Non-existence of solutions for non-autonomous elliptic systems

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Abstract

We extend the classical Pohozaev's identity to semilinear elliptic systems of Hamiltonian type, providing a simpler approach, and a generalization, of the results of E. Mitidieri [6], R.C.A.M. Van der Vorst [14], and Y. Bozhkov and E. Mitidieri [1].

Key words: Pohozaev's identity, non-existence of solutions.

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1 Introduction

Any solution $u(x)$ of semilinear Dirichlet problem on a bounded domain $\Omega \subset R^n$

$$\Delta u + f(x,u) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \tag{1.1}$$

satisfies the well known Pohozaev's identity

$$\int_\Omega \left[2nF(x,u) + (2-n)uf(x,u) + 2\Sigma_{i=1}^n x_i F_{x_i}(x,u)\right] dx = \int_{\partial\Omega} (x \cdot \nu)|\nabla u|^2 \, dS \, . \tag{1.2}$$

Here $F(x,u) = \int_0^u f(x,t)\,dt$, and ν is the unit normal vector on $\partial\Omega$, pointing outside. (From the equation (1.1), $\int_\Omega uf(x,u)\,dx = \int_\Omega |\nabla u|^2\,dx$, which gives an alternative form of the Pohozaev's identity.) Pohozaev's identity is usually written for the case $f = f(u)$, but the present version is also known, see e.g., K. Schmitt [13]. A standard use of this identity is to conclude that if Ω is a star-shaped domain with respect to the origin, i.e., $x \cdot \nu \geq 0$ for all $x \in \partial\Omega$, and $f(u) = u|u|^{p-1}$, for some constant p, then the problem (1.1) has no non-trivial solutions in the super-critical case, when $p > \frac{n+2}{n-2}$. In this note we present a proof of Pohozaev's identity, which appears a little more straightforward than the usual one, see e.g., L. Evans [2], and then use a similar idea for systems, generalizing the well-known results of E. Mitidieri [6], see also R.C.A.M. Van der Vorst [14], and of Y. Bozhkov and E. Mitidieri [1], by allowing explicit dependence on x in the Hamiltonian function.

Let $z = x \cdot \nabla u = \Sigma_{i=1}^n x_i u_{x_i}$. It is straightforward to verify that z satisfies

$$\Delta z + f_u(x,u)z = -2f(x,u) - \Sigma_{i=1}^n x_i f_{x_i}(x,u) \, . \tag{1.3}$$

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We multiply the equation (1.1) by z, and subtract from that the equation (1.3) multiplied by u, obtaining

(1.4) $\Sigma_{i=1}^n \left(zu_{x_i} - uz_{x_i}\right)_{x_i} + \Sigma_{i=1}^n \left(f(x,u) - uf_u(x,u)\right) x_i u_{x_i} = 2f(x,u)u + \Sigma_{i=1}^n x_i f_{x_i}(x,u)u \, .$

We have

$$\Sigma_{i=1}^n \left(f(x,u) - uf_u(x,u)\right) x_i u_{x_i} = \Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i}(2F - uf) - 2\Sigma_{i=1}^n x_i F_{x_i} + \Sigma_{i=1}^n x_i f_{x_i}(x,u)u =$$
$$\Sigma_{i=1}^n \frac{\partial}{\partial x_i}\left[x_i(2F - uf)\right] - n(2F - uf) - 2\Sigma_{i=1}^n x_i F_{x_i} + \Sigma_{i=1}^n x_i f_{x_i}(x,u)u \, .$$

We then rewrite (1.4)

(1.5) $\Sigma_{i=1}^n \left[(zu_{x_i} - uz_{x_i}) + x_i(2F(x,u) - uf(x,u))\right]_{x_i} = 2nF(x,u) + (2-n)uf(x,u) + 2\Sigma_{i=1}^n x_i F_{x_i} \, .$

Integrating over Ω, we conclude the Pohozaev's identity (1.2). (The only non-zero boundary term is $\Sigma_{i=1}^n \int_{\partial\Omega} z u_{x_i} \nu_i \, dS$. Since $\partial\Omega$ is a level set of u, $\nu = \pm\frac{\nabla u}{|\nabla u|}$, i.e., $u_{x_i} = \pm|\nabla u|\nu_i$. Then $z = \pm(x \cdot \nu)|\nabla u|$, and $\Sigma_{i=1}^n u_{x_i}\nu_i = \pm|\nabla u|$.)

We refer to (1.5) as a *differential form* of Pohozaev's identity. For radial solutions on a ball, the corresponding version of (1.5) played a crucial role in the study of exact multiplicity of solutions, see T. Ouyang and J. Shi [7], and also P. Korman [5], which shows the potential usefulness of this identity.

2 Non-existence of solutions for a class of systems

The following class of systems has attracted considerable attention recently

(2.1)
$$\begin{aligned} &\Delta u + H_v(u,v) = 0 \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega \\ &\Delta v + H_u(u,v) = 0 \ \text{ in } \Omega, \ \ v = 0 \ \text{ on } \partial\Omega \, , \end{aligned}$$

where $H(u,v)$ is a given differentiable function, see e.g., the following surveys: D.G. de Figueiredo [3], P. Quittner and P. Souplet [11], B. Ruf [12], see also P. Korman [4]. This system is of *Hamiltonian* type, so that it has some of the properties of scalar equations.

More generally, let $H = H(x, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m)$, with integer $m \geq 1$, and consider the Hamiltonian system of $2m$ equations

(2.2)
$$\begin{aligned} &\Delta u_k + H_{v_k} = 0 \ \text{ in } \Omega, \ \ u_k = 0 \ \text{ on } \partial\Omega, \ \ k = 1, 2, \ldots, m \\ &\Delta v_k + H_{u_k} = 0 \ \text{ in } \Omega, \ \ v_k = 0 \ \text{ on } \partial\Omega, \ \ k = 1, 2, \ldots, m \, . \end{aligned}$$

We call solution of (2.2) to be positive, if $u_k(x) > 0$ and $v_k(x) > 0$ for all $x \in \Omega$, and all k. We consider only the classical solutions, with u_k and v_k of class $C^2(\Omega) \cap C^1(\bar{\Omega})$. We have the following generalization of the results of [1] and [6].

Theorem 2.1 *Assume that* $H(x, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m) \in C^2(\Omega \times R_+^m \times R_+^m) \cap C(\bar{\Omega} \times \bar{R}_+^m \times \bar{R}_+^m)$ *satisfies*

(2.3)
$$H(x, 0, \ldots, 0, 0, \ldots, 0) = 0 \ \text{ for all } x \in \partial\Omega \, .$$

Then for any positive solution of (2.2), and any real numbers $a_1, \ldots, a_m$*, one has*

(2.4)
$$\begin{aligned} &\int_\Omega \left[2nH + (2-n)\Sigma_{k=1}^m \left(a_k u_k H_{u_k} + (2 - a_k) v_k H_{v_k}\right) + 2\Sigma_{i=1}^n x_i H_{x_i}\right] dx \\ &\qquad = 2\Sigma_{k=1}^m \int_{\partial\Omega} (x \cdot \nu)|\nabla u_k||\nabla v_k| \, dS \, . \end{aligned}$$

Proof: Define $p_k = x \cdot \nabla u_k = \Sigma_{i=1}^n x_i u_{kx_i}$, and $q_k = x \cdot \nabla v = \Sigma_{i=1}^n x_i v_{kx_i}$, $k = 1, 2, \ldots, m$. These functions satisfy the system

$$\begin{aligned}
(2.5) \qquad & \Delta p_k + \Sigma_{j=1}^m H_{v_k u_j} p_j + \Sigma_{j=1}^m H_{v_k v_j} q_j = -2H_{v_k} - \Sigma_{i=1}^n x_i H_{v_k x_i}, \quad k = 1, 2, \ldots, m \\
& \Delta q_k + \Sigma_{j=1}^m H_{u_k u_j} p_j + \Sigma_{j=1}^m H_{u_k v_j} q_j = -2H_{u_k} - \Sigma_{i=1}^n x_i H_{u_k x_i}, \quad k = 1, 2, \ldots, m \,.
\end{aligned}$$

We multiply the first equation in (2.2) by q_k, and subtract from that the first equation in (2.5) multiplied by v_k. The result can be written as

$$\begin{aligned}
(2.6) \qquad & \Sigma_{i=1}^n \left[(u_{kx_i} q_k - p_{kx_i} v_k)_{x_i} + (-u_{kx_i} q_{kx_i} + v_{kx_i} p_{kx_i}) \right] \\
& + H_{v_k} q_k - \Sigma_{j=1}^m H_{v_k u_j} p_j v_k - \Sigma_{j=1}^m H_{v_k v_j} q_j v_k = 2 v_k H_{v_k} + v_k \Sigma_{i=1}^n x_i H_{v_k x_i} \,.
\end{aligned}$$

Similarly, we multiply the second equation in (2.2) by p_k, and subtract from that the second equation in (2.5) multiplied by u_k, and write the result as

$$\begin{aligned}
(2.7) \qquad & \Sigma_{i=1}^n \left[(v_{kx_i} p_k - q_{kx_i} u_k)_{x_i} + (-v_{kx_i} p_{kx_i} + u_{kx_i} q_{kx_i}) \right] \\
& + H_{u_k} p_k - \Sigma_{j=1}^m H_{u_k u_j} p_j u_k - \Sigma_{j=1}^m H_{u_k v_j} q_j u_k = 2 u_k H_{u_k} + u_k \Sigma_{i=1}^n x_i H_{u_k x_i} \,.
\end{aligned}$$

Adding the equations (2.6) and (2.7), we get

$$\begin{aligned}
& \Sigma_{i=1}^n \left[u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k \right]_{x_i} + H_{u_k} p_k + H_{v_k} q_k - \Sigma_{j=1}^m H_{u_k u_j} p_j u_k \\
& - \Sigma_{j=1}^m H_{u_k v_j} q_j u_k - \Sigma_{j=1}^m H_{v_k u_j} p_j v_k - \Sigma_{j=1}^m H_{v_k v_j} q_j v_k \\
& = 2 u_k H_{u_k} + 2 v_k H_{v_k} + u_k \Sigma_{i=1}^n x_i H_{u_k x_i} + v_k \Sigma_{i=1}^n x_i H_{v_k x_i} \,.
\end{aligned}$$

We now sum in k, putting the result into the form

$$\begin{aligned}
& \Sigma_{k=1}^m \Sigma_{i=1}^n \left[u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k \right]_{x_i} \\
& + \Sigma_{i=1}^n x_i \left(2H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k} \right)_{x_i} = 2 \Sigma_{k=1}^m u_k H_{u_k} + 2 \Sigma_{k=1}^m v_k H_{v_k} + 2 \Sigma_{i=1}^n x_i H_{x_i} \,.
\end{aligned}$$

Writing,

$$\begin{aligned}
\Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i} (2H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k}) &= \Sigma_{i=1}^n \frac{\partial}{\partial x_i} \left[x_i (2H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k}) \right] \\
& - n (2H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k}) \,,
\end{aligned}$$

we obtain the differential form of Pohozaev's identity

$$\begin{aligned}
& \Sigma_{k=1}^m \Sigma_{i=1}^n \left[u_{kx_i} q_k - p_{kx_i} v_k + v_{kx_i} p_k - q_{kx_i} u_k + x_i \left(2H - \Sigma_{k=1}^m u_k H_{u_k} - \Sigma_{k=1}^m v_k H_{v_k} \right) \right]_{x_i} \\
& = 2nH + (2 - n) \left(\Sigma_{k=1}^m u_k H_{u_k} + \Sigma_{k=1}^m v_k H_{v_k} \right) + 2 \Sigma_{i=1}^n x_i H_{x_i} \,.
\end{aligned}$$

Integrating, we obtain, in view of (2.3),

$$\begin{aligned}
(2.8) \qquad & \int_\Omega \left[2nH(u, v) + (2 - n) \left(\Sigma_{k=1}^m u_k H_{u_k} + \Sigma_{k=1}^m v_k H_{v_k} \right) + 2 \Sigma_{i=1}^n x_i H_{x_i} \right] dx \\
& = 2 \Sigma_{k=1}^m \int_{\partial\Omega} (x \cdot \nu) |\nabla u_k| |\nabla v_k| \, dS \,.
\end{aligned}$$

(Since we consider positive solutions, and $\partial\Omega$ is a level set for both u_k and v_k, we have $\nu = -\frac{\nabla u_k}{|\nabla u_k|} = -\frac{\nabla v_k}{|\nabla v_k|}$, i.e., $u_{ki} = -|\nabla u_k| \nu_i$ and $v_{ki} = -|\nabla v_k| \nu_i$ on the boundary $\partial\Omega$.) From the first equation in (2.2), $\int_\Omega v_k H_{v_k} \, dx = \int_\Omega \nabla u_k \cdot \nabla v_k \, dx$, while from the second equation $\int_\Omega u_k H_{u_k} \, dx = \int_\Omega \nabla u_k \cdot \nabla v_k \, dx$, i.e., for each k

$$\int_\Omega v_k H_{v_k} \, dx = \int_\Omega u_k H_{u_k} \, dx \,.$$

Using this in (2.8), we conclude the proof. $\diamondsuit$

Remarks

1. We consider only the classical solutions. Observe that by our conditions and elliptic regularity, classical solutions are in fact of class $C^3(\Omega)$, so that all quantities in the above proof are well defined.

2. In case H is independent of x, the condition (2.3) can be assumed without loss of generality.

As a consequence, we have the following non-existence result.

Proposition 1 *Assume that Ω is a star-shaped domain with respect to the origin, and for some real constants $\alpha_1, \dots, \alpha_m$, all $u_k > 0$, $v_k > 0$, and all $x \in \Omega$, we have*

$$nH + (2-n)\Sigma_{k=1}^{m} \left(\alpha_k u_k H_{u_k} + (1-\alpha_k) v_k H_{v_k}\right) + \Sigma_{i=1}^{n} x_i H_{x_i} < 0 \,. \tag{2.9}$$

Then the problem (2.2) has no positive solutions.

Proof: We use the identity (2.4), with $a_k/2 = \alpha_k$. Then, assuming existence of positive solution, the left hand side of (2.4) is negative, while the right hand side is non-negative, a contradiction. $\diamondsuit$

Observe, that it suffices to assume that Ω is star-shaped with respect to any one of its points (which we then take to be the origin).

In case $m = 1$, and $H = H(u, v)$, we recover the following condition of E. Mitidieri [6].

Proposition 2 *Assume that Ω is a star-shaped domain with respect to the origin, and for some real constant α, and all $u > 0$, $v > 0$ we have*

$$\alpha u H_u(u,v) + (1-\alpha) v H_v(u,v) > \frac{n}{n-2} H(u,v) \,. \tag{2.10}$$

Then the problem (2.1) has no positive solutions.

Comparing this result to E. Mitidieri [6], observe that we do not require that $H_u(0,0) = H_v(0,0) = 0$.

An important subclass of (2.1) is

$$\begin{aligned} \Delta u + f(v) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \\ \Delta v + g(u) = 0 \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega \,, \end{aligned} \tag{2.11}$$

which corresponds to $H(u,v) = F(v) + G(u)$, where $F(v) = \int_0^v f(t)\,dt$, $G(u) = \int_0^u g(t)\,dt$. Unlike [6], we do not require that $f(0) = g(0) = 0$. The Theorem 2.1 now reads as follows.

Theorem 2.2 *Let $f,\, g \in C(\bar{R}_+)$. For any positive solution of (2.11), and any real number a, one has*

$$\begin{aligned} \int_\Omega \left[2n(F(v) + G(u)) + (2-n)\left(avf(v) + (2-a)ug(u)\right)\right] dx \\ = 2\int_{\partial\Omega} (x\cdot\nu)|\nabla u||\nabla v|\, dS \,. \end{aligned} \tag{2.12}$$

More generally, we consider

$$\begin{aligned} \Delta u + f(x,v) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \\ \Delta v + g(x,u) = 0 \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega \,, \end{aligned} \tag{2.13}$$

with $H(x,u,v) = F(x,v) + G(x,u)$, where $F(x,v) = \int_0^v f(x,t)\,dt$, $G(x,u) = \int_0^u g(x,t)\,dt$.

Theorem 2.3 *Let $f,\, g \in C(\Omega \times \bar{R}_+)$. For any positive solution of (2.13), and any real number a, one has*

$$(2.14)\int_\Omega \left[2n(F(x,v)+G(x,u)) + (2-n)\,(avf(x,v)+(2-a)ug(x,u)) + 2\Sigma_{i=1}^n x_i\,(F_{x_i}+G_{x_i})\right]\,dx \\ = 2\int_{\partial\Omega}(x\cdot\nu)|\nabla u||\nabla v|\,dS\,.$$

We now consider a particular system

$$(2.15) \qquad \begin{aligned} \Delta u + v^p &= 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \\ \Delta v + g(x,u) &= 0 \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega\,, \end{aligned}$$

with $g(x,u) \in C(\Omega \times \bar{R}_+)$, and a constant $p > 0$.

Theorem 2.4 *Assume that Ω is a star-shaped domain with respect to the origin, and*

$$(2.16)\, nG(x,u) + (2-n)\left(1 - \frac{n}{(n-2)(p+1)}\right)ug(x,u) + \Sigma_{i=1}^n x_i G_{x_i} < 0\,, \ \ \textit{for } x \in \Omega, \textit{ and } u > 0\,.$$

Then the problem (2.15) has no positive solutions.

Proof: We use the identity (2.14), with $f(v) = v^p$. We select the constant a, so that

$$2nF(v) + (2-n)avf(v) = 0\,,$$

i.e., $a = \frac{2n}{(n-2)(p+1)}$. Then, assuming existence of a positive solution, the left hand side of (2.14) is negative, while the right hand side is non-negative, a contradiction. ♢

Observe that in case $p = 1$, the Theorem 2.4 provides a non-existence result for a biharmonic problem with Navier boundary conditions

$$(2.17) \qquad \Delta^2 u = g(x,u) \ \text{ in } \Omega, \quad u = \Delta u = 0 \ \text{ on } \partial\Omega\,.$$

Proposition 3 *Assume that Ω is a star-shaped domain with respect to the origin, and the condition (2.16), with $p = 1$, holds. Then the problem (2.17) has no positive solutions.*

Finally, we consider the system

$$(2.18) \qquad \begin{aligned} \Delta u + v^p &= 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \\ \Delta v + u^q &= 0 \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega\,. \end{aligned}$$

The curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}$ is called a *critical hyperbola*. We recover the following well known result of E. Mitidieri [6], see also R.C.A.M. Van der Vorst [14]. (Observe that we relax the restriction $p,\, q > 1$ from [6].)

Proposition 4 *Assume that $p,\, q > 0$, and*

$$(2.19) \qquad \frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}\,.$$

Then the problem (2.18) has no positive solutions.

Proof: Condition (2.19) implies (2.16), and then the Theorem 2.4 applies. ♢

In case $p = 1$, we recover the following known result, see E. Mitidieri [6].

Proposition 5 *Assume that Ω is a star-shaped domain with respect to the origin, and $q > \frac{n+4}{n-4}$. Then the problem*

$$(2.20) \qquad \Delta^2 u = u^q \ \ \textit{in } \Omega, \quad u = \Delta u = 0 \ \ \textit{on } \partial\Omega$$

has no positive solutions.

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