A global solution curve for a class of free boundary value problems arising in plasma physics

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Abstract

We study the existence and multiplicity of solutions and the global solution curve of the following free boundary value problem, arising in plasma physics, see R. Temam [18], and H. Berestycki and H. Brezis [3]: find a function $u(x)$ and a constant b, satisfying

$$\Delta u + g(x, u) = p(x) \ \text{ in } D$$
$$u \mid_{\partial D} = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0 \,.$$

Here $D \subset R^n$, is a bounded domain, with a smooth boundary. This problem can be seen as a PDE generalization of the periodic problem for one-dimensional pendulum-like equations. We use continuation techniques. Our approach is suitable for numerical computations.

Key words: Free boundary value problem, global solution curve.

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1 Introduction

We study the existence and multiplicity of solutions, and the global solution structure of the following free boundary value problem, arising in plasma physics, see R. Temam [18], H. Berestycki and H. Brezis [3], and also K.-C. Chang [6]. On a bounded domain $D \subset R^n$, with a smooth boundary, we seek to find a function $u(x)$ and a constant b, satisfying

(1.1)
$$\Delta u + g(x, u) = p(x) \ \text{ in } D$$
$$u \mid_{\partial D} = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0 \,.$$

Here $g(x, u)$ and $p(x)$ are given functions, n denotes the unit normal vector on ∂D, pointing outside. More recently, this problem was studied in P. Amster, P.L. De Napoli, and M.C. Mariani [2]. As explained in [2], this problem can be seen as a PDE generalization of the periodic problem for the one-dimensional pendulum-like equation:

(1.2)
$$u'' + g(t, u) = p(t)$$
$$u(0) = u(T) = b, \quad u'(0) = u'(T).$$

with $p(t)$ being T-periodic, and $g(t, u)$ is T-periodic in t. Indeed, writing the second boundary condition in (1.2) as $\int_0^T u'' \, dt = 0$, we see that a similar condition for (1.1) is $\int_D \Delta u \, dx = 0$, which by the divergence theorem is equivalent to the second boundary condition in (1.1). There exists an enormous literature for the periodic problem (1.2), see e.g., the review paper of J. Mawhin [14], and [4], [5], [8], [9], [10], [15], [16], [17]. By constrast, not much is known for the more general problem (1.1).

We use continuation techniques to study the solution curves for the problem (1.1), similarly to our approach to the pendulum-like equations [10]. One can think of the problem (1.1) as being "at resonance", i.e., some conditions on $p(x)$ are necessary for its solvability. Indeed, decompose $p(x) = \mu_0 + \theta(x)$, with $\mu_0 = \frac{1}{|D|} \int_D p(x) \, dx$, and $\int_D \theta(x) \, dx = 0$. Integrating the equation in (1.1), we get

(1.3)
$$\int_D g(x, u(x)) \, dx = \mu_0 |D|,$$

where $|D|$ is the volume of D. Assume that the following limits exist: $g(x, \pm\infty) = \lim_{u \to \pm\infty} g(x, u)$, uniformly in $x \in \bar{D}$, with $g(x, \pm\infty) \in L^\infty(D)$, and suppose that

(1.4)
$$g(x, -\infty) < g(x, u) < g(x, \infty) \quad \text{for all } x \in D.$$

Then, from (1.3) and (1.4), we obtain

(1.5)
$$\frac{1}{|D|} \int_D g(x, -\infty) \, dx < \mu_0 < \frac{1}{|D|} \int_D g(x, \infty) \, dx$$

to be a necessary condition for existence of solutions. Similarly to the classical paper of E.M. Landesman and A.C. Lazer [13], we give conditions under which the condition (1.5) is sufficient for existence.

In order to use continuation techniques, we will embed the problem (1.1) into a family of problems

(1.6)
$$\Delta u + k g(x, u) = p(x) \text{ in } D$$
$$u \mid_{\partial D} = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0,$$

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with $0 \leq k \leq 1$. At $k = 1$, one recovers the original problem (1.1), while at $k = 0$ the equation is linear. It turns out that at $k = 0$, the problem has infinitely many solutions, and one may fix a unique solution, by additionally prescribing any value of $\frac{1}{|D|} \int_D u(x)\, dx \equiv \xi_1$. One then performs continuation in k on curves of fixed average. Namely, for any $\xi_1 \in R$, one solves the following problem: find (u, b, μ), as a function of k, solving

(1.7)
$$\Delta u + k g(x, u) = \mu + \theta(x) \ \text{ in } D$$
$$u\,|\,_{\partial D} = b, \quad \int_{\partial D} \tfrac{\partial u}{\partial n}\, ds = 0$$
$$\tfrac{1}{|D|} \int_D u(x)\, dx = \xi_1 \,.$$

To prove the solvability of (1.7), we set up the appropriate function spaces, and show that the corresponding differential operator is one-to-one, and onto, so that the Implicit Function Theorem applies. "Onto" is the harder part. Once the continuation process is completed, and one has a solution of the problem (1.7) at $k = 1$, it remains to show that one can select a value of ξ_1, at which $\mu = \mu_0$, thus giving us a solution of the original problem (1.1). This part is also accomplished by continuation.

In addition to a result of Landesman-Lazer type, we obtain an existence result of D.G. de Figueiredo and W.-M. Ni [7] type, which does not require that limits at infinity exist. We also give a condition for the existence of at least two solutions for the problem (1.1). We show that ξ_1, the average value of the solution, is a global parameter, and then we study the curve $\mu = \mu(\xi_1)$, yielding a multiplicity result.

The continuation approach of this paper is similar to our paper on the pendulum-like equations, see [10]. In addition to its conceptual clarity, this approach is suitable for numerical computation of all solutions of (1.1). Every solution can be obtained by two continuations, first in k, $0 \leq k \leq 1$, and then in ξ_1. These solutions curves are easy to compute numerically e.g., by the predictor-corrector method, since we prove that no turning points (or other singularities) are encountered. We had performed similar numerical computations in [11].

2 Continuation of solutions of any fixed average

We begin with the following linear problem

(2.1)
$$\Delta u = \theta(x) \ \text{ in } D$$
$$u\,|\,_{\partial D} = b, \quad \int_{\partial D} \tfrac{\partial u}{\partial n}\, ds = 0$$
$$\tfrac{1}{|D|} \int_D u(x)\, dx = \xi_1 \,.$$

Lemma 2.1 *Given any $\theta(x) \in L^2(D)$, with $\int_D \theta(x)\,dx = 0$, and any $\xi_1 \in R$, we can find a unique pair $(u, b) \in W^{2,2}(D) \cap W_0^{1,2}(D) \times R$, solving (2.1).*

Proof: Let $v(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$ be the solution of

$$\Delta v = \theta(x) \ \text{ in } D, \ \ v = 0 \ \text{ on } \partial D.$$

By the divergence theorem, $\int_{\partial D} \frac{\partial v}{\partial n}\,ds = 0$. Then $u = v + b$ will give us solution of (2.1), if we select the constant b, so that $\frac{1}{|D|} \int_D (v + b)\,dx = \xi_1$. Uniqueness follows from the fact that any harmonic in D function, satisfying $\int_{\partial D} \frac{\partial u}{\partial n}\,ds = 0$, is a constant. $\diamond$

Corollary 1 *Assume in addition that $\theta(x) \in L^\infty(D)$, and $\xi_1 = 0$. Then there is a constant c, such that*

$$\|u\|_{L^\infty(\bar{D})} \leq c.$$

Proof: Using the $W^{2,p}$ estimates and the Sobolev embedding theorem, we conclude an estimate for $\|v\|_{L^\infty(\bar{D})}$, and then for $|b|$, giving us an estimate of $u = v + b$. $\diamond$

The following Poincaré's inequality is well known. By c_0 we denote the best (largest) constant.

Lemma 2.2 *Assume that $u(x) \in W_0^{1,2}(D)$ is any function satisfying $\int_D u(x)\,dx = 0$. Then there is a constant c_0, depending only on D, such that*

$$\int_D |\nabla u|^2 \, dx \geq c_0 \int_D u^2 \, dx.$$

We consider next the following linear problem: given the function $a(x)$, find a triple $(w(x), \mu^*, b) \in W^{2,2}(D) \times R \times R$, satisfying

(2.2)
$$\Delta w + a(x)w = \mu^*, \ \text{ in } D$$
$$w\,|\,\partial D = b, \quad \int_{\partial D} \frac{\partial w}{\partial n}\,ds = 0$$
$$\int_D w(x)\,dx = 0.$$

Lemma 2.3 *Assume that*

$$a(x) < c_0, \ \text{ for all } x \in D.$$

Then the only solution of the problem (2.2) is $w(x) \equiv 0$, $\mu^* = 0$, $b = 0$.

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Proof: Multiply the equation in (2.2) by w, and integrate over D. Since

$$-\int_D w\Delta w \, dx = \int_D |\nabla w|^2 \, dx + b\int_{\partial D} \frac{\partial w}{\partial n} \, ds = \int_D |\nabla w|^2 \, dx \,,$$

we have by Poincaré's inequality

$$c_0 \int_D w^2 \, dx > \int_D a(x) w^2 \, dx = -\int_D w\Delta w \, dx = \int_D |\nabla w|^2 \, dx \geq c_0 \int_D w^2 \, dx \,,$$

from which we conclude that $w(x) \equiv 0$, and then $\mu^* = 0$, and $b = 0$. $\quad\diamond$

Theorem 2.1 *Consider the problem*

(2.3)
$$\Delta u + kg(x, u) = \mu + \theta(x) \ \ in \ D$$
$$u \,|\, \partial D = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0$$
$$\frac{1}{|D|} \int_D u(x) \, dx = \xi_1 \,,$$

where $\theta(x) \in L^2(D)$ is a given function. The function $u(x)$ and the constants μ and b are unknown. Assume that $g(x, u) \in C^1(\bar{D} \times R)$ satisfies

(2.4)
$$|g_u(x, u)| \leq M, \ \ for \ all \ x \in \bar{D}, \ and \ u \in R \,,$$

and

(2.5)
$$M < \min(c_0, \lambda_2) \,.$$

Then for any $0 \leq k \leq 1$, and $\xi_1 \in R$, there exists a unique solution $(u(x), b, \mu) \in W^{2,2}(D) \times R \times R$, solving (2.3).

Proof: We perform continuation in k. When $k = 0$, we take $\mu = 0$, and obtain the unique solution $(u(x), 0, b)$ by Lemma 2.1. We consider two cases.

Case I: $\xi_1 = 0$.

We shall recast the problem (2.3) in the operator form, after we define the appropriate spaces. We shall denote by $W^{2,2}(D) \cap W_0^{1,2}(D) + R$ the functions of the form $v(x) + b$, with $v(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$, and $b \in R$. Define

$$\bar{H}^2 = \left\{ u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D) + R \ | \ \int_D u(x) \, dx = 0, \ \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0 \right\} \,.$$

One checks that $\bar{H}^2$ is a Banach space. Consider the map

$$F(u, \mu, k) = \Delta u + kg(x, u) - \mu \ : \ \bar{H}^2 \times R \times R \to L^2(D) \,.$$

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Then the problem (2.3) can be written in the operator form

$$(2.6) \qquad\qquad F(u, \mu, k) = \theta(x).$$

Observe that the unknown constant b is now "placed" in the definition of the space $\bar{H}^2$. We wish to apply the Implicit Function Theorem to perform continuation in k. For that we need to show that the linearized operator

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = \Delta w + k g_u(x, u)w - \mu^*$$

is one-to-one and onto.

It is one-to-one, because the only solution of the problem

$$(2.7) \qquad\qquad \Delta w + k g_u(x, u)w - \mu^* = 0, \quad \text{in } D$$
$$w \,|\, \partial D = b, \quad \int_{\partial D} \frac{\partial w}{\partial n} \, ds = 0$$
$$\int_D w(x) \, dx = 0$$

is $w(x) \equiv 0$, $\mu^* = 0$, in view of Lemma 2.3.

Turning to the "onto" part, we need to show that given any $\theta(x) \in L^2(D)$, one can find $w \in \bar{H}^2$, and $\mu^* \in R$, solving

$$(2.8) \qquad\qquad \Delta w + k g_u(x, u)w = \mu^* + \theta(x), \quad \text{in } D$$
$$w \,|\, \partial D = b, \quad \int_{\partial D} \frac{\partial w}{\partial n} \, ds = 0$$
$$\int_D w(x) \, dx = 0 \,.$$

We consider two further cases.

Case 1. The operator $L \,:\, W^{2,2}(D) \cap W_0^{1,2}(D) \to L^2(D)$, defined by

$$Lw \equiv \Delta w + k g_u(x, u)w, \quad \text{subject to } w = 0 \text{ on } \partial D$$

is invertible. Let $w_1 = L^{-1}(1)$, i.e., w_1 satisfies

$$(2.9) \qquad \Delta w_1 + k g_u(x, u)w_1 = 1, \quad \text{in } D, \quad w_1 = 0 \quad \text{on } \partial D \,.$$

We also consider $z = 1 - L^{-1}\left(k g_u(x, u)\right)$, i.e., z satisfies

$$(2.10) \qquad \Delta z + k g_u(x, u)z = 0, \quad \text{in } D, \quad z = 1 \quad \text{on } \partial D \,.$$

We shall build the solution of (2.8), by using w_1 and z. Multiplying (2.9) by z, subtracting the equation (2.10) multiplied by w_1, and integrating

$$(2.11) \qquad\qquad \int_D z \, dx = \int_{\partial D} \frac{\partial w_1}{\partial n} \, ds \,.$$

Sub-case i. The integrals in (2.11) are both non-zero. We now construct the solution of (2.8) in the form

$$w = \mu^*(w_1 + bz) + L^{-1}(\theta(x)) + b_1 z \,,$$

with the constants μ^*, b and b_1 to be selected. Clearly, w satisfies the equation in (2.8), while $w \,|\, \partial D = b + b_1$. Since $\int_D z \, dx \neq 0$, we can select b and b_1, such that

$$\int_D (w_1 + bz) \, dx = \int_D (L^{-1}(\theta) + b_1 z) \, dx = 0 \,.$$

It follows that $\int_D w(x) \, dx = 0$, for any μ^*. The function $W \equiv w_1 + bz$ satisfies

$$\Delta W + k g_u(x, u)W = 1, \ \text{ in } D$$
$$W \,|\, \partial D = b,$$
$$\int_D W(x) \, dx = 0 \,.$$

By Lemma 2.3, it follows that $\int_{\partial D} \frac{\partial W}{\partial n} \, ds \neq 0$. Hence, we can select μ^*, so that

$$\int_{\partial D} \frac{\partial w}{\partial n} \, ds = 0 \,.$$

Sub-case ii. The integrals in (2.11) are both zero. Since $\int_D z(x) \, dx = 0$, it follows by Lemma 2.3 that

(2.12) $$\int_{\partial D} \frac{\partial z}{\partial n} \, ds \neq 0 \,.$$

Similarly, since $\int_{\partial D} \frac{\partial w_1}{\partial n} \, ds = 0$, we conclude by Lemma 2.3 that

(2.13) $$\int_D w_1(x) \, dx \neq 0 \,.$$

We now construct the solution of (2.8) in the form

$$w = \mu^* w_1 + L^{-1}(\theta(x)) + bz \,.$$

Clearly, w satisfies the equation in (2.8), while $w \,|\, \partial D = b$. By (2.13) we can choose μ^*, so that $\int_D w(x) \, dx = 0$, for all b, and then we choose b, so that $\int_{\partial D} \frac{\partial w}{\partial n} \, ds = 0$, by using (2.12).

Case 2. The operator L is not invertible.

Since we have assumed $kg_u(x, u) < \lambda_2$, it follows that the null space of L is one dimensional, spanned by some $\bar{w}$, with $\bar{w}(x) > 0$ on D. By the Fredholm alternative, given any $f(x) \in L^2(D)$, $L^{-1}(f(x))$ is defined, i.e., the problem

$$\Delta w + kg_u(x, u)w = f(x), \quad \text{in } D, \quad w = 0 \quad \text{on } \partial D$$

is solvable, if and only if $\int_D f(x)\bar{w}(x)\,dx = 0$. We can choose a constant s, so that $\int_D (s + \theta(x))\,\bar{w}(x)\,dx = 0$, which implies that $L^{-1}(s + \theta(x))$ is defined. Similarly, we select a constant t, so that $L^{-1}(-kg_u(x, u) + t)$ is defined, and we set $z = 1 + L^{-1}(-kg_u(x, u) + t)$, i.e., z satisfies

$$(2.14) \qquad \Delta z + kg_u(x, u)z = t, \quad \text{in } D, \quad z = 1 \quad \text{on } \partial D.$$

Sub-case i. $\int_D z\,dx \neq 0$. We construct the solution of (2.8) in the form

$$w = L^{-1}(s + \theta(x)) + b_1 z + a(\bar{w} + b_2 z),$$

with s fixed above, and the constants a, b_1 and b_2 to be selected. We choose b_1 and b_2, so that

$$\int_D \left[L^{-1}(s + \theta(x)) + b_1 z \right] dx = \int_D (\bar{w} + b_2 z)\,dx = 0.$$

Then $\int_D w(x)\,dx = 0$, for any constant a. The function $W(x) = \bar{w} + b_2 z$ satisfies

$$\Delta W + kg_u(x, u)W = b_2 t, \quad \text{in } D$$
$$W\,|_{\partial D} = b_2,$$
$$\int_D W(x)\,dx = 0.$$

By Lemma 2.3, it follows that $\int_{\partial D} \frac{\partial W}{\partial n}\,ds \neq 0$. We select a so that $\int_{\partial D} \frac{\partial w}{\partial n}\,ds = 0$. Then w satisfies (2.8). (Observe that $Lw = s + (b_1 + ab_2)t + \theta(x) \equiv \mu^* + \theta(x)$, $w\,|_{\partial D} = b_1 + ab_2 \equiv b$.)

Sub-case ii. $\int_D z\,dx = 0$. By Lemma 2.3, and (2.14), it follows that $\int_{\partial D} \frac{\partial z}{\partial n}\,ds \neq 0$. Let

$$w(x) = L^{-1}(s + \theta(x)) + a\bar{w} + bz.$$

Choose the constant a, so that

$$\int_D \left[L^{-1}(s + \theta(x)) + a\bar{w} \right] dx = 0.$$

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Then, $\int_D w(x)\,dx = 0$, for any b. By Lemma 2.3, $\int_{\partial D} \frac{\partial z}{\partial n}\,ds \neq 0$. Then choosing b, so that $\int_{\partial D} \frac{\partial w}{\partial n}\,ds = 0$, we conclude that $w(x)$ satisfies (2.8). (Here $Lw = s + bt + \theta(x) \equiv \mu^* + \theta(x)$, $w\,|\,_{\partial D} = b$.)

Case II: $\xi_1 \neq 0$. We reduce it to the case $\xi_1 = 0$, by letting $v = u - \xi_1$. Then $v(x)$ satisfies

$$(2.15) \qquad \Delta v + kg(x, v + \xi_1) = \mu + \theta(x) \ \text{ in } D$$
$$v\,|\,_{\partial D} = b - \xi_1, \quad \int_{\partial D} \frac{\partial v}{\partial n}\,ds = 0$$
$$\tfrac{1}{|D|} \int_D v(x)\,dx = 0\,.$$

We perform the continuation in k exactly the same way as before, since the bound on $|g_v(x, v + \xi_1)|$ remains the same.

We conclude that each solution of (2.3) can be continued locally in k. To show that solutions can be continued for all $0 \leq k \leq 1$, we need an a priori bound on $(u, b, \mu)(k)$. Write $u(x) = \xi_1 + U(x)$, with $\int_D U(x)\,dx = 0$. We have

$$(2.16) \qquad \Delta U + kg(x, \xi_1 + U) = \mu + \theta(x) \ \text{ in } D$$
$$U\,|\,_{\partial D} = b - \xi_1, \quad \int_{\partial D} \frac{\partial U}{\partial n}\,ds = 0\,.$$

By our conditions (2.4) and (2.5), we can find positive constants c_1 and c_2, with $c_1 < c_0$, such that

$$(2.17) \quad |g(x, \xi_1 + U)| \leq c_1|\xi_1 + U| + c_2, \ \text{ for all } x \in \bar{D}, \text{ and } U \in R\,.$$

Multiplying (2.16) by U, and using the Lemma 2.2, we conclude a bound on $||U||_{L^2(D)}$ (keep in mind that ξ_1 is arbitrary, but fixed). Using this bound in (2.16), and the estimate (2.17), we obtain a bound on $||U||_{W^{2,2}(D)}$. We conclude that
$$(2.18) \qquad\qquad ||u||_{W^{2,2}(D)} \leq c_3|\xi_1| + c_4 \equiv c_5\,.$$
Integrating the equation (2.16) over D, using (2.17) and the bound on $||U||_{L^2(D)}$, we obtain a bound on μ. Writing $u = b + V$, and proceeding similarly, we conclude an L^2 bound on V, which implies a bound on b, since $||u||_{L^2(D)}$ is bounded. $\diamondsuit$

This theorem implies that for any ξ_1 we have a curve of solutions $(u, b, \mu)(k)$ $\in W^{2,2}(D) \times R \times R$ solving (2.3), and at $k = 1$, we have a solution of

$$(2.19) \qquad \Delta u + g(x, u) = \mu + \theta(x) \ \text{ in } D$$
$$u\,|\,_{\partial D} = b, \quad \int_{\partial D} \frac{\partial u}{\partial n}\,ds = 0$$
$$\tfrac{1}{|D|} \int_D u(x)\,dx = \xi_1\,.$$

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Recall that the original problem (1.1), after we decomposed $p(x) = \mu_0 + \theta(x)$, is

(2.20)
$$\Delta u + g(x, u) = \mu_0 + \theta \ \text{ in } D$$
$$u \,|\, \partial D = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0 \,.$$

Hence, it remains to show that one can choose ξ_1, so that $\mu = \mu_0$ in (2.19). The corresponding $u(x)$ (from (2.19)) is then a solution of (2.20).

Remark If the domain D is an interval in one dimension, then $c_0 = \lambda_2$. Indeed, suppose $D = (-L, L)$. Represent $u(x) \in W_0^{1,2}(-L, L)$ by its Fourier series
$$u(x) = a_0 + \Sigma_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \,.$$

If $\int_{-L}^{L} u(x) \, dx = 0$, then $a_0 = 0$, and hence
$$\int_{-L}^{L} u'^2(x) \, dx \geq \frac{\pi^2}{L^2} \int_{-L}^{L} u^2(x) \, dx = 0 \,,$$

i.e., $c_0 = \frac{\pi^2}{L^2}$. The Dirichlet eigenvalues are $\lambda_n = \frac{n^2 \pi^2}{(2L)^2}$, so that $c_0 = \lambda_2$.

3   Continuation in ξ_1

We have just seen that for each ξ_1 the problem

(3.1)
$$\Delta u + g(x, u) = \mu + \theta(x) \ \text{ in } D$$
$$u \,|\, \partial D = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0$$

has a solution $(u, b, \mu) \in W^{2,2}(D) \times R \times R$, with the average value of $u(x)$ equal to ξ_1. We now show that all of these solutions lie on a unique continuous solution curve, which is globally parameterized by ξ_1.

Theorem 3.1 *Assume that the conditions of the Theorem 2.1 hold. Then any solution of (3.1) can be continued in ξ_1, for all $-\infty < \xi_1 < \infty$, giving us a curve of solutions $(u, b, \mu)(\xi_1) \in W^{2,2}(D) \times R \times R$. All solutions of the problem (3.1) lie on this continuous solution curve. Moreover, for each ξ_1 there exists a unique solution (u, b, μ) of (3.1), i.e., ξ_1 is a global parameter.*

Proof: We use the Implicit Function Theorem to show that any solution of (3.1) can be continued in ξ_1. The proof is essentially the same, as the

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one above for continuation in k. We let $u = \xi_1 + v$, with $\int_D v(x)\, dx = 0$. Defining the map

$$F(v, \mu, \xi_1) = \Delta v + g(x, v + \xi_1) - \mu \; : \; \bar{H}^2 \times R \times R \to L^2(D)\,,$$

we recast the problem (3.1) in the operator form

(3.2) $$F(v, \mu, \xi_1) = \theta(x)\,.$$

Similarly to the Theorem 2.1, we apply the Implicit Function Theorem to perform local continuation in ξ_1, by showing that the linearized operator

$$F_{(v,\mu)}(u, \mu, \xi_1)(w, \mu^*) = \Delta w + g_u(x, v + \xi_1)w - \mu^*$$

is one-to-one and onto. By the estimate (2.18), $||v||_{W^{2,2}(D)}$ does not grow more than linearly in $|\xi_1|$, so that we can continue the solutions for all $-\infty < \xi_1 < \infty$.

Turning to the uniqueness, assume that there are two different triples of solutions $(u_1(x), b_1, \mu_1)$ and $(u_2(x), b_2, \mu_2)$, with $u_1(x)$ and $u_2(x)$ having the same average value ξ_1^0. By the Theorem 2.1, we can continue both triples for decreasing k on two curves of solutions, with fixed average ξ_1^0. These curves do not intersect, since at all points the Implicit Function Theorem applies. At $k = 0$, we obtain a contradiction with the uniqueness part of Lemma 2.1. $\diamondsuit$

4 Existence and multiplicity of solutions

We now discuss which values of μ are covered in (3.1), as we continue solutions in $\xi_1 \in (-\infty, \infty)$, which translates into existence results for our problem (1.1). In this section we shall assume $g(x, u)$ to be bounded, which will provide us with uniform in ξ_1 estimates of solutions in (3.1). We have the following result of Landesman-Lazer type.

Theorem 4.1 *Assume that the function $g(x, u) \in L^\infty(\bar{D} \times R)$ satisfies the conditions (1.4), (2.4) and (2.5). Assume also that $\theta(x) \in L^\infty(D)$. Then the condition (1.5) is necessary and sufficient for the existence of solution for the problem (1.1).*

Proof: We proved above that the condition (1.5) is necessary for solvability, we now prove its sufficiency. By the Theorem 2.1, for any $\xi_1 \in R$, there

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exists a triple $(u, b, \mu) = (u, b, \mu)(\xi_1)$ solving the problem (1.1), with the average of $u(x)$ equal to ξ_1. As before, we write $u = \xi_1 + U$, with $U = U(\xi_1)$ satisfying $\int_D U \, dx = 0$. By the Theorem 3.1, $\mu = \mu(\xi_1)$ is continuous in ξ_1, and integrating the equation (3.1), we express

$$(4.1) \qquad \mu = \frac{1}{|D|} \int_D g\left(x, \xi_1 + U(\xi_1)\right) dx.$$

We need to show that we can choose $\xi_1 = \xi_1^0$, so that $\mu(\xi_1^0) = \mu_0$. (Recall that in (1.1) we set $p(x) = \mu_0 + \theta(x)$.) This will follow, once we prove the existence of the limits

$$(4.2) \qquad \lim_{\xi_1 \to \infty} \int_D g\left(x, \xi_1 + U(\xi_1)\right) dx = \int_D g\left(x, \infty\right) dx,$$

$$(4.3) \qquad \lim_{\xi_1 \to -\infty} \int_D g\left(x, \xi_1 + U(\xi_1)\right) dx = \int_D g\left(x, -\infty\right) dx,$$

since μ_0 is assumed to lie in between these limits. In view of (4.1), $U = U(\xi_1)$ satisfies

$$\Delta U = -g\left(x, \xi_1 + U\right) + \frac{1}{|D|} \int_D g\left(x, \xi_1 + U\right) dx + \theta(x), \text{ in } D$$

$$U \mid \partial D = b_1, \quad \int_{\partial D} \frac{\partial U}{\partial n} \, ds = 0$$

$$\frac{1}{|D|} \int_D U(x) \, dx = 0,$$

with $b_1 = b - \xi_1$, a new unknown constant. Since $g(x, u)$ is bounded, by Corollary 1 to Lemma 2.1, there is a constant c, so that

$$\|U(\xi_1)\|_{L^\infty(\bar{D})} \le c, \text{ uniformly in } \xi_1 \in R,$$

from which the limits (4.2) and (4.3) follow. $\qquad\qquad\qquad \diamondsuit$

The following result is similar to the one in D.G. de Figueiredo and W.-M. Ni [7], who considered a Dirichlet problem.

Theorem 4.2 *Consider the problem*

$$(4.4) \qquad \Delta u + g(x, u) = \theta(x) \text{ in } D$$

$$u \mid \partial D = b, \quad \int_{\partial D} \frac{\partial u}{\partial n} \, ds = 0,$$

where $\theta(x) \in L^\infty(D)$ is a given function, satisfying $\int_D \theta(x) \, dx = 0$. Assume that the function $g(x, u) \in L^\infty(\bar{D} \times R)$ satisfies the conditions (2.4) and (2.5), and in addition assume there is a constant $G > 0$ such that

$$(4.5) \qquad ug(x, u) > 0, \quad \text{for all } |u| > G, \text{ and } x \in D.$$

Then the problem (4.4) has a solution $(u, b) \in W^{2,p}(D) \times R$, for any $p > 1$.

Proof: The proof is similar to the one above. Here we have $\mu^0 = 0$. Using (4.5), we see from (4.1) that $\mu > 0$ (< 0) for ξ_1 large and positive (negative). By continuity, we have $\mu(\xi_1^0) = 0$ for some ξ_1^0. $\diamondsuit$

We also have the following multiplicity result.

Theorem 4.3 *Consider again the problem (4.4), and assume that the conditions of the Theorem 4.2 hold. Assume, in addition, that the limits $g(x, \pm\infty) = \lim_{u \to \pm\infty} g(x, u)$ exist, uniformly in $x \in \bar{D}$, and*

$$(4.6) \qquad\qquad\qquad g(x, \pm\infty) \equiv 0 \,.$$

Then there exists an interval (μ_-, μ_+), with $\mu_- < 0$ and $\mu_+ > 0$, so that for $\mu \in (\mu_-, \mu_+) \setminus \{0\}$ the problem (4.4) has at least two solutions (u_1, b_1) and (u_2, b_2).

Proof: By the Theorem 2.1, for any $\xi_1 \in R$, there exists a triple $(u, b, \mu) = (u, b, \mu)(\xi_1)$ solving the problem (4.4), with average of $u(x)$ equal to ξ_1. As in Theorem 4.1, we continue this solution in ξ_1, paying particular attention to the curve $\mu = \mu(\xi_1)$. From the formula (4.1) and the condition (4.5) it follows that $\mu(\xi_1)$ is positive (negative) for $|\xi_1|$ large and positive (negative). The condition (4.6) implies that $\lim_{|\xi_1| \to \infty} \mu(\xi_1) = 0$, i.e., both sides of the curve $\mu = \mu(\xi_1)$ tend to zero. Hence, we can find some intervals $(\mu_-, 0)$ and $(0, \mu_+)$, which are covered at least twice by the function $\mu = \mu(\xi_1)$. $\diamondsuit$

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