array}\right].$$
Apply $R_3 - R_1$:
$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right].$$
Apply $R_3 + R_2$:
$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{array}\right].$$
Perform back-substitution: $x_3 = \frac{3}{4}$, $x_2 = \frac{3}{2}$, $x_1 = \frac{1}{4}$.

2. Form a system of equations with the augmented matrix $[C_1\, C_2\, C_3\, |b]$. Solve it to get $x_1 = 0$, $x_2 = 1$ and $x_3 = 1$. It follows that $b = C_2 + C_3$.

3. Any linear combination of C_1, C_2, C_3 has the first component equal to 0, and hence it cannot be equal to b, which has the first component 5.

5. a. Form a system of equations with the augmented matrix $[C_1\, C_2\, |b]$, and determine $x_1 = 1$, $x_2 = -2$. It follows that $b = C_1 - 2C_2$, so that the vector b lies in the plane spanned by C_1 and C_2.

5. b. The system of equations with the augmented matrix $[C_1\, C_2\, |b]$ is inconsistent. It follows that the vector b does not lie in the plane spanned by C_1 and C_2.

6. a. Span of C_1, C_2, C_3 has the third component zero, while the third component of b is 1.

6. b. $b = C_1 + C_2 + C_3$, hence b is in span of C_1, C_2, C_3.

7. Vector $x \in R^4$ is a 4×1 matrix. Since A is of size 4×5, the product Ax is not defined.

8. $x \in R^8$ is an 8×1 matrix. Hence, Ax is defined, and Ax is of size 7×1, or $Ax \in R^7$.

Section 1.4

1. All three systems have the same matrix. The same sequence of row operations is used in each case. Therefore we form a "long" augmented matrix $\left[\begin{array}{c|c|c|c} A & 0 & b_1 & b_2 \end{array}\right]$ and perform the Gaussian elimination on the entire long rows. When A is reduced to the row echelon form, one restores separately each system, to perfom back substitution on each one.

Apply $R_2 - R_1$ and $R_3 - R_1$:

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & -1 & 0 & 2 & -1 \\ 1 & 2 & 0 & 0 & 3 & 0 \\ 1 & 2 & -1 & 0 & 2 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} \textcircled{1} & 2 & -1 & 0 & 2 & -1 \\ 0 & 0 & \textcircled{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

Restore separately each system. The variable x_2 is free, therefore $Ax = 0$ and $Ax = b_1$ have each infinitely many solutions. For $Ax = b_2$ the third equation says $0 = 3$, and the system is inconsistent. Indeed, the restored system for $Ax = 0$ is

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_3 &= 0\,. \end{aligned} \tag{0.1}$$

Then $x_3 = 0$, $x_1 = -2x_2$ and x_2 is free. (x_2 is pivot variable.) For the system $Ax = b_1$ get

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_3 &= 1\,. \end{aligned} \tag{0.2}$$

Then $x_3 = 1$, $x_1 = -2x_2 + 3$ and x_2 is free. The system $Ax = b_2$ is inconsistent.

2. A has at most 4 pivots, and hence at least one free variable. There are infinitely many solutions.

3. No free variables. There is only the trivial solution.

4. Solutions of non-homogeneous system $Ax = b$ can be written as $x = p+y$, where p is any particular solution of that system, and y is the general solution of the corresponding homogeneous system $Ax = 0$. We are given that y is the line of slope -3 through the origin (or a set of vectors $t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$), and $p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. It follows that $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, or the line of slope -3 through the point $(2, 1)$.

5. If x_1 and x_2 are two solutions of $Ax = b$, then $Ax_1 = b$ and $Ax_2 = b$. Subtracting

$$A\,(x_1 - x_2) = 0\,.$$

It follows that $x_1 - x_2$ is a solution of the corresponding homogeneous equation. Since the homogeneous system has only the trivial solution, conclude that $x_1 - x_2 = 0$, or $x_1 = x_2$, so that $Ax = b$ can have at most one solution.

6. a. Since x_1 and x_2 are solutions of homogeneous system, $Ax_1 = 0$ and $Ax_2 = 0$. Then

$$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0.$$

6. b. Similarly,

$$A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2 = 0 + 0 = 0\,,$$

so that $c_1x_1 + c_2x_2$ is also solution of $Ax = 0$.

7. If x and y are two solutions, $Ax = b$ and $Ay = b$. Adding:

$$A(x + y) = 2b\,.$$

Since $2b \neq b$ for $b \neq 0$, it follows that $x + y$ is not a solution of the system $Ax = b$.

8. a. True. If $Ax = b$ has trivial solution, then $A0 = b$ or $b = 0$ and the system is homegeneous.

b. True. There is one free variable.

c. False. There are two pivots. The solution set involves two arbitrary constants.

d. False. To show that a statement is false, it is enough to provide one example to the contrary. We now exhibit a system of 5 equations with 6 unknowns that is inconsistent. The first two equations are:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 5 \,. \end{aligned}$$

This system is inconsistent, since the same sum on the left cannot be equal to both 0 and 5. Add to this system three more arbitrary equations in $x_1, \ldots, x_6$. Obtain an inconsistent 5×6 system.

Section 1.5

1. a. The second vector is twice the first one. Dependent.

1. b. The second vector is not a constant multiple of the first one. Independent.

1. c. One of the vectors is the zero vector. Dependent.

1. f. Any 3 vectors in R^2 are linearly dependent.

1. k. Form a matrix using these vectors as columns, and then apply $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 4 & -1 \end{bmatrix}.$$

Perform $R_2 \leftrightarrow R_4$. ($R_2 \leftrightarrow R_3$ is also possible, but that will require another row exchange down the road.) Obtain:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, apply $R_3 + \frac{1}{4}R_2$:

$$\begin{bmatrix} \textcircled{1} & -1 & 2 \\ 0 & \textcircled{4} & -1 \\ 0 & 0 & -\frac{9}{4} \\ 0 & 0 & 0 \end{bmatrix}.$$

There are three pivots (the third one is $-\frac{9}{4}$), so that the vectors are linearly independent.

2. a. Set a linear combination of these vectors to zero

$$x_1(u_1 + u_2) + x_2(u_1 - u_2) = 0\,.$$

Rearrange:

$$(x_1 + x_2)u_1 + (x_1 - x_2)u_2 = 0\,.$$

Since u_1 and u_2 are linearly independent, it follows that

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0\,. \end{aligned}$$

The only solution of the last system is $x_1 = x_2 = 0$. The vectors $u_1 + u_2$ and $u_1 - u_2$ are linearly independent.

3. Since the vectors $u_1 + u_2$ and $u_1 - u_2$ are linearly dependent, one of them is a scalar multiple of the other, so that

$$u_1 + u_2 = a\,(u_1 - u_2)\;,$$

for some number a. Rearrange:

$$(1 - a)u_1 + (1 + a)u_2 = 0\,.$$

Since the coefficients $1 - a$ and $1 + a$ cannot be both zero, it follows that the vectors u_1 and u_2 are also linearly dependent.

4. Take a linear combination of these vectors, and set it equal to the zero vector

$$(*) \qquad x_1u_1 + x_2\,(u_1 + u_2) + x_3\,(u_1 + u_2 + u_3) + x_4\,(u_1 + u_2 + u_3 + u_4) = 0\,.$$

Rearrange:

$$(x_1 + x_2 + x_3 + x_4)\,u_1 + (x_2 + x_3 + x_4)\,u_2 + (x_3 + x_4)\,u_3 + x_4u_4 = 0\,.$$

Since the vectors u_1, u_2, u_3, u_4 are linearly independent the coefficients of the last linear combination must be all zero:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_4 &= 0\,. \end{aligned}$$

Solving this system of equations gives $x_1 = x_2 = x_3 = x_4 = 0$. Since the formula (*) holds true only when all coefficients are zero, it follows that the vectors u_1, u_1+u_2, $u_1+u_2+u_3$ and $u_1+u_2+u_3+u_4$ are linearly independent.

5. No. Consider three vectors that lie in the same plane, but no pair of them lies on the same line. Then they are linearly dependent, but linearly independent pairwise.

6. Clearly

$$1 \cdot u_1 + 1 \cdot u_2 + (-1) \cdot (u_1 + u_2) + 0 \cdot u_4 = \mathbf{0}\,,$$

and the coefficients $1, 1, (-1), 0$ are not all zero.

7. Since u_1, u_2, u_3 are linearly dependent

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0\,,$$

with a non-trivial combination of the coefficients x_1, x_2, x_3 (at least one of them is non-zero). Then for any u_4

$$x_1 u_1 + x_2 u_2 + x_3 u_3 + 0 \cdot u_4 = 0\,,$$

with a non-trivial combination of the coefficients $x_1, x_2, x_3, 0$ (at least one of them is non-zero).

8. Suppose that, on the contrary, the vectors u_1, u_2, u_3 are linearly dependent. Then

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = 0\,,$$

with at least one of the coefficients non-zero. But then

$$x_1 u_1 + x_2 u_2 + x_3 u_3 + 0 \cdot u_4 = 0\,,$$

with at least one of the coefficients non-zero. It follows that the vectors u_1, u_2, u_3, u_4 are linearly dependent, contrary to what is given.

9. Since $u_2 = 0$, obtain

$$0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 + 0 \cdot u_4 = 0\,,$$

and one of the coefficients (the second one) is non-zero. (Remark: the vectors are considered to be in R^5 to make this problem non-trivial. For example, in R^3 four vectors would be automatically linearly dependent.)

10. The formula

$$n^2 = n + n + \cdots + n$$

holds only at integer values of n, while the definition of differentiation requires that functions be defined on some interval. Hence, it is not admissible to differentiate this formula.

Chapter 2

Section 2.1

2. $3X = -I$, $X = -\frac{1}{3}I$.

3. e. and f. The matrices B are diagonal. Multiply the columns of A by the diagonal entries of B. (The first column of A is multiplied by b_{11}, the second column of A is multiplied by b_{22}, etc.)

3. g. Since B is diagonal, multiply the first column of A by 2, the second column by -1, the third column by 0 to get

$$AB = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

4. All three formulas are not true in general. The correct formulas are:

a. $(A - B)(A + B) = (A - B)A + (A - B)B = A^2 - BA + AB + B^2$.

b. $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2$.

c. $(AB)^2 = ABAB$.

If the matrices A and B commute ($BA = AB$), then indeed we have:

a. $(A - B)(A + B) = A^2 - B^2$.

b. $(A + B)^2 = A^2 + 2AB + B^2$.

c. $(AB)^2 = A^2B^2$.

5. Apply the formula $(AB)^T = B^T A^T$ to two matrices at a time:

$$(ABC)^T = (A\,(BC))^T = (BC)^T A^T = C^T B^T A^T \,.$$

6. Apply the formula $(AB)^T = B^T A^T$:

$$\left(A^2\right)^T = (AA)^T = A^T A^T = \left(A^T\right)^2 \,.$$

8. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = A^2 A = O$.

10 a. Vectors in R^n are $n \times 1$ matrices. Hence x^T is a $1 \times n$ matrix, or a row vector.

10. b. If $x \neq 0$, then at least one of its components is non-zero. Hence, $x^T x = x_1^2 + x_2^2 + \cdots + x_n^2 > 0$.

Section 2.2

2. It is $E_3(-5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

4. a., b., c. Let B be any matrix of the same size as A. Show that $AB \neq I$.

5. All of the matrices in parts a.-e. are either elementary or diagonal ones, for which we have formulas to write down inverse matrices.

5. g. Use the formula for the inverse of a 2×2 matrix to obtain

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}.$$

6. a. Apply $R_3 - R_1$:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & -4 & 1 & -1 & 0 & 1 \end{array}\right].$$

Apply $R_3 - 4R_2$ to get

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & -4 & 1 \end{array}\right].$$

Apply $-R_2$ and $-\frac{1}{3}R_3$ to get

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{array}\right].$$

Apply $R_2 + R_3$ to get

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{array}\right].$$

Apply $R_1 - 2R_2$ to get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{array}\right].$$

7. The columns of this matrix are linearly dependent. By Theorem 2.2.2, this matrix is not invertible.

8. By the definition of the square of a matrix, $(AB)^2 = ABAB$. We are given that

$$ABAB = AABB\,.$$

Multiply both sides by A^{-1} from the left:

$$BAB = ABB\,.$$

Multiply both sides by B^{-1} from the right:

$$BA = AB\,.$$

9. c. Observe that

$$E_{13}E_{24} = E_{24}E_{13}\,,$$

because it does not matter if one switches rows 1 and 3 first, and rows 2 and 4 second, or the other way around. Then

$$P^2 = E_{13}E_{24}E_{24}E_{13} = E_{13}IE_{13} = I\,,$$

because both matrices E_{24} and E_{13} are their own inverses.

11. Since $A^k = O$,

$$\left(I + A + A^2 + \cdots + A^{k-1}\right)(I - A) = I - A^k = I\,,$$

so that the matrix $I + A + A^2 + \cdots + A^{k-1}$ gives the inverse of $I - A$.

Section 2.3

1. a. $B\,(AB)^{-1}\,A = BB^{-1}A^{-1}A = IA^{-1}A = I$.

1. b. $(2A)^{-1}\,A^2 = \dfrac{1}{2}A^{-1}AA = \dfrac{1}{2}A$.

1. c. $\left[4\,(AB)^{-1}\,A\right]^{-1} = \dfrac{1}{4}\left[(AB)^{-1}\,A\right]^{-1} = \dfrac{1}{4}A^{-1}AB = \dfrac{1}{4}B$.

2. Inverses of elementary matrices are elementary matrices of the same type.

a. $E_{13}(2)^{-1} = E_{13}(-2)$.

c. $E_{13}^{-1} = E_{13}$.

3. a. The matrix A is obtained from I by switching row 2 and row 4. Therefore, $A = E_{24}$.

3. b. The matrix B is obtained from I by applying $R_4 - 5R_3$. Therefore, $B = E_{34}(-5)$.

3. c. The matrix C is obtained from I by multiplying row 4 by 7. Therefore, $C = E_4(7)$, and $C^{-1} = E_4(\frac{1}{7})$.

4. a. Restore the elementary matrices and perform multiplication from right to left: $E_{12}(-3)E_{13}(-1)E_{23}(4) = E_{12}(-3)\,[E_{13}(-1)E_{23}(4)]$. Obtain

$$E_{13}(-1)E_{23}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix},$$

by applying $R_3 - R_1$ to the second matrix. Then

$$E_{12}(-3)\,[E_{13}(-1)E_{23}(4)] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix},$$

obtained by by applying $R_2 - 3R_1$ to the second matrix.

4. b. Spell out the elementary matrices, and perform multiplication from right to left: $E_{12}E_{13}(-1)E_{23}(4) = E_{12}\,[E_{13}(-1)E_{23}(4)]$. The product of the last two matrices

$$E_{13}(-1)E_{23}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

is obtained by applying $R_3 - R_1$ to the second matrix. Then

$$E_{12}\left[E_{13}(-1)E_{23}(4)\right] = E_{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

is obtained by switching rows 1 and 2 of the second matrix.

4. e. Again, $E_3(3)E_{13}(-1)E_{12} = E_3(3)\left[E_{13}(-1)E_{12}\right]$.

$$E_{13}(-1)E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

applying $R_3 - R_1$ to the second matrix. Then

$$E_3(3)\left[E_{13}(-1)E_{12}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 3 \end{bmatrix},$$

applying $3R_3$ to the second matrix.

5. a. $R_2 - 3R_1$ takes this matrix into U, while $L = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$.

5. b. Apply $R_2 - R_1$ and $R_3 - R_1$. Then

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Apply Apply $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

Forward elimination gave U, while

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

5. e. Apply $R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 2 & 4 & 3 & 1 \\ 0 & -2 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 2 \end{bmatrix}.$$

Apply $R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, $R_4 - R_3$ gives

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

The last matrix is U, while

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$

Observe that zeroes under the diagonal correspond to row operations that were not used.

6. a. Row exchange is needed for Gaussian elimination, therefore the LU decomposition is not possible.

6. b. The multiplication by permutation matrix PA interchanges the rows of A so that no row exchanges are needed in forward elimination.

7. a. $A^{-1} = E_{23}^{-1}E_3(-2)^{-1}E_{12}(3)^{-1} = E_{23}E_3(-\frac{1}{2})E_{12}(-3)$.

7. b. Restore the 3×3 elementary matrices, and perform multiplication from right to left: $E_{23}\left(E_3(-\frac{1}{2})E_{12}(-3)\right)$. Begin with

$$E_3(-\frac{1}{2})E_{12}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix},$$

obtained by performing $-\frac{1}{2}R_3$ on the second matrix. Then

$$E_{23} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ -3 & 1 & 0, \end{bmatrix}$$

obtained by performing $R_2 \leftrightarrow R_3$ on the second matrix.

9. Taking inverses of both sides, we get an equivalent statement to prove

$$A^{-1} + B^{-1} = B^{-1}(A+B)A^{-1} \,.$$

Distributing B^{-1}, and then distributing A^{-1} on the right

$$B^{-1}(A+B)A^{-1} = (B^{-1}A + I)A^{-1} = B^{-1} + A^{-1} = A^{-1} + B^{-1} \,.$$

Section 2.4

1. a. Not a subspace, because the zero vector, with $x_1 = x_2 = 0$, does not belong to this subset of R^2.

1. b. Multiplying a vector of say length $\frac{1}{2}$ lying inside the unit sphere by say 5, produces a vector of length $\frac{5}{2}$ lying outside of the unit sphere. The subset is not closed under multiplication by a scalar. Not a subspace.

1. c. Yes, a subspace. For $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ we are given that $x_1 + x_4 = 0$. Any $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ belonging to this subset satisfies $y_1 + y_4 = 0$. Their sum $x+y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ x_4+y_4 \\ x_5+y_5 \end{bmatrix}$ also has the sum of the first and the fourth components zero:

$$x_1 + y_1 + x_4 + y_4 = x_1 + x_4 + y_1 + y_4 = 0 + 0 = 0 \,.$$

Similarly for $cx = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ cx_4 \\ cx_5 \end{bmatrix}$ one has the sum of the first and the fourth components:

$$cx_1 + cx_4 = c\,(x_1 + x_4) = 0 \,.$$

1. f. Vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ belong to this subset, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not. The subset is not closed under addition. Not a subspace.

1. g. The subset is a line through the origin, or the span of any vector going along this line. A subspace.

1. h. Zero vector belongs to the set $x = \begin{bmatrix} 0 \\ x_2 \\ x_2^2 \end{bmatrix}$ (when $x_2 = 0$, $x = \mathbf{0}$), so that we cannot quickly conclude that this set is not a subspace. However, this set is indeed not a subspace, because $2x$ does not belong to this set if $x \neq 0$.

4. a. The vectors b_1 and b_2 are linearly independent. Therefore they form a basis of R^2. To find the coordinates of e_1, solve the system with the augmented matrix

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 2 & 1 & 0 \end{array}\right]$$

to get $x_1 = \frac{1}{3}$, $x_2 = -\frac{2}{3}$.

4. b. $1b_1 + 3b_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

5. Three linearly independent vectors b_1, b_2, b_3 form a basis of R^3. The coordinates of v_1 and v_2 with respect to this basis can be calculated in parallel by working with the augmented matrix

$$\left[\begin{array}{ccc|c|c} b_1 & b_2 & b_3 & v_1 & v_2 \end{array}\right].$$

6. a. Solve the system with the augmented matrix

$$\left[\begin{array}{cc|c} b_1 & b_2 & b_3 \end{array}\right]$$

to get $x_1 = -1$, $x_2 = 1$.

7. $x = x_1 e_1 + x_2 e_2 + x_3 e_3$.

8. c. Draw the vector x in the first quadrant of the $x_1 x_2$-plane, for simplicity. Rotate x by the angle θ and reflect the result with respect to the x_1 axis. Then rotate just obtained result by the angle θ and reflect the last result with respect to the x_1 axis. Obtain x. So that $PPx = x$ for any x.

Section 2.5

1. g. To solve the system $Ax = 0$, perform $R_2 - R_1$ and $R_3 + R_1$

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 0 \\ 2 & 0 & 4 & 1 & 0 \\ -2 & -1 & -3 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} \textcircled{2} & 1 & 3 & 0 & 0 \\ 0 & \textcircled{-1} & 1 & 1 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right].$$

The variable x_3 is free, so set $x_3 = t$. Back substitution gives: $x_4 = 0$, $x_2 = t$, $x_1 = -2t$, so that $x = t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. The null space $N(A)$ is spanned by the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\dim N(A) = 1$.

1. h. $Hx = 0$ gives one equation with four unknowns

$$-x_1 + x_2 + 3x_3 = 0\,.$$

x_1 is the pivot variable, while x_2, x_3, x_4 are free. Express $x_1 = x_2 + 3x_3$, and the solution is

$$x = \begin{bmatrix} x_2 + 3x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

2. If a 4×5 matrix has two pivots, it has three free variables. The dimension of its null space is 3.

3. Since the rank is 3, there are 3 pivots. There are 4 free variables, and the dimension of the null space is 4.

4. a. The system $Ax = 0$ has only the trivial solution, so that the null space is the trivial subspace.

b. The column space is R^4 because the system $Ax = b$ has a (unique) solution for any vector $b \in R^4$.

5. There is one free variable. The null space consists of multiples of a three dimensional vector. The column space is a span of two of the columns.

6. The matrix A has at most 3 pivots (each pivot occupies its own row). Therefore, there is at least 2 free variables.

7. There are no pivots. Only the zero matrix O has this property.

8. a. The matrix is already in the row echelon form. Columns one and two have pivots, so C_1 and C_2 form a basis of the column space $C(A)$. The rank of A is 2. To express C_3, do back-substitution on

$$\left[\begin{array}{cc:c} \textcircled{-1} & 1 & -1 \\ 0 & \textcircled{2} & 4 \end{array}\right]$$

to obtain $x_2 = 2$ and $x_1 = 3$. Conclusion: $C_3 = 3C_1 + 2C_2$.

8. c. $R_2 + 3R_1$ gives

$$\begin{bmatrix} \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Only column one has pivot, and hence C_1 spans $C(A)$. Indeed, $C_2 = C_1$, and $C_3 = 3C_1$.

8. d. Apply $R_2 - R_1$ and $R_3 + 2R_1$. Obtain:

$$A = \begin{bmatrix} -1 & 2 & 5 \\ -1 & 2 & 5 \\ 2 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 4 & 8 \end{bmatrix}.$$

Apply $R_2 \leftrightarrow R_3$.

$$\begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{-1} & 2 & 5 \\ 0 & \textcircled{4} & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Span of C_1 and C_2 gives the basis of $C(A)$. To express C_3 through C_1 and C_2, do back-substitution on

$$\left[\begin{array}{cc:c} \textcircled{-1} & 2 & 5 \\ 0 & \textcircled{4} & 8 \\ 0 & 0 & 0 \end{array}\right].$$

Obtain $x_2 = 2$ and $x_1 = -1$, so that $C_3 = -C_1 + 2C_2$.

8. e. Perform $R_1 \leftrightarrow R_3$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 5 \\ -1 & 0 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{-1} & 0 & -3 \\ 0 & \textcircled{2} & 5 \\ 0 & 0 & \textcircled{1} \end{bmatrix}.$$

The columns of this matrix are linearly independent. Since any three linearly independent vectors in R^3 form a basis in R^3, it follows that $C(A) = R^3$.

8. f. Perform $R_2 - R_1$ and $R_3 + R_1$

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 2 & 0 & 4 & 1 \\ -2 & -1 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \textcircled{2} & 1 & 3 & 0 \\ 0 & \textcircled{-1} & 1 & 1 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}.$$

The column space is spanned by C_1, C_2 and C_4. To express C_3 through C_1, C_2 and C_4, do a back substitution on

$$\left[\begin{array}{ccc:c} \textcircled{2} & 1 & 0 & 3 \\ 0 & \textcircled{-1} & 1 & 1 \\ 0 & 0 & \textcircled{1} & 0 \end{array}\right].$$

Obtain $x_3 = 0$, $x_2 = -1$, $x_1 = 2$. Conclude $C_3 = 2C_1 - C_2$.

10. b. Both $N(A)$ and $C(A)$ have dimension 1, and therefore both are arbitrary multiples of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which belongs to both spaces.

10. c. Observe that $A^2 = O$. All $x \in R^2$ satisfy $Ox = 0$. Hence $N(A^2) = R^2$.

11. b. Try the matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ from the preceding exercise.

12. a. The difference of any two solutions satisfies the homogeneous system $Ax = 0$. If $C_1, C_2, \ldots, C_n$ are the columns of A, and $x_1, x_2, \ldots, x_n$ are the components of x, then $x_1C_1 + x_2C_2 + \ldots + x_nC_n = 0$. By the linear independence of the columns, $x = 0$, and hence any two solutions of $Ax = b$ are identical.

Chapter 3

Section 3.1

1. Evaluation of both determinants gives

$$2x + 3 = -x\,,$$

so that $x = -1$.

3. b. Determinant of a diagonal matrix matrix is equal to the product of the diagonal entries: $1(-2)(-3)(-4) = -24$.

3. g. Expand in the first row to get

$$a \begin{vmatrix} 0 & b \\ c & -2 \end{vmatrix} = -abc\,.$$

3. i. Expand in the third column, to take advantage of the two zeros it contains.

3. l. All entries of the third column are zero. Expanding in the third column one shows that the determinant is zero.

4. In both cases $|A^2| = |A|^2$, which is a general fact, which will be justified in the next section.

5. Expansion in the first column gives

$$|A| = (-1)^{n-1} \begin{vmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{vmatrix}\,.$$

The new $(n-1) \times (n-1)$ determinant is expanded in the first row to get

$$|A| = (-1)^{n-1}(-1)^{n-2}|I| = (-1)^{2n-3} = -1\,,$$

since the number $2n-3$ is odd.

7. All elements of the third row are zero, since $a_{ij} = 0$ for $i = 3$. Then $|A| = 0$.

8. When computing a determinant, one performs multiplications, additions, and subtractions that turn integers into integers. If all entries of the matrix are integers, its determinant is an integer. The converse statement is "if the determinant is an integer then all entries of the matrix are integers". An example of

$$\begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{vmatrix} = -2$$

proves it wrong.

Section 3.2

1. b. Perform $R_1 \leftrightarrow R_3$, followed by $R_2 - 3R_1$. After that expand in the first column.

1. g. Perform $R_2 - aR_1$ and $R_3 - a^2R_1$, then expand in the first column. Obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a)(c+a) - (b-a)(b+a)(c-a)\,.$$

Factoring $(b-a)(c-a)$ out, this simplifies to $(b-a)(c-a)(c-b)$.

2. a. Apply $R_2 - 3R_1$ to obtain

$$\begin{vmatrix} a & b & c \\ d+3a & e+3b & f+3c \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 5\,.$$

2. b. Factor 2 out of the second row to obtain

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & k \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 10\,.$$

2. c. Factor 3 out of the first row, and 2 out of the second row to obtain

$$\begin{vmatrix} 3a & 3b & 3c \\ 2d & 2e & 2f \\ g & h & k \end{vmatrix} = 6\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 30\,.$$

2. d. Apply $R_2 - 3R_1$ to obtain the determinant in part b. Answer. 10.

2. e. Perform $R_1 \leftrightarrow R_2$ to obtain

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -5\,.$$

2. f. Perform $R_2 \leftrightarrow R_3$, followed by $R_1 \leftrightarrow R_2$ to obtain

$$\begin{vmatrix} d & e & f \\ g & h & k \\ a & b & c \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = 5\,.$$

2. g. Factor -1 out of the third column.

2. h. A column of zeros makes the determinant zero.

3. a. Adding all rows to the last one, produces all entries equal to zero in the last row. The new determinant is equal to zero, and it is equal to the original one.

4. b. Since A is 4×4, $|2A| = 2^4|A| = 16 \cdot 3 = 48$.

4. c. $|B^2| = |B|^2 = \dfrac{1}{4}$.

4. f. $|2AB^{-1}| = 2^4|A||B^{-1}| = 16|A|\,\dfrac{1}{|B|} = 96$.

4. g. $|A^2(-B)^T| = |A^2|\,|(-B)^T| = |A|^2\,|(-B)| = 3^2(-1)^4|B| = \dfrac{9}{2}$.

5. $|-A| = (-1)^7|A| = -|A|$, so that $|A| = -|A|$, and then $|A| = 0$.

7. Expanding the determinant in the first column obtain a linear equation of the type

$$A + Bx + Cy = 0\,,$$

with some numbers A, B, C. This line passes through the point (a, b), because when $x = a$ and $y = b$ the determinant is zero, since the columns one and two are identical.

8. Expanding the determinant in the first column obtain a linear equation of the type

$$A + Bx + Cy + Dz = 0\,,$$

with some numbers A, B, C, D. This equation represents a plane. The point (a_1, a_2, a_3) lies on this plane, because when $x = a_1$, $y = a_2$, $z = a_3$ the determinant is zero (its first two columns are identical).

9. For B one has $R_2 = 2R_1$ (also, columns one and three are identical), so that $|B| = 0$. Then $|A^3B| = |A^3|\,|B| = 0$.

10. Apply $R_2 - 2R_1, R_3 - 2R_1, \ldots, R_n - 2R_1$. Obtain an upper triangular determinant, with the diagonal entries $1, 1, 2, 3, \ldots, n-1$. Their product is $(n-1)!$.

11. It takes $n-1$ row exchanges to put the last row back into the first position. Then use $n-2$ row exchanges to put the next to last row back into the second position. The total number of row exchanges

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n(n-1)}{2}$$

is equal to the number of sign changes of the determinant, as $|B|$ is transformed into $|A|$.

12. $|A^T| = |-A|$, implies that $|A| = (-1)^n|A|$, giving $|A| = -|A|$ since n is odd, so that $|A| = 0$.

13. b. Apply $R_n - R_{n-1}$, then $R_{n-1} - R_{n-2}$, and so on ending with $R_2 - R_1$, obtain determinant of an upper triangular matrix with all diagonal entries equal to 1.

14. If $A^2 = -I$ for some matrix A, then

$$|A^2| = |-I|\,.$$

But $|A^2| = |A|^2 \geq 0$, while for n odd, $|-I| = (-1)^n = -1 < 0$, a contradiction.

15. If rows are linearly dependent, one of them is a linear combination of the others. Suppose that the matrix is 4×4, and

$$R_4 = aR_1 + bR_2 + cR_3\,.$$

Perform the elementary operations $R_4 - aR_1$, $R_4 - bR_2$, $R_4 - cR_3$. On one hand the determinant is unchanged, and on the other hand the row 4 has all zeros, so that $|A| = 0$.

Section 3.3

1. b. $|A| = 0$, no inverse matrix exists.

1. g. Expand $|A|$ in the third row.

1. h. Use Gaussian elimination on the first column of $|A|$.

2. c. Determinant of the system is zero, so that Cramer's rule does not apply. Gaussian elimination shows that this system is inconsistent.

2. d. The second row can be discarded. The variable x_2 is free, there are infinitely many solutions.

3. a. Recall that $A \operatorname{Adj} A = |A|I$, and then

$$|A \operatorname{Adj} A| = \det\left(|A|I\right)\,.$$

On the left one has determinant of a product of two matrices, on the right determinant of a constant $|A|$ times the unit matrix I. Then

$$|A|\,|\operatorname{Adj} A| = |A|^n\,,$$

$$|\text{Adj A}| = |A|^{n-1}.$$

3. b. By part (a), $|\text{Adj}\, A| = 0$ if and only if $|A| = 0$. So that either both matrices are singular, or both are non-singular.

4. a. Determinant of a lower triangular matrix equals to the product of the diagonal entries. If one of the diagonal entries is zero, the determinant is zero, and the matrix is not invertible.

4. b. In the adjugate matrix $C_{21}, C_{31}, \ldots$ (all cofactors below the main diagonal) are determinants of triangular matrices, with one of the diagonal entries zero. It follows that $C_{21} = 0, C_{31} = 0, \ldots$, so that A^{-1} is lower triangular.

6. Since $\det A = 0$, the matrix A has fewer than n pivots. So that either the system $Ax = b$ is inconsistent, or it has infinitely many solutions, since there are free variables.

7. Write all three vectors in components, and show that both sides of each identity contain the same expressions. For Part b. observe that *vector product is not associative*, with $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ being different from $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, in general. Part c. is rather long.

8. a. A is a block-diagonal matrix, with blocks of dimensions 2×2, 2×2, and the scalar 4. Invert each block separately to obtain A^{-1}.

8. b. The first two components of the vector Ay are obtained by multiplying $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and the last three components of the vector Ay are zero. The vector Az has zeros in the first, second and and fifth components, while the third and fourth components are calculated by multiplying $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$. Similarly, $Aw = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5x_5 \end{bmatrix}$.

This example shows how the three blocks of A act separately on vectors in R^5. Other block matrices act similarly.

Chapter 4

Section 4.1

2. a. b. c. d. If a matrix is upper or lower triangular, or diagonal, then its diagonal entries give the eigenvalues.

2. e. Expand

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

to get

$$\begin{gathered} (3-\lambda)(1-\lambda) - 8 = 0\,, \\ \lambda^2 - 4\lambda - 5 = 0\,, \\ (\lambda+1)(\lambda-5) = 0\,. \end{gathered}$$

The roots (the eigenvalues) are $\lambda_1 = -1$, $\lambda_2 = 5$.

2. g. Expand

$$\begin{vmatrix} -2-\lambda & -1 & 4 \\ 3 & 2-\lambda & -5 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

in the third row to get

$$\begin{gathered} (1-\lambda)\left[(-2-\lambda)(2-\lambda)+3\right] = 0\,, \\ (1-\lambda)(\lambda^2-1) = 0\,. \end{gathered}$$

Setting the first factor to zero gives $\lambda_1 = 1$. Setting the second factor to zero gives $\lambda_2 = 1$, $\lambda_3 = -1$.

2. h. This example is covered in the text, in Section 4.2.

3. a. The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 3$.

To find eigenvectors corresponding to $\lambda_1 = -3$ we need to solve the system $(A+3I)x = 0$, or

$$\begin{aligned} 5x_1 + x_2 &= 0 \\ 5x_1 + x_2 &= 0\,. \end{aligned}$$

Discard the second equation:

$$5x_1 + x_2 = 0\,.$$

Set $x_2 = 5$, to avoid fractions, and then $x_1 = -1$. Obtained an eigenvector $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$, or any of its multiples $c \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

To find eigenvectors corresponding to $\lambda_1 = 3$ we need to solve the system $(A - 3I)x = 0$, or

$$\begin{aligned} -x_1 + x_2 &= 0 \\ 5x_1 - 5x_2 &= 0\,. \end{aligned}$$

Discard the second equation:

$$-x_1 + x_2 = 0\,.$$

Set $x_2 = 1$, and then $x_1 = 1$. Obtained an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or any of its multiples $c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. e. The eigenvalues are $2, -3, 0, 5$, the diagonal entries. The eigenvectors are e_1, e_2, e_3, e_4 the coordinate vectors. Indeed, to find eigenvectors corresponding to $\lambda_1 = 2$, one needs to solve $(A - 2I)\,x = 0$. Since

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

the corresponding system is

$$\begin{aligned} 0 &= 0 \\ -5x_2 &= 0 \\ -2x_3 &= 0 \\ 3x_4 &= 0\,. \end{aligned}$$

The solution is $x_2 = x_3 = x_4 = 0$, while $x_1 = c$, arbitrary. In the vector form $x = ce_1$. Proceed similarly to find other eigenvectors.

3. f. Building on the solution to 3. e., it follows that the eigenvalues of any $n \times n$ diagonal matrix are its diagonal entries. The eigenvectors are $e_1, e_2, \ldots, e_n$ the coordinate vectors.

3. g. The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & -2-\lambda & 1 \\ 3 & 3 & -\lambda \end{vmatrix} = 0\,.$$

Expand in the third row

$$3\begin{vmatrix} 1 & 1 \\ -2-\lambda & 1 \end{vmatrix} - 3\begin{vmatrix} 2-\lambda & 1 \\ -1 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 0\,,$$

$$3(3+\lambda) - 3(3-\lambda) - \lambda\,[(2-\lambda)(-2-\lambda)+1] = 0\,,$$

$$6\lambda - \lambda(\lambda^2 - 3) = 0\,.$$

Factor λ:

$$\lambda(9-\lambda^2) = 0\,.$$

The roots (or the eigenvalues) are $\lambda = 0$ and $\lambda = \pm 3$.

4. Sum of the eigenvalues is equal to the trace:

$$\lambda_1 + \lambda_2 = 6\,.$$

Given that $\lambda_1 = -1$, it follows that $\lambda_2 = 7$, and then

$$|A| = \lambda_1\,\lambda_2 = -7\,.$$

5. You may begin with, say $\begin{vmatrix} 2 & a \\ b & 3 \end{vmatrix}$, which has trace 5, and then choose the numbers a and b, so that the determinant is 4.

6. a. The eigenvalues of A^3 are $(-2)^3 = -8$, $1^3 = 1$, $(\frac{1}{4})^3 = \frac{1}{64}$. The determinant $|A^3|$ is their product,

$$|A^3| = (-2) \times 1 \times \frac{1}{64} = -\frac{1}{8}\,.$$

6. b. $|A| = -\frac{1}{2}$, the product of its eigenvalues. Then

$$|A^{-1}| = \frac{1}{|A|} = -2\,.$$

7. If A is invertible, so is A^{-1} (its inverse is A). Hence, A^{-1} cannot have zero eigenvalues.

8. Since A has zero eigenvalue, $|A| = 0$ ($|A|$ is the product of eigenvalues). Then $|AB| = |A||B| = 0$, therefore AB is not invertible.

9. If $Ax = \lambda x$, then $(kA)x = k\lambda x$, so that x is an eigenvector of kA, and $k\lambda$ is the corresponding eigenvalue.

10. a. Since A and A^T have identical characteristic polynomials (by the Hint), all of the eigenvalues are the same.

11. b. If $Ax = \lambda x$, then

$$(3A^2 + 5I)x = (3\lambda^2 + 5)x \,.$$

12. b. $\quad \mathrm{tr}\,(AB) = \sum_{i,j=1}^{n} a_{ij} b_{ji} = \sum_{i,j=1}^{n} b_{ji} a_{ij} .$

Here i and j are "dummy" variables of summation. Rename i to be j, and j to be i. Then

$$\sum_{i,j=1}^{n} b_{ji} a_{ij} = \sum_{i,j=1}^{n} b_{ij} a_{ji} = \mathrm{tr}\,(BA) \ .$$

12. c. Use part (b) of this problem:

$$\mathrm{tr}\ (AB - BA) = \mathrm{tr}\, I \,,$$

$$0 = n \,,$$

a contradiction, proving that the equality $AB - BA = I$ is not possible for any two matrices A and B.

13. Similar matrices have the same eigenvalues. Therefore they have the same trace, since the trace equals to the sum of eigenvalues.

14. Assume that $Ax = \lambda x$ and $Bx = \mu x$. Then

$$(AB - BA)\, x = ABx - BAx = \mu Ax - \lambda Bx = \mu\lambda x - \lambda\mu x = 0 \,.$$

It follows that x is an eigenvector of $AB - BA$, corresponding to zero eigenvalue. Hence, $|AB - BA| = 0$.

15. Add to the last row all other rows. The last row will consist of zeroes, so that $|A - bI| = 0$. Then $\lambda = b$ is a root of the characteristic equation, or an eigenvalue of A.

Section 4.2

2. b. The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 3 & 2 \\ 1 & 1-\lambda & -2 \\ -3 & -1 & -\lambda \end{vmatrix} = 0 \,.$$

Expand in the third column and simplify the first two terms:

$$2(2 - 3\lambda) + 2(\lambda + 6) - \lambda \left[(3 - \lambda)(1 - \lambda) - 3\right] = 0 \,,$$

$$-4\lambda + 16 - \lambda\left[(3-\lambda)(1-\lambda) - 3\right] = 0\,.$$

Now expand the expression in the square bracket

$$-4\lambda + 16 - \lambda(\lambda^2 - 4\lambda) = 0\,,$$

$$-4(\lambda - 4) - \lambda^2(\lambda - 4) = 0\,,$$

$$(\lambda - 4)(\lambda^2 + 4) = 0\,.$$

The roots, or the eigenvalues, are $\lambda_1 = -2i$, $\lambda_2 = 2i$, $\lambda_3 = 4$.

To find the eigenvectors corresponding to $\lambda_1 = -2i$, need to solve

$$(A + 2iI)x = 0\,,$$

with

$$A + 2iI = \begin{bmatrix} 3+2i & 3 & 2 \\ 1 & 1+2i & -2 \\ -3 & -1 & 2i \end{bmatrix}.$$

We know that the rows of this matrix are linearly dependent. The second row is not a multiple of the first, therefore the third row is a linear combination of the first two, although the exact complex coefficients are not easy to find. Therefore, discard the third equation to obtain

$$\begin{aligned} (3+2i)x_1 + 3x_2 + 2x_3 &= 0 \\ x_1 + (1+2i)x_2 - 2x_3 &= 0\,. \end{aligned}$$

Setting $x_3 = 1$ gives

$$\begin{aligned} (3+2i)x_1 + 3x_2 &= -2 \\ x_1 + (1+2i)x_2 &= 2\,. \end{aligned}$$

Use Cramer's rule: $x_1 = \frac{-8-4i}{-4+8i} = i$, $x_2 = \frac{8+4i}{-4+8i} = -i$. The eigenvectors are $\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$, and any of its multiples. The eigenvectors corresponding to $\lambda_2 = 2i$ are the complex conjugates: $c\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$.

3. The characteristic polynomial $|A - \lambda I|$ is a polynomial of degree n. If n is odd, this polynomial has at least one real root by the intermediate value theorem. (If this polynomial tends to $-\infty$ as $\lambda \to -\infty$, then it tends to ∞ as $\lambda \to \infty$.)

5. Since $\lambda_1 + \lambda_2 = \operatorname{tr} A = 2$ and $\lambda_1\lambda_2 = \det(A) = 2$, it follows that the eigenvalues are $1 \pm i$.

6. The matrix A has eigenvalues $\pm i$ and $\pm 2i$. Hence the size of A is at least 4×4.

Section 4.3

1. a. A has eigenvalues $\lambda_1 = 3$ with an eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\lambda_2 = 2$ with an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use these eigenvectors as columns to get $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the eigenvalues to form $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

1. b. $\lambda = 2$ is a double eigenvalue, but it has only one linearly independent eigenvector, namely $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the first coordinate vector in R^2. This matrix does not have a full set of eigenvectors, and therefore it is not diagonalizable.

1. d. $\lambda = 2$ is a triple eigenvalue, but it has only one linearly independent eigenvector, which is $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in R^3$. This matrix is not diagonalizable.

1. f. Verify that the columns of P, given in the answer, are the eigenvectors of A, corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$.

1. g. This matrix has a double eigenvalue $\lambda_1 = \lambda_2 = 0$ with two linearly independent eigenvectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and eigenvalue $\lambda_3 = 3$ corresponding to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This matrix is diagonalizable. Use the eigenvectors as columns to produce the diagonalizing matrix $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Then

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

1. h. $\lambda = 1$ is an eigenvalue of multiplicity four, but it has only one linearly independent eigenvector, which is $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in R^4$. This matrix is not diagonalizable.

1. i. The eigenvalues are $\lambda_1 = a$ corresponding to an eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\lambda_1 = b$ corresponding to an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

2. We have $A = PDP^{-1}$, with P and D from the preceding exercise. Then

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^k & b^k - a^k \\ 0 & b^k \end{bmatrix}.$$

3. a. Since the eigenvalues are different, the corresponding eigenvectors are linearly independent, and the matrix is diagonalizable.

3. b. $\left(\sqrt{A}\right)^2 = P \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}^2 P^{-1} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} = A.$

3. c. Diagonalize B, then $\sqrt{B} = P \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} P^{-1}$.

3. d. As in 3. a., one shows that $C^2 = A$,

4. The eigenvalues of A are 0 and 1. They are different so that A is diagonalizable. Write

$$A = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1},$$

with an appropriate diagonalizing matrix P and its inverse P^{-1}. Then

$$A^k = P \begin{bmatrix} 1^k & 0 \\ 0 & 0^k \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = A.$$

5. The eigenvalues of A are $-\frac{1}{2}$ and $\frac{1}{2}$. They are different so that A is diagonalizable. Write

$$A = P \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} P^{-1},$$

with an appropriate diagonalizing matrix P and its inverse P^{-1}. Then

$$A^k = P \begin{bmatrix} (-\frac{1}{2})^k & 0 \\ 0 & (\frac{1}{2})^k \end{bmatrix} P^{-1} \to POP^{-1} = O\,,$$

as $k \to \infty$.

6. The eigenvalues of A are distinct so that A is diagonalizable. Write

$$A = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\,,$$

with an appropriate diagonalizing matrix P and its inverse P^{-1}. Then

$$A^7 = P \begin{bmatrix} 0^7 & 0 & 0 \\ 0 & (-1)^7 & 0 \\ 0 & 0 & 1^7 \end{bmatrix} P^{-1} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} = I\,.$$

7. The eigenvalues of A are distinct so that A is diagonalizable. Write

$$A = P \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}\,,$$

with an appropriate diagonalizing matrix P and its inverse P^{-1}. Then

$$A^4 = P \begin{bmatrix} (-i)^4 & 0 & 0 & 0 \\ 0 & i^4 & 0 & 0 \\ 0 & 0 & (-1)^4 & 0 \\ 0 & 0 & 0 & 1^4 \end{bmatrix} P^{-1} = PIP^{-1} = I\,.$$

9. In the 2×2 case $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$. Then

$$q(A) = P \begin{bmatrix} q(\lambda_1) & 0 \\ 0 & q(\lambda_2) \end{bmatrix} P^{-1} = POP^{-1} = O\,,$$

since eigenvalues are roots of the characteristic equation $q(\lambda) = 0$.

Chapter 5

Section 5.1

1. e. Between any two non-parallel vectors there is an acute angle (less than $\pi/2$) and an obtuse angle (greater than $\pi/2$), and these angles add up to π. Recall also that the range of the arc cosine function is $[0, \pi]$, so that arc cosine of a negative number is an obtuse angle. Here $||y_1|| = 3$, $||y_3|| = 2$, $y_1 \cdot y_3 = -1$, $\cos\theta = -\frac{1}{6}$. The acute angle is $\pi - \arccos\left(-\frac{1}{6}\right) \approx \pi - 1.738 \approx 1.403$ in radian measure.

1. g. $\text{Proj}_{x_1} x_3 = \frac{x_1 \cdot x_3}{||x_1||^2} x_1 = -x_1$, since $x_1 \cdot x_3 = -9$ and $||x_1|| = 3$.

1. i. The vectors v_1 and v_2 are orthogonal, hence the projection of v_2 on v_1 is the zero vector.

2. $(x+y) \cdot (x-y) = x \cdot x - x \cdot y + y \cdot x - y \cdot y = ||x||^2 - ||y||^2$.

3. Vectors $x+y$ and $x-y$ give the diagonals of the parallelogram with sides x and y. If the sides are equal, $||x|| = ||y||$, then

$$(x+y) \cdot (x-y) = ||x||^2 - ||y||^2 = 0\,,$$

and the diagonals are orthogonal. Conversely, if the diagonals are orthogonal, it follows from the same formula that the sides are equal.

4. $||x+y||^2 = (x+y) \cdot (x+y) = x \cdot x + x \cdot y + y \cdot x + y \cdot y = ||x||^2 + 2x \cdot y + ||y||^2 = 16 - 2 + 9 = 23$.

5. a. Since $\cos\theta_i = \frac{x \cdot e_i}{||x||\,||e_i||} = \frac{x_i}{||x||}$, obtain

$$\cos^2\theta_1 + \cos^2\theta_2 + \cdots + \cos^2\theta_n = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{||x||^2} = 1\,.$$

5. b. In case $n = 2$, $\theta_2 = \frac{\pi}{2} - \theta_1$, so that $\cos\theta_2 = \sin\theta_1$, and the formula becomes

$$\cos^2\theta_1 + \sin^2\theta_1 = 1\,.$$

6. Consider the triangle formed by the vectors x, y and $x+y$ for the geometrical interpretation.

8. Ae_j equals to the column j of A. Taking the inner product with e_i picks out the element i of this column, which is a_{ij}.

9. a. Using the Cauchy-Schwarz inequality

$$||\text{Proj}_a b|| = ||\frac{a \cdot b}{||a||^2}\, a|| = \frac{|a \cdot b|}{||a||^2}||a|| \le \frac{||a||\,||b||}{||a||^2}||a|| = ||b||\,.$$

9. c. True:

$$\text{Proj}_{2a}\, b = \frac{2a \cdot b}{||2a||^2}\, 2a = \frac{a \cdot b}{||a||^2}\, a = \text{Proj}_a\, b\,.$$

9. b. Part 9. c. shows that $\text{Proj}_a\, b$ does not change if vector a is multiplied by any number. If this number c is chosen small, then $||\text{Proj}_{ca}\, b|| > ||ca||$.

10. Just observe the derivation in the text works for rectangular matrices as well.

Section 5.2

1. $u_1 \cdot u_2 = 0$, hence the vectors are orthogonal. They are orthonormal because $||u_1|| = 1$ and $||u_2|| = 1$. Two linearly independent vectors form a basis of R^2. To find the coordinates of e_1 and e_2 with respect to the basis $B = \{u_1, u_2\}$, form the augmented matrix

$$\left[\begin{array}{cc|c|c} u_1 & u_2 & e_1 & e_2 \end{array}\right],$$

and do Gaussian elimination on the entire long matrix. Obtain $e_1 = \frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_2$, and $e_2 = -\frac{1}{\sqrt{2}}u_1 + \frac{1}{\sqrt{2}}u_2$, so that $[e_1]_B = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $[e_2]_B = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

2. Since the vectors u_1, u_2, u_3 are orthonormal, can use the following formulas to the coordinates with respect to the basis $B = \{u_1, u_2, u_3\}$:

$$[w_1]_B = \begin{bmatrix} w_1 \cdot u_1 \\ w_1 \cdot u_2 \\ w_1 \cdot u_3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix},$$

$$[w_2]_B = \begin{bmatrix} w_2 \cdot u_1 \\ w_2 \cdot u_2 \\ w_2 \cdot u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{6}{\sqrt{2}} \end{bmatrix},$$

$$[e_2]_B = \begin{bmatrix} e_2 \cdot u_1 \\ e_2 \cdot u_2 \\ e_2 \cdot u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} \\ 0 \end{bmatrix}.$$

3. a. Any set of linearly independent vectors form a basis in the subspace that they span.

3. b. Since the vectors v_1 and v_2 are orthogonal

$$\text{Proj}_W\, b = \frac{b \cdot v_1}{||v_1||^2}\, v_1 + \frac{b \cdot v_2}{||v_2||^2}\, v_2 = \frac{3}{9}v_1 + \frac{0}{2}v_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

Since $\text{Proj}_W\, b \neq b$, b does not belong to W.

3. c. Before calculating the coordinates of w, we need to make sure that w belongs to W (so that w can be expressed through the basis of W). To this end, calcluate the projection

$$\text{Proj}_W\, w = \frac{w \cdot v_1}{||v_1||^2} v_1 + \frac{w \cdot v_2}{||v_1||^2} v_2 = -3v_1 + v_2 = w\,,$$

and hence $w \in W$. The same calculation shows that $[w]_B = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

3. d. *There is a misprint in the book.* The correct statement is " Calculate $\text{Proj}_W\, w$. Does w belong to W?"

Solution. Since w belongs to W (by part c.), $\text{Proj}_W\, w = w$.

3. e. W is the plane passing through the vectors v_1 and v_2.

f. $W^\perp$ is the straight line perpendicular to the plane W.

4. a. Since u_1, u_2, u_3 are orthonormal, they are linearly independent, and hence they form a basis of their span.

4. b. $\text{Proj}_W\, b = (b \cdot u_1)u_1 + (b \cdot u_2)u_2 + (b \cdot u_3)u_3 = -\frac{1}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3$.

5. Let $w_1, w_2, \ldots, w_k$ be some basis of W. Observe that $k \leq n$. A vector $x \in R^n$ belongs to $W^\perp$ when $w_1 \cdot x = 0, w_2 \cdot x = 0, \ldots, w_k \cdot x = 0$. So that we have a system of k equations with n unknowns to determine x. The matrix of this homogeneous system has rows $w_1^T, w_2^T, \ldots, w_k^T$. Since the rows are linearly independent, there are k pivots, and the the solution space (which is $W^\perp$) has dimension $n - k$.

6. We will show that every vector in $\left(W^\perp\right)^\perp$ belongs also to W, and conversely that any vector in W is in $\left(W^\perp\right)^\perp$.

Assume that $x \in W$. Then x is orthogonal to any vector in $W^\perp$, by the definition of $W^\perp$. Hence, $x \in \left(W^\perp\right)^\perp$.

Conversely, assume that $x \in \left(W^\perp\right)^\perp$. Decompose

$$x = \text{Proj}_W x + z\,,$$

with $z \in W^\perp$. Since x is orthogonal to $W^\perp$, $z = 0$. Then $x = \text{Proj}_W x$, which implies that $x \in W$.

7. Since the vectors $q_1, q_2, \ldots, q_k$ are orthonormal

$$||a||^2 = a \cdot a = (a_1\, q_1 + a_2\, q_2 + \cdots + a_k\, q_k) \cdot (a_1\, q_1 + a_2\, q_2 + \cdots + a_k\, q_k)$$
$$= a_1^2 + a_2^2 + \cdots + a_k^2 \,.$$

9. A^T is of size $n \times m$, and so $A^T A$ is a square $n \times n$ matrix. $A^T A$ is symmetric because

$$\left(A^T A\right)^T = A^T A \,.$$

To show that $A^T A$ is invertible, follow the Hint in the book to show that $A^T A x = 0$ implies that $x = 0$. This means that $A^T A$ has n pivots, and therefore is invertible.

10. Assume that w_1, w_2, w_3 are linearly dependent, so that $x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$ with some numbers x_1, x_2, x_3 that are not all zero. Then

$$x_1 x_2 x_3 G = \begin{vmatrix} x_1 w_1 \cdot w_1 & x_1 w_1 \cdot w_2 & x_1 w_1 \cdot w_3 \\ x_2 w_2 \cdot w_1 & x_2 w_2 \cdot w_2 & x_2 w_2 \cdot w_3 \\ x_3 w_3 \cdot w_1 & x_3 w_3 \cdot w_2 & x_3 w_3 \cdot w_3 \end{vmatrix} = \begin{vmatrix} x_1 w_1 \cdot w_1 & x_1 w_1 \cdot w_2 & x_1 w_1 \cdot w_3 \\ x_2 w_2 \cdot w_1 & x_2 w_2 \cdot w_2 & x_2 w_2 \cdot w_3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \,.$$

On the second step we added the first and the second row to the third row, producing a row of zeroes. Indeed,

$$x_1 w_1 \cdot w_1 + x_2 w_2 \cdot w_1 + x_3 w_3 \cdot w_1 = (x_1 w_1 + x_2 w_2 + x_3 w_3) \cdot w_1 = 0 \,,$$

and similarly the other two sums are zero.

Conversely, assume that the Gramian $G = 0$. Then its columns C_1, C_2, C_3 are linearly dependent, so that

$$(1) \qquad x_1 C_1 + x_2 C_2 + x_3 C_3 = 0 \,,$$

with some numbers x_1, x_2, x_3 that are not all zero. The first component of (1) is

$$x_1 w_1 \cdot w_1 + x_2 w_1 \cdot w_2 + x_3 w_1 \cdot w_3 = 0 \,,$$

or

$$(2) \qquad w_1 \cdot (x_1 w_1 + x_2 w_2 + x_3 w_3) = 0 \,.$$

Express similarly the second and the third components of (1):

$$(3) \qquad w_2 \cdot (x_1 w_1 + x_2 w_2 + x_3 w_3) = 0 \,,$$
$$(4) \qquad w_3 \cdot (x_1 w_1 + x_2 w_2 + x_3 w_3) = 0 \,.$$

Multiply the equation (2) by x_1, the equation (3) by x_2, the equation (4) by x_3 and add the results:

$$(x_1 w_1 + x_2 w_2 + x_3 w_3) \cdot (x_1 w_1 + x_2 w_2 + x_3 w_3) = 0\,,$$

so that $||x_1 w_1 + x_2 w_2 + x_3 w_3|| = 0$, or $x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$, proving that the vectors w_1, w_2, w_3 are linearly dependent.

The proof is similar for the general case of n vectors.

11. b. Here $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$, and a calculation gives the least squares solution

$$\bar{x} = \left(A^T A\right)^{-1} A^T b = \frac{1}{50} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since $A^T b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

11. c. $p = A\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence b is orthogonal to $C(A)$.

Section 5.3

1. a. $v_1 = w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Normalize:

$$u_1 = \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$u_2 = \frac{1}{||v_2||} v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

1 1. c. Here $v_1 = w_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$,

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 3 \\ 2 \\ -4 \\ 1 \end{bmatrix} - \frac{12}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

2

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{w_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-3}{6} \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ -3 \end{bmatrix}.$$

3 Normalize:

$$u_1 = \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

4

$$u_2 = \frac{1}{||v_2||} v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

5

$$u_2 = \frac{1}{||v_2||} v_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 1 \\ -1 \\ -3 \end{bmatrix}.$$

6

7 1. e. This example is similar to 1.b., only vectors have more components.

8 Here $v_1 = w_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$,

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{16} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

Normalize v_1, v_2 to obtain u_1, u_2.

1. f. Since the vectors u_1 and u_2 form an orthonormal basis of the subspace W,

$$\text{Proj}_W b = \text{Proj}_{u_1} b + \text{Proj}_{u_2} b = (b \cdot u_1)\, u_1 + (b \cdot u_2)\, u_2 = u_1 - u_2\,.$$

2. a. The null-space $N(A)$ is spanned by the vectors $w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 0 \end{bmatrix}$. Apply the Gram-Schmidt process to these vectors to produce an orthogonal basis for the null-space $N(A)$: $u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}$.

2. c. The null-space $N(A)$ is spanned by the vectors

$$w_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Apply the Gram-Schmidt process to these vectors to produce an orthogonal basis for the null-space $N(A)$:

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

3. *Any $m \times n$ matrix A with linearly independent columns can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns, and R is a square $n \times n$ upper triangular matrix.* If A is a square $n \times n$ matrix, so is Q.

3. a. $|A| = |Q||R|$. If $|A| \neq 0$, then $|R| \neq 0$, so that R is non-singular. The diagonal entries of R are positive because they contain the magnitudes of the vectors $v_1, v_2, \ldots$.

3. b. Multiply $A = QR$ from the left by A^T: $Q^TA = Q^TQR = Q^{-1}QR = R$.

4. a. The columns of the matrix A are $w_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $w_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Apply Gram-Schmidt: $v_1 = w_1$,

$$v_2 = w_2 + \frac{3}{25}w_1 = \frac{1}{25}\begin{bmatrix} -16 \\ 12 \end{bmatrix} = \frac{4}{25}\begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

Hence, $u_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $u_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Then $Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$.

Also, $w_1 = 5u_1$, and $w_2 = -\frac{3}{5}u_1 + \frac{4}{5}u_2$, giving R. Alternatively, $R = \begin{bmatrix} w_1 \cdot u_1 & w_2 \cdot u_1 \\ 0 & w_2 \cdot u_2 \end{bmatrix} = \begin{bmatrix} 5 & -\frac{3}{5} \\ 0 & \frac{4}{5} \end{bmatrix}$.

4. e. The columns of the matrix A are $w_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$, $w_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Apply Gram-Schmidt: $v_1 = w_1$,

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{||v_1||^2}v_1 = w_2 - \frac{4}{4}v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{||v_1||^2}v_1 - \frac{w_3 \cdot v_2}{||v_2||^2}v_2 = w_3 - \frac{-2}{4}v_1 - \frac{-2}{2}v_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Normalize $u_1 = \frac{1}{2}v_1$, $u_2 = \frac{1}{\sqrt{2}}v_2$, $u_3 = v_3$. Hence, $Q = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$.

To calculate the matrix R we use the above computations to express w_1, w_2, w_3 through u_1, u_2, u_3. Obtain

$$w_1 = v_1 = ||v_1||u_1 = 2u_1\,,$$

$$w_2 = v_1 + v_2 = ||v_1||u_1 + ||v_2||u_2 = 2u_1 + \sqrt{2}u_2\,,$$

$$w_3 = -\frac{1}{2}v_1 - v_2 + v_3 = -\frac{1}{2}||v_1||u_1 - ||v_2||u_2 + ||v_3||u_3 = -u_1 - \sqrt{2}u_2 + u_3\,.$$

Hence, $R = \begin{bmatrix} 2 & 2 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$.

5. a. Q is orthogonal if and only if $Q^T = Q^{-1}$. Then

$$\left(Q^T\right)^T = \left(Q^{-1}\right)^T = \left(Q^T\right)^{-1}\,.$$

It follows that Q^T is orthogonal.

b. Since Q^T is orthogonal, the rows of Q are orthonormal.

c. Since Q is orthogonal, $Q^T = Q^{-1}$. To prove that Q^{-1} is orthogonal, need to show that

$$\left(Q^{-1}\right)^T = \left(Q^{-1}\right)^{-1}\,.$$

Both sides are equal to Q.

6. Since columns of Q are unit vectors, the entries $Q_{31} = Q_{32} = 0$. Similarly, $Q_{13} = Q_{23} = 0$, because the rows of Q are unit vectors. The third column of Q is also a unit vector. Answer. $Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$.

7. a. Take inner product of $Qx = \lambda x$ with another copy of the same formula:

$$Qx \cdot Qx = \lambda x \cdot \lambda x\,,$$

or

$$\lambda^2 x \cdot x = Qx \cdot Qx = x \cdot Q^T Qx = x \cdot Q^{-1}Qx = x \cdot x\,,$$

so that $\lambda^2 = 1$, $\lambda = \pm 1$ (since the eigenvector $x \neq 0$).

7. b. The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is an orthogonal matrix with the eigenvalues $\lambda = \pm i$.

7. c. If Q is upper triangular then Q^{-1} is upper triangular, while Q^T is lower triangular. Since $Q^T = Q^{-1}$, it follows that Q is diagonal. The diagonal entries of Q are ± 1, because they are eigenvalues of an orthogonal matrix.

8. The eigenspace of $\lambda = -2$ is spanned by $w_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Applying the Gram-Schmidt process to these vectors produces an orthonormal basis of this eigenspace: $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

9. In case $n = 3$, this formula for R was developed in the text. Follow the same derivation.

Section 5.4

1. a. $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, which is not the zero vector. The transformation $T(x)$ is not linear.

1. b. $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, giving the first column of A. Similarly, $T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ gives the second column of A.

1. f. This transformation is neither homogeneous nor additive. It is easier to show that it is not homogeneous. For example, $T(2x) = 4T(x) \neq 2T(x)$.

Conclusion. If all components of $T(x)$ are linear functions of $x_1, x_2, \ldots, x_n$, and $T(0) = 0$ holds, then $T(x)$ is a linear transformation. Its matrix A can be found by inspection (just by looking), similarly to matrices of linear systems.

2. a. $T(e_1) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ gives the first column of A, and so on.

The other three columns of A are given by $T(e_2), T(e_3), T(e_4)$. The matrix A can be also found by inspection, as explained in the Conclusion above.

2. b.,c.,d. Try to use the short-cut from the Conclusion above.

2. e. $T(e_1) = -2e_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, giving the first column of the matrix A. (Indeed, projection of e_1 on the x_1x_2-plane leaves e_1 unchanged, then reflection with respect to the origin produces $-e_1$, and finally doubling the length gives $-2e_1$.) Similarly, $T(e_2) = -2e_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$, giving the second column of the matrix A. Since the projection of e_3 on the x_1x_2-plane is the zero vector, $T(e_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which gives the third column. Obtain $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2. f. The projection of $x \in R^3$ on the x_1x_2-plane is $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$. When this vector is rotated by the angle θ counterclockwise, the third component stays zero, while the first two components are rotated. For $x = e_1$, the projection on the x_1x_2-plane is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The first two components of this vector represent the vector e_1 in R^2. Its rotation is $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, as was established in our discussion of the rotation matrix. It follows that $T(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$. Similarly, $T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$. Finally, $T(e_3) = 3e_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. Hence, $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

2. g. Here $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Indeed,

$$T(x) = \begin{bmatrix} 2x_1 \\ -2x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

2. h. Here $||a|| = 2$. By (4.2) the projection matrix is

$$P = \frac{1}{4} a a^T = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} .$$

(Use the first definition of matrix product.)

3. Since

$$T_2 \left(T_1(x_1 + x_2) \right) = T_2 \left(T_1(x_1) + T_1(x_2) \right) = T_2 \left(T_1(x_1) \right) + T_2 \left(T_1(x_2) \right) ,$$

it follows that the composition $T_2 \left(T_1(x) \right)$ is additive. Similarly,

$$T_2 \left(T_1(cx) \right) = T_2 \left(c T_1(x) \right) = c T_2 \left(T_1(x) \right) ,$$

so that the composition $T_2 \left(T_1(x) \right)$ is homogeneous.

4. a. Assume that $T(u) = 0$ implies that $u = 0$. If now $T(u_1) = T(u_2)$, then $T(u_1 - u_2) = 0$ and hence $u_1 = u_2$, so that $T(u)$ is one-to-one. The converse statement is proved similarly.

4. b. Represent $T(u) = Au$ with an $m \times n$ matrix A. The homogeneous system $Au = 0$ has non-trivial solutions. It follows that $T(u) = 0$ *does not imply* that $u = 0$. Hence $T(u)$ is not one-to-one by the part a.

5. If a linear transformation $T(x) : R^n \to R^m$ has a matrix representation $T(x) = Ax$, then the range of $T(x)$ is the same as the column space $C(A)$. Then $T(x)$ is onto if and only if $C(A) = R^m$.

5. a. One has rank $A = m$ if and only if $C(A) = R^m$. Indeed, if $C(A)$ is spanned by m linearly independent vectors, these vectors also span R^m.

5. b. If $n < m$, the matrix A has fewer than m pivots. Hence dimension of $C(A)$ is less than m, and then $C(A)$ is a proper subspace of R^m ($C(A) \neq R^m$).

6. c. Let $T(x_1) = y_1$, $T(x_2) = y_2$. By linearity of $T(x)$

$$T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2) = c_1y_1 + c_2y_2\,,$$

for any scalars c_1 and c_2. It follows that

$$T^{-1}(c_1y_1 + c_2y_2) = c_1x_1 + c_2x_2 = c_1T^{-1}(y_1) + c_2T^{-1}(y_2)\,,$$

proving that $T^{-1}(y)$ is linear.

7. a. There are infinitely many vectors that share the same projection.

b. $T(x)$ is not onto, its range consists of a line.

8. b. The columns of P are $T(e_1)$ and $T(e_2)$.

8. c. To see that $PP = I$, draw a vector x in the first quadrant of x_1x_2-plane. Px is obtained by rotating of x followed by reflection with respect to x_1 axis. To get PPx one rotates Px and reflects the result with respect to x_1 axis. This brings one back to x. Hence $PPx = Ix$ for any x, so that $PP = I$.

f. As in part c, two reflections and two rotations bring any $x \in R^2$ back to the same x.

Section 5.5

1. Matrix AA^T is symmetric because

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T\,.$$

To see that AA^T is positive definite, we shall show that $AA^Tx \cdot x > 0$ for any $x \neq 0$. So assume that $x \neq 0$. We claim that $A^Tx \neq 0$. Indeed, if $A^Tx = 0$, then $x = \left(A^T\right)^{-1} 0 = 0$, a contradiction. ($A^T$ is invertible because A is.) Conclude:

$$AA^Tx \cdot x = A^Tx \cdot A^Tx = ||A^Tx||^2 > 0\,.$$

2. a. Since $B^T = B$, and $\left(A^T\right)^T = A$,

$$\left(A^TBA\right)^T = A^TB^T\left(A^T\right)^T = A^TBA\,,$$

and hence A^TBA is symmetric.

3. Eigenvalues of a positive definite matrix are all positive. Determinant is equal to the product of eigenvalues.

4. b. The eigenvalues are $\lambda_1 = -2$, with the normalized eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $\lambda_1 = 3$, with the normalized eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. These eigenvectors form an orthonormal set, and they are the columns of the orthogonal diagonalizing matrix P.

5. b. Since $A^T = -A$,

$$|A^T| = |-A| \,.$$

Using that $|A^T| = |A|$, and $|-A| = (-1)^n |A| = -|A|$ because n is odd, obtain

$$|A| = -|A| \,,$$

so that $|A| = 0$.

5. c. By part a, the eigenvalues of A are of the form iq, with real q. The eigenvalues of $I + A$ are $1 + iq$. Since $1 + iq$ cannot be zero, the matrix $I + A$ is non-singular.

5. d. To justify that $(I - A)(I + A)^{-1}$ is orthogonal, we show that its transpose is equal to its inverse. Indeed,

$$\left[(I - A)(I + A)^{-1}\right]^T = (I + A^T)^{-1}(I - A^T) = (I - A)^{-1}(I + A) \,,$$

$$\left[(I - A)(I + A)^{-1}\right]^{-1} = (I + A)(I - A)^{-1} \,.$$

To see that

$$(I - A)^{-1}(I + A) = (I + A)(I - A)^{-1} \,,$$

multiply from both the left and from the right by $I - A$, to get an equivalent and correct expression

$$(I + A)(I - A) = (I - A)(I + A) \,.$$

(Both sides are equal to $I - A^2$.)

6. The matrix $A^T A + I$ is symmetric because

$$\left(A^T A + I\right)^T = \left(A^T A\right)^T + I^T = A^T A + I \,.$$

This matrix is positive definite because

$$\left(A^T A + I\right) x \cdot x = A^T A x \cdot x + I x \cdot x = Ax \cdot Ax + ||x||^2 = ||Ax||^2 + ||x||^2 > 0 \,,$$

for all $x \neq 0$.

7. Since $A^T = A$, obtain

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}\,,$$

and hence A^{-1} is symmetric.

8. a. Since

$$\left(u_i u_i^T\right)^T = \left(u_i^T\right)^T u_i^T = u_i u_i^T\,,$$

it follows that $A^T = A$.

8. b. Since $u_i^T u_j = u_i \cdot u_j = 0$ for $i \neq j$, it follows that $Au_j = \lambda_j u_j$.

9. a. $Ae_1 \cdot e_1 = -5 < 0$, therefore A is not positive definite.

10. For any non-zero vector $x \in R^n$, the vector $y = Sx$ is also non-zero. Indeed, if $y = 0$, then $x = S^{-1}0 = 0$, a contradiction. Hence

$$S^T ASx \cdot x = ASx \cdot Sx = Ay \cdot y > 0\,,$$

and hence the matrix $S^T AS$ is positive definite. (This matrix is symmetric, since $\left(S^T AS\right)^T = S^T AS$.)

12. Calculate

$$A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 144 \end{bmatrix}.$$

We call $\lambda_1 = 144$ and $\lambda_2 = 9$, in order to arrange the singular values $\sigma_1 = \sqrt{\lambda_1} = 12$ and $\sigma_2 = \sqrt{\lambda_2} = 3$ to be in decreasing order. The corresponding unit eigenvectors are $x_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (another possibility is $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$). Calculate $Ax_1 = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$, $Ax_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and then

$$q_1 = \frac{Ax_1}{\sigma_1} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix},$$

$$q_2 = \frac{Ax_2}{\sigma_2} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Calculate $q_3 = q_1 \times q_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$. All of the pieces are in place for singular value decomposition:

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T .$$

Section 5.6

2. c. Here the $-2x_1x_2$ term gives $a_{12} = a_{21} = -1$, the $8x_2x_3$ term gives $a_{23} = a_{32} = 4$, while $3x_1^2$ produces $a_{11} = 3$, x_2^2 produces $a_{22} = 1$, $-5x_3^2$ produces $a_{33} = -5$. The quadratic form does not have a x_1x_3 term, therefore $a_{13} = a_{31} = 0$.

3. b. Here $a_{38} = a_{83} = 11$. Therefore the coefficient in x_3x_8 is 22.

3. c. The purely quadratic terms correspond to the diagonal entries of the $n \times n$ matrix A, while the x_ix_j terms can be identified with the terms above the diagonal in A. There a total of $\frac{n(n+1)}{2}$ of terms that lie on or above the diagonal. (Counting such terms from first, second and other columns: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.)

4. a. The matrix of this quadratic form $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has an eigenvalue $\lambda_1 = 2$ with the normalized eigenvector $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = 4$ with the normalized eigenvector $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Using these eigenvectors as columns, obtain the diagonalizing matrix $P = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The change of variables $x = Py$ takes the form

$$\begin{aligned} x_1 &= \tfrac{1}{\sqrt{2}} \left(-y_1 + y_2\right) \\ x_2 &= \tfrac{1}{\sqrt{2}} \left(y_1 + y_2\right) . \end{aligned}$$

Substituting these expressions into our quadratic form $3x_1^2 + 4x_1x_2 + 3x_2^2$, gives the diagonalized form $2y_1^2 + 4y_2^2$.

4. b. The matrix of this quadratic form $A = \begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}$ has an eigenvalue $\lambda_1 = -1$ with the normalized eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = 4$ with the normalized eigenvector $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Using these eigenvectors as columns, obtain the diagonalizing matrix $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. The change of variables $x = Py$ takes the form

$$\begin{aligned} x_1 &= \tfrac{1}{\sqrt{5}} \left(2y_1 - y_2\right) \\ x_2 &= \tfrac{1}{\sqrt{5}} \left(y_1 + 2y_2\right) . \end{aligned}$$

Substituting these expressions into our quadratic form $-4x_1x_2 + 3x_2^2$, gives the diagonalized form $-y_1^2 + 4y_2^2$.

4. d. The matrix of the quadratic form $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = \lambda_2 = -2$ with the eigenspace spanned by $w_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\lambda_3 = 1$ with the eigenvector $w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The vectors w_1 and w_2 are not orthogonal. Apply the Gram-Schmidt process: $v_1 = w_1$,

$$v_2 = w_2 - \frac{w_2 \cdot w_1}{w_1 \cdot w_1} w_1 = w_2 - \frac{1}{2} w_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} .$$

Normalize $u_1 = \frac{1}{\sqrt{2}} v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \frac{2}{\sqrt{6}} v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. The vectors u_1 and u_2 give the first and the second columms of P. Since w_3 is orthogonal to u_1 and u_2, its normanlzation $\frac{1}{\sqrt{3}} w_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the third column of P.

Conclude:

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

The change of variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or in coordinates, $x_1 = -\frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$, $x_2 = \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$, $x_3 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$, produces $-2y_1^2 - 2y_2^2 + y_3^2$.

5. Since A has zero eigenvalue, $|A| = 0$. It follows that $|S^TAS| = |A||S|^2 = 0$, and hence S^TAS also has zero eigenvalue. The multiplicity of zero eigenvalue is the same for A and S^TAS, since by law of inertia both matrices have exactly the same number of non-zero eigenvalues.

6. a. If x_0 is an eigenvector corresponding to $\lambda = 0$, then $Ax_0 = 0x_0 = 0$, and then $Ax_0 \cdot x_0 = 0$.

6. b. Diagonalizing as in 6. a., conclude that all eigenvalues of a positive semidefinite matrix are nonnegative.

6. c. Since determinant is nonzero, there is no zero eigenvalues. Hence, all eigenvalues are positive, and the matrix is positive definite.

7. Following the Hint given in the text,

$$Ax \cdot x = \int_0^1 \left(\sum_{i=1}^{n} x_i t^{i-1} \right)^2 dt \geq 0\,.$$

It remains to rule out the possibility that this integral is zero. This can happen only if $\sum_{i=1}^n x_i t^{i-1} = 0$ for all $t \in (0,1)$, which in turn will require that all $x_i = 0$. But the vector x, with components x_i, is assumed to be non-zero. Hence, $Ax \cdot x > 0$.

Section 5.7

2. a. Consider the linear combination

$$x_1A_1 + x_2A_2 + x_3A_3 = O\,.$$

In components

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_2 + 2x_3 &= 0 \\ 3x_3 &= 0\,, \end{aligned}$$

giving $x_1 = x_2 = x_3 = 0$.

2. b. To express D need to solve

$$x_1A_1 + x_2A_2 + x_3A_3 = D\,.$$

In components

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ 3x_3 &= 3\,, \end{aligned}$$

giving $x_1 = x_2 = x_3 = 1$.

2. c. The vectors A_1, A_2, A_3, A_4 are linearly independent because

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = O$$

implies that $x_1 = x_2 = x_3 = x_4 = 0$. Four linearly independent vectors form a basis of four dimensional space $M_{2\times 2}$.

2. d. The coordinates of F are the solutions

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = F\,.$$

In components obtain a system of four equations with four unknowns, which is solved by back substitution:

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ 3x_3 &= 0 \\ x_4 &= -7\,, \end{aligned}$$

giving $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, $x_4 = -7$.

3. b. Obtain

$$||x^2-1||^2 = \left(x^2-1\right)\cdot\left(x^2-1\right) = \int_{-1}^{1}\left(x^2-1\right)^2\,dx = \frac{16}{15}\,.$$

3. c. Similarly

$$||\sqrt{2}||^2 = \sqrt{2}\cdot\sqrt{2} = \int_{-1}^{1} 2\,dx = 4\,.$$

4. Denote $w_1 = 1, w_2 = x+2, w_3 = x^2-x$. Then $v_1 = w_1 = 1$,

$$v_2 = w_2 - \frac{w_2\cdot v_1}{||v_1||^2}v_1 = x+2-2 = x\,,$$

since $w_2\cdot v_1 = \int_{-1}^{1}(x+2)\,dx = 4$, and $||v_1||^2 = \int_{-1}^{1} 1\,dx = 2$. Then

$$v_3 = w_3 - \frac{w_3\cdot v_1}{||v_1||^2}v_1 - \frac{w_3\cdot v_2}{||v_2||^2}v_2 = x^2 - \frac{1}{3}\,,$$

because $w_3\cdot v_1 = \int_{-1}^{1}(x^2-x)\,dx = \frac{2}{3}$, $w_3\cdot v_2 = \int_{-1}^{1}(x^2-x)x\,dx = -\frac{2}{3}$, $||v_2||^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$. Standardization produces $u_1 = 1$, $u_2 = x$, $u_3 = \frac{1}{2}(3x^2-1)$.

5. a. The transformation I is integration $I(p(x)) = \int_0^x p(t)\,dt$, which is taking the antiderivative with $c = 0$. I is linear because the integration is linear.

Let $T(x)$ be a linear transformation $T : V_1 \to V_2$. Assume that $B_1 = \{w_1, w_2, \ldots, w_p\}$ is a basis of V_1, and $B_2 = \{z_1, z_2, \ldots, z_s\}$ is a basis of V_2. Then the matrix of $T(x)$ is $A = [\ [T(w_1)]_{B_2}\ [T(w_2)]_{B_2}\ \ldots\ [T(w_p)]_{B_2}\ ]$, of size $s\times p$, obtained by using the vectors $[T(w_i)]_{B_2}$ as its columns. (*There is a misprint in the book on* A.)

5. b. The standard basis of P_3 is $1, x, x^2, x^3$, the standard basis of P_4 is $1, x, x^2, x^3, x^4$. Calculate

$$I(1) = x = 0\times 1 + 1\times x + 0\times x^2 + 0\times x^3 + 0\times x^4\,,$$

so that the first column of the matrix of I is $\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}$. Proceed similarly with

$I(x) = \frac{1}{2}x^2$, so that the second column is $\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$, $I(x^2) = \frac{1}{3}x^3$, so that the third column is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$, and $I(x^3) = \frac{1}{4}x^4$, so that the fourth column is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \end{bmatrix}$. The matrix of $I(x)$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

6. b. Using the standard basis in the vector space of 2×2 matrices

$$T(E_{11}) = E_{21} = 0 \times E_{11} + 0 \times E_{12} + 1 \times E_{21} + 0 \times E_{22}\,,$$

so that the first column is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Similarly

$$T(E_{12}) = E_{22} = 0 \times E_{11} + 0 \times E_{12} + 0 \times E_{21} + 1 \times E_{22}\,,$$

so that the second column is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$,

$$T(E_{21}) = 2E_{11} = 2 \times E_{11} + 0 \times E_{12} + 0 \times E_{21} + 0 \times E_{22}\,,$$

so that the third column is $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$$T(E_{22}) = 2E_{12} = 0 \times E_{11} + 2 \times E_{12} + 0 \times E_{21} + 0 \times E_{22}\,,$$

so that the fourth column is $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. The matrix of $T(x)$ is

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

7. The transformation is not linear because $T(O) \neq O$.

8. Legendre polynomials are polynomials of degree n, satisfying $P_n(1) = 1$ (*there is a misprint in the book on this condition*), and orthogonal on $(-1, 1)$. Differentiating n times a polynomial of degree $2n$, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1\right)^n \right]$, indeed produces a polynomial of degree n. Repeated differentiations produce many terms, but all except one vanish when $x = 1$. That happens when all n derivatives "fall" on $\left(x^2 - 1\right)^n$, which produces a coefficient of $2^n n!$. To prove orthogonality, follow the Hint in the book.

Chapter 6

Section 6.1

1. a. The matrix $\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$ has an eigenvalue $\lambda_1 = -1$, with the corresponding eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = 2$, with the corresponding eigenvector $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

1. b. The matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ has an eigenvalue $\lambda_1 = 5$, with the corresponding eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = 0$, with the corre-

sponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} .$$

1. d. To avoid a tedious calculation of eigenvalues and eigenvectors, one may enter the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & -2 & 1 \end{bmatrix}$ into *Mathematica* as the following "row of rows": $A = \{\{1, 1, 1\}, \{2, 2, 1\}, \{4, -2, 1\}\}$. The command Eigensystem[A] produces the eigenvalues of A, and the corresponding eigenvectors. The eigenvalues are $\lambda_1 = -1$, corresponding to $\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\lambda_2 = 2$, corresponding to $\xi_2 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, and $\lambda_3 = 3$, corresponding to $\xi_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} .$$

1. (e) The eigenvalues are $\lambda_1 = -1$, corresponding to $\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 3$, corresponding to $\xi_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $\lambda_3 = 0$, corresponding to $\xi_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} .$$

1. (f) The eigenvalues are $-1, -1, 1, 3$. (This matrix is block diagonal.) The eigenvalue -1 is repeated, but it has two linearly independent eigenvectors

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix},$$

where $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigevector corresponding to $\lambda = 1$, and $\begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix}$ corresponds to $\lambda = 3$.

2. (b) The eigenvalues are $\lambda_1 = 0$ with an eigenvector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$ with an eigenvector $\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\lambda_3 = 3$ with an eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The general solution is

$$x(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The initial condition implies

$$x(0) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Solving this system of three equations, $c_1 = 1$, $c_2 = -1$, $c_3 = 3$.

3. (a) The first component of the vector $\frac{x(t+h)-x(t)}{h}$ is $\frac{x_1(t+h)-x_1(t)}{h} \to x'(t)$.

3. (b) Differentiate the first component of $x(t)$, and then other components.

5. (a) The matrix of this system has a double eigenvalue $\lambda_1 = \lambda_2 = -1$, and only one linearly independent eigenvector $\xi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We have one solution:

$x_1(t) = e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The system $(A - \lambda_1 I)\eta = \xi$, or $(A+I)\eta = \xi$, to determine the generalized eigenvector $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ takes the form

$$2\eta_1 - \eta_2 = 1$$
$$4\eta_1 - 2\eta_2 = 2\,.$$

Discard the second equation, then set $\eta_1 = 0$ in the first equation, to obtain a generalized eigenvector $\eta = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right).$$

5. (b) Using the initial conditions

$$x(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then $c_1 = 1$, $c_2 = 3$.

6. Expanding $|A - \lambda I|$ in the second row shows that the characteristic equation has a factor $(-1 - \lambda)$, and hence $\lambda = -1$ is an eigenvalue. The second factor is a cubic polynomial, for which we guess a root $\lambda_2 = -1$. Then the cubic can be factored as $(\lambda + 1)$ times a quadratic polynomial. The quadratic polynomial has roots $\lambda_3 = -2$ and $\lambda_4 = -4$. Calculation shows that the repeated eigenvalue $\lambda = -1$ has only one linearly independent eigenvector $\xi = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. If η denotes the corresponding generalized eigenvector, and ξ_3, ξ_4 are eigenvectors correponding to λ_3, λ_4 respectively, then the general solution is

$$x(t) = c_1 e^{-t}\xi + c_2 e^{-t}\left(t\xi + \eta\right) + c_3 e^{-2t}\xi_3 + c_4 e^{-4t}\xi_4\,.$$

Using the L'Hospital rule, $x(t) \to 0$ as $t \to \infty$. Observe that the exact knowledge of vectors η, ξ_3, ξ_4 is not needed here.

7. The eigenvalues satisfy $\lambda_1 \lambda_2 = \det A = -a^2 - 2 < 0$, $\lambda_1 + \lambda_2 = \mathrm{trA} = 0$. Hence the eigenvalues are non-zero, and have opposite sign.

8. $(A - \lambda I)(2\eta) = 2\xi \neq \xi$, since the eigenvector $\xi \neq 0$.

9. Follows using that $(A - \lambda I)(c\xi) = 0$.

10. If $A^T = A$, $A\xi = \lambda\xi$, and η is a generalized eigenvector satisfying $(A - \lambda I)\eta = \xi$, then

$$\xi \cdot \xi = (A - \lambda I)\eta \cdot \xi = \eta \cdot (A^T - \lambda I)\xi = \eta \cdot (A\xi - \lambda x) = \eta \cdot 0 = 0\,,$$

and hence $\xi = 0$, which is not possible for an eigenvector. It follows that a generalized eigenvector η does not exist.

If A is symmetric there it has a complete set of eigenvectors, providing the general solution of $x' = Ax$. Conclusion: symmetric matrices do not have generalized eigenvectors, but they are not needed for solving $x' = Ax$.

Section 6.2

1. a. The eigenvalues are $\lambda = 1 \pm i$. An eigenvector corresponding to $\lambda = 1 + i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$, leading to a complex valued solution

$$e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} = (\cos t + i \sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Since both the real and the imaginary parts of the complex valued solution are also solutions, the general solution of our system is

$$x(t) = c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

2. The general solution is

$$x(t) = c_1 e^{(-1+2i)t}\xi_1 + c_2 e^{(-1-2i)t}\xi_2\,,$$

where ξ_1, ξ_2 the corresponding complex-valued eigenvectors. Observe that

$$e^{(-1+2i)t} = e^{-t}e^{2it} = e^{-t}\left(\cos 2t + i \sin 2t\right) \to 0\,,$$

as $t \to \infty$. Similarly, $e^{(-1-2i)t} \to 0$, $t \to \infty$. Hence $x(t)$ tends to zero.

3. The solution is

$$x(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

which is rotation of the initial vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

5. The eigenvalues of this system satisfy

$$\lambda_1 + \lambda_2 = \text{tr}(A) = a + d < 0\,,$$
$$\lambda_1 \lambda_2 = \det(A) = ad - bc > 0\,.$$

If the eigenvalues are real, they are of the same sign by the second formula, and therefore they are both negative by the first formula. If the eigenvalues are complex, $p \pm iq$, their real part is negative, because $\lambda_1 + \lambda_2 = 2p < 0$. In either case, solution tends to zero as $t \to \infty$.

6. (a) The characteristic polynomial of a 3×3 matrix is a cubic, and hence one of its roots is real. That root λ must be zero, in order for $e^{\lambda t}$ to remain bounded, as $t \to \pm\infty$. The root $\lambda = 0$ must be simple, otherwise the solution contains an unbounded factor of t. The other two roots must be purely imaginary $\lambda = \pm i\, q$, for the corresponding solutions to remain bounded as $t \to \pm\infty$. Then the general solution has the form

$$x(t) = c_1 \xi_1 + c_2 \cos qt\, \xi_2 + c_3 \sin qt\, \xi_3\,,$$

where ξ_1, ξ_2 and ξ_3 are constant, real valued three dimensional vectors. The solution is periodic, of period $\frac{2\pi}{q}$.

6. (b) Observe that $a_{ji} = -a_{ij}$, and then $a_{ii} = 0$ for any skew-symmetric matrix. Then any 3×3 skew-symmetric matrix is of the form $\begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}$, with some real p, q and r. Compute the eigenvalues $\lambda = 0$, $\lambda = \pm i\,\sqrt{p^2 + q^2 + r^2}$.

6. (c) Use part (a) to show that all solutions have period $\frac{2\pi}{\sqrt{p^2+q^2+r^2}}$.

7. We are given that the eigenvalues of A satisfy $\lambda_1 \lambda_2 < 0$, hence we may assume that $\lambda_1 < 0$ and $\lambda_2 > 0$. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2\,,$$

where ξ_1, ξ_2 the corresponding eigenvectors. The numbers c_1, c_2 depend on the initial conditions. If $c_2 \neq 0$, the solution tends to infinity, and if $c_2 = 0$, $x(t) \to 0$ as $t \to \infty$. There are no periodic solutions.

Section 6.3

1. a. Here $A^2 = O$, $A^3 = O, \ldots, A^n = O$ for all $n \geq 2$. Hence

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

1. c. The matrix $Dt = \begin{bmatrix} 2t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3t \end{bmatrix}$ is diagonal. Just exponentiate the diagonal elements:

$$e^{Dt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}.$$

1. d. Here $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = O, \ldots, A^n = O$ for all $n \geq 3$. Hence

$$e^{At} = I + At + \frac{1}{2}A^2t^2 = I + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

1. e. Write

$$A = -2I + J\,,$$

where $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Using 1. d.

$$e^{At} = e^{-2tI + tJ} = e^{-2tI}e^{Jt} = e^{-2t}e^{Jt} = e^{-2t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

1. f. The matrix A is a block matrix, consisting of a 2×2 and 1×1 blocks. Calculate the exponentials of each block separately.

2. Since the matrices A and $-A$ commute

$$e^A\, e^{-A} = e^{A-A} = e^O = I\,.$$

Hence, e^{-A} is the inverse of e^A.

3. Since the matrices A and A commute

$$\left(e^A\right)^2 = e^A e^A = e^{2A}\,,$$

and similarly $\left(e^A\right)^m = e^{mA}$, for any integer m.

5. a. If $Ax = \lambda x$, then

$$e^A x = \sum_{k=0}^{\infty} \frac{A^k x}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x = e^{\lambda} x \,.$$

It follows that e^{λ} is an eigenvalue of e^A corresponding to an eigenvector x.

5. b. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$ are the eigenvalues of e^A, as follows by 5. a. Then

$$\det e^A = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = e^{\operatorname{tr} A} \,.$$

5. c. By 5. b., $\det e^A > 0$, and hence e^A is non-singular.

6. The matrix e^A is symmetric, as a sum of symmetric matrices. Similarly, the matrix $e^{A/2}$ is symmetric. Then for any $x \neq 0$,

$$e^A x \cdot x = e^{A/2} e^{A/2} x \cdot x = e^{A/2} x \cdot e^{A^T/2} x = e^{A/2} x \cdot e^{A/2} x = ||e^{A/2} x||^2 > 0 \,,$$

because in case $e^{A/2} x = 0$, it follows that $x = 0$, a contradiction. (Recall that $e^{A/2}$ is non-singular, by the exercise 5. c.)

8. b. With $K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, calculate $K^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $K^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $K^4 = O$. Since $K^m = O$, for $m \geq 4$,

$$\sin Kt = Kt - \frac{1}{6} K^3 t^3 \,.$$

11. By the definition, $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. Apply the triangle inequality to a partial sum (the triangle inequality holds for arbitrary number of terms)

$$||\sum_{k=0}^{N} \frac{A^k}{k!}|| \leq \sum_{k=0}^{N} \frac{||A||^k}{k!} < \sum_{k=0}^{\infty} \frac{||A||^k}{k!} = e^{||A||} \,.$$

The numerical sequence $\{||\sum_{k=0}^{N} \frac{A^k}{k!}||\}$ converges to $||e^A||$ as $N \to \infty$, and all terms of this sequence are less than $e^{||A||}$. It follows that

$$||e^A|| \le e^{||A||}\,.$$

Section 6.4

1. d. The matrix of this system has an eigenvalue $\lambda_1 = -1$ with corresponding eigenvector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and a repeated eigenvalue $\lambda_2 = \lambda_3 = 1$ with with two linearly independent eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using the initial conditions, obtain $c_1 = 1$, $c_2 = 2$, $c_3 = 2$. *The answer in the book is wrong* (the second and the third components are switched in the book).

2. a. The solution of this system with the initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $x(t) = \begin{bmatrix} \cos 2t \\ \frac{1}{2}\sin 2t \end{bmatrix}$, and it gives the first column of the fundamental solution matrix $X(t)$. The solution with the initial condition $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x(t) = \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix}$, and it gives the second column of $X(t)$.

3. a. Using that $A^T = -A$, calculate

$$\frac{d}{dt} x(t)\cdot y(t) = x'(t)\cdot y(t) + x(t)\cdot y'(t) = Ax\cdot y + x\cdot Ay = x\cdot A^T y + x\cdot Ay = 0\,,$$

so that $x(t)\cdot y(t)$ is independent of t, and hence $x(t)\cdot y(t) = x(0)\cdot y(0)$.

3. b. Letting $y(t) = x(t)$ in the last formula, conclude that $||x(t)||^2 = ||x(0)||^2$ for all t.

3. c. Column i of the fundamental matrix is the solution of $x' = A(t)x$, $x(0) = e_i$. Column j of the fundamental matrix is the solution of $y' = A(t)y$,

$y(0) = e_i$. Since the coordinate vectors e_i and e_j are orthogonal, so are $x(t)$ and $y(t)$ for all t, by 3. a. All columns of the fundamental matrix are of unit length, by 3. b. Hence, the fundamental matrix is orthogonal.

7. a. Write $J_0 = \lambda I + J$, with the matrix J satisfying $J^2 = O$. Then the binomial formula simplifies:

$$J_0^n = (\lambda I + J)^n = \lambda^n I + n\lambda^{n-1} J + \frac{n(n-1)}{2}\lambda^{n-2} J^2 \,.$$

7. b. By L'Hospital rule, if $|\lambda| < 1$, then $n\lambda^n \to 0$ as $n \to \infty$. It follows that all elements of the matrix tend to zero, $J_0^n \to O$ as $n \to \infty$.

7. c. To see that $\lim_{n\to\infty} A^n = O$, write A in the Jordan normal form, and apply part 7. b. to each block. Then

$$(I - A)\sum_{k=0}^{n} A^k = I - A^{n+1} \to I \,, \quad \text{as } n \to \infty,$$

so that $I - A$ is the inverse matrix of $\sum_{k=0}^{\infty} A^k$.

Section 6.5

1. a. Search for a particular solution in the form $x_1(t) = Ae^{2t}$, $x_2(t) = Be^{2t}$. Substitution into the system gives (after dividing both equations by e^{2t})

$$\begin{aligned} 2A &= B + 2 \\ 2B &= A - 1 \,. \end{aligned}$$

Solve this system: $A = 1$, $B = 0$. It follows that $Y(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ is a particular solution. The general solution is the sum of this particular solution and the general solution of the corresponding homogeneous system

$$x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \,,$$

which is

$$c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

1. b. Search for a particular solution in the form $x_1(t) = Ae^{2t}$, $x_2(t) = Be^{2t}$. Substitution into the system gives $A = \frac{2}{3}$, $B = \frac{1}{3}$. Add this particular solution and the general solution of the corresponding homogeneous system.

2. a. Search for a particular solution in the form $Y(t) = \begin{bmatrix} A \\ B \end{bmatrix}$, and calculate $Y(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution of the corresponding homogeneous system

$$x'(t) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} x(t)$$

is $x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The general solution of the non-homogeneous system is

$$x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} .$$

Use the initial conditions to calculate $c_1 = c_2 = \frac{1}{3}$.

6. b. Multiplication of block matrices gives $JJ = -I$, so that $-J$ is the inverse of J.

6. c. Let J_n denote the determinant of $2n \times 2n$ matrix J. Expanding J first in the first row, and then in the last row, gives

$$J_n = (-1)^{2n} \cdot 1 \cdot (-1)^{2n-1} \cdot (-1) \cdot J_{n-1} = J_{n-1} \,,$$

so that J_n is independent of n. Since

$$J_1 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1 \,,$$

it follows that $J_n = 1$, for all n.

Section 6.6

1. The Fibonacci numbers are: odd,odd,even,odd,odd,even,odd,odd,even and so on. Every third number is even.

2. The second term of the Binet's formula tends to zero as $n \to \infty$. Hence Fibonacci numbers are approximated by a geometric progression given by the first term of Binet's formula, for large n.

3. Search for solution in the form $x_n = r^n$. Substitution into the difference equation gives

$$r^n = 3r^{n-1} - 2r^{n-2} \,.$$

Division by r^{n-2} gives a quadratic equation

$$r^2 - 3r + 2 = 0$$

with roots $r_1 = 1$, $r_2 = 3$. The general solution of the difference equation is

$$x_n = c_1 + c_2 3^n \,.$$

From the initial conditions $c_1 = c_2 = 1$.

4. This approach to deriving Binet's formula is explained in the book of G. Strang [16].

6. a. Since the columns of A are linearly dependent, it follows that the determinant of A is zero, so that $\lambda = 0$ is one of the eigenvalues.

6. c. Since A is Markov matrix, one of its eigenvalues is $\lambda = 1$. The third eigenvalue is $\lambda = \frac{1}{6}$, since the sum of eigenvalues is equal to the trace of A.

8. a. The entry i of Ax is $\sum_{j=1}^{n} a_{ij} x_j$ and it is positive because all a_{ij} are positive while all x_j are non-negative with at least one of them positive.

8. b. Look for all numbers $t > 0$ such that $Ax \geq tx$ for some vector $x \geq \mathbf{0}$, $x \neq \mathbf{0}$. The largest possible value of such t's we call $t_{\max}$. We claim that

$$Ax = t_{\max} x \,,$$

so that $t_{\max}$ is an eigenvalue of A. Assume, on the contrary, that

$$Ax \geq t_{\max} x \,, \quad \text{not an equality} \,.$$

By part a:

$$A \left(Ax - t_{\max} x \right) > 0 \,,$$

giving

$$A^2 x > t_{\max} Ax \,.$$

Denoting $Ax = y > 0$ obtain

$$Ay > t_{\max} y \,.$$

We can then choose $\epsilon > 0$ small so that

$$Ay > \left(t_{\max} + \epsilon \right) y \,,$$

contradicting the maximality of $t_{\max}$, proving that $t_{\max}$ is an eigenvalue of A.

Using part a again, the corresponding eigenvector satisfies $x > 0$.

We claim that any other eigenvalue λ satisfies

$$|\lambda| \le t_{\max} .$$

Begin with

$$Az = \lambda z ,$$

and use the Cauchy-Schwarz inequality:

$$|\lambda||z| = |Az| \le |A||z| = A|z| .$$

(Since $A > 0$, $|A| = A$.) Hence

$$A|z| \ge |\lambda||z| , \quad |z| > 0 .$$

It follows that $|\lambda|$ is one of the eligible t's, and hence it cannot exceed $t_{\max}$.

To prove that the eigenvalue $t_{\max}$ is simple, one needs a strict inequality $|\lambda| < t_{\max}$. Please find this remaining piece on the internet.

9. The component i of Ax is $\sum_{j=1}^n a_{ij} x_j$. The sum of all entries of Ax

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j = \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_j = \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} = \sum_{j=1}^n x_j ,$$

after switching the order of summation, using that $\sum_{i=1}^n a_{ij} = 1$ by definition of Markov matrix. (Elements of a matrix can be added up by calculating either column totals first, or calculating row totals first.)

10. a. Other terms in $A^n x_0$ tend to zero as $n \to \infty$, by using (6.4) in the text.

11. The matrix A is diagonally dominant. The second and the third Gershgorin's circles are identical.

Chapter 7

Section 7.1

Sylvester's criterion provides a third way to determine if a symmetric matrix is positive definite (in addition to all eigenvalues being positive, and to $Ax \cdot x > 0$ holding for all $x \neq 0$).

1. a. Since A is positive definite, $Ae_i \cdot e_i > 0$. Then $a_{ii} = Ae_i \cdot e_i > 0$.

1. b. Denote $B = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$ and $z = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$. Then for any $x = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{bmatrix} \in$ R^5, by the positive difiniteness of A conclude:

$$0 < Ax \cdot x = Bz \cdot z\,.$$

Since z is an arbitrary vector in R^2, it follows that B is positive definite.

2. a. Here $a_{33} < 0$, and hence $Ae_3 \cdot e_3 = a_{33} < 0$.

2. b. Here $a_{33} = 0$, and hence $Ae_3 \cdot e_3 = 0$.

2. c. The matrix is not symmetric (the notion of positive definiteness applies only to symmetric matrices).

2. d. The second principal minor is zero. Use Sylvester's criterion to conclude that the matrix is not is positive definite.

3. d. Here $Ax \cdot x = 4x_1^2 + 4x_1x_2 + x_2^2 = (2x_1 + x_2)^2 \geq 0$, but $Ax \cdot x = 0$ if $x_2 = 2x_1$. A is positive semidefinite.

4. a. The first Gershgorin's circle is centered at the point $x = 4$ on the x-axis of the complex plane. Its radius is 3, and so it does not include the origin, and stays in the right half of the complex plane. Similarly, with other Gershgorin's circles. Hence all eigenvalues lie in the right half of the complex plane. Since A is symmetric, all of its eigenvalues are real, and hence positive. Then A is positive definite.

5. a. To find the critical points one needs to solve the system

$$\begin{aligned} f_x &= 3x^2 + 30y = 0 \\ f_y &= 30x + 6y = 0 \\ f_z &= 2z = 0\,. \end{aligned}$$

From the third equation $z = 0$. From the second equation express $y = -5x$, and use this in the first equation to obtain

$$x^2 - 50x = 0\,.$$

Obtain $x = 0$ and $x = 50$, so that the critical points are $(0,0,0)$ and $(50,-250,0)$. Calculate the Hessian at $(0,0,0)$

$$H(0,0,0) = \begin{bmatrix} 0 & 30 & 0 \\ 30 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

It has one negative eigenvalue $\lambda_1 = 3 - 3\sqrt{101}$, and two positive eigenvalues $\lambda_2 = 3 + 3\sqrt{101}$ and $\lambda_3 = 2$. One has a saddle point at $(0,0,0)$. Calculate the Hessian at $(50,-250,0)$

$$H(50,-250,0) = \begin{bmatrix} 300 & 30 & 0 \\ 30 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

By Sylvester's criterion, this matrix is positive definite, and hence $(50,-250,0)$ is a point of minimum.

5. b. To find the critical points one needs to solve the system

$$\begin{aligned} f_x &= -2x + y + 2z = 0 \\ f_y &= x - 4y = 0 \\ f_z &= 2x - 2z = 0\,. \end{aligned}$$

This linear homogeneous system has only the trivial solution $x = y = z = 0$, so that $(0,0,0)$ is the only critical point. Calculate the Hessian at the critical point:

$$H(0,0,0) = \begin{bmatrix} -2 & 1 & 2 \\ 1 & -4 & 0 \\ 2 & 0 & -2 \end{bmatrix}.$$

Mathematica approximately calculates the eigenvalues. Turns out that one of the eigenvalues is negative and two are positive, and hence $(0,0,0)$ is a saddle point.

Without computer assistance, one may proceed as follows. By Sylvester's criterion $H(0,0,0)$ is not positive definite, and not negative definite, so that it cannot have all eigenvalues of the same sign. This matrix is non-singular, so that it cannot have a zero eigenvalue. Hence, eigenvalues are non-zero, and of different signs. It follows that $(0,0,0)$ is a saddle point.

5. c. Similarly to 5. b., $(0,0,0)$ is the only critical point. Calculate the Hessian at the critical point:

$$H\,(0,0,0) = \begin{bmatrix} -2 & 1 & 2 \\ 1 & -4 & 0 \\ 2 & 0 & -8 \end{bmatrix}.$$

By Sylvester's criterion $H\,(0,0,0)$ is negative definite ($-H\,(0,0,0)$ is positive definite), and hence $(0,0,0)$ is a point of maximum.

5. e. To find the critical points one needs to solve the system

$$\begin{aligned} f_x &= 2 - \tfrac{y^2}{2x^2} = 0 \\ f_y &= \tfrac{y}{x} - \tfrac{2z^2}{y^2} = 0 \\ f_z &= \tfrac{4z}{y} - \tfrac{4}{z^2} = 0\,. \end{aligned}$$

From the first equation $\frac{y}{x} = \pm 2$. Using this relation, conclude from the second equation that $\frac{y}{x} = 2$. Then the second equation implies that $\frac{z}{y} = \pm 1$. The third equation implies that $\frac{z}{y} = 1$. Then the third equation gives $z = \pm 1$. Assume first that $z = 1$. Then the second equation takes the form

$$2 - \frac{2}{y^2} = 0\,.$$

Then $y = \pm 1$, and in view of the third equation, $y = 1$. Since $\frac{y}{x} = 2$, obtain $x = \frac{1}{2}$. So that $\left(\frac{1}{2}, 1, 1\right)$ is a critical point. Since $f(x,y,z)$ is an odd function, it follows that $\left(-\frac{1}{2}, -1, -1\right)$ is also a critical point. Calculate the Hessian at $\left(\frac{1}{2}, 1, 1\right)$

$$H\left(\frac{1}{2}, 1, 1\right) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 6 & -4 \\ 0 & -4 & 12 \end{bmatrix}.$$

By Sylvester's criterion, this matrix is positive definite, and hence $\left(\frac{1}{2}, 1, 1\right)$ is a point of minimum. Since $f(x,y,z)$ is an odd function, it follows that $\left(-\frac{1}{2}, -1, -1\right)$ is a point of maximum.

5. f. Set the first partials to zero. From

$$f_{x_1} = 1 - \frac{x_2}{x_1^2} = 0$$

obtain $x_2 = x_1^2$. From

$$f_{x_2} = \frac{1}{x_1} - \frac{x_3}{x_2^2} = 0$$

obtain $x_3 = \frac{x_2^2}{x_1} = x_1^3$. Continue, to get $x_i = x_1^i$, $i = 2, 3, \ldots, n$. (The last relation, $x_n = x_1^n$ follows from $f_{x_{n-1}} = 0$.) Using these relations in $f(x_1, x_2, \ldots, x_n)$ obtain that

$$f = f(x_1) = nx_1 + \frac{2}{x_1^n}\,.$$

at any critical point. This function has a global minimum at $x_1 = 2^{\frac{1}{n+1}}$.

6. Set the first partials to zero

$$\begin{aligned}\cos x - \cos(x + y + z) &= 0\\ \cos y - \cos(x + y + z) &= 0\\ \cos z - \cos(x + y + z) &= 0\,.\end{aligned}$$

It follows that

$$\cos x = \cos y = \cos z\,.$$

Since $\cos x$ is decreasing on $(0, \pi)$, conclude that

$$x = y = z\,,$$

and then

$$\cos 3x - \cos x = 0\,.$$

Using the trig identity $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, write the last equation as

$$-2\sin 2x \sin x = 0\,.$$

$x = \frac{\pi}{2}$ is the only solution inside $(0, \pi)$. Hence the function $f(x, y, z) = \sin x + \sin y + \sin z - \sin(x + y + z)$ has only one critical point, $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$, on $(0, \pi)$.

Calculate the Hessian at the critical point

$$H\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}.$$

This matrix is negative definite, since its negative $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is positive definite by Sylvester's criterion. Hence, $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ is a point of maximum of $f(x, y, z)$.

7. a. The Hessian is positive definite.

7. b. The Hessian is negative definite.

7. c. The Hessian is indefinite.

8. a. Apply $R_2 - 3R_1$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

So that $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Factor:

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The first factor on the right is D, and the second one is U. (The $A = LDU$ decomposition involves "a new U", when compared with the $A = LU$ decomposition.)

9. Calculate the $A = LDU$ decomposition, and just observe that $U = L^T$, since the matrix A is symmetric.

Section 7.2

1. a. The Jacobian

$$J(0,0) = \begin{vmatrix} u_x(0,0) & u_y(0,0) \\ v_x(0,0) & v_y(0,0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0\,.$$

The implicit function theorem does not apply.

1. b. The Jacobian

$$J(0,1) = \begin{vmatrix} u_x(0,1) & u_y(0,1) \\ v_x(0,1) & v_y(0,1) \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \neq 0\,.$$

The implicit function theorem applies.

1. c. The Jacobian

$$J(1,0) = \begin{vmatrix} u_x(1,0) & u_y(1,0) \\ v_x(1,0) & v_y(1,0) \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & e \end{vmatrix} = 3e \neq 0\,.$$

The implicit function theorem applies.

2. a. To show that $1, 1$ components are the same on the left and on the right, one needs

$$x_p = x_u u_p + x_v v_p \, ,$$

which follows by the multivariable chain rule. Similarly, the other components are equal.

3. b. Make a change of variables $x = au$, $y = bv$, $z = cw$. Instead of using the Jacobian, one may simply write $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$. Then

$$\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dxdydz = abc \iiint_B \sqrt{1 - u^2 - v^2 - w^2}\, dudvdw \, ,$$

where B is the unit ball $u^2 + v^2 + w^2 \le 1$. Use spherical coordinates in the last integral to obtain

$$abc \int_0^{2\pi}\int_0^{\pi}\int_0^1 \sqrt{1-\rho^2}\rho^2 \sin\varphi \, d\rho d\varphi d\theta = 4\pi abc \int_0^1 \sqrt{1-\rho^2}\,\rho^2 \, d\rho = \frac{\pi^2}{4} abc \, .$$

(The integral $\int_0^1 \sqrt{1-\rho^2}\,\rho^2\,d\rho$ is computed by a trig substitution $x = \sin\theta$.)

2. c. The volume is given by $\iiint_V \, dxdydz$. Proceeding as in part b, obtain

$$\iiint_V \, dxdydz = abc \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \frac{4}{3}\pi abc \, .$$

Section 7.3

1. a. Here $x = 2\cos t$, $y = 3\sin t$, or

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \, .$$

2. With $\gamma(t) = (x(t), y(t), 0)$, calculate $\gamma'(t) = (x'(t), y'(t), 0)$, $\gamma''(t) = (x''(t), y''(t), 0)$, $||\gamma'(t)|| = \left(x'^2 + y'^2\right)^{\frac{1}{2}}$, and

$$\gamma'(t) \times \gamma''(t) = \left(0, 0, x'(t)y''(t) - x''(t)y'(t)\right) \, .$$

By Theorem 7.3.2

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(x'^2(t) + y'^2(t)\right)^{\frac{3}{2}}} \, .$$

1. b. If t is a polar angle, then $t=\frac{\pi}{4}$ is the line $y=x$. On the torus, $t=\frac{\pi}{4}$ gives the point $\left(2\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}\right)$ that is not on the line $y=x$.

3. a. Since s is arc-length, $x'^2(s)+y'^2(s)=1$ for all s. Then use the formula from exercise 2.

4. With $\gamma(x)=(x,f(x),0)$, calculate $\gamma'(x)=(1,f'(x),0)$, $\gamma''(x)=(0,f''(x),0)$, $||\gamma'(x)||=\left(1+f'^2(x)\right)^{\frac{1}{2}}$, and

$$\gamma'(x)\times\gamma''(x)=\left(0,0,f''(x)\right)\,.$$

By Theorem 7.3.2

$$\kappa(x)=\frac{|f''(x)|}{\left(1+f'^2(x)\right)^{\frac{3}{2}}}\,.$$

5. d. Use the definitions of $\tanh u$ and $\operatorname{sech} u$, and 5. b.

6. a. Write the unit sphere as

$$x^2+y^2=1-z^2=1-\sin^2\varphi\,.$$

When $\varphi=\frac{\pi}{4}$, obtain the circle

$$x^2+y^2=\frac{1}{2}\,,$$

which is a circle on the plane $z=\frac{\sqrt{2}}{2}$.

6. b. Once the curve $\sigma(\theta,\frac{\pi}{4})$ has been identified as a circle, there is no need for integration to find its length. It is $2\pi r=2\pi\frac{\sqrt{2}}{2}=\sqrt{2}\pi$.

6. c. The point on the sphere is $\sigma(\frac{\pi}{4},\frac{\pi}{4})=\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$. Calculate $\sigma_\theta=(-\sin\theta\sin\varphi,\cos\theta\sin\varphi,0)$, $\sigma_\theta(\frac{\pi}{4},\frac{\pi}{4})=\left(-\frac{1}{2},\frac{1}{2},0\right)$, $\sigma_\varphi=(\cos\theta\cos\varphi,\sin\theta\cos\varphi,-\sin\varphi)$, $\sigma_\varphi(\frac{\pi}{4},\frac{\pi}{4})=\left(\frac{1}{2},\frac{1}{2},-\frac{1}{\sqrt{2}}\right)$. The normal to the tangent plane is

$$\bar{N}=\sigma_\theta(\frac{\pi}{4},\frac{\pi}{4})\times\sigma_\varphi(\frac{\pi}{4},\frac{\pi}{4})=\left(-\frac{1}{2\sqrt{2}},-\frac{1}{2\sqrt{2}},-\frac{1}{2}\right)\,.$$

The equation of the tangent plane at the point $\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ is

$$-\frac{1}{2\sqrt{2}}(x-\frac{1}{2})-\frac{1}{2\sqrt{2}}(y-\frac{1}{2})-\frac{1}{2}(z-\frac{1}{\sqrt{2}})=0\,.$$

Section 7.4

1. a. Here $x = u - v$, $y = u + v$, so that

$$x^2 + y^2 = (u - v)^2 + (u + v)^2 = 2(u^2 + v^2) = 2z\,.$$

Calculate

$$\sigma_u = (1, 1, 2u)\ ,$$

$$\sigma_v = (-1, 1, 2v)\ ,$$

$$E = \sigma_u \cdot \sigma_u = 2 + 4u^2\,,$$

$$F = \sigma_u \cdot \sigma_v = 4uv\,,$$

$$G = \sigma_v \cdot \sigma_v = 2 + 4v^2\,.$$

1. e. Here $x^2 + y^2 = u^2 = z^2$. Calculate

$$\sigma_u = (\cos v, \sin v, 1)\ ,$$

$$\sigma_v = (-u \sin v, u \cos v, 0)\ ,$$

$$E = \sigma_u \cdot \sigma_u = 2\,,$$

$$F = \sigma_u \cdot \sigma_v = 0\,,$$

$$G = \sigma_v \cdot \sigma_v = u^2\,.$$

2. The projection of this curve on the xy-plane is

$$x^2 + y^2 = u^2 = e^{4t}\,.$$

Write this projection in polar coordinates:

$$r = e^{2t}\,,$$

which is an expanding spiral. Since $z = u = e^{2t}$, the curve is climbing. The curve is somewhat similar to helix (although expanding and climbing fast).

Write this curve as

$$\gamma(t) = \left(e^{2t} \cos t, e^{2t} \sin t, e^{2t}\right).$$

Calculate $||\gamma'(t)|| = 3e^{2t}$, and then the length is

$$\int_0^{2\pi} ||\gamma'(t)||\, dt = \int_0^{2\pi} 3e^{2t}\, dt = \frac{3}{2}\left(e^{4\pi} - 1\right) .$$

3. Calculate

$$\sigma_x = (1, 0, f_x) ,$$
$$\sigma_y = (0, 1, f_y) ,$$
$$E = \sigma_x \cdot \sigma_x = 1 + f_x^2 ,$$
$$F = \sigma_x \cdot \sigma_y = f_x f_y ,$$
$$G = \sigma_y \cdot \sigma_y = 1 + f_y^2 .$$

4. The surface is $z = x^2 + y^2 + 2x$. Write this surface as

$$z = (x-1)^2 + y^2 - 1 ,$$

a paraboloid with the vertex at the point $(1, 0, -1)$.

Calculate

$$\sigma_u = (1, 0, 2u + 2) ,$$
$$\sigma_v = (0, 1, 2v) ,$$
$$E = \sigma_u \cdot \sigma_u = 1 + 4(u+1)^2 ,$$
$$F = \sigma_u \cdot \sigma_v = 4(u+1)v ,$$
$$G = \sigma_v \cdot \sigma_v = 1 + 4v^2 .$$

Then

$$\cos\theta = \frac{4(u+1)v}{\sqrt{[1 + 4(u+1)^2]\,(1 + 4v^2)}} .$$

Here θ is the angle between the coordinate curves at the point $\sigma(u, v)$.

5. a. Write the vectors in components: $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. Then both sides of the vector identity are equal to $a_2b_1c_2d_1 - a_1b_2c_2d_1 + a_3b_1c_3d_1 - a_1b_3c_3d_1 - a_2b_1c_1d_2 + a_1b_2c_1d_2 + a_3b_2c_3d_2 - a_2b_3c_3d_2 - a_3b_1c_1d_3 + a_1b_3c_1d_3 - a_3b_2c_2d_3 + a_2b_3c_2d_3$.

I used *Mathematica*.

5. d. Since the surface is regular, $E = \sigma_u \cdot \sigma_u > 0$ (otherwise the vectors σ_u and σ_v are linearly dependent). By part c, $EG - F^2 > 0$. By Sylverster's criterion, the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ of the first fundamental form is positive definite.

6. a. Consider the surface $\sigma(u, v) = (x(u, v), y(u, v), 0)$. Calculate

$$\sigma_u = (x_u(u, v), y_u(u, v), 0) \; ,$$
$$\sigma_v = (x_v(u, v), y_v(u, v), 0) \; ,$$
$$E = \sigma_u \cdot \sigma_u = x_u^2 + y_u^2 \, ,$$
$$G = \sigma_v \cdot \sigma_v = x_v^2 + y_v^2 \, ,$$
$$F = \sigma_u \cdot \sigma_v = x_u x_v + y_u y_v \, ,$$
$$EG - F^2 = (x_u^2 + y_u^2)(x_v^2 + y_v^2) - (x_u x_v + y_u y_v)^2 = (x_u y_v - y_u x_v)^2 \; ,$$
$$\sqrt{EG - F^2} = |x_u y_v - y_u x_v| = | \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} | \, ,$$

the absolute value of the Jacobian. (Recall that $\sqrt{z^2} = |z|$.) Then the area of the region R is

$$\iint_D \sqrt{EG - F^2} \, dudv = \iint_D | \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} | \, dudv \, .$$

7. a. Write $\sigma(u(t), v(t)) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$. The derivative of the vector function $\sigma(u(t), v(t))$ is obtained by differentiation of each component, for which the "usual chain rule" applies.

Section 7.5

1. a. With $\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, calculate

$$\sigma_u(u, v) = \left(f'(u) \cos v, f'(u) \sin v, g'(u)\right) ,$$
$$\sigma_v(u, v) = (-f(u) \sin v, f(u) \cos v, 0) \, ,$$
$$\sigma_{uu}(u, v) = \left(f''(u) \cos v, f''(u) \sin v, g''(u)\right) ,$$
$$\sigma_{uv}(u, v) = \left(-f'(u) \sin v, f'(u) \cos v, 0\right) ,$$
$$\sigma_{vv}(u, v) = (-f(u) \cos v, -f(u) \sin v, 0) \, ,$$

$$\sigma_u(u,v) \times \sigma_v(u,v) = \left(-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u)\right),$$
$$||\sigma_u(u,v) \times \sigma_v(u,v)||^2 = f^2(u)\left(f'^2(u) + g'^2(u)\right) = f^2(u),$$
$$\bar{N} = \frac{\sigma_u(u,v) \times \sigma_v(u,v)}{||\sigma_u(u,v) \times \sigma_v(u,v)||} = \left(-g'(u)\cos v, -g'(u)\sin v, f'(u)\right),$$
$$L = \sigma_{uu}(u,v) \cdot \bar{N} = f'g'' - f''g',$$
$$M = \sigma_{uv}(u,v) \cdot \bar{N} = 0,$$
$$N = \sigma_{vv}(u,v) \cdot \bar{N} = fg'.$$

The second fundamental form is $\left(f'g'' - f''g'\right) du^2 + fg'dv^2$.

2. a. The characteristic equation $|A - \lambda B| = 0$ takes the form

$$\begin{vmatrix} -1-3\lambda & 0 \\ 0 & 2-4\lambda \end{vmatrix} = 0,$$

or

$$(1+3\lambda)(1-2\lambda) = 0.$$

The roots (the generalized eigenvalues) are $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = \frac{1}{2}$.

The generalized eigenvectors corresponding to $\lambda_1 = -\frac{1}{3}$ are solutions of

$$(A - \frac{1}{3}B)x = 0.$$

The first equation of this system is $0 = 0$, and it is discarded. The second equation becomes

$$\frac{10}{3}x_2 = 0.$$

Then $x_2 = 0$, while x_1 is arbitrary. The generalized eigenvectors corresponding to $\lambda_1 = -\frac{1}{3}$ are multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The generalized eigenvectors corresponding to $\lambda_2 = \frac{1}{2}$ are solutions of

$$(A + \frac{1}{2}B)x = 0.$$

The second equation of this system is $0 = 0$, and it is discarded. The first equation becomes

$$-\frac{5}{2}x_1 = 0.$$

Then $x_1 = 0$, while x_2 is arbitrary. The generalized eigenvectors corresponding to $\lambda_2 = \frac{1}{2}$ are multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In general if the matrices A and B are both diagonal, of the form $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$, the characteristic equation $|A - \lambda B| = 0$ takes the form

$$\begin{vmatrix} a_1 - \lambda b_1 & 0 \\ 0 & a_2 - \lambda b_2 \end{vmatrix} = 0,$$

$$(a_1 - \lambda b_1)(a_2 - \lambda b_2) = 0.$$

Its roots, the generalized eigenvalues, are $\lambda_1 = \frac{a_1}{b_1}$, $\lambda_2 = \frac{a_2}{b_2}$. The corresponding generalized eigenvectors are the coordinate vectors e_1 and e_2.

2. b. The characteristic equation $|A - \lambda B| = 0$ takes the form

$$\begin{vmatrix} 1 - 2\lambda & 2 - \lambda \\ 2 - \lambda & 1 - 2\lambda \end{vmatrix} = 0\,,$$

$$(1 - 2\lambda)^2 - (2 - \lambda)^2 = 0\,,$$

$$3(\lambda^2 - 1) = 0\,.$$

The roots (the generalized eigenvalues) are $\lambda_1 = -1$ and $\lambda_2 = 1$. The generalized eigenvectors corresponding to $\lambda_1 = -1$ are solutions of

$$(A + B)x = 0\,,$$

which are multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The generalized eigenvectors corresponding to $\lambda_2 = 1$ are solutions of

$$(A - B)x = 0\,,$$

which are multiples of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. a. Obtain

$$B\left(\frac{x}{\sqrt{Bx \cdot x}}\right) \cdot \frac{x}{\sqrt{Bx \cdot x}} = \frac{Bx \cdot x}{Bx \cdot x} = 1\,.$$

4. Multiply by B and divide by λ:

$$BA^{-1}x = \frac{1}{\lambda}x\,.$$

Hence, $\frac{1}{\lambda}$ is an eigenvalue of BA^{-1}.

Section 7.6

1. If A and B are the matrices of the second and the first fundamental forms respectively, then the characteristic equation $|A - kB| = 0$ takes the form

$$\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0\,,$$

$$(L-kE)(N-kG) - (M-kF)^2 = 0\,,$$

$$\left(EG - F^2\right)k^2 + (-GL + 2FM - EN)\,k + LN - M^2 = 0\,.$$

If k_1 and k_2 are roots of the last quadratic equation, it can be factored as

$$\left(EG - F^2\right)(k-k_1)(k-k_2) = 0\,.$$

Compare the constant terms of the last two equations

$$\left(EG - F^2\right)k_1k_2 = LN - M^2\,.$$

It follows that the Gaussian curvature satisfies $K = k_1k_2 = \frac{LN-M^2}{EG-F^2}$.

2. For the torus $\sigma(\theta,\varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta)$, calculate

$$\sigma_\theta(\theta,\varphi) = (-b\sin\theta\cos\varphi, -b\sin\theta\sin\varphi, b\cos\theta)\,,$$

$$\sigma_\varphi(\theta,\varphi) = (-(a + b\cos\theta)\sin\varphi, (a + b\cos\theta)\cos\varphi, 0)\,.$$

$$E = \sigma_\theta \cdot \sigma_\theta = b^2,$$

$$F = \sigma_\theta \cdot \sigma_\varphi = 0,$$

$$G = \sigma_\varphi \cdot \sigma_\varphi = (a + b\cos\theta)^2\,.$$

The first fundamental form is $b^2 d\theta^2 + (a + b\cos\theta)^2\,d\varphi^2$.

Calculate further

$$\sigma_{\theta\theta}(\theta,\varphi) = (-b\cos\theta\cos\varphi, -b\cos\theta\sin\varphi, -b\sin\theta)\,,$$

$$\sigma_{\theta\varphi}(\theta,\varphi) = (b\sin\theta\sin\varphi, -b\sin\theta\cos\varphi, 0)\,,$$

$$\sigma_{\varphi\varphi}(\theta,\varphi) = (-(a + b\cos\theta)\cos\varphi, -(a + b\cos\theta)\sin\varphi, 0)\,.$$

$$\sigma_\theta \times \sigma_\varphi = (-b\cos\theta\cos\varphi(a + b\cos\theta), -b\cos\theta\sin\varphi(a + b\cos\theta), -b\sin\theta(a + b\cos\theta))\,,$$

$$||\sigma_\theta \times \sigma_\varphi|| = \sqrt{(\sigma_\theta \times \sigma_\varphi)\cdot(\sigma_\theta \times \sigma_\varphi)} = b(a + b\cos\theta),$$

$$\bar{N} = \frac{\sigma_\theta \times \sigma_\varphi}{||\sigma_\theta \times \sigma_\varphi||} = (-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, -b\sin\theta)\,,$$

$$L = \sigma_{\theta\theta} \cdot \bar{N} = b\cos^2\theta\cos^2\varphi + b\cos^2\theta\sin^2\varphi + b\sin^2\theta = b,$$

$$M = \sigma_{\theta\varphi} \cdot \bar{N} = 0,$$

$$N = \sigma_{\varphi\varphi} \cdot \bar{N} = (a + b\cos\theta)\cos\theta.$$

The second fundamental form is $bd\theta^2 + (a + b\cos\theta)\cos\theta d\varphi^2$.

The matrices of the first and the second fundamental form are both diagonal of the form $A = \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix}$, $B = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$. The characteristic equation $|A - kB| = 0$ takes the form

$$\begin{vmatrix} L - kE & 0 \\ 0 & N - kG \end{vmatrix} = 0,$$

$$(L - kE)(N - kG) = 0.$$

Its roots are the principal curvatures

$$k_1 = \frac{L}{E} = \frac{1}{b},$$

$$k_2 = \frac{N}{G} = \frac{\cos\theta}{a + b\cos\theta}.$$

When $k_2 > 0$, or $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the points on the torus are elliptic (corresponding to the right half of the circle that is being rotated, when producing the torus). Hyperbolic points correspond to $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.