Lectures on Differential Equations

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Contents

Introduction

This book is based on several courses that I taught at the University of Cincinnati. Chapters 1-4 are based on the course "Differential Equations" for sophomores in science and engineering. Only some basic concepts of multivariable calculus are used (functions of two variables and partial derivatives), and they are reviewed in the text. Chapters 7 and 8 are based on the course "Fourier Series and PDE", and they should provide a wide choice of material for the instructors. Chapters 5 and 6 were used in graduate ODE courses, providing most of the needed material. Some of the sections of this book are outside of the scope of usual courses, but I hope they will be of interest to students and instructors alike. The book has a wide range of problems.

I attempted to share my enthusiasm for the subject, and write a textbook that students will like to read. While some theoretical material is either quoted, or just mentioned without proof, my goal was to show all of the details when doing problems. I tried to use plain language and not to be too wordy. I think that an extra word of explanation has often as much potential to confuse a student, as to be helpful. I also tried not to overwhelm students with new information. I forgot who said it first: "one should teach the truth, nothing but the truth, but not the whole truth".

I hope that experts will find this book useful as well. It presents several important topics that are hard to find in the literature: Massera's theorem, Lyapunov's inequality, Picone's form of Sturm's comparison theorem, "sideways" heat equation, periodic population models, "hands on" numerical solution of nonlinear boundary value problems, the isoperimetric inequality, etc. The book also contains new exposition of some standard topics. We have completely revamped the presentation of the Frobenius method for series solution of differential equations, so that the "regular singular points" are now hopefully in the past. In the proof of the existence and uniqueness theorem, we replaced the standard Picard iterations with monotone iterations, which should be easier for students to absorb. There are many other fresh touches throughout the book. The book contains a number of interesting non-standard problems, including some original ones, published by the author over the years in the Problem Sections of SIAM Review, EJDE, and other journals. All of the challenging problems are provided with hints, making them easy to solve for instructors. We use asterisk (or star) to identify non-standard problems.

How important are differential equations? Here is what Isaac Newton said: "It is useful to solve differential equations". And what he knew was just the beginning. Today differential equations are used widely in science and engineering. This book presents many applications as well. Some of these applications are very old, like the tautochrone problem considered by Christian Huygens in 1659. Some applications, like when a drone is targeting a car, are modern. Differential Equations is also a beautiful subject, which lets students see Calculus "in action".

I attempted to start each topic with simple examples, to keep the presentation focused, and to show all of the details. I think this book is suitable for self-study. However, instructor can help in many ways. He (she) will present the subject with the enthusiasm it deserves, draw more pictures, talk about the history, and his jokes will supplement the lame ones in the book.

I am very grateful to the MAA Book Board, including Steve Kennedy, Stan Seltzer and the whole group of anonymous reviewers, for providing me with detailed lists of corrections and suggested changes. Their help was crucial in making considerable improvements of the manuscript.

It is a pleasure to thank Ken Meyer and Dieter Schmidt for constant encouragement while I was writing this book. I also wish to thank Ken for reading the entire book, and making a number of useful suggestions, like doing Fourier series early, with applications to periodic vibrations and radio tuning. I wish to thank Roger Chalkley, Tomasz Adamowicz, Dieter Schmidt, and Ning Zhong for a number of useful comments. Many useful comments came from students in my classes. They liked the book, and that provided me with the biggest encouragement.

Chapter 1

First Order Equations

First order equations occur naturally in many applications, making them an important object to study. They are also used throughout this book, and are of great theoretical importance. Linear first order equations, the first class of the equations we study, turns out to be of particular importance. Separable, exact and homogeneous equations are also used throughout the book. Applications are made to population modeling, and to various physical and geometrical problems. If a solution cannot be found by a formula, we prove that solution still exists, and indicate how it can be computed numerically.

1.1 Integration by Guess-and-Check

Many problems in differential equations end with a computation of an integral. One even uses the term "integration of a differential equation" instead of "solution". We need to be able to compute integrals quickly, which can be done by using the approach of this section. For example, one can write down

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$$

very quickly, avoiding three integrations by parts.

Recall the product rule

$$(fg)' = fg' + f'g \,.$$

Example 1 $\int xe^x \, dx$. We need to find the function, with the derivative equal to xe^x. If we try a guess: xe^x, then its derivative

$$(xe^x)' = xe^x + e^x$$

has an extra term e^x. To remove this extra term, we subtract e^x from the initial guess, so that

$$\int xe^x\,dx = xe^x - 6e^x + c.$$

By differentiation, we verify that this is correct. Of course, integration by parts may also be used.

Example 2 $\int x\cos 3x\,dx$. Starting with the initial guess $\frac{1}{3}x\sin 3x$, with the derivative equal to $x\cos 3x + \frac{1}{3}\sin 3x$, we compute

$$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c.$$

Example 3 $\int_0^\pi x\cos 3x\,dx = \left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]\Big|_0^\pi = -\frac{2}{9}.$

We see that the initial guess is the product $f(x)g(x)$, chosen in such a way that $f(x)g'(x)$ gives the integrand.

Example 4 $\int xe^{-5x}\,dx$. Starting with the initial guess $-\frac{1}{5}xe^{-5x}$, we compute

$$\int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + c.$$

Example 5 $\int x^2\sin 3x\,dx$. The initial guess is $-\frac{1}{3}x^2\cos 3x$. Its derivative

$$\left(-\frac{1}{3}x^2\cos 3x\right)' = x^2\sin 3x - \frac{2}{3}x\cos 3x$$

has an extra term $-\frac{2}{3}x\cos 3x$. To remove this term, we modify our guess: $-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x$. Its derivative

$$\left(-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x\right)' = x^2\sin 3x + \frac{2}{9}\sin 3x$$

still has an extra term $\frac{2}{9}\sin 3x$. So we make the final adjustment

$$\int x^2\sin 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + c\,.$$

This is easier than integrating by parts twice.

Example 6 $\int x\sqrt{x^2+4}\,dx$. We begin by rewriting the integral as $\int x\left(x^2+4\right)^{1/2}\,dx$. One usually computes this integral by a substitution $u = x^2+4$, with $du = 2x\,dx$. Forgetting a constant multiple for now, the integral becomes $\int u^{1/2}\,du$. Ignoring a constant multiple again, this evaluates to $u^{3/2}$. Returning to the original variable, we have our initial guess $\left(x^2+4\right)^{3/2}$. Differentiation

$$\frac{d}{dx}\left(x^2+4\right)^{3/2} = 3x\left(x^2+4\right)^{1/2}$$

gives us the integrand with an extra factor of 3. To fix that, we multiply the initial guess by $\frac{1}{3}$:

$$\int x\sqrt{x^2+4}\,dx = \frac{1}{3}\left(x^2+4\right)^{3/2} + c.$$

Example 7 $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}\,dx$. Instead of using partial fractions, let us try to split the integrand as

$$\frac{1}{x^2+1} - \frac{1}{x^2+4}\,.$$

This is off by a factor of 3. The correct formula is

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)\,.$$

Then

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + c\,.$$

Sometimes one can guess the splitting twice, as in the following case.

Example 8 $\displaystyle\int \frac{1}{x^2\left(1-x^2\right)}\,dx$.

$$\frac{1}{x^2\left(1-x^2\right)} = \frac{1}{x^2} + \frac{1}{1-x^2} = \frac{1}{x^2} + \frac{1}{(1-x)(1+x)} = \frac{1}{x^2} + \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x}\,.$$

Then (for $|x| < 1$)

$$\int \frac{1}{x^2\left(1-x^2\right)}\,dx = -\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + c\,.$$

1.2 First Order Linear Equations

Background

Suppose we need to find a function $y(x)$ so that

$$y'(x) = x.$$

We have a *differential equation*, because it involves a derivative of the unknown function. This is a *first order equation*, as it only involves the first derivative. Solution is, of course,

$$y(x) = \frac{x^2}{2} + c, \tag{2.1}$$

where c is an arbitrary constant. We see that differential equations have infinitely many solutions. The formula (2.1) gives us the *general solution*. Then we can select the solution that satisfies an extra *initial condition*. For example, for the problem

$$\begin{aligned} y'(x) &= x \\ y(0) &= 5 \end{aligned} \tag{2.2}$$

we begin with the general solution given in formula (2.1), and then evaluate it at $x = 0$

$$y(0) = c = 5.$$

So that $c = 5$, and solution of the problem (14.10) is

$$y(x) = \frac{x^2}{2} + 5.$$

The problem (14.10) is an example of an *initial value problem*. If the variable x represents time, then the value of $y(x)$ at the initial time $x = 0$ is prescribed to be 5. The initial condition may be prescribed at other values of x, as in the following example:

$$\begin{aligned} y' &= y \\ y(1) &= 2e\,. \end{aligned}$$

Here the initial condition is prescribed at $x = 1$, e denotes the Euler number $e \approx 2.718$. Observe that while y and y' are both functions of x, we do not spell this out. This problem can also be solved using calculus. Indeed, we are looking for a function $y(x)$, with the derivative equal to $y(x)$. This is a property of the function e^x, and its constant multiples. The general solution is

$$y(x) = ce^x,$$

and the initial condition gives

$$y(1) = ce = 2e,$$

so that $c = 2$. The solution is then

$$y(x) = 2e^x.$$

We see that the main effort is in finding the general solution. Selecting c, to satisfy the initial condition, is usually easy.

Recall from calculus that

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x).$$

In case $g(x)$ is an integral, we have

$$\frac{d}{dx} e^{\int p(x)\,dx} = p(x)e^{\int p(x)\,dx}, \tag{2.3}$$

because the derivative of the integral $\int p(x)\,dx$ is $p(x)$.

1.2.1 The Integrating Factor

Let us find the general solution of the important class of equations

$$y' + p(x)y = g(x), \tag{2.4}$$

where $p(x)$ and $g(x)$ are given functions. This is a *linear equation*, because $y' + p(x)y$ is a linear combination of the unknown functions y and y', for each fixed x.

Calculate the function

$$\mu(x) = e^{\int p(x)\,dx},$$

and its derivative

$$\mu'(x) = p(x)e^{\int p(x)\,dx} = p(x)\mu\,. \tag{2.5}$$

We now multiply the equation (2.4) by $\mu(x)$, giving

$$\mu y' + p(x)\mu y = \mu g(x)\,. \tag{2.6}$$

Let us use the product rule and the formula (2.5) to calculate the derivative

$$\frac{d}{dx}[\mu y] = \mu y' + \mu' y = \mu y' + p(x)\mu y\,.$$

So that we may rewrite the equation (2.6) in the form

$$\frac{d}{dx}[\mu y] = \mu g(x)\,. \tag{2.7}$$

This relation allows us to compute the general solution. Indeed, we know the function on the right. By integration, we express $\mu(x)y(x) = \int \mu(x)g(x)\,dx$, and then solve for $y(x)$.

In practice one needs to memorize the formula for the *integrating factor* $\mu(x)$, and the form (2.7) of our equation (2.4). When computing $\mu(x)$, we shall always take the constant of integration to be zero, $c = 0$, because the method works for any c.

Example 1 Solve

$$y' + 2xy = x$$
$$y(0) = 2\,.$$

Here $p(x) = 2x$, and $g(x) = x$. Compute

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}.$$

The equation (2.7) takes the form

$$\frac{d}{dx}\left[e^{x^2} y\right] = xe^{x^2}.$$

Integrate both sides, and then perform integration by a substitution $u = x^2$ (or use guess-and-check)

$$e^{x^2} y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c\,.$$

Solving for y, gives

$$y(x) = \frac{1}{2} + ce^{-x^2}.$$

From the initial condition

$$y(0) = \frac{1}{2} + c = 2,$$

so that $c = \frac{3}{2}$. Answer: $y(x) = \frac{1}{2} + \frac{3}{2}e^{-x^2}$.

Example 2 Solve

$$y' + \frac{1}{t}y = \cos 2t\,, \quad y(\pi/2) = 1\,.$$

Here the independent variable is t, $y = y(t)$, but the method is, of course, the same. Compute (for $t > 0$)

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t\,,$$

and then by (2.7)

$$\frac{d}{dt}\left[ty\right] = t\cos 2t\,.$$

Integrate both sides, and perform integration by parts

$$ty = \int t\cos 2t\,dt = \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t + c\,.$$

Divide by t

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}\frac{\cos 2t}{t} + \frac{c}{t}\,.$$

The initial condition gives

$$y(\pi/2) = -\frac{1}{4}\frac{1}{\pi/2} + \frac{c}{\pi/2} = 1\,.$$

Solve for c (multiplying by $\pi/2$)

$$c = \pi/2 + \frac{1}{4}\,,$$

and the solution is

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}\frac{\cos 2t}{t} + \frac{\pi/2 + \frac{1}{4}}{t}\,. \tag{2.8}$$

The solution $y(t)$ defines a curve, called the *integral curve*, for this intial-value problem. The initial condition tells us that $y = 1$ when $t = \pi/2$, so that the point $(\pi/2, 1)$ lies on the integral curve. What is the maximal interval on which the solution (2.8) is valid? I.e., starting with the initial point $t = \pi/2$, how far can we continue the solution to the left and to the right of the initial point? We see from (2.8) that the maximal interval is

$(0, \infty)$. As t tends to 0 from the right, $y(t)$ tends to $+\infty$. At $t = 0$, the solution $y(t)$ is undefined.

Example 3 Solve

$$x\frac{dy}{dx} + 2y = \sin x\,, \quad y(-\pi) = -2\,.$$

Here the equation is not in the form (2.4), for which the theory applies. We divide the equation by x

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x}.$$

Now the equation is in the right form, with $p(x) = \dfrac{2}{x}$ and $g(x) = \dfrac{\sin x}{x}$. Using the properties of logarithms, compute

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

And then

$$\frac{d}{dx}\left[x^2 y\right] = x^2\frac{\sin x}{x} = x\sin x.$$

Integrate both sides, and perform integration by parts

$$x^2 y = \int x \sin x\,dx = -x\cos x + \sin x + c,$$

giving us the general solution

$$y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{c}{x^2}\,.$$

The initial condition implies

$$y(-\pi) = -\frac{1}{\pi} + \frac{c}{\pi^2} = -2.$$

Solve for c:

$$c = -2\pi^2 + \pi.$$

Answer: $y(x) = -\dfrac{\cos x}{x} + \dfrac{\sin x}{x^2} + \dfrac{-2\pi^2 + \pi}{x^2}$. This solution is valid on the interval $(-\infty, 0)$ (that is how far it can be continued to the left and to the right, starting from the initial point $x = -\pi$).

Example 4 Solve

$$\frac{dy}{dx} = \frac{1}{y-x}, \quad y(1) = 0.$$

We have a problem: not only this equation is not in the right form, this is a *nonlinear equation*, because $\dfrac{1}{y-x}$ is not a linear function of y (it is not of the form $ay+b$, for any fixed x). We need a little trick. Let us pretend that dy and dx are numbers, and take the reciprocals of both sides of the equation, getting

$$\frac{dx}{dy} = y - x,$$

or

$$\frac{dx}{dy} + x = y.$$

Let us now think of y as independent variable, and x as a function of y, $x = x(y)$. Then the last equation is linear, with $p(y) = 1$ and $g(y) = y$. We proceed as usual: $\mu(y) = e^{\int 1\,dy} = e^y$, and

$$\frac{d}{dy}\left[e^y x\right] = ye^y.$$

Integration gives

$$e^y x = \int ye^y\,dy = ye^y - e^y + c,$$

and solving for x we obtain

$$x(y) = y - 1 + ce^{-y}.$$

To find c, we need an initial condition. The original initial condition tells us that $y = 0$ for $x = 1$. For the inverse function $x(y)$ this translates to $x(0) = 1$. So that $c = 2$.

Answer: $x(y) = y - 1 + 2e^{-y}$ (see the Figure 1.1).

Rigorous justification of this method is based on the formula for the derivative of the inverse function, that we recall next. Let $y = y(x)$ be some function, and $y_0 = y(x_0)$. Let $x = x(y)$ be its inverse function. Then $x_0 = x(y_0)$, and we have

$$\frac{dx}{dy}(y_0) = \frac{1}{\frac{dy}{dx}(x_0)}.$$

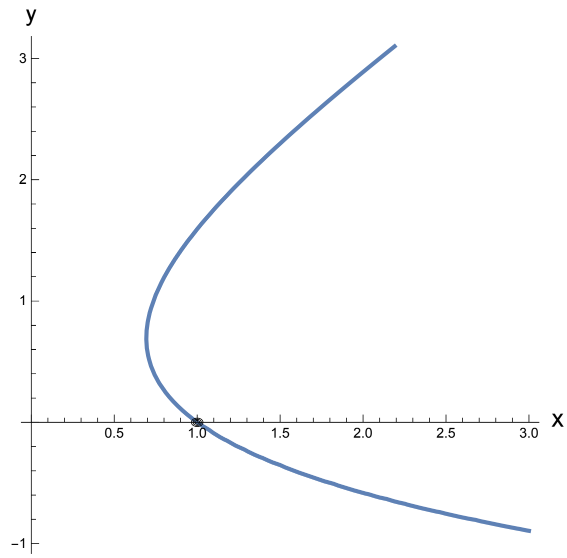


Figure 1.1: The integral curve $x = y - 1 + 2e^{-y}$, with the initial point $(1, 0)$ marked

1.3 Separable Equations

Background

Suppose we have a function $F(y)$, and y in turn depends on x, $y = y(x)$. So that, in effect, F depends on x. To differentiate F with respect to x, we use the chain rule from calculus:

$$\frac{d}{dx}F(y(x)) = F'(y(x))\frac{dy}{dx}.$$

The Method

Given two functions $F(y)$ and $G(x)$, let us use the corresponding lower case letters to denote their derivatives, so that $F'(y) = f(y)$ and $G'(x) = g(x)$, and correspondingly $\int f(y)\, dy = F(y) + c$, $\int g(x)\, dx = G(x) + c$. Our goal is to solve the following equation

$$f(y)\frac{dy}{dx} = g(x). \tag{3.1}$$

This is a nonlinear equation.

Using the upper case functions, this equation becomes

$$F'(y)\frac{dy}{dx} = G'(x).$$

By the chain rule, we rewrite this as

$$\frac{d}{dx}F(y) = \frac{d}{dx}G(x).$$

If derivatives of two functions are the same, these functions differ by a constant, so that

$$F(y) = G(x) + c. \tag{3.2}$$

This gives the desired general solution! If one is lucky, it may be possible to solve this relation for y as a function of x. If not, maybe one can solve for x as a function of y. If both attempts fail, one can use a computer implicit plotting routine to draw the integral curves, given by (3.2).

We now describe a simple procedure, which leads from the equation (3.1) to its solution (3.2). Let us pretend that $\frac{dy}{dx}$ is not a notation for the derivative, but a ratio of two numbers dy and dx. Clearing the denominator in (3.1)

$$f(y)\,dy = g(x)\,dx.$$

We have separated the variables, everything involving y is now on the left, while x appears only on the right. Integrate both sides:

$$\int f(y)\,dy = \int g(x)\,dx,$$

which gives us immediately the solution (3.2).

Example 1 Solve

$$\frac{dy}{dx} = x\left(y^2+9\right).$$

To separate the variables, we multiply by dx, and divide by y^2+9

$$\int \frac{dy}{y^2+9}\,dy = \int x\,dx\,.$$

So that the general solution is

$$\frac{1}{3}\arctan\frac{y}{3} = \frac{1}{2}x^2 + c\,,$$

which can be solved for y

$$y = 3\tan\left(\frac{3}{2}x^2 + 3c\right) = 3\tan\left(\frac{3}{2}x^2 + c\right) .$$

On the last step we replaced $3c$, which is an arbitrary constant, by c.

Example 2 Solve

$$\left(xy^2 + x\right)\, dx + e^x\, dy = 0 .$$

This is an example of a differential equation, written *in differentials*. (Dividing through by dx, we can put it into a familiar form $xy^2+x+e^x\dfrac{dy}{dx} = 0$, although there is no need to do that.)

By factoring, we are able to separate the variables:

$$e^x\, dy = -x(y^2+1)\, dx ,$$

$$\int \frac{dy}{y^2+1} = -\int xe^{-x}\, dx ,$$

$$\tan^{-1} y = xe^{-x} + e^{-x} + c .$$

Answer: $y(x) = \tan\left(xe^{-x} + e^{-x} + c\right)$.

Example 3 Find all solutions of

$$\frac{dy}{dx} = y^2 .$$

We separate the variables, and obtain

$$\int \frac{dy}{y^2} = \int dx , \quad -\frac{1}{y} = x + c , \quad y = -\frac{1}{x+c} .$$

However, division by y^2 is possible only if $y^2 \neq 0$. The case when $y^2 = 0$ leads to another solution: $y = 0$. Answer: $y = -\dfrac{1}{x+c}$, and $y = 0$.

When performing a division by a non-constant expression, one needs to check if any solutions are lost, when this expression is zero. (If you divide the quadratic equation $x(x-1) = 0$ by x, the root $x = 0$ is lost. If you divide by $x-1$, the root $x = 1$ is lost.)

Recall the fundamental theorem of calculus:

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x) ,$$

for any constant a. The integral $\int_a^x f(t)\,dt$ gives us an antiderivative of $f(x)$, so that we may write

$$\int f(x)\,dx = \int_a^x f(t)\,dt + c\,. \tag{3.3}$$

Here we can let c be an arbitrary constant, and a fixed, or the other way around.

Example 4 Solve

$$\frac{dy}{dx} = e^{x^2} y^2\,, \quad y(1) = -2\,.$$

Separation of variables

$$\int \frac{dy}{y^2} = \int e^{x^2}\,dx$$

gives on the right an integral that cannot be evaluated in elementary functions. We shall change it to a definite integral, as in (3.3). It is convenient to choose $a = 1$, because the initial condition is given at $x = 1$:

$$\int \frac{dy}{y^2} = \int_1^x e^{t^2}\,dt + c\,,$$

$$-\frac{1}{y} = \int_1^x e^{t^2}\,dt + c\,.$$

When $x = 1$, we have $y = -2$, which gives $c = \frac{1}{2}$ (using that $\int_1^1 e^{t^2}\,dt = 0$). Answer: $y(x) = -\dfrac{1}{\int_1^x e^{t^2}\,dt + \frac{1}{2}}$. For any x, the integral $\int_1^x e^{t^2}\,dt$ can be quickly computed by a numerical integration method, for example, by using the trapezoidal rule.

1.3.1 Problems

I. Integrate by Guess-and-Check.

1. $\displaystyle\int x e^{5x}\,dx$. Answer. $x\dfrac{e^{5x}}{5} - \dfrac{e^{5x}}{25} + c$.

2. $\displaystyle\int x \cos 2x\,dx$. Answer. $x\dfrac{\sin 2x}{2} + \dfrac{\cos 2x}{4} + c$.

3. $\displaystyle\int (2x+1)\sin 3x\,dx$. Answer. $-(2x+1)\dfrac{\cos 3x}{3} + \dfrac{2}{9}\sin 3x + c$.

4. $\int xe^{-\frac{1}{2}x}\,dx$. Answer. $e^{-x/2}(-4-2x)+c$.

5. $\int x^2 e^{-x}\,dx$. Answer. $-x^2e^{-x}-2xe^{-x}-2e^{-x}+c$.

6. $\int x^2\cos 2x\,dx$. Answer. $\frac{1}{2}x\cos 2x+\left(\frac{1}{2}x^2-\frac{1}{4}\right)\sin 2x+c$.

7. $\int \frac{x}{\sqrt{x^2+1}}\,dx$. Answer. $\sqrt{x^2+1}+c$.

8. $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$. Answer. $\sqrt{2}-1$.

9. $\int \frac{1}{(x^2+1)(x^2+9)}\,dx$. Answer. $\frac{1}{8}\tan^{-1}x-\frac{1}{24}\tan^{-1}\frac{x}{3}+c$.

10. $\int \frac{x}{(x^2+1)(x^2+2)}\,dx$. Answer. $\frac{1}{2}\ln\left(x^2+1\right)-\frac{1}{2}\ln\left(x^2+2\right)+c$.

11. $\int \frac{dx}{x^3+4x}$. Answer. $\frac{1}{4}\ln x-\frac{1}{8}\ln\left(x^2+4\right)+c$.

12. $\int \frac{(\ln x)^5}{x}\,dx$. Answer. $\frac{1}{6}(\ln x)^6+c$.

13. $\int x^2e^{x^3}\,dx$. Answer. $\frac{1}{3}e^{x^3}+c$.

14. $\int_0^\pi x\sin nx\,dx$, where n is a positive integer.
Answer. $-\frac{\pi}{n}\cos n\pi=\frac{\pi}{n}(-1)^{n+1}$.

15. $\int e^{2x}\sin 3x\,dx$. Answer. $e^{2x}\left(\frac{2}{13}\sin 3x-\frac{3}{13}\cos 3x\right)+c$.

Hint: Look for the antiderivative in the form $Ae^{2x}\sin 3x+Be^{2x}\cos 3x$, and determine the constants A and B by differentiation.

16. $\int_2^\infty \frac{dx}{x\,(\ln x)^2}$. Answer. $\frac{1}{\ln 2}$.

17. $\int x^3e^{-x}\,dx$. Answer. $-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}-6e^{-x}+c$.

II. Find the general solution of the linear problems.

1. $y'-y\sin x=\sin x$. Answer. $y=-1+ce^{-\cos x}$.

2. $y'+\frac{1}{x}y=\cos x$. Answer. $y=\frac{c}{x}+\sin x+\frac{\cos x}{x}$.

3. $xy' + 2y = e^{-x}$. Answer. $y = \dfrac{c}{x^2} - \dfrac{(x+1)e^{-x}}{x^2}$.

4. $x^4y' + 3x^3y = x^2e^x$. Answer. $y = \dfrac{c}{x^3} + \dfrac{(x-1)e^x}{x^3}$.

5. $\dfrac{dy}{dx} = 2x(x^2 + y)$. Answer. $y = ce^{x^2} - x^2 - 1$.

6. $xy' - 2y = xe^{1/x}$. Answer. $y = cx^2 - x^2e^{1/x}$.

7. $y' + 2y = \sin 3x$. Answer. $y = ce^{-2x} + \dfrac{2}{13}\sin 3x - \dfrac{3}{13}\cos 3x$.

8. $x\,(yy' - 1) = y^2$.

Hint: Set $v = y^2$. Then $v' = 2yy'$, and one obtains a linear equation for $v = v(x)$. Answer. $y^2 = -2x + cx^2$.

III. Find the solution of the initial value problem, and state the maximum interval on which this solution is valid.

1. $y' - 2y = e^x$, $y(0) = 2$. Answer. $y = 3e^{2x} - e^x$; $(-\infty, \infty)$.

2. $y' + \dfrac{1}{x}y = \cos x$, $y(\dfrac{\pi}{2}) = 1$. Answer. $y = \dfrac{\cos x + x\sin x}{x}$; $(0, \infty)$.

3. $xy' + 2y = \dfrac{\sin x}{x}$, $y(\dfrac{\pi}{2}) = -1$. Answer. $y = -\dfrac{\pi^2 + 4\cos x}{4x^2}$; $(0, \infty)$.

4. $xy' + (2 + x)y = 1$, $y(-2) = 0$.

Answer. $y = \dfrac{1}{x} + \dfrac{3e^{-x-2}}{x^2} - \dfrac{1}{x^2}$; $(-\infty, 0)$.

5. $x(y' - y) = e^x$, $y(-1) = \dfrac{1}{e}$. Answer. $y = e^x \ln|x| + e^x$; $(-\infty, 0)$.

6. $(t+2)\dfrac{dy}{dt} + y = 5$, $y(1) = 1$. Answer. $y = \dfrac{5t-2}{t+2}$; $(-2, \infty)$.

7. $ty' - 2y = t^4\cos t$, $y(\pi/2) = 0$.

Answer. $y = t^3\sin t + t^2\cos t - \dfrac{\pi}{2}t^2$; $(-\infty, \infty)$. Solution is valid for all t.

8. $t\ln t\,\dfrac{dr}{dt} + r = 5te^t$, $r(2) = 0$. Answer. $r = \dfrac{5e^t - 5e^2}{\ln t}$; $(1, \infty)$.

9. $xy' + 2y = y' + \dfrac{1}{(x-1)^2}$, $y(-2) = 0$.

Answer. $y = \dfrac{\ln|x-1| - \ln 3}{(x-1)^2} = \dfrac{\ln(1-x) - \ln 3}{(x-1)^2}$; $(-\infty, 1)$.

10. $\dfrac{dy}{dx} = \dfrac{1}{y^2 + x}$, $y(2) = 0$.

Hint: Consider $\dfrac{dx}{dy}$, and obtain a linear equation for $x(y)$.

Answer. $x = -2 + 4e^y - 2y - y^2$.

11*. Find a solution ($y = y(t)$) of $y' + y = \sin 2t$, which is a periodic function.

Hint: Look for a solution in the form $y(t) = A \sin 2t + B \cos 2t$, substitute this expression into the equation, and determine the constants A and B.

Answer. $y = \dfrac{1}{5} \sin 2t - \dfrac{2}{5} \cos 2t$.

12*. Show that the equation $y' + y = \sin 2t$ has no other periodic solutions.

Hint: Consider the equation that the difference of any two solutions satisfies.

13*. For the equation

$$y' + a(x)y = f(x)$$

assume that $a(x) \geq a_0 > 0$, where a_0 is a constant, and $f(x) \to 0$ as $x \to \infty$. Show that any solution tends to zero as $x \to \infty$.

Hint: Write the integrating factor as $\mu(x) = e^{\int_0^x a(t)\,dt} \geq e^{a_0 x}$, so that $\mu(x) \to \infty$ as $x \to \infty$. Then express

$$y = \frac{\int_0^x \mu(t) f(t)\,dt + c}{\mu(x)},$$

and use L'Hospital's rule.

14*. Assume that in the equation (for $y = y(t)$)

$$y' + ay = f(t)$$

the continuous function $f(t)$ satisfies $|f(t)| \leq M$ for all $-\infty < t < \infty$, where M and a are positive constants. Show that there is only one solution, call it $y_0(t)$, which is bounded for all $-\infty < t < \infty$. Show that $\lim_{t\to\infty} y_0(t) = 0$, provided that $\lim_{t\to\infty} f(t) = 0$, and $\lim_{t\to-\infty} y_0(t) = 0$, provided that

$\lim_{t\to-\infty} f(t) = 0$. Show also that $y_0(t)$ is a periodic function, provided that $f(t)$ is a periodic function.

Hint: Using the integrating factor e^{at}, express

$$e^{at}y(t) = \int_\alpha^t e^{as} f(s)\, ds + c\,.$$

Select $c = 0$, and $\alpha = -\infty$. Then $y_0(t) = e^{-at}\int_{-\infty}^t e^{as} f(s)\, ds$, and $|y_0(t)| \le e^{-at}\int_{-\infty}^t e^{as}|f(s)|\, ds \le \frac{M}{a}$. In case $\lim_{t\to-\infty} f(t) = 0$, a similar argument shows that $|y_0(t)| \le \frac{\epsilon}{a}$, for $-t$ large enough. In case $\lim_{t\to\infty} f(t) = 0$, use L'Hospital's rule.

IV. Solve by separating the variables.

1. $\dfrac{dy}{dx} = \dfrac{2}{x(y^3+1)}$. Answer. $\dfrac{y^4}{4} + y - 2\ln|x| = c$.

2. $e^x\, dx - y\, dy = 0$, $y(0) = -1$. Answer. $y = -\sqrt{2e^x - 1}$.

3. $(x^2y^2 + y^2)\, dx - yx\, dy = 0$.

Answer. $y = e^{\frac{x^2}{2} + \ln|x| + c} = c\,|x|\, e^{\frac{x^2}{2}}$ (writing $e^c = c$).

4. $y' = x^2\sqrt{4 - y^2}$. Answer. $y = 2\sin\left(\dfrac{x^3}{3} + c\right)$, and $y = \pm 2$.

5. $y'(t) = ty^2(1+t^2)^{-1/2}$, $y(0) = 2$. Answer. $y = -\dfrac{2}{2\sqrt{t^2+1} - 3}$.

6. $(y - xy + x - 1)\, dx + x^2\, dy = 0$, $y(1) = 0$. Answer. $y = \dfrac{e - e^{\frac{1}{x}}x}{e}$.

7. $x^2y^2y' = y - 1$. Answer. $\dfrac{y^2}{2} + y + \ln|y-1| = -\dfrac{1}{x} + c$, and $y = 1$.

8. $y' = e^{x^2}y$, $y(2) = 1$. Answer. $y = e^{\int_2^x e^{t^2}\, dt}$.

9. $y' = xy^2 + xy$, $y(0) = 2$. Answer. $y = \dfrac{2e^{\frac{x^2}{2}}}{3 - 2e^{\frac{x^2}{2}}}$.

10. $y' - 2xy^2 = 8x$, $y(0) = -2$.

Hint: There are infinitely many choices for c, but they all lead to the same solution.

Answer. $y = 2\tan\left(2x^2 - \dfrac{\pi}{4}\right)$.

11. $y'(t) = y - y^2 - \dfrac{1}{4}$.

Hint: Write the right hand side as $-\frac{1}{4}(2y-1)^2$.

Answer. $y = \dfrac{1}{2} + \dfrac{1}{t+c}$, and $y = \frac{1}{2}$.

12. $\dfrac{dy}{dx} = \dfrac{y^2 - y}{x}$.

Answer. $\left|\dfrac{y-1}{y}\right| = e^c|x|$, and also $y = 0$ and $y = 1$.

13. $\dfrac{dy}{dx} = \dfrac{y^2 - y}{x}$, $y(1) = 2$. Answer. $y = \dfrac{2}{2-x}$.

14. $y' = (x+y)^2$, $y(0) = 1$.

Hint: Set $x + y = z$, where $z = z(x)$ is a new unknown function.

Answer. $y = -x + \tan(x + \frac{\pi}{4})$.

15. Show that one can reduce

$$y' = f(ax + by)$$

to a separable equation. Here a and b are constants, $f = f(z)$ is an arbitrary function.

Hint: Set $ax + by = z$, where $z = z(x)$ is a new unknown function.

16. A particle is moving on a polar curve $r = f(\theta)$. Find the function $f(\theta)$ so that the speed of the particle is 1, for all θ.

Hint: $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$, and then

$$\text{speed}^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = f'^2(\theta) + f^2(\theta) = 1\,,$$

or $f' = \pm\sqrt{1 - f^2}$.

Answer. $f(\theta) = \pm 1$, or $f(\theta) = \pm\sin(\theta + c)$. ($r = \sin(\theta + c)$ is a circle of radius $\frac{1}{2}$ with center on the ray $\theta = \dfrac{\pi}{2} - c$, and passing through the origin.)

17*. Find the differentiable function $f(x)$ satisfying the following *functional equation* (for all x and y)

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}\,.$$

Hint: By setting $x = y = 0$, conclude that $f(0) = 0$. Then $f'(x) = \lim\limits_{y\to 0} \dfrac{f(x+y)-f(x)}{y} = c\left(1+f^2(x)\right)$, where $c = f'(0)$.

Answer. $f(x) = \tan c\,x$.

1.4 Some Special Equations

Differential equations that are not linear are called *nonlinear*. In this section we encounter several classes of nonlinear equations that can be reduced to linear ones.

1.4.1 Homogeneous Equations

Let $f(t)$ be a given function. Setting here $t = \dfrac{y}{x}$, we obtain a function $f(\dfrac{y}{x})$, which is a function of two variables x and y, but it depends on them in a special way. One calls functions of the form $f(\dfrac{y}{x})$ *homogeneous*. For example, $\dfrac{y-4x}{x-y}$ is a homogeneous function, because we can put it into the form (dividing both the numerator and the denominator by x)

$$\frac{y-4x}{x-y} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}\,,$$

so that here $f(t) = \frac{t-4}{1-t}$.

Our goal is to solve *homogeneous equations*

$$\frac{dy}{dx} = f(\frac{y}{x}). \tag{4.1}$$

Set $v = \dfrac{y}{x}$. Since y is a function of x, the same is true of $v = v(x)$. Solving for y, $y = xv$, we express by the product rule

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Switching to v in (4.1), gives

$$v + x\frac{dv}{dx} = f(v). \tag{4.2}$$

This is a separable equation! Indeed, after taking v to the right, we can separate the variables

$$\int \frac{dv}{f(v) - v}\, dv = \int \frac{dx}{x}\,.$$

After solving this equation for $v(x)$, we can express the original unknown $y = xv(x)$.

In practice, one should try to remember the formula (4.2).

Example 1 Solve

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$
$$y(1) = -2\,.$$

To see that this equation is homogeneous, we rewrite it as (dividing both the numerator and the denominator by x^2)

$$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\frac{y}{x}}\,.$$

Set $v = \dfrac{y}{x}$, or $y = xv$. Using that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, obtain

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}\,.$$

Simplify:

$$x\frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v = \frac{1 + v^2}{2v}\,.$$

Separating the variables gives

$$\int \frac{2v}{1 + v^2}\, dv = \int \frac{dx}{x}\,.$$

We now obtain the solution, by performing the following steps (observe that $\ln c$ is another way to write an arbitrary constant):

$$\ln\left(1 + v^2\right) = \ln x + \ln c = \ln cx\,,$$

$$1 + v^2 = cx\,,$$
$$v = \pm\sqrt{cx-1}\,,$$
$$y(x) = xv = \pm x\sqrt{cx-1}\,.$$

From the initial condition

$$y(1) = \pm\sqrt{c-1} = -2\,.$$

It follows that we need to select "minus", and $c = 5$.

Answer: $y(x) = -x\sqrt{5x-1}$.

There is an alternative (equivalent) definition: a function $f(x, y)$ is called *homogeneous* if

$$f(tx, ty) = f(x, y), \quad \text{for all constants } t.$$

If this condition holds, then setting $t = \dfrac{1}{x}$, we see that

$$f(x, y) = f(tx, ty) = f(1, \frac{y}{x})\,,$$

so that $f(x, y)$ is a function of $\dfrac{y}{x}$, and the old definition applies. It is easy to check that $f(x, y) = \frac{x^2+3y^2}{2xy}$ from the Example 1 satisfies the new definition.

Example 2 Solve

$$\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}, \quad \text{with } x > 0,\ y \geq 0\,.$$

It is more straightforward to use the new definition to verify that the function $f(x, y) = \frac{y}{x+\sqrt{xy}}$ is homogeneous. For all $t > 0$, we have

$$f(tx, ty) = \frac{(ty)}{(tx) + \sqrt{(tx)(ty)}} = \frac{y}{x + \sqrt{xy}} = f(x, y)\,.$$

Letting $y/x = v$, or $y = xv$, we rewrite this equation as

$$v + xv' = \frac{xv}{x + \sqrt{x\,xv}} = \frac{v}{1 + \sqrt{v}}\,.$$

We proceed to separate the variables:

$$x\frac{dv}{dx} = \frac{v}{1+\sqrt{v}} - v = -\frac{v^{3/2}}{1+\sqrt{v}}\,,$$

$$\int \frac{1+\sqrt{v}}{v^{3/2}}\,dv = -\int \frac{dx}{x}\,,$$

$$-2v^{-1/2} + \ln v = -\ln x + c\,.$$

The integral on the left was evaluated by performing division, and splitting it into two pieces. Finally, we replace v by y/x, and simplify:

$$-2\sqrt{\frac{x}{y}} + \ln\frac{y}{x} = -\ln x + c\,,$$

$$-2\sqrt{\frac{x}{y}} + \ln y = c\,.$$

We obtained an implicit representation of a family of solutions. One can solve for x, $x = \frac{1}{4}y\,(c - \ln y)^2$.

When separating the variables, we had to assume that $v \neq 0$ (in order to divide by $v^{3/2}$). In case $v = 0$, we obtain another solution: $y = 0$.

1.4.2 The Logistic Population Model

Let $y(t)$ denote the number of rabbits on a tropical island at time t. The simplest model of population growth is

$$y' = ay$$
$$y(0) = y_0\,.$$

Here $a > 0$ is a given constant, called the *growth rate*. This model assumes that initially the number of rabbits was equal to some number $y_0 > 0$, while the rate of change of population, given by $y'(t)$, is proportional to the number of rabbits. The population of rabbits grows, which results in a faster and faster rate of growth. One expects an explosive growth. Indeed, solving the equation, we get

$$y(t) = ce^{at}\,.$$

From the initial condition $y(0) = c = y_0$, which gives us $y(t) = y_0e^{at}$, an exponential growth. This is the notorious *Malthusian model* of population growth. Is it realistic? Yes, sometimes, for a limited time. If the initial number of rabbits y_0 is small, then for a while their number may grow exponentially.

A more realistic model, which can be used for a long time, is the *logistic model*:

$$
\begin{aligned}
y' &= ay - by^2 \\
y(0) &= y_0 \,.
\end{aligned}
\tag{4.3}
$$

Here a, b and y_0 are given positive constants, and $y = y(t)$. Writing this equation in the form

$$y' = by\left(\frac{a}{b} - y\right),$$

we see that when $0 < y < \frac{a}{b}$, we have $y'(t) > 0$ and $y(t)$ is increasing, while in the case $y > \frac{a}{b}$ we have $y'(t) < 0$ and $y(t)$ is decreasing.

If y_0 is small, then for small t, $y(t)$ is small, so that the by^2 term is negligible, and we have exponential growth. As $y(t)$ increases, the by^2 term is not negligible anymore, and we can expect the rate of growth of $y(t)$ to get smaller and smaller, and $y(t)$ to tend to a finite limit. (Writing the equation as $y' = (a - b\,y)y$, we can regard the $a - b\,y$ term as the rate of growth.) In case the initial number y_0 is large (when $y_0 > a/b$), the quadratic on the right in (4.3) is negative, so that $y'(t) < 0$, and the population decreases. If $y_0 = a/b$, then $y'(0) = 0$, and we expect that $y(t) = a/b$ for all t. We now solve the equation (4.3) to confirm our guesses.

This equation can be solved by separating the variables. Instead, we use another technique that will be useful in the next section. Divide both sides of the equation by y^2:

$$y^{-2}y' = ay^{-1} - b.$$

Introduce a new unknown function $v(t) = y^{-1}(t) = \dfrac{1}{y(t)}$. By the generalized power rule, $v' = -y^{-2}y'$, so that we can rewrite the last equation as

$$-v' = av - b,$$

or

$$v' + av = b.$$

This is a linear equation for $v(t)$! To solve it, we follow the familiar steps, and then we return to the original unknown function $y(t)$:

$$\mu(t) = e^{\int a\,dt} = e^{at},$$

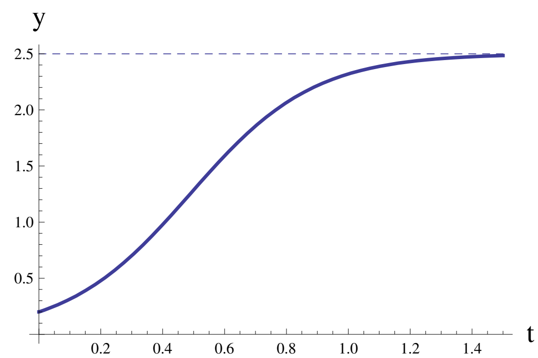


Figure 1.2: The solution of $y' = 5y - 2y^2$, $y(0) = 0.2$

$$\frac{d}{dt}\left[e^{at}v\right] = be^{at},$$

$$e^{at}v = b\int e^{at}\,dt = \frac{b}{a}e^{at} + c,$$

$$v = \frac{b}{a} + ce^{-at},$$

$$y(t) = \frac{1}{v} = \frac{1}{\frac{b}{a} + ce^{-at}}\,.$$

To find the constant c, we use the initial condition

$$y(0) = \frac{1}{\frac{b}{a} + c} = y_0\,,$$

$$c = \frac{1}{y_0} - \frac{b}{a}\,.$$

We conclude:

$$y(t) = \frac{1}{\frac{b}{a} + \left(\frac{1}{y_0} - \frac{b}{a}\right)e^{-at}}\,.$$

Observe that $\lim_{t\to+\infty} y(t) = a/b$, no matter what initial value y_0 we take. The number a/b is called the *carrying capacity.* It tells us the number of rabbits, in the long run, that our island will support. A typical solution curve, called the *logistic curve* is given in Figure 1.2.

1.4.3 Bernoulli's Equations

Let us solve the equation

$$y'(t) = p(t)y(t) + g(t)y^n(t).$$

Here $p(t)$ and $g(t)$ are given functions, n is a given constant. The logistic model above is just a particular example of Bernoulli's equation.

Proceeding similarly to the logistic equation, we divide this equation by y^n:

$$y^{-n}y' = p(t)y^{1-n} + g(t).$$

Introduce a new unknown function $v(t) = y^{1-n}(t)$. Compute $v' = (1-n)y^{-n}y'$, so that $y^{-n}y' = \frac{1}{1-n}v'$, and rewrite the equation as

$$v' = (1-n)p(t)v + (1-n)g(t).$$

This is a linear equation for $v(t)$! After solving for $v(t)$, we calculate the solution $y(t) = v^{\frac{1}{1-n}}(t)$.

Example Solve

$$y' = y + \frac{t}{\sqrt{y}}\,.$$

Writing this equation in the form $y' = y + ty^{-1/2}$, we see that this is Bernoulli's equation, with $n = -1/2$, so that we need to divide through by $y^{-1/2}$. But that is the same as multiplying through by $y^{1/2}$, which we do, obtaining

$$y^{1/2}y' = y^{3/2} + t\,.$$

We now let $v(t) = y^{3/2}$, $v'(t) = \frac{3}{2}y^{1/2}y'$, obtaining a linear equation for v, which is solved as usual:

$$\frac{2}{3}v' = v + t\,, \quad v' - \frac{3}{2}v = \frac{3}{2}t,$$

$$\mu(t) = e^{-\int \frac{3}{2}\,dt} = e^{-\frac{3}{2}t}\,, \quad \frac{d}{dt}\left[e^{-\frac{3}{2}t}v\right] = \frac{3}{2}te^{-\frac{3}{2}t},$$

$$e^{-\frac{3}{2}t}v = \int \frac{3}{2}te^{-\frac{3}{2}t}\,dt = -te^{-\frac{3}{2}t} - \frac{2}{3}e^{-\frac{3}{2}t} + c,$$

$$v = -t - \frac{2}{3} + ce^{\frac{3}{2}t}\,.$$

Returning to the original variable y, gives the answer: $y = \left(-t - \dfrac{2}{3} + ce^{\frac{3}{2}t}\right)^{2/3}$.

1.4.4* Riccati's Equations

Let us try to solve the equation

$$y'(t) + a(t)y(t) + b(t)y^2(t) = c(t).$$

Here $a(t)$, $b(t)$ and $c(t)$ are given functions. In case $c(t) = 0$, this is Bernoulli's equation, which we can solve. For general $c(t)$, one needs some luck to solve this equation. Namely, one needs to guess some solution $p(t)$, which we refer to as a *particular solution*. Then a substitution $y(t) = p(t) + z(t)$ produces Bernoulli's equation for $z(t)$

$$z' + (a + 2bp)\, z + bz^2 = 0\,,$$

which can be solved.

There is no general way to find a particular solution, which means that one cannot always solve Riccati's equation. Occasionally one can get lucky.

Example 1 Solve

$$y' + y^2 = t^2 - 2t\,.$$

We see a quadratic polynomial on the right, which suggests to look for a particular solution in the form $y = at + b$. Substitution into the equation produces a quadratic polynomial on the left too. Equating the coefficients in t^2, t and constant terms, gives three equations to find a and b. In general, three equations with two unknowns will have no solutions, but this is a lucky case, with the solution $a = -1$, $b = 1$, so that $p(t) = -t + 1$ is a particular solution. Substituting $y(t) = -t+1+v(t)$ into the equation, and simplifying, we get

$$v' + 2(1-t)v = -v^2\,.$$

This is Bernoulli's equation. Divide through by v^2, and then set $z = \dfrac{1}{v}$, $z' = -\dfrac{v'}{v^2}$, to get a linear equation:

$$v^{-2}v' + 2(1-t)v^{-1} = -1\,, \quad z' - 2(1-t)z = 1\,,$$

$$\mu = e^{-\int 2(1-t)\,dt} = e^{t^2-2t}\,, \quad \frac{d}{dt}\left[e^{t^2-2t}z\right] = e^{t^2-2t}\,,$$

$$e^{t^2-2t}z = \int e^{t^2-2t}\,dt\,.$$

The last integral cannot be evaluated through elementary functions (*Mathematica* can evaluate it through a special function, called *Erfi*). So we leave this integral unevaluated. One gets z from the last formula, after which one expresses v, and finally y. The result is a family of solutions: $y(t) = -t + 1 + \dfrac{e^{t^2-2t}}{\int e^{t^2-2t}\,dt}$. (The usual arbitrary constant c is now "inside" of the integral. Replacing $\int e^{t^2-2t}\,dt$ by $\int_a^t e^{s^2-2s}\,ds$ will give a formula for $y(t)$ that can be used for computations and graphing.) Another solution: $y = -t + 1$ (corresponding to $v = 0$).

Example 2 Solve

$$y' + 2y^2 = \frac{6}{t^2}\,. \tag{4.4}$$

We look for a particular solution in the form $y(t) = a/t$, and calculate $a = 2$, so that $p(t) = 2/t$ is a particular solution ($a = -3/2$ is also a possibility). The substitution $y(t) = 2/t + v(t)$ produces Bernoulli's equation

$$v' + \frac{8}{t}v + 2v^2 = 0\,.$$

Solving it, gives $v(t) = \dfrac{7}{c\,t^8 - 2t}$, and $v = 0$. The solutions of (4.4) are $y(t) = \dfrac{2}{t} + \dfrac{7}{c\,t^8 - 2t}$, and also $y = \dfrac{2}{t}$.

Let us outline an alternative approach to the last problem. Setting $y = 1/z$ in (4.4), then clearing the denominators, gives

$$\begin{aligned} -\frac{z'}{z^2} + 2\frac{1}{z^2} &= \frac{6}{t^2}, \\ -z' + 2 &= \frac{6z^2}{t^2}\,. \end{aligned}$$

This is a homogeneous equation, which we can solve.

There are some important ideas that we learned in this subsection. Knowledge of one particular solution may help to "crack open" the equation, and get other solutions. Also, the form of this particular solution depends on the equation.

1.4.5* Parametric Integration

Let us solve the initial value problem (here $y = y(x)$)

$$\begin{aligned} y &= \sqrt{1 - y'^2} \\ y(0) &= 1\,. \end{aligned} \tag{4.5}$$

This equation is not solved for the derivative $y'(x)$. Solving for $y'(x)$, and then separating the variables, one can indeed find the solution. Instead, let us assume that

$$y'(x) = \sin t,$$

where t is a parameter (upon which both x and y will depend). From the equation (4.5):

$$y = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t,$$

assuming that $\cos t \geq 0$. Recall the differentials: $dy = y'(x)\,dx$, or

$$dx = \frac{dy}{y'(x)} = \frac{-\sin t\,dt}{\sin t} = -dt,$$

so that $\dfrac{dx}{dt} = -1$, which gives

$$x = -t + c.$$

We obtained a family of solutions in parametric form (valid if $\cos t \geq 0$)

$$\begin{aligned} x &= -t + c \\ y &= \cos t\,. \end{aligned}$$

Solving for t, $t = -x + c$, gives $y = \cos(-x + c)$. From the initial condition, calculate that $c = 0$, giving us the solution $y = \cos x$. This solution is valid on infinitely many disjoint intervals where $\cos x \geq 0$ (because we see from the equation (4.5) that $y \geq 0$). This problem admits another solution: $y = 1$.

For the equation

$$y'^5 + y' = x$$

we do not have an option of solving for $y'(x)$. Parametric integration appears to be the only way to solve it. We let $y'(x) = t$, so that from the equation, $x = t^5 + t$, and $dx = \frac{dx}{dt}\,dt = (5t^4 + 1)\,dt$. Then

$$dy = y'(x)\,dx = t(5t^4 + 1)\,dt\,,$$

so that $\dfrac{dy}{dt} = t(5t^4+1)$, which gives $y = \frac{5}{6}t^6 + \frac{1}{2}t^2 + c$. We obtained a family of solutions in parametric form:

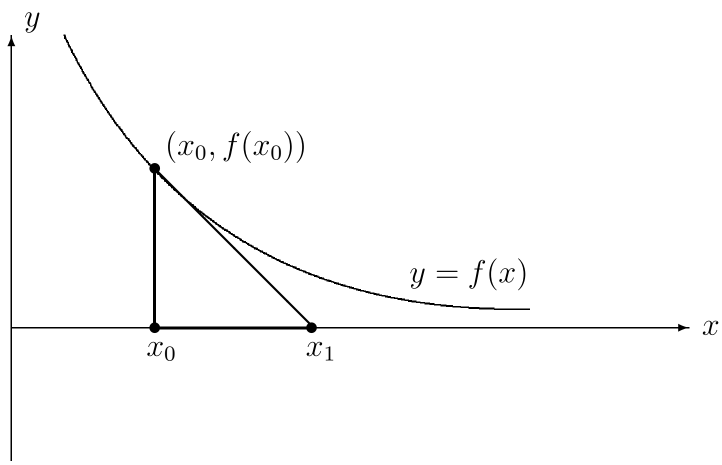
$$\begin{aligned} x &= t^5 + t \\ y &= \tfrac{5}{6}t^6 + \tfrac{1}{2}t^2 + c\,. \end{aligned}$$

If an initial condition is given, one can determine the value of c, and then plot the solution.

1.4.6 Some Applications

Differential equations arise naturally in geometric and physical problems.

Example 1 Find all positive decreasing functions $y = f(x)$, with the following property: the area of the triangle formed by the vertical line going down from the curve, the x-axis and the tangent line to this curve is constant, equal to $a > 0$.



The triangle formed by the tangent line, the line $x = x_0$, and the x-axis

Let $(x_0, f(x_0))$ be an arbitrary point on the graph of $y = f(x)$. Draw the triangle in question, formed by the vertical line $x = x_0$, the x-axis, and the tangent line to this curve. The tangent line intersects the x-axis at some point x_1, lying to the right of x_0, because $f(x)$ is decreasing. The slope of the tangent line is $f'(x_0)$, so that the point-slope equation of the tangent line is

$$y = f(x_0) + f'(x_0)(x - x_0)\,.$$

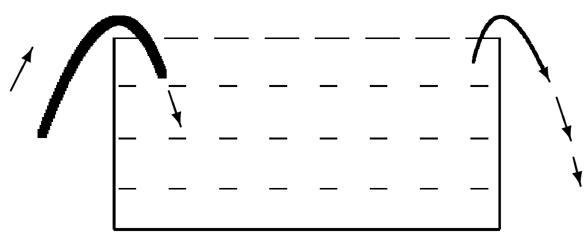
At x_1, we have $y = 0$, so that

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)\,.$$

Solve this for x_1, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. It follows that the horizontal side of our triangle is $-\dfrac{f(x_0)}{f'(x_0)}$, while the vertical side is $f(x_0)$. The area of this right

triangle is then
$$-\frac{1}{2}\frac{f^2(x_0)}{f'(x_0)} = a\,.$$
(Observe that $f'(x_0) < 0$, so that the area is positive.) The point x_0 was arbitrary, so that we replace it by x, and then we replace $f(x)$ by y, and $f'(x)$ by y':
$$-\frac{1}{2}\frac{y^2}{y'} = a\,, \quad \text{or} \quad -\frac{y'}{y^2} = \frac{1}{2a}\,.$$
We solve this differential equation by taking the antiderivatives of both sides:
$$\frac{1}{y} = \frac{1}{2a}x + c\,.$$
Answer: $y(x) = \dfrac{2a}{x+2ac}$. This is a family of hyperbolas. One of them is $y = \dfrac{2a}{x}$.

Example 2 A tank holding 10L (liters) originally is completely filled with water. A salt-water mixture is pumped into the tank at a rate of 2L per minute. This mixture contains 0.3 kg of salt per liter. The excess fluid is flowing out of the tank at the same rate (2L per minute). How much salt does the tank contain after 4 minutes?



Salt-water mixture pumped into a full tank

Let t be the time (in minutes) since the mixture started flowing, and let $y(t)$ denote the amount of salt in the tank at time t. The derivative, $y'(t)$, approximates the rate of change of salt per minute, and it is equal to the difference between the rate at which salt flows in, and the rate it flows out. The salt is pumped in at a rate of 0.6 kg per minute. The density of salt at time t is $\dfrac{y(t)}{10}$ (so that each liter of the solution in the tank contains $\dfrac{y(t)}{10}$ kg of salt). Then, the salt flows out at the rate $2\,\frac{y(t)}{10} = 0.2\,y(t)$ kg/min. The difference of these two rates gives $y'(t)$, so that

$$y' = 0.6 - 0.2y\,.$$

This is a linear differential equation. Initially, there was no salt in the tank, so that $y(0) = 0$ is our initial condition. Solving this equation together with the initial condition, we have $y(t) = 3 - 3e^{-0.2t}$. After 4 minutes, we have $y(4) = 3 - 3e^{-0.8} \approx 1.65$ kg of salt in the tank.

Now suppose a patient has alcohol poisoning, and doctors are pumping in water to flush his stomach out. One can compute similarly the weight of poison left in the stomach at time t. (An example is included in the Problems.)

1.5 Exact Equations

This section covers *exact equations*. While this class of equations is rather special, it often occurs in applications.

Let us begin by recalling partial derivatives. If a function $f(x) = x^2 + a$ depends on a parameter a, then $f'(x) = 2x$. If $g(x) = x^2 + y^3$, with a parameter y, we have $\frac{dg}{dx} = 2x$. Another way to denote this derivative is $g_x = 2x$. We can also regard g as a function of two variables, $g = g(x,y) = x^2 + y^3$. Then the *partial derivative* with respect to x is computed by regarding y to be a parameter, $g_x = 2x$. Alternative notation: $\frac{\partial g}{\partial x} = 2x$. Similarly, a partial derivative with respect to y is $g_y = \frac{\partial g}{\partial y} = 3y^2$. The derivative g_y gives us the rate of change in y, when x is kept fixed.

The equation (here $y = y(x)$)

$$y^2 + 2xyy' = 0$$

can be easily solved, if we rewrite it in the equivalent form

$$\frac{d}{dx}\left(xy^2\right) = 0.$$

Then $xy^2 = c$, and the solution is

$$y(x) = \pm\frac{c}{\sqrt{x}}.$$

We wish to play the same game for general equations of the form

$$M(x,y)+N(x,y)y'(x)=0. \tag{5.1}$$

Here the functions $M(x,y)$ and $N(x,y)$ are given. In the above example, $M=y^2$ and $N=2xy$.

Definition The equation (5.1) is called *exact* if there is a function $\psi(x,y)$, with continuous derivatives up to second order, so that we can rewrite (5.1) in the form

$$\frac{d}{dx}\psi(x,y)=0. \tag{5.2}$$

The solution of the exact equation is (c is an arbitrary constant)

$$\psi(x,y)=c. \tag{5.3}$$

There are two natural questions: what conditions on $M(x,y)$ and $N(x,y)$ will force the equation (5.1) to be exact, and if the equation (5.1) is exact, how does one find $\psi(x,y)$?

Theorem 1.5.1 *Assume that the functions $M(x,y)$, $N(x,y)$, $M_y(x,y)$ and $N_x(x,y)$ are continuous in some disc $D:(x-x_0)^2+(y-y_0)^2<r^2$, around some point (x_0,y_0). Then the equation (5.1) is exact in D if and only if the following partial derivatives are equal*

$$M_y(x,y)=N_x(x,y)\,, \quad \text{for all points } (x,y) \text{ in } D. \tag{5.4}$$

This theorem makes two claims: if the equation is exact, then the partials are equal, and conversely, if the partials are equal, then the equation is exact.

Proof: 1. Assume that the equation (5.1) is exact, so that it can be written in the form (5.2). Performing the differentiation in (5.2), using the chain rule, gives

$$\psi_x+\psi_y y'=0\,.$$

But this equation is the same as (5.1), so that

$$\begin{aligned}\psi_x&=M\\ \psi_y&=N\,.\end{aligned}$$

Taking the second partials

$$\begin{aligned}\psi_{xy}&=M_y\\ \psi_{yx}&=N_x\,.\end{aligned}$$

We know from calculus that $\psi_{xy} = \psi_{yx}$, therefore $M_y = N_x$.

2. Assume that $M_y = N_x$. We will show that the equation (5.1) is then exact by producing $\psi(x, y)$. We have just seen that $\psi(x, y)$ must satisfy

$$\begin{aligned} \psi_x &= M(x, y) \\ \psi_y &= N(x, y)\,. \end{aligned} \tag{5.5}$$

Take the antiderivative in x of the first equation

$$\psi(x, y) = \int_{x_0}^{x} M(t, y)\, dt + h(y)\,, \tag{5.6}$$

where $h(y)$ is an arbitrary function of y, and x_0 is an arbitrary number. To determine $h(y)$, substitute the last formula into the second line of (5.5)

$$\psi_y(x, y) = \int_{x_0}^{x} M_y(t, y)\, dt + h'(y) = N(x, y)\,,$$

or

$$h'(y) = N(x, y) - \int_{x_0}^{x} M_y(t, y)\, dt \equiv p(x, y)\,. \tag{5.7}$$

Observe that we denoted by $p(x, y)$ the right side of the last equation. It turns out that $p(x, y)$ does not really depend on x! Indeed, taking the partial derivative in x,

$$\frac{\partial}{\partial x} p(x, y) = N_x(x, y) - M_y(x, y) = 0\,,$$

because it was given to us that $M_y(x, y) = N_x(x, y)$. So that $p(x, y)$ is a function of y only, or $p(y)$. The equation (5.7) takes the form

$$h'(y) = p(y)\,.$$

We determine $h(y)$ by integration, and use it in (5.6) to get $\psi(x, y)$. $\diamondsuit$

Recall that the equation in differentials

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is an alternative form of (5.1), so that it is exact if and only if $M_y = N_x$, for all x and y.

Example 1 Consider

$$e^x \sin y + y^3 - (3x - e^x \cos y)\, \frac{dy}{dx} = 0\,.$$

Here $M(x, y) = e^x \sin y + y^3$, $N(x, y) = -3x + e^x \cos y$. Compute

$$M_y = e^x \cos y + 3y^2$$
$$N_x = e^x \cos y - 3\,.$$

The partials are not the same, this equation is not exact, and our theory does not apply.

Example 2 Solve (for $x > 0$)

$$\left(\frac{y}{x} + 6x\right)\, dx + (\ln x - 2)\, dy = 0\,.$$

Here $M(x, y) = \dfrac{y}{x} + 6x$ and $N(x, y) = \ln x - 2$. Compute

$$M_y = \frac{1}{x} = N_x\,,$$

and so the equation is exact. To find $\psi(x, y)$, we observe that the equations (5.5) take the form

$$\psi_x = \frac{y}{x} + 6x$$
$$\psi_y = \ln x - 2\,.$$

Take the antiderivative in x of the first equation

$$\psi(x, y) = y \ln x + 3x^2 + h(y)\,,$$

where $h(y)$ is an arbitrary function of y. Substitute this $\psi(x, y)$ into the second equation

$$\psi_y = \ln x + h'(y) = \ln x - 2\,,$$

which gives

$$h'(y) = -2\,.$$

Integrating, $h(y) = -2y$, and so $\psi(x, y) = y \ln x + 3x^2 - 2y$, giving us the solution

$$y \ln x + 3x^2 - 2y = c\,.$$

We can solve this relation for y, $y(x) = \dfrac{c - 3x^2}{\ln x - 2}$. Observe that when solving for $h(y)$, we chose the integration constant to be zero, because at the next step we set $\psi(x, y)$ equal to c, an arbitrary constant.

Example 3 Find the constant b, for which the equation

$$\left(2x^3e^{2xy} + x^4ye^{2xy} + x\right)\,dx + bx^5e^{2xy}\,dy = 0$$

is exact, and then solve the equation with that b.

Here $M(x,y) = 2x^3e^{2xy} + x^4ye^{2xy} + x$, and $N(x,y) = bx^5e^{2xy}$. Setting equal the partials M_y and N_x, we have

$$5x^4e^{2xy} + 2x^5ye^{2xy} = 5bx^4e^{2xy} + 2bx^5ye^{2xy}\,.$$

One needs $b = 1$ for this equation to be exact. When $b = 1$, the equation becomes

$$\left(2x^3e^{2xy} + x^4ye^{2xy} + x\right)\,dx + x^5e^{2xy}\,dy = 0\,,$$

and we already know that it is exact. We look for $\psi(x,y)$ by using (5.5), as in Example 2

$$\psi_x = 2x^3e^{2xy} + x^4ye^{2xy} + x$$

$$\psi_y = x^5e^{2xy}\,.$$

It is easier to begin this time with the second equation. Taking the antiderivative in y, in the second equation,

$$\psi(x,y) = \frac{1}{2}x^4e^{2xy} + h(x)\,,$$

where $h(x)$ is an arbitrary function of x. Substituting $\psi(x,y)$ into the first equation gives

$$\psi_x = 2x^3e^{2xy} + x^4ye^{2xy} + h'(x) = 2x^3e^{2xy} + x^4ye^{2xy} + x\,.$$

This tells us that $h'(x) = x$, $h(x) = \frac{1}{2}x^2$, and then $\psi(x,y) = \frac{1}{2}x^4e^{2xy} + \frac{1}{2}x^2$.
Answer: $\dfrac{1}{2}x^4e^{2xy} + \dfrac{1}{2}x^2 = c$, or $y = \dfrac{1}{2x}\ln\left(\dfrac{2c - x^2}{x^4}\right)$.

Exact equations are connected with conservative vector fields. Recall that a vector field $\mathbf{F}(\mathbf{x},\mathbf{y}) =< M(x,y), N(x,y) >$ is called *conservative* if there is a function $\psi(x,y)$, called the *potential*, such that $\mathbf{F}(\mathbf{x},\mathbf{y}) = \nabla\psi(x,y)$. Recalling that the gradient $\nabla\psi(x,y) =< \psi_x, \psi_y >$, we have $\psi_x = M$, and $\psi_y = N$, the same relations that we had for exact equations.

1.6 Existence and Uniqueness of Solution

We consider a general initial value problem

$$y' = f(x, y)$$
$$y(x_0) = y_0 ,$$

with a given function $f(x, y)$, and given numbers x_0 and y_0. Let us ask two basic questions: is there a solution of this problem, and if there is, is the solution unique?

Theorem 1.6.1 *Assume that the functions $f(x, y)$ and $f_y(x, y)$ are continuous in some neighborhood of the initial point (x_0, y_0). Then there exists a solution, and there is only one solution. The solution $y = y(x)$ is defined on some interval (x_1, x_2) that includes x_0.*

One sees that the conditions of this theorem are not too restrictive, so that the theorem tends to apply, providing us with the existence and uniqueness of solution. But not always!

Example 1 Solve

$$y' = \sqrt{y}$$
$$y(0) = 0 .$$

The function $f(x, y) = \sqrt{y}$ is continuous (for $y \geq 0$), but its partial derivative in y, $f_y(x, y) = \frac{1}{2\sqrt{y}}$, is not even defined at the initial point $(0, 0)$. The theorem does not apply. One checks that the function $y = \frac{x^2}{4}$ solves our initial value problem (for $x \geq 0$). But here is another solution: $y(x) = 0$. (Having two different solutions of the same initial value problem is like having two primadonnas in the same theater.)

Observe that the theorem guarantees existence of solution only on some interval (it is not "happily ever after").

Example 2 Solve for $y = y(t)$

$$y' = y^2$$
$$y(0) = 1 .$$

Here $f(t, y) = y^2$, and $f_y(t, y) = 2y$ are continuous functions. The theorem applies. By separation of variables, we determine the solution $y(t) = \dfrac{1}{1-t}$. As time t approaches 1, this solution disappears, by going to infinity. This phenomenon is sometimes called the *blow up in finite time*.

1.7 Numerical Solution by Euler's method

We have learned a number of techniques for solving differential equations, however the sad truth is that most equations cannot be solved (by a formula). Even a simple looking equation like

$$y' = x + y^3 \tag{7.1}$$

is totally out of reach. Fortunately, if you need a specific solution, say the one satisfying the initial condition

$$y(0) = 1\,, \tag{7.2}$$

it can be easily approximated using the method developed in this section (by the Theorem 1.6.1, such solution exists, and it is unique, because $f(x,y) = x + y^3$ and $f_y(x,y) = 3y^2$ are continuous functions).

In general, we shall deal with the problem

$$\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0\,. \end{aligned}$$

Here the function $f(x,y)$ is given (in the example above we had $f(x,y) = x + y^3$), and the initial condition prescribes that solution is equal to a given number y_0 at a given point x_0. Fix a step size h, and let $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh$. We will approximate $y(x_n)$, the value of the solution at x_n. We call this approximation y_n. To go from the point (x_n, y_n) to the point (x_{n+1}, y_{n+1}) on the graph of solution $y(x)$, we use the tangent line approximation:

$$y_{n+1} \approx y_n + y'(x_n)(x_{n+1} - x_n) = y_n + y'(x_n)h = y_n + f(x_n, y_n)h\,.$$

(We expressed $y'(x_n) = f(x_n, y_n)$ from the differential equation. Because of the approximation errors, the point (x_n, y_n) is not exactly lying on the solution curve $y = y(x)$, but we pretend that it does.) The resulting formula is easy to implement, it is just one computational loop, starting with the initial point (x_0, y_0).

One continues the computations until the points x_n go as far as needed. Decreasing the step size h, will improve the accuracy. Smaller h's will require more steps, but with the power of modern computers, that is not a problem, particularly for simple examples, like the problem (7.1), (7.2), which is

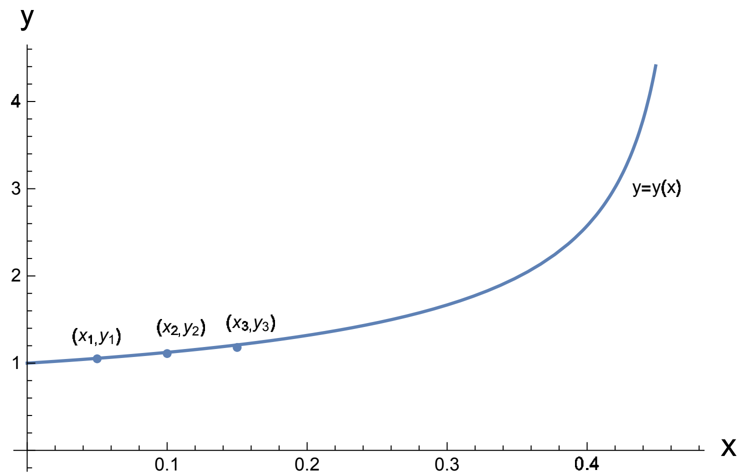


Figure 1.3: The numerical solution of $y' = x + y^3$, $y(0) = 1$

discussed next. In that example $x_0 = 0$, $y_0 = 1$. If we choose $h = 0.05$, then $x_1 = 0.05$, and

$$y_1 = y_0 + f(x_0, y_0)h = 1 + (0 + 1^3)\, 0.05 = 1.05.$$

Continuing, we have $x_2 = 0.1$, and

$$y_2 = y_1 + f(x_1, y_1)h = 1.05 + (0.05 + 1.05^3)\, 0.05 \approx 1.11\,.$$

Next, $x_3 = 0.15$, and

$$y_3 = y_2 + f(x_2, y_2)h = 1.11 + (0.1 + 1.11^3)\, 0.05 \approx 1.18\,.$$

These computations imply that $y(0.05) \approx 1.05$, $y(0.1) \approx 1.11$, and $y(1.15) \approx 1.18$. If you need to approximate the solution on the interval $(0, 0.4)$, you have to make five more steps. Of course, it is better to program a computer. A computer computation reveals that this solution tends to infinity (blows up) at $x \approx 0.47$. The Figure 1.3 presents the solution curve, computed by *Mathematica*, as well as the three points we computed by Euler's method.

Euler's method is using the tangent line approximation, or the first two terms of the Taylor series approximation. One can use more terms of the Taylor series, and develop more sophisticated methods (which is done in books on numerical methods, and implemented in software packages, like *Mathematica*). But here is a question: if it is so easy to compute numerical approximation of solutions, why bother learning analytical solutions? The reason is that we seek not just to solve a differential equation, but to understand it. What happens if the initial condition changes? The equation may include some parameters, what happens if they change? What happens to solutions in the long term?

1.7.1 Problems

I. Determine if the equation is homogeneous, and if it is, solve it.

1. $\frac{dy}{dx} = \frac{y+2x}{x}$, with $x > 0$. Answer. $y = x\,(2\ln x + c)$.

2. $(x+y)\;dx - x\,dy = 0$. Answer. $y = x\,(\ln|x| + c)$.

3. $\frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$. Answer. $y = x\left(1 - \frac{1}{\ln|x| + c}\right)$, and $y = x$.

4. $\frac{dy}{dx} = \frac{y^2 + 2x}{y}$.

5. $y' = \frac{y^2}{x^2} + \frac{y}{x}$, $y(1) = 1$. Answer. $y = \frac{x}{1 - \ln x}$.

6. $y' = \frac{y^2}{x^2} + \frac{y}{x}$, $y(-1) = 1$. Answer. $y = -\frac{x}{1 + \ln|x|}$.

7. $\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$, $y(1) = 2$. Answer. $y = \frac{2x^2}{3 - 2x}$.

8. $xy' - y = x\tan\frac{y}{x}$. Answer. $\sin\frac{y}{x} = cx$.

9. $xy' = \frac{x^2}{x+y} + y$. Answer. $y = -x \pm x\sqrt{2\ln|x| + c}$.

10. $y' = \frac{x^2 + y^2}{xy}$, $y(1) = -2$. Answer. $y = -x\sqrt{2\ln x + 4}$.

11. $y' = \frac{y + x^{-1/2}y^{3/2}}{\sqrt{xy}}$, with $x > 0$, $y > 0$. Answer. $2\sqrt{\frac{y}{x}} = \ln x + c$.

12. $x^3 y' = y^2\left(y - xy'\right)$. Answer. $\ln|y| + \frac{1}{2}\left(\frac{y}{x}\right)^2 = c$, and $y = 0$.

13*. A function $f(x,y)$ is called *quasi-homogeneous* if for any constant α

$$f(\alpha x, \alpha^p y) = \alpha^{p-1} f(x,y)\,,$$

with some constant p.

(i) Letting $\alpha = \frac{1}{x}$, and $v = \frac{y}{x^p}$, verify that

$$f(x,y) = x^{p-1} g(v)\,,$$

where $g(v)$ is some function of one variable.

(ii) Consider a *quasi-homogeneous equation*

$$y' = f(x,y)\,,$$

where $f(x,y)$ is a quasi-homogeneous function. Show that a change of variables $v = \frac{y}{x^p}$ produces a separable equation.

(iii) Solve

$$y' = x + \frac{y^2}{x^3}\,.$$

Hint: Denoting $f(x,y) = x + \dfrac{y^2}{x^3}$, we have $f(\alpha x, \alpha^2 y) = \alpha f(x,y)$, so that $p = 2$. Letting $v = \frac{y}{x^2}$, or $y = x^2 v$, we get

$$xv' = 1 - 2v + v^2\,.$$

Answer. $y = x^2 \left(1 - \frac{1}{\ln|x| + c}\right)$.

II. Solve the following Bernoulli's equations.

1. $y'(t) = 3y - y^2$. Answer. $y = \dfrac{3}{1 + ce^{-3t}}$, and $y = 0$.

2. $y' - \dfrac{1}{x} y = y^2$, $y(2) = -2$. Answer. $y = \dfrac{2x}{2 - x^2}$.

3. $xy' + y + xy^2 = 0$, $y(1) = 2$. Answer. $y = \dfrac{2}{x\,(1 + 2\ln x)}$.

4. $y' + y = xy^3$, $y(0) = -1$. Answer. $y = -\dfrac{\sqrt{2}}{\sqrt{2x + e^{2x} + 1}}$.

5. $\dfrac{dy}{dx} = \dfrac{y^2 + 2x}{y}$. Answer. $y = \pm\sqrt{-1 - 2x + ce^{2x}}$.

6. $y' + x\sqrt[3]{y} = 3y$. Answer. $y = \pm\left(\dfrac{x}{3} + \dfrac{1}{6} + ce^{2x}\right)^{\frac{3}{2}}$, and $y = 0$.

Hint: When dividing the equation by $\sqrt[3]{y}$, one needs to check if $y = 0$ is a solution, and indeed it is.

7. $y' + y = -xy^2$. Answer. $y = \dfrac{1}{ce^x - x - 1}$, and $y = 0$.

8. $y' + xy = y^3$, $y(1) = -\dfrac{1}{e}$. Answer. $y = -\dfrac{1}{\sqrt{-2e^{x^2}\int_1^x e^{-t^2}\,dt + e^{(x^2+1)}}}$.

9. The equation

$$\frac{dy}{dx} = \frac{y^2 + 2x}{y}$$

could not be solved in the preceding problem set, because it is not homogeneous. Can you solve it now? Answer. $y = \pm\sqrt{ce^{2x} - 2x - 1}$.

10. $y' = \dfrac{x}{y}e^{2x} + y$. Answer. $y = \pm e^x\sqrt{x^2 + c}$.

11. Solve the *Gompertz population model* (a and b are positive constants)

$$\frac{dx}{dt} = x\,(a - b\ln x)\ , \quad x > 1\,.$$

Hint: Setting $y = \ln x$, obtain $y' = a - by$.

Answer. $x(t) = e^{a/b}e^{c\,e^{-bt}}$.

12. Solve

$$x(y' - e^y) + 2 = 0\,.$$

Hint: Divide the equation by e^y, then set $v = e^{-y}$, obtaining a linear equation for $v = v(x)$. Answer. $y = -\ln\left(x + cx^2\right)$.

13. $\dfrac{dy}{dx} = \dfrac{y}{x + x^2y}$.

Hint: Consider $\dfrac{dx}{dy}$, and obtain Bernoulli's equation for $x(y)$.

Answer. $x = \dfrac{2y}{c - y^2}$.

III. 1. Use parametric integration to solve

$$y'^3 + y' = x\,.$$

Answer. $x = t^3 + t$, $y = \dfrac{3}{4}t^4 + \dfrac{1}{2}t^2 + c$.

2. Use parametric integration to solve

$$y = \ln(1 + y'^2) \, .$$

Answer. $x = 2\tan^{-1} t + c$, $y = \ln(1 + t^2)$. Another solution: $y = 0$.

3. Use parametric integration to solve

$$y' + \sin(y') = x \, , \quad y(0) = 0 \, .$$

Answer. $x = t + \sin t$, $y = \frac{1}{2}t^2 + t \sin t + \cos t - 1$.

4. Solve the logistic model (for $0 < y < 3$)

$$y'(t) = 3y - y^2 \, , \quad y(0) = 1$$

as a separable equation. What is the carrying capacity? What is $\lim_{t \to \infty} y(t)$?

Answer. $y(t) = \dfrac{3}{1 + 2e^{-3t}}$, $\lim_{t \to \infty} y(t) = 3$.

5. A tank is completely filled with 100L of water-salt mixture, which initially contains 10 kg of salt. Water is flowing in at a rate of 5L per minute. The new mixture flows out at the same rate. How much salt remains in the tank after an hour?

Answer. Approximately 0.5 kg.

6. A tank is completely filled with 100L of water-salt mixture, which initially contains 10 kg of salt. A water-salt mixture is flowing in at a rate of 3L per minute, and each liter of it contains 0.1 kg of salt. The new mixture flows out at the same rate. How much salt is contained in the tank after t minutes?

Answer. 10 kg.

7. Water is being pumped into patient's stomach at a rate of 0.5L per minute to flush out 300 grams of alcohol poisoning. The excess fluid is flowing out at the same rate. The stomach holds 3L. The patient can be discharged when the amount of poison drops to 50 grams. How long should this procedure last?

Answer. $t = 6 \ln 6 \approx 10.75$ minutes.

8. Temperature in a room is maintained at 70°. If an object at 100° is placed in this room, it cools down to 80° in 5 minutes. A bowl of soup at

190° is placed in this room. The soup is ready to eat at 130°. How many minutes one should wait?

Hint: If $y(t)$ is the temperature after t minutes, it is natural to assume that the speed of cooling is proportional to the difference of temperatures, so that

$$y' = -k(y - 70)$$

for some constant $k > 0$. We are given that $y(5) = 80$, provided that $y(0) = 100$. This allows us to calculate $k = \frac{\ln 3}{5}$. Then assuming that $y(0) = 190$, one calculates t such that $y(t) = 130$.

Answer. $t = 5\,\dfrac{\ln 2}{\ln 3} \approx 3.15$ minutes.

9. Find all curves $y = f(x)$ with the following property: if you draw a tangent line at any point $(x_0, f(x_0))$ on this curve, and continue the tangent line until it intersects the x-axis, then the point of intersection is $x_0/2$.

Answer. $y = cx^2$. (A family of parabolas.)

10. Find all positive decreasing functions $y = f(x)$, with the following property: in the triangle formed by the vertical line going down from the curve, the x-axis and the tangent line to this curve, the sum of two sides adjacent to the right angle is a constant, equal to $b > 0$.

Answer. $y - b \ln y = x + c$.

11. Find all positive decreasing functions $y = f(x)$, with the following property: for the tangent line at $(x_0, f(x_0))$, the length of the segment between the point $(x_0, f(x_0))$ and the y-axis is equal to 1, for all $0 < x_0 \le 1$.

Answer. $y = -\sqrt{1-x^2} - \ln x + \ln\left[1 + \sqrt{1-x^2}\right] + c$. This historic curve (first studied by Huygens in 1692) is called the *tractrix*.

12. Find all curves $y = f(x)$ such that the point of intersection of the tangent line at $(x_0, f(x_0))$ with the x-axis is equidistant from the origin and the point $(x_0, f(x_0))$, at any x_0.

Answer. $x^2 + y^2 = cy$, a family of circles. (Hint: The differential equation $y' = \dfrac{2xy}{x^2 - y^2}$ is homogeneous.)

13. Solve Riccati's equation

$$y' + 2e^x y - y^2 = e^x + e^{2x}\,.$$

Answer. $y = e^x$, and $y = e^x - \dfrac{1}{x+c}$.

14. Solve Riccati's equation

$$y' + (2e^x + 2)\, y - e^x y^2 = e^x + 2\,.$$

Answer. $y = 1$, and $y = 1 + \dfrac{1}{e^x + ce^{2x}}$.

15*. (From the Putnam competition, 2009) Show that any solution of

$$y' = \frac{x^2 - y^2}{x^2(y^2+1)}$$

satisfies $\lim\limits_{x\to\infty} y(x) = \infty$.

Hint: Using "partial fractions", rewrite this equation as

$$y' = \frac{1 + 1/x^2}{y^2+1} - \frac{1}{x^2}\,.$$

Then $y'(x) > -\frac{1}{x^2}$, which precludes $y(x)$ from going to $-\infty$. So, either $y(x)$ is bounded, or it goes to $+\infty$, as $x \to \infty$ (possibly along some sequence). If $y(x)$ is bounded when x is large, then $y'(x)$ exceeds a positive constant for all large x, and therefore $y(x)$ tends to infinity, a contradiction (observe that $1/x^2$ becomes negligible for large x). Finally, if $y(x)$ failed to tend to infinity as $x \to \infty$ (while going to infinity over a subsequence), it would have infinitely many points of local minimum, at which $y = x$, a contradiction.

16. Solve the *integral equation*

$$y(x) = \int_1^x y(t)\, dt + x + 1\,.$$

Hint: Differentiate the equation, and also evaluate $y(1)$.

Answer. $y = 3e^{x-1} - 1$.

IV. Determine if the equation is exact, and if it is, solve it.

1. $(2x + 3x^2 y)\, dx + (x^3 - 3y^2)\, dy = 0$. Answer. $x^2 + x^3 y - y^3 = c$.

2. $(x + \sin y)\, dx + (x \cos y - 2y)\, dy = 0$. Answer. $\frac{1}{2}x^2 + x \sin y - y^2 = c$.

3. $\dfrac{x}{x^2+y^4}\, dx + \dfrac{2y^3}{x^2+y^4}\, dy = 0$. Answer. $x^2 + y^4 = c$.

4. Find a simpler solution for the preceding problem.

5. $(6xy-\cos y)\, dx+(3x^2+x\sin y+1)\, dy = 0.$ Answer. $3x^2y-x\cos y+y = c.$

6. $(2x-y)\, dx+(2y-x)\, dy = 0,\ y(1) = 2.$ Answer. $x^2+y^2-xy = 3.$

7. $2x\left(1+\sqrt{x^2-y}\right)\, dx-\sqrt{x^2-y}\, dy = 0.$ Answer. $x^2+\dfrac{2}{3}\left(x^2-y\right)^{\frac{3}{2}} = c.$

8. $\left(ye^{xy}\sin 2x+2e^{xy}\cos 2x+2x\right)\, dx+\left(xe^{xy}\sin 2x-2\right)\, dy = 0,\ \ y(0) = -2.$

Answer. $e^{xy}\sin 2x+x^2-2y = 4.$

9. Find the value of b for which the following equation is exact, and then solve the equation, using that value of b

$$(ye^{xy}+2x)\, dx+bxe^{xy}\, dy = 0\,.$$

Answer. $b = 1$, $y = \dfrac{1}{x}\ln(c-x^2).$

10. Verify that the equation

$$(2\sin y+3x)\ dx+x\cos y\, dy = 0$$

is not exact, however if one multiplies it by x, the equation becomes exact, and it can be solved. Answer. $x^2\sin y+x^3 = c.$

11. Verify that the equation

$$(x-3y)\ dx+(x+y)\ dy = 0$$

is not exact, however it can be solved as a homogeneous equation.

Answer. $\ln|y-x|+\dfrac{2x}{x-y} = c.$

V. 1. Find three solutions of the initial value problem

$$y' = (y-1)^{1/3},\ \ y(1) = 1\,.$$

Is it desirable in applications to have three solutions of the same initial value problem? What "went wrong"? (Why the existence and uniqueness Theorem 1.6.1 does not apply here?)

Answer. $y(x) = 1$, and $y(x) = 1 \pm \left(\frac{2}{3}x - \frac{2}{3}\right)^{\frac{3}{2}}$.

2. Find all y_0, for which the following problem has a unique solution

$$y' = \frac{x}{y^2 - 2x}, \quad y(2) = y_0 .$$

Hint: Apply the existence and uniqueness Theorem 1.6.1.

Answer. All y_0 except ± 2.

3. Show that the function $\frac{x|x|}{4}$ solves the problem

$$\begin{aligned} y' &= \sqrt{|y|} \\ y(0) &= 0 , \end{aligned}$$

for all x. Can you find another solution?

Hint: Consider separately the cases when $x > 0$, $x < 0$, and $x = 0$.

4. Show that the problem (here $y = y(t)$)

$$\begin{aligned} y' &= y^{2/3} \\ y(0) &= 0 \end{aligned}$$

has infinitely many solutions.

Hint: Consider $y(t)$ that is equal to zero for $t < a$, and to $\frac{(t-a)^3}{27}$ for $t \geq a$, where $a > 0$ is any constant.

5. (i) Apply Euler's method to

$$y' = x(1 + y), \;\; y(0) = 1 .$$

Take $h = 0.25$, and do four steps, obtaining an approximation for $y(1)$.

(ii) Take $h = 0.2$, and do five steps of Euler's method, obtaining another approximation for $y(1)$.

(iii) Solve the above problem exactly, and determine which one of the two approximations is better.

6. Write a computer program to implement Euler's method for

$$y' = f(x, y), \;\; y(x_0) = y_0 .$$

It involves a simple loop: $y_{n+1} = y_n + hf(x_0 + nh, y_n)$, $n = 0, 1, 2, \ldots$.

1.8 * The Existence and Uniqueness Theorem

In this section, for the initial value problem

$$
\begin{aligned}
&y' = f(x, y) \\
&y(x_0) = y_0 \, ,
\end{aligned}
\tag{8.3}
$$

we prove a more general existence and uniqueness theorem than the Theorem 1.6.1 stated above.

Define a rectangular box B around the initial point (x_0, y_0) to be the set of points (x, y), satisfying $x_0 - a \le x \le x_0 + a$ and $y_0 - b \le y \le y_0 + b$, for some positive a and b. It is known from calculus that in case $f(x, y)$ is continuous on B, it is bounded on B, so that for some constant $M > 0$

$$|f(x, y)| \le M, \quad \text{for all points } (x, y) \text{ in } B \, . \tag{8.4}$$

Theorem 1.8.1 *Assume that the function $f(x, y)$ is continuous on B, and for some constant $L > 0$, it satisfies (the Lipschitz condition)*

$$|f(x, y_2) - f(x, y_1)| \le L|y_2 - y_1|, \tag{8.5}$$

for any two points (x, y_1) and (x, y_2) in B. Then the initial value problem (8.3) has a unique solution, which is defined for x on the interval $(x_0 - \frac{b}{M}, x_0 + \frac{b}{M})$, in case $\frac{b}{M} < a$, and on the interval $(x_0 - a, x_0 + a)$ if $\frac{b}{M} \ge a$.

Proof: Assume, for definiteness, that $\frac{b}{M} < a$, and the other case is similar. We shall prove the existence of solutions first, and let us restrict to the case $x > x_0$ (the case when $x < x_0$ is similar). Integrating the equation in (8.3) over the interval (x_0, x), we convert the initial value problem (8.3) into an equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt \, . \tag{8.6}$$

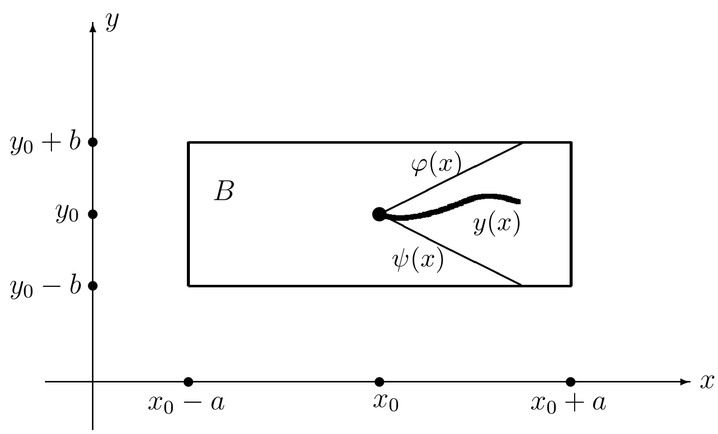
(If $y(x)$ solves (8.6), then $y(x_0) = y_0$, and by differentiation $y' = f(x, y)$.) By (8.4), obtain

$$-M \le f(t, y(t)) \le M \, , \tag{8.7}$$

and then any solution of (8.6) lies between two straight lines

$$y_0 - M(x - x_0) \le y(x) \le y_0 + M(x - x_0) \, .$$

For $x_0 \le x \le x_0 + \frac{b}{M}$ these lines stay in the box B, reaching its upper and lower boundaries at $x = x_0 + \frac{b}{M}$. (In the other case, when $\frac{b}{M} \ge a$, these lines stay in B for all $x_0 \le x \le x_0 + a$.) We denote $\varphi(x) = y_0 + M(x - x_0)$, and call this function a *supersolution*, while $\psi(x) = y_0 - M(x - x_0)$ is called a *subsolution*.



The functions $\varphi(x)$ and $\psi(x)$ exiting the box B

1. *A special case.* Let us make an additional assumption that $f(x, y)$ is increasing in y, so that if $y_2 > y_1$, then $f(x, y_2) > f(x, y_1)$, for any two points (x, y_1) and (x, y_2) in B. We shall construct a solution of (8.3) as the limit of a sequence of iterates $\psi(x)$, $y_1(x), y_2(x), \ldots, y_n(x), \ldots$, defined as follows

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt\,,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt\,,\ \ldots\,,\ y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt\,.$$

We claim that for all x on the interval $x_0 < x \le x_0 + \frac{b}{M}$, the following inequalities hold

$$\psi(x) \le y_1(x) \le y_2(x) \le \cdots \le y_n(x) \le \cdots\,. \tag{8.8}$$

Indeed, $f(t, \psi(t)) \ge -M$, by (8.7), and then

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt \ge y_0 - M(x - x_0) = \psi(x)\,,$$

giving us the first of the inequalities in (8.8). Then

$$y_2(x) - y_1(x) = \int_{x_0}^{x} [f(t, y_1(t)) - f(t, \psi(t))] \, dt \geq 0 \,,$$

using the just established inequality $\psi(x) \leq y_1(x)$, and the monotonicity of $f(x, y)$. So that $y_1(x) \leq y_2(x)$, and the other inequalities in (8.8) are established similarly. Next, we claim that for any x on the interval $x_0 < x \leq x_0 + \frac{b}{M}$, all of these iterates lie below the supersolution $\varphi(x)$, so that

$$\psi(x) \leq y_1(x) \leq y_2(x) \leq \cdots \leq y_n(x) \leq \cdots \leq \varphi(x) \,. \tag{8.9}$$

Indeed, $f(t, \psi(t)) \leq M$, by (8.7), giving

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t)) \, dt \leq y_0 + M(x - x_0) = \varphi(x) \,,$$

proving the first inequality in (8.9), and that the graph of $y_1(x)$ stays in the box B, for $x_0 < x \leq x_0 + \frac{b}{M}$. Then

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t)) \, dt \leq y_0 + M(x - x_0) = \varphi(x) \,,$$

and so on, for all $y_n(x)$.

At each x in $(x_0, x_0 + \frac{b}{M})$, the numerical sequence $\{y_n(x)\}$ is non-decreasing, bounded above by the number $\varphi(x)$. Hence, this sequence has a limit which we denote by $y(x)$. The sequence $f(x, y_n(x))$ is also non-decreasing, and it converges to $f(x, y(x))$. By the monotone convergence theorem, we may pass to the limit in the recurrence relation

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t)) \, dt \,, \tag{8.10}$$

concluding that $y(x)$ gives the desired solution of the integral equation (8.6).

(If one starts the recurrence relation (8.10) with the supersolution $\phi(x)$, one obtains similarly a decreasing sequence of iterates converging to a solution of (8.6).)

2. *The general case.* Define $g(x, y) = f(x, y) + Ay$. If we choose the constant A large enough, then the new function $g(x, y)$ will be increasing in y, for $(x, y) \in B$. Indeed, using the Lipschitz condition (8.5),

$$g(x, y_2) - g(x, y_1) = f(x, y_2) - f(x, y_1) + A\,(y_2 - y_1)$$

$$\geq -L\,(y_2 - y_1) + A\,(y_2 - y_1) = (A - L)\,(y_2 - y_1) > 0\,,$$

for any two points (x, y_1) and (x, y_2) in B, provided that $A > L$, and $y_2 > y_1$. We now consider an equivalent equation (recall that $g(x,y) = f(x,y) + Ay$)

$$y' + Ay = f(x,y) + Ay = g(x,y)\,.$$

Multiplying both sides by the integrating factor e^{Ax}, we put this equation into the form

$$\frac{d}{dx}\left[e^{Ax}y\right] = e^{Ax}g(x,y)\,.$$

Set $z(x) = e^{Ax}y(x)$, then $y(x) = e^{-Ax}z(x)$, and the new unknown function $z(x)$ satisfies

$$\begin{aligned} z' &= e^{Ax}g\left(x, e^{-Ax}z\right) \\ z(x_0) &= e^{Ax_0}y_0\,. \end{aligned} \tag{8.11}$$

The function $e^{Ax}g\left(x, e^{-Ax}z\right)$ is increasing in z. The special case applies, so that the solution $z(x)$ of (8.11) exists. Then $y(x) = e^{-Ax}z(x)$ gives the desired solution of (8.3).

Finally, we prove the uniqueness of solution. Let $u(x)$ be another solution of (8.6) on the interval $(x_0, x_0 + \frac{b}{M})$, so that

$$u(x) = y_0 + \int_{x_0}^{x} f(t, u(t))\,dt\,.$$

Subtracting this from (8.6), gives

$$y(x) - u(x) = \int_{x_0}^{x} \left[f(t, y(t)) - f(t, u(t))\right]\,dt\,.$$

Assume first that x is in $[x_0, x_0 + \frac{1}{2L}]$. Then using the Lipschitz condition (8.5), we estimate

$$\begin{aligned} |y(x) - u(x)| &\leq \int_{x_0}^{x} |f(t, y(t)) - f(t, u(t))|\,dt \leq L\int_{x_0}^{x} |y(t)) - u(t)|\,dt \\ &\leq L(x - x_0)\max_{[x_0, x_0 + \frac{1}{2L}]} |y(x) - u(x)| \leq \frac{1}{2}\max_{[x_0, x_0 + \frac{1}{2L}]} |y(x) - u(x)|\,. \end{aligned}$$

It follows that

$$\max_{[x_0, x_0 + \frac{1}{2L}]} |y(x) - u(x)| \leq \frac{1}{2}\max_{[x_0, x_0 + \frac{1}{2L}]} |y(x) - u(x)|\,.$$

But then $\max_{[x_0,x_0+\frac{1}{2L}]}|y(x)-u(x)|=0$, so that $y(x)=u(x)$ on $[x_0,x_0+\frac{1}{2L}]$. Let $x_1=x_0+\frac{1}{2L}$. We just proved that $y(x)=u(x)$ on $[x_0,x_0+\frac{1}{2L}]$, and in particular $y(x_1)=u(x_1)$. Repeating (if necessary) the same argument on $[x_1,x_1+\frac{1}{2L}]$, and so on, we will eventually conclude that $y(x)=u(x)$ on $(x_0,x_0+\frac{b}{M})$. ♢

Observe that the Lipschitz condition (8.5) follows from the easy to check requirement that the function $f_y(x,y)$ is continuous in the box B.

We shall need the following important tool.

Lemma 1.8.1 *(Bellman-Gronwall Lemma) Assume that for $x\geq x_0$, the functions $u(x)$ and $a(x)$ are continuous, and satisfy $u(x)\geq 0$, $a(x)\geq 0$. Assume that for some number $K>0$ we have*

$$u(x)\leq K+\int_{x_0}^{x}a(t)u(t)\,dt,\quad \text{for } x\geq x_0\,. \tag{8.12}$$

Then

$$u(x)\leq Ke^{\int_{x_0}^{x}a(t)\,dt},\quad \text{for } x\geq x_0\,. \tag{8.13}$$

Proof: Divide the inequality (8.12) by its right hand side (which is positive)

$$\frac{a(x)u(x)}{K+\int_{x_0}^{x}a(t)u(t)\,dt}\leq a(x)\,.$$

Integrating both sides over (x_0,x) (the numerator of the fraction on the left is equal to the derivative of its denominator), gives

$$\ln\left(K+\int_{x_0}^{x}a(t)u(t)\,dt\right)-\ln K\leq\int_{x_0}^{x}a(t)\,dt\,,$$

which implies that

$$K+\int_{x_0}^{x}a(t)u(t)\,dt\leq Ke^{\int_{x_0}^{x}a(t)\,dt}\,.$$

Using the inequality (8.12) once more, we get (8.13). ♢

In addition to the initial value problem (8.3), with $f(x,y)$ satisfying the Lipschitz condition (8.5), consider

$$\begin{aligned} z'&=f(x,z)\\ z(x_0)&=z_0\,. \end{aligned} \tag{8.14}$$

If $z_0 = y_0$, then $z(x) = y(x)$ for all $x \in B$, by the Theorem 1.8.1 (observe that the Lipschitz condition (8.5) implies the continuity of $f(x, y)$ on B). Now suppose that $z_0 \neq y_0$, but $|z_0 - y_0|$ is small. We claim that $z(x)$ and $y(x)$ will remain close over any bounded interval $(x_0, x_0 + p)$, provided that both solutions exist on that interval, and $|z_0 - y_0|$ is small enough. This fact is known as the *continuous dependence of solutions, with respect to the initial condition*.

We begin the proof of the claim by observing that $z(x)$ satisfies

$$z(x) = z_0 + \int_{x_0}^{x} f(t, z(t))\, dt\,.$$

From this formula we subtract (8.6), and then estimate

$$z(x) - y(x) = z_0 - y_0 + \int_{x_0}^{x} \left[f(t, z(t)) - f(t, y(t))\right]\, dt\,;$$

$$|z(x) - y(x)| \leq |z_0 - y_0| + \int_{x_0}^{x} |f(t, z(t)) - f(t, y(t))|\, dt$$

$$\leq |z_0 - y_0| + \int_{x_0}^{x} L\, |z(t) - y(t)|\, dt\,.$$

(We used the triangle inequality for numbers: $|a + b| \leq |a| + |b|$, the triangle inequality for integrals: $|\int_{x_0}^{x} g(t)\, dt| \leq \int_{x_0}^{x} |g(t)|\, dt$, and the condition (8.5).) By the Bellman-Gronwall lemma

$$|z(x) - y(x)| \leq |z_0 - y_0|\, e^{L(x - x_0)} \leq |z_0 - y_0|\, e^{Lp}, \;\; \text{for } x \in (x_0, x_0 + p)\,,$$

so that $z(x)$ and $y(x)$ remain close over the interval $(x_0, x_0 + p)$, provided that $|z_0 - y_0|$ is small enough.

1.8.1 Problems

1. Assume that the function $u(x) \geq 0$ is continuous for $x \geq 1$, and for some number $K > 0$, we have

$$x u(x) \leq K + \int_{1}^{x} u(t)\, dt, \;\; \text{for } x \geq 1\,.$$

Show that $u(x) \leq K$, for $x \geq 1$.

2. Assume that the functions $a(x) \geq 0$, and $u(x) \geq 0$ are continuous for $x \geq x_0$, and we have

$$u(x) \leq \int_{x_0}^{x} a(t) u(t)\, dt, \;\; \text{for } x \geq x_0\,.$$

Show that $u(x) = 0$, for $x \geq x_0$. Then give an alternative proof of the uniqueness part of the Theorem 1.8.1.

Hint: Let $K \to 0$ in the Bellman-Gronwall lemma.

3. Assume that the functions $a(x) \geq 0$, and $u(x) \geq 0$ are continuous for $x \geq x_0$, and we have

$$u(x) \leq \int_{x_0}^{x} a(t)u^2(t)\,dt, \ \ \text{for } x \geq x_0\,.$$

Show that $u(x) = 0$, for $x \geq x_0$.

Hint: Observe that $u(x_0) = 0$. When t is close to x_0, $u(t)$ is small. But then $u^2(t) < u(t)$. (Alternatively, one may treat the function $a(t)u(t)$ as known, and use the preceding problem.)

4. Show that if a function $x(t)$ satisfies

$$0 \leq \frac{dx}{dt} \leq x^2 \ \ \text{for all } t, \ \ \text{and } x(0) = 0,$$

then $x(t) = 0$ for all $t \in (-\infty, \infty)$.

Hint: Show that $x(t) = 0$ for $t > 0$. In case $t < 0$, introduce new variables y and s, by setting $x = -y$ and $t = -s$, so that $s > 0$.

5. Assume that the functions $a(x) \geq 0$, and $u(x) \geq 0$ are continuous for $x \geq x_0$, and we have

$$u(x) \leq K + \int_{x_0}^{x} a(t)\,[u(t)]^m\,dt\,, \ \ \text{for } x \geq x_0\,, \tag{8.15}$$

with some constants $K > 0$ and $0 < m < 1$. Show that

$$u(x) \leq \left[K^{1-m} + (1-m)\int_{x_0}^{x} a(t)\,dt\right]^{\frac{1}{1-m}}\,, \ \ \text{for } x \geq x_0\,.$$

This fact is known as *Bihari's inequality*. Show also that the same inequality holds in case $m > 1$, under an additional assumption that

$$K^{1-m} + (1-m)\int_{x_0}^{x} a(t)\,dt > 0\,, \quad \text{for all } x \geq x_0\,.$$

Hint: Denote the right hand side of (8.15) by $w(x)$. Then $w(x_0) = K$, and

$$w' = a(x)u^m \leq a(x)w^m\,.$$

Divide by w^m, and integrate over (x_0, x).

6. For the initial value problem

$$y' = f(x, y)\,, \;\; y(x_0) = y_0\,,$$

or the corresponding integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt\,,$$

the *Picard iterations* are defined by the recurrence relation

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt\,, \;\; n = 0, 1, 2, \ldots\,,$$

starting with $y_0(x) = y_0$. (Picard's iterations are traditionally used to prove the existence and uniqueness Theorem 1.8.1.)

(i) Compute the Picard iterations for

$$y' = y\,, \;\; y(0) = 1\,,$$

and compare them with the exact solution.

(ii) Compute the Picard iterates $y_1(x)$ and $y_2(x)$ for

$$y' = 2xy^2\,, \;\; y(0) = 1\,,$$

and compare them with the exact solution, for $|x|$ small.

Hint: The exact solution may be written as a series $y(x) = 1 + x^2 + x^4 + x^6 + \cdots$.

Answer. $y_0(x) = 1$, $y_1(x) = 1 + x^2$, $y_2(x) = 1 + x^2 + x^4 + \frac{x^6}{3}$. The difference $|y(x) - y_2(x)|$ is very small, for $|x|$ small.

7. Let $y(x)$ be the solution for $x > 0$ of the equation

$$y' = f(x, y)\,, \;\; y(0) = y_0\,.$$

Assume that $|f(x, y)| \le a(x)|y| + b(x)$, with positive functions $a(x)$ and $b(x)$ satisfying $\int_0^\infty a(x)\, dx < \infty$, $\int_0^\infty b(x)\, dx < \infty$. Show that $|y(x)|$ is bounded for all $x > 0$.

Hint: Apply the Bellman-Gronwall lemma to the corresponding integral equation.

8. Assume that for $x \geq x_0$ the continuous functions $y(x)$, $f(x)$ and $g(x)$ are non-negative, and

$$y(x) \leq f(x) + \int_{x_0}^{x} g(t)y(t)\,dt\,, \quad \text{for } x \geq x_0\,.$$

Show that

$$y(x) \leq f(x) + \int_{x_0}^{x} g(t)f(t)e^{\int_t^x g(u)\,du}\,dt\,, \quad \text{for } x \geq x_0\,.$$

Hint: Denote $I(x) = \int_{x_0}^{x} g(t)y(t)\,dt$. Since $I'(x) = g(x)y(x) \leq g(x)I(x) + g(x)f(x)$, it follows that

$$\frac{d}{dx}\left[e^{-\int_{x_0}^{x} g(u)\,du} I(x)\right] \leq e^{-\int_{x_0}^{x} g(u)\,du} g(x)f(x)\,.$$

Integration over $[x_0, x]$ gives $I(x) \leq \int_{x_0}^{x} g(t)f(t)e^{\int_t^x g(u)\,du}\,dt$.

Chapter 2

Second Order Equations

The central topic of this chapter involves linear second order equations with constant coefficients. These equations, while relatively easy to solve, are of great importance, particularly for their role in modeling mechanical and electrical oscillations. Several sections deal with such applications. Some non-standard applications are also included: the motion of a meteor, coupled pendulums, and the path of a military drone. Then we study Euler's equation with variable coefficients, and higher order equations. The chapter concludes with a more advanced topic of oscillation theory.

2.1 Special Second Order Equations

Probably the simplest second order equation is

$$y''(x) = 0.$$

Taking the antiderivative

$$y'(x) = c_1 \,.$$

We denoted an arbitrary constant by c_1, because we expect another arbitrary constant to make an appearance. Indeed, taking another antiderivative, we get the general solution

$$y(x) = c_1 x + c_2 \,.$$

This example suggests that general solutions of second order equations depend on two arbitrary constants.

General second order equations for the unknown function $y = y(x)$ can often be written as

$$y'' = f(x, y, y'),$$

where f is a given function of its three variables. One cannot expect all such equations to be solvable, as we could not even solve all first order equations. In this section we study special second order equations, which are reducible to first order equations, greatly increasing their chances to be solved.

2.1.1 y is not present in the equation

Let us solve for $y(t)$ the equation

$$ty'' - y' = t^2 \,.$$

The derivatives of y are present in this equation, but not the function y itself. We denote $y'(t) = v(t)$, and $v(t)$ is our new unknown function. Clearly, $y''(t) = v'(t)$, and the equation becomes

$$tv' - v = t^2 \,.$$

This is a first order equation for $v(t)$! This equation is linear, so that we solve it as usual. Once $v(t)$ is calculated, the solution $y(t)$ is determined by integration. Details:

$$v' - \frac{1}{t}v = t \,,$$

$$\mu(t) = e^{-\int \frac{1}{t}\,dt} = e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t} \,,$$

$$\frac{d}{dt}\left[\frac{1}{t}v\right] = 1 \,,$$

$$\frac{1}{t}v = t + c_1 \,,$$

$$y' = v = t^2 + c_1 t \,,$$

$$y(t) = \frac{t^3}{3} + c_1\frac{t^2}{2} + c_2 \,.$$

Here c_1 and c_2 are arbitrary constants.

For the general case of equations with y not present

$$y'' = f(x, y') \,,$$

the change of variables $y' = v$ results in the first order equation

$$v' = f(x, v) \,.$$

Let us solve the following equation for $y(x)$:

$$y'' + 2xy'^2 = 0 \,.$$

Again, y is missing in this equation. Setting $y' = v$, with $y'' = v'$, gives a first order equation:

$$v' + 2xv^2 = 0 \,,$$

$$\frac{dv}{dx} = -2xv^2 \,.$$

The last equation has a solution $v = 0$, or $y' = 0$, giving $y = c$, the first family of solutions. Assuming that $v \neq 0$, we separate the variables

$$\int \frac{dv}{v^2}\, dv = -\int 2x\, dx \,,$$

$$-\frac{1}{v} = -x^2 - c_1 \,,$$

$$y' = v = \frac{1}{x^2 + c_1} \,.$$

Let us now assume that $c_1 > 0$. Then

$$y(x) = \int \frac{1}{x^2 + c_1}\, dx = \frac{1}{\sqrt{c_1}} \arctan \frac{x}{\sqrt{c_1}} + c_2 \,,$$

the second family of solutions. If $c_1 = 0$ or $c_1 < 0$, we get two more different formulas for solutions! Indeed, in case $c_1 = 0$, or $y' = \frac{1}{x^2}$, an integration gives $y = -\frac{1}{x} + c_3$, the third family of solutions. In case $c_1 < 0$, we can write (replacing c_1 by $-c_1^2$, with a new c_1)

$$y' = \frac{1}{x^2 - c_1^2} = \frac{1}{(x - c_1)(x + c_1)} = \frac{1}{2c_1}\left[\frac{1}{x - c_1} - \frac{1}{x + c_1}\right] \,.$$

Integration gives the fourth family of solutions

$$y = \frac{1}{2c_1} \ln|x - c_1| - \frac{1}{2c_1} \ln|x + c_1| + c_4 \,.$$

Prescribing two initial conditions is appropriate for second order equations. Let us solve

$$y'' + 2xy'^2 = 0 \,, \;\; y(0) = 0 \,, \; y'(0) = 1 \,.$$

We just solved this equation, so that as above

$$y'(x) = \frac{1}{x^2 + c_1} \,.$$

From the second initial condition $y'(0) = \frac{1}{c_1} = 1$, giving $c_1 = 1$. It follows that $y'(x) = \frac{1}{x^2+1}$, and then $y(x) = \arctan x + c_2$. From the first initial condition $y(0) = c_2 = 0$. Answer: $y(x) = \arctan x$.

2.1.2 x is not present in the equation

Let us solve for $y(x)$

$$y'' + yy'^3 = 0 \,.$$

All three functions appearing in the equation are functions of x, but x itself is not present in the equation. On the curve $y = y(x)$, the slope y' is a function of x, but it is also a function of y. We set $y' = v(y)$, and $v(y)$ will be the new unknown function. By the chain rule

$$y''(x) = \frac{d}{dx}\, v(y) = v'(y)\frac{dy}{dx} = v'v \,,$$

and our equation takes the form

$$v'v + yv^3 = 0 \,.$$

This is a first order equation! To solve it, we begin by factoring

$$v\left(v' + yv^2\right) = 0.$$

If the first factor is zero, $y' = v = 0$, we obtain a family of solutions $y = c$. Setting the second factor to zero

$$\frac{dv}{dy} + yv^2 = 0 \,,$$

gives a separable equation. We solve it by separating the variables

$$-\int \frac{dv}{v^2} = \int y \, dy \,,$$

$$\frac{1}{v} = \frac{y^2}{2} + c_1 = \frac{y^2 + 2c_1}{2} \,,$$

$$\frac{dy}{dx} = v = \frac{2}{y^2 + 2c_1} \,.$$

To find $y(x)$ we need to solve another first order equation ($\frac{dy}{dx} = \frac{2}{y^2+2c_1}$). Separating the variables:

$$\int \left(y^2 + 2c_1\right)\, dy = \int 2\, dx\,,$$

$$y^3/3 + 2c_1 y = 2x + c_2\,,$$

giving a second family of solutions.

For the general case of equations with x not present

$$y'' = f(y, y')\,,$$

the change of variables $y' = v(y)$ produces a first order equation for $v = v(y)$

$$vv' = f(y, v)\,.$$

Let us solve for $y(x)$:

$$y'' = yy'\,, \quad y(0) = -2\,,\; y'(0) = 2\,.$$

In this equation x is missing, and we could solve it as in the preceding example. Instead, write this equation as

$$\frac{d}{dx}y' = \frac{d}{dx}\left(\frac{1}{2}y^2\right)\,,$$

and integrate, to get

$$y'(x) = \frac{1}{2}y^2(x) + c_1\,.$$

Evaluate the last equation at $x = 0$, and use the initial conditions:

$$2 = \frac{1}{2}(-2)^2 + c_1\,,$$

so that $c_1 = 0$. Then

$$\frac{dy}{dx} = \frac{1}{2}y^2\,.$$

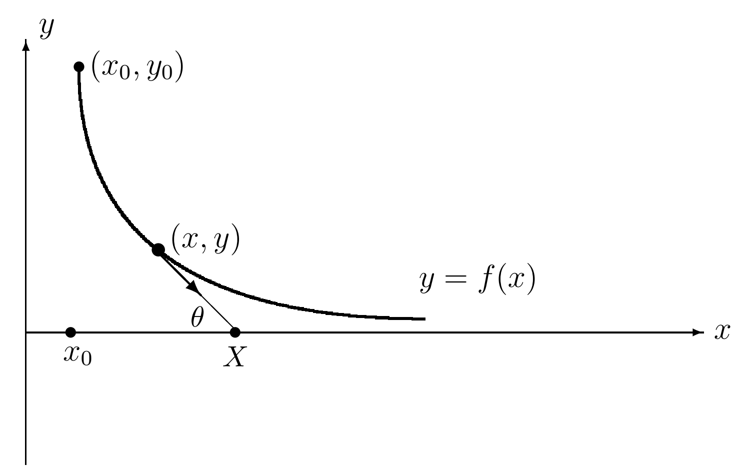
Solving this separable equation gives $y = -\dfrac{1}{\frac{1}{2}x + c_2}$. Using the first initial condition again, calculate $c_2 = \frac{1}{2}$. Answer: $y = -\dfrac{2}{x+1}$.

2.1.3* The Trajectory of Pursuit

Problem. A car is moving along a highway (the x axis) with a constant speed a. A drone is flying in the skies (at a point (x, y), which depends on time t), with a constant speed v. Find the trajectory for the drone, so that the tangent line always passes through the car. Assume that the drone starts at a point (x_0, y_0), and the car starts at x_0.

Solution. If X gives the position of the car, we can express the slope of the tangent line as follows ($\frac{dy}{dx} = -\tan\theta$, where θ is the angle the tangent line makes with the x-axis)

$$\frac{dy}{dx} = -\frac{y}{X - x}\,. \tag{1.1}$$



Since the velocity of the car is constant, $X = x_0 + at$. Then (1.1) gives

$$x_0 + at - x = -y\,\frac{dx}{dy}\,.$$

Differentiate this formula with respect to y, and simplify (here $x = x(y)$)

$$\frac{dt}{dy} = -\frac{1}{a}\,y\,\frac{d^2x}{dy^2}\,. \tag{1.2}$$

On the other hand, $v = \dfrac{ds}{dt}$, and $ds = \sqrt{dx^2 + dy^2}$, so that

$$dt = \frac{1}{v}\,ds = \frac{1}{v}\sqrt{dx^2 + dy^2} = -\frac{1}{v}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\;dy\,,$$

and then

$$\frac{dt}{dy} = -\frac{1}{v}\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,. \tag{1.3}$$

(Observe that $\dfrac{dt}{dy} < 0$, so that minus is needed in front of the square root.) Comparing (1.2) with (1.3), and writing $x'(y) = \dfrac{dx}{dy}$, $x''(y) = \dfrac{d^2x}{dy^2}$, we arrive at the equation of motion for the drone

$$y\,x''(y) = \frac{a}{v}\sqrt{x'^2(y) + 1}\,.$$

The unknown function $x(y)$ is not present in this equation. Therefore set $x'(y) = p(y)$, with $x''(y) = p'(y)$, obtaining a first order equation for $p(y)$, which is solved by separating the variables

$$y\frac{dp}{dy} = \frac{a}{v}\sqrt{p^2+1}\,,$$

$$\int \frac{dp}{\sqrt{p^2+1}} = \frac{a}{v}\int \frac{dy}{y}\,.$$

The integral on the left is computed by a substitution $p = \tan\theta$, giving

$$\ln\left(p + \sqrt{p^2+1}\right) = \frac{a}{v}\left(\ln y + \ln c\right)\,,$$

$$p + \sqrt{p^2+1} = cy^{\frac{a}{v}} \qquad \text{(with a new } c\text{)}\,.$$

Write the last formula as

$$\sqrt{p^2+1} = cy^{\frac{a}{v}} - p\,, \tag{1.4}$$

and square both sides, getting

$$1 = c^2 y^{\frac{2a}{v}} - 2cy^{\frac{a}{v}}p\,.$$

Solve this for $p = x'(y)$:

$$x'(y) = \frac{1}{2}cy^{\frac{a}{v}} - \frac{1}{2c}y^{-\frac{a}{v}}\,.$$

The constant c we determine from (1.4). At $t = 0$, $p = x'(y_0) = 0$, and so $c = y_0^{-\frac{a}{v}}$. (At $t = 0$, the drone is pointed vertically down, because the car is directly under it.) Then

$$x'(y) = \frac{1}{2}\left(\frac{y}{y_0}\right)^{\frac{a}{v}} - \frac{1}{2}\left(\frac{y}{y_0}\right)^{-\frac{a}{v}}\,.$$

Integrating, and using that $x(y_0) = x_0$, we finally obtain (assuming $v \neq a$)

$$x(y) = \frac{y_0}{2(1+a/v)}\left[\left(\frac{y}{y_0}\right)^{1+\frac{a}{v}} - 1\right] - \frac{y_0}{2(1-a/v)}\left[\left(\frac{y}{y_0}\right)^{1-\frac{a}{v}} - 1\right] + x_0 .$$

2.2 Linear Homogeneous Equations with Constant Coefficients

We wish to find solution $y = y(t)$ of the equation

$$ay'' + by' + cy = 0 , \tag{2.1}$$

where a, b and c are given numbers. This is arguably the most important class of differential equations, because it arises when applying Newton's second law of motion (or when modeling electric oscillations). If $y(t)$ denotes displacement of an object at time t, then this equation relates the displacement with velocity $y'(t)$ and acceleration $y''(t)$. The equation (2.1) is *linear*, because it involves a linear combination of the unknown function $y(t)$, and its derivatives $y'(t)$ and $y''(t)$. The term *homogeneous* refers to the right hand side of this equation being zero.

Observe that if $y(t)$ is a solution, then so is $2y(t)$. Indeed, substitute $2y(t)$ into the equation:

$$a(2y)'' + b(2y)' + c(2y) = 2\left(ay'' + by' + cy\right) = 0 .$$

The same argument shows that $c_1y(t)$ is a solution for any constant c_1. If $y_1(t)$ and $y_2(t)$ are two solutions, a similar argument will show that $y_1(t) + y_2(t)$ and $y_1(t) - y_2(t)$ are also solutions. More generally, a *linear combination* of two solutions, $c_1y_1(t) + c_2y_2(t)$, is also a solution, for any constants c_1 and c_2. Indeed,

$$\begin{aligned} &a\left(c_1y_1(t) + c_2y_2(t)\right)'' + b\left(c_1y_1(t) + c_2y_2(t)\right)' + c\left(c_1y_1(t) + c_2y_2(t)\right) \\ &= c_1\left(ay_1'' + by_1' + cy_1\right) + c_2\left(ay_2'' + by_2' + cy_2\right) = 0 . \end{aligned}$$

This fact is called the *linear superposition property of solutions.*

We now try to find a solution of the equation (2.1) in the form $y = e^{rt}$, where r is a constant to be determined. We have $y' = re^{rt}$ and $y'' = r^2e^{rt}$, so that the substitution into the equation (2.1) gives

$$a(r^2e^{rt}) + b(re^{rt}) + ce^{rt} = e^{rt}\left(ar^2 + br + c\right) = 0 .$$

Dividing by a positive quantity e^{rt}, obtain

$$ar^2 + br + c = 0\,.$$

This is a quadratic equation for r, called the *characteristic equation*. If r is a root (solution) of this equation, then e^{rt} solves our differential equation (2.1). When solving a quadratic equation, it is possible to encounter two real roots, one (repeated) real root, or two complex conjugate roots. We shall look at these cases in turn.

2.2.1 The Characteristic Equation Has Two Distinct Real Roots

Assume that the roots r_1 and r_2 are real, and $r_2 \neq r_1$. Then $e^{r_1 t}$ and $e^{r_2 t}$ are two solutions, and their linear combination gives us the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}\,.$$

As there are two constants to play with, one can prescribe two additional conditions for the solution to satisfy.

Example 1 Solve

$$\begin{gathered} y'' + 4y' + 3y = 0 \\ y(0) = 2 \\ y'(0) = -1\,. \end{gathered}$$

Assuming that $y(t)$ gives displacement of a particle, we prescribe that at time zero the displacement is 2, and the velocity is -1. These two conditions are usually referred to as the *initial conditions*, and together with the differential equation, they form an *initial value problem*. The characteristic equation is

$$r^2 + 4r + 3 = 0\,.$$

Solving it (say by factoring as $(r+1)(r+3) = 0$), gives the roots $r_1 = -1$, and $r_2 = -3$. The general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}\,.$$

Calculate $y(0) = c_1 + c_2$. Compute $y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$, and therefore $y'(0) = -c_1 - 3c_2$. The initial conditions tell us that

$$\begin{gathered} c_1 + c_2 = 2 \\ -c_1 - 3c_2 = -1\,. \end{gathered}$$

We have two equations to find two unknowns c_1 and c_2. Obtain: $c_1 = 5/2$, and $c_2 = -1/2$ (say by adding the equations).
Answer: $y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

Example 2 Solve

$$y'' - 4y = 0\,.$$

The characteristic equation is

$$r^2 - 4 = 0\,.$$

Its roots are $r_1 = -2$, and $r_2 = 2$. The general solution is then

$$y(t) = c_1 e^{-2t} + c_2 e^{2t}\,.$$

More generally, for the equation

$$y'' - a^2 y = 0 \quad (a \text{ is a given constant}),$$

the general solution is

$$y(t) = c_1 e^{-at} + c_2 e^{at}\,.$$

This should become automatic, because such equations appear often.

Example 3 Find the constant a, so that the solution of the initial value problem

$$9y'' - y = 0, \quad y(0) = 2, \;\; y'(0) = a$$

is bounded as $t \to \infty$, and find that solution.

We begin by writing down (automatically!) the general solution

$$y(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{\frac{1}{3}t}\,.$$

Compute $y'(t) = -\frac{1}{3}c_1 e^{-\frac{1}{3}t} + \frac{1}{3}c_2 e^{\frac{1}{3}t}$, and then the initial conditions give

$$\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= -\tfrac{1}{3}c_1 + \tfrac{1}{3}c_2 = a\,. \end{aligned}$$

Solving this system of two equation for c_1 and c_2 (by multiplying the second equation through by 3, and adding the result to the first equation), gives $c_2 = 1 + \frac{3}{2}a$, and $c_1 = 1 - \frac{3}{2}a$. The solution is

$$y(t) = \left(1 - \frac{3}{2}a\right) e^{-\frac{1}{3}t} + \left(1 + \frac{3}{2}a\right) e^{\frac{1}{3}t}\,.$$

In order for this solution to stay bounded as $t \to \infty$, the coefficient in front of $e^{\frac{1}{3}t}$ must be zero. So that $1 + \frac{3}{2}a = 0$, and $a = -\frac{2}{3}$. The solution then becomes $y(t) = 2e^{-\frac{1}{3}t}$.

Finally, recall that if r_1 and r_2 are roots of the characteristic equation, then we can factor the characteristic polynomial as

$$ar^2 + br + c = a(r - r_1)(r - r_2)\,. \tag{2.2}$$

2.2.2 The Characteristic Equation Has Only One (Repeated) Real Root

This is the case when $r_2 = r_1$. We still have one solution $y_1(t) = e^{r_1 t}$. Of course, any constant multiple of this function is also a solution, but to form a general solution we need another truly different solution, as we saw in the preceding case. It turns out that $y_2(t) = te^{r_1 t}$ is that second solution, and the general solution is then

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}\,.$$

To justify that $y_2(t) = te^{r_1 t}$ is a solution, we begin by observing that in this case the formula (2.2) becomes

$$ar^2 + br + c = a(r - r_1)^2\,.$$

Square out the quadratic on the right as $ar^2 - 2ar_1 r + ar_1^2$. Because it is equal to the quadratic on the left, the coefficients of both polynomials in r^2, r, and the constant terms are the same. We equate the coefficients in r:

$$b = -2ar_1\,. \tag{2.3}$$

To substitute $y_2(t)$ into the equation, we compute its derivatives $y_2'(t) = e^{r_1 t} + r_1 t e^{r_1 t} = e^{r_1 t}(1 + r_1 t)$, and similarly $y_2''(t) = e^{r_1 t}\left(2r_1 + r_1^2 t\right)$. Then

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= ae^{r_1 t}\left(2r_1 + r_1^2 t\right) + be^{r_1 t}(1 + r_1 t) + cte^{r_1 t} \\ &= e^{r_1 t}(2ar_1 + b) + te^{r_1 t}\left(ar_1^2 + br_1 + c\right) = 0\,. \end{aligned}$$

In the last line, the first bracket is zero because of (2.3), and the second bracket is zero because r_1 solves the characteristic equation.

Example 1 Solve $\;9y'' + 6y' + y = 0$.

The characteristic equation

$$9r^2 + 6r + 1 = 0$$

has a double root $r = -\frac{1}{3}$. The general solution is then

$$y(t) = c_1 e^{-\frac{1}{3}t} + c_2 t e^{-\frac{1}{3}t} .$$

Example 2 Solve

$$\begin{gathered} y'' - 4y' + 4y = 0 \\ y(0) = 1, \;\; y'(0) = -2 . \end{gathered}$$

The characteristic equation

$$r^2 - 4r + 4 = 0$$

has a double root $r = 2$. The general solution is then

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} .$$

Here $y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$, and from the initial conditions

$$\begin{gathered} y(0) = c_1 = 1 \\ y'(0) = 2c_1 + c_2 = -2 . \end{gathered}$$

From the first equation $c_1 = 1$, and then $c_2 = -4$. Answer: $y(t) = e^{2t} - 4te^{2t}$.

2.3 The Characteristic Equation Has Two Complex Conjugate Roots

In this section we complete the theory of linear equations with constant coefficients. The following important fact will be needed.

2.3.1 Euler's Formula

Recall Maclauren's formula

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots .$$

Let $z = i\,\theta$, where $i = \sqrt{-1}$ is the imaginary unit, and θ is a real number. Calculating the powers, and separating the real and imaginary parts, gives

$$\begin{aligned} e^{i\,\theta} &= 1 + i\,\theta + \tfrac{1}{2!}(i\,\theta)^2 + \tfrac{1}{3!}(i\,\theta)^3 + \tfrac{1}{4!}(i\,\theta)^4 + \tfrac{1}{5!}(i\,\theta)^5 + \cdots \\ &= 1 + i\,\theta - \tfrac{1}{2!}\theta^2 - \tfrac{1}{3!}i\,\theta^3 + \tfrac{1}{4!}\theta^4 + \tfrac{1}{5!}i\,\theta^5 + \cdots \\ &= \left(1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 + \cdots\right) + i\,\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5 + \cdots\right) \\ &= \cos\theta + i\,\sin\theta\,. \end{aligned}$$

We derived *Euler's formula*:

$$e^{i\,\theta} = \cos\theta + i\,\sin\theta\,. \tag{3.1}$$

Replacing θ by $-\theta$, gives

$$e^{-i\,\theta} = \cos(-\theta) + i\,\sin(-\theta) = \cos\theta - i\sin\theta\,. \tag{3.2}$$

Adding the last two formulas, we express

$$\cos\theta = \frac{e^{i\,\theta} + e^{-i\,\theta}}{2}\,. \tag{3.3}$$

Subtracting from (3.1) the formula (3.2), and dividing by $2i$

$$\sin\theta = \frac{e^{i\,\theta} - e^{-i\,\theta}}{2i}\,. \tag{3.4}$$

The last two formulas are also known as *Euler's formulas.*

2.3.2 The General Solution

Recall that to solve the equation

$$ay'' + by' + cy = 0 \tag{3.5}$$

one needs to solve the characteristic equation

$$ar^2 + br + c = 0\,.$$

Assume now that its roots are complex. Complex roots come in *conjugate pairs*: if $p+iq$ is one root, then $p-iq$ is the other, and we may assume that $q > 0$. These roots are, of course, different, so that we have two solutions $z_1 = e^{(p+iq)t}$, and $z_2 = e^{(p-iq)t}$. The problem with these solutions is that they are complex-valued. Adding $z_1 + z_2$, gives another solution of (3.5).

Dividing this new solution by 2, we get yet another solution. So that the function $y_1(t) = \dfrac{z_1 + z_2}{2}$ is a solution of our equation (3.5), and similarly the function $y_2(t) = \dfrac{z_1 - z_2}{2i}$ is another solution. Using the formula (3.3), compute

$$y_1(t) = \frac{e^{(p+iq)t} + e^{(p-iq)t}}{2} = e^{pt}\frac{e^{iqt} + e^{-iqt}}{2} = e^{pt}\cos qt\,.$$

This is a real valued solution of our equation! Similarly,

$$y_2(t) = \frac{e^{(p+iq)t} - e^{(p-iq)t}}{2i} = e^{pt}\frac{e^{iqt} - e^{-iqt}}{2i} = e^{pt}\sin qt$$

is our second solution. The general solution is then

$$y(t) = c_1 e^{pt}\cos qt + c_2 e^{pt}\sin qt\,.$$

Example 1 Solve $y'' + 4y' + 5y = 0$.

The characteristic equation

$$r^2 + 4r + 5 = 0$$

can be solved quickly by completing the square:

$$(r+2)^2 + 1 = 0\,, \quad (r+2)^2 = -1\,,$$
$$r + 2 = \pm i\,, \quad r = -2 \pm i\,.$$

Here $p = -2$, $q = 1$, and the general solution is

$$y(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t\,.$$

Example 2 Solve $y'' + y = 0$.

The characteristic equation

$$r^2 + 1 = 0$$

has roots $\pm i$. Here $p = 0$ and $q = 1$, and the general solution is $y(t) = c_1\cos t + c_2\sin t$.

More generally, for the equation

$$y'' + a^2 y = 0 \quad (a \text{ is a given constant}),$$

the general solution is

$$y(t) = c_1 \cos at + c_2 \sin at\,.$$

This should become automatic, because such equations appear often.

Example 3 Solve

$$y'' + 4y = 0, \quad y(\pi/3) = 2, \;\; y'(\pi/3) = -4\,.$$

The general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t\,.$$

Compute $y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$. From the initial conditions

$$y(\pi/3) = c_1 \cos \frac{2\pi}{3} + c_2 \sin \frac{2\pi}{3} = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 2$$

$$y'(\pi/3) = -2c_1 \sin \frac{2\pi}{3} + 2c_2 \cos \frac{2\pi}{3} = -\sqrt{3}c_1 - c_2 = -4\,.$$

This gives $c_1 = \sqrt{3} - 1$, $c_2 = \sqrt{3} + 1$. Answer:

$$y(t) = (\sqrt{3} - 1) \cos 2t + (\sqrt{3} + 1) \sin 2t\,.$$

2.3.3 Problems

I. Solve the second order equations, with y missing.

1. $2y'\,y'' = 1$. Answer. $y = \pm\frac{2}{3}(x + c_1)^{3/2} + c_2$.

2. $xy'' + y' = x$. Answer. $y = \frac{x^2}{4} + c_1 \ln x + c_2$.

3. $y'' + y' = x^2$. Answer. $y = \frac{x^3}{3} - x^2 + 2x + c_1 e^{-x} + c_2$.

4. $xy'' + 2y' = (y')^2$, $y(1) = 0$, $y'(1) = 1$. Answer. $y = 2\tan^{-1} x - \frac{\pi}{2}$.

5. $y'' + 2xy'^2 = 0\,$, $y(0) = 0\,$, $y'(0) = -4$. Answer. $y = \ln\left|\frac{2x - 1}{2x + 1}\right|$.

II. Solve the second order equations, with x missing.

1. $yy'' - 3\,(y')^3 = 0$. Answer. $3\,(y \ln y - y) + c_1 y = -x + c_2$, and $y = c$.

2. $yy''+(y')^2=0$. Answer. $y^2=c_1+c_2x$ (this includes the $y=c$ family).

3. $y''=2yy'$, $y(0)=0$, $y'(0)=1$. Answer. $y=\tan x$.

4. $y''=3y^2y'+y'$, $y(0)=1$, $y'(0)=2$. Answer. $y=\sqrt{\dfrac{e^{2x}}{2-e^{2x}}}$.

5*. $y''y=2xy'^2$, $y(0)=1$, $y'(0)=-4$.

Hint: Write:

$$y''y-y'^2=(2x-1)y'^2; \qquad \frac{y''y-y'^2}{y'^2}=2x-1; \qquad -\left(\frac{y}{y'}\right)'=2x-1\,.$$

Integrating, and using the initial conditions

$$-\frac{y}{y'}=x^2-x+\frac{1}{4}=\frac{(2x-1)^2}{4}\,.$$

Answer. $y=e^{\frac{4x}{2x-1}}$.

III. Solve the linear second order equations, with constant coefficients.

1. $y''+4y'+3y=0$. Answer. $y=c_1e^{-t}+c_2e^{-3t}$.

2. $y''-3y'=0$. Answer. $y=c_1+c_2e^{3t}$.

3. $2y''+y'-y=0$. Answer. $y=c_1e^{-t}+c_2e^{\frac{1}{2}t}$.

4. $y''-3y=0$. Answer. $y=c_1e^{-\sqrt{3}t}+c_2e^{\sqrt{3}t}$.

5. $3y''-5y'-2y=0$. Answer. $y=c_1e^{-\frac{1}{3}t}+c_2e^{2t}$.

6. $y''-9y=0$, $y(0)=3$, $y'(0)=3$. Answer. $y=e^{-3t}+2e^{3t}$.

7. $y''+5y'=0$, $y(0)=-1$, $y'(0)=-10$. Answer. $y=-3+2e^{-5t}$.

8. $y''+y'-6y=0$, $y(0)=-2$, $y'(0)=3$. Answer. $y=-\dfrac{7}{5}e^{-3t}-\dfrac{3e^{2t}}{5}$.

9. $4y''-y=0$. Answer. $y=c_1e^{-\frac{1}{2}t}+c_2e^{\frac{1}{2}t}$.

10. $3y''-2y'-y=0$, $y(0)=1$, $y'(0)=-3$. Answer. $y=3e^{-t/3}-2e^{t}$.

11. $3y''-2y'-y=0$, $y(0)=1$, $y'(0)=a$.
Find the value of a for which the solution is bounded, as $t\to\infty$.

Answer. $y = \frac{1}{4}(3a+1)e^{t} - \frac{3}{4}(a-1)e^{-t/3}$, $a = -\frac{1}{3}$.

IV. Solve the linear second order equations, with constant coefficients.

1. $y'' + 6y' + 9y = 0$. Answer. $y = c_1 e^{-3t} + c_2 t e^{-3t}$.

2. $4y'' - 4y' + y = 0$. Answer. $y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$.

3. $y'' - 2y' + y = 0$, $y(0) = 0$, $y'(0) = -2$. Answer. $y = -2te^{t}$.

4. $9y'' - 6y' + y = 0$, $y(0) = 1$, $y'(0) = -2$. Answer. $y = \frac{1}{3}e^{t/3}(3-7t)$.

V.
1. Using Euler's formula, compute: (i) $e^{i\pi}$ (ii) $e^{-i\pi/2}$ (iii) $e^{i\frac{3\pi}{4}}$
(iv) $e^{2\pi i}$ (v) $\sqrt{2}e^{i\frac{9\pi}{4}}$ (vi) $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^5$.

2. Show that $\sin 2\theta = 2\sin\theta\cos\theta$, and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Hint: Begin with $e^{i2\theta} = (\cos\theta + i\sin\theta)^2$. Apply Euler's formula on the left, and square out on the right. Then equate the real and imaginary parts.

3*. Show that

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta, \quad \text{and} \quad \cos 3\theta = -3\sin^2\theta\cos\theta + \cos^3\theta\,.$$

Hint: Begin with $e^{i3\theta} = (\cos\theta + i\sin\theta)^3$. Apply Euler's formula on the left, and "cube out" on the right. Then equate the real and imaginary parts.

VI. Solve the linear second order equations, with constant coefficients.

1. $y'' + 4y' + 8y = 0$. Answer. $y = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t$.

2. $y'' + 16y = 0$. Answer. $y = c_1\cos 4t + c_2\sin 4t$.

3. $y'' - 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = -2$. Answer. $y = e^{2t}\cos t - 4e^{2t}\sin t$.

4. $y'' + 4y = 0$, $y(0) = -2$, $y'(0) = 0$. Answer. $y = -2\cos 2t$.

5. $9y'' + y = 0$, $y(0) = 0$, $y'(0) = 5$. Answer. $y = 15\sin\frac{1}{3}t$.

6. $y'' - y' + y = 0$. Answer. $y = e^{\frac{t}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right)$.

7. $4y'' + 8y' + 5y = 0$, $y(\pi) = 0$, $y'(\pi) = 4$. Answer. $y = -8e^{\pi - t}\cos\frac{1}{2}t$.

8. $y'' + y = 0$, $y(\pi/4) = 0$, $y'(\pi/4) = -1$. Answer. $y = -\sin(t - \pi/4)$.

VII.
1. Consider the equation ($y = y(t)$)

$$y'' + by' + cy = 0\,,$$

with positive constants b and c. Show that all of its solutions tend to zero, as $t \to \infty$.

2. Consider the equation

$$y'' + by' - cy = 0\,,$$

with positive constants b and c. Assume that some solution is bounded, as $t \to \infty$. Show that this solution tends to zero, as $t \to \infty$.

3. Explain why $y_1 = te^{-t}$ and $y_2 = e^{3t}$ cannot be both solutions of

$$ay'' + by' + cy = 0\,,$$

no matter what the constants a, b and c are.

4. Solve the non-linear equation

$$ty''y + y'y - ty'^2 = 0\,.$$

Hint: Consider the derivative of $\dfrac{ty'}{y}$. Answer. $y = c_2 t^{c_1}$.

2.4 Linear Second Order Equations with Variable Coefficients

In this section we present some theory of second order linear equations with *variable coefficients*. Several applications will appear in the following section. Also, this theory explains why the general solutions from the preceding sections give *all solutions* of the corresponding equations.

Linear Systems

Recall that a system of two equations (here the numbers a, b, c, d, g and h are given, while x and y are the unknowns)

$$\begin{aligned} a\,x + b\,y &= g \\ c\,x + d\,y &= h \end{aligned}$$

has a unique solution, if and only if the determinant of this system is non-zero, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$. This fact is justified by explicitly solving the system:

$$x = \frac{dg - bh}{ad - bc}, \quad y = \frac{ah - cg}{ad - bc}.$$

It is also easy to justify that a determinant is zero, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, if and only if its columns are proportional, so that $a = \gamma b$ and $c = \gamma d$, for some constant γ.

General Theory

We consider an initial value problem for linear second order equations

$$\begin{aligned} y'' + p(t)y' + g(t)y &= f(t) \\ y(t_0) &= \alpha \\ y'(t_0) &= \beta\,. \end{aligned} \tag{4.1}$$

The coefficient functions $p(t)$ and $g(t)$, and the function $f(t)$ are assumed to be given. The constants t_0, α and β are also given, so that at some initial "time" $t = t_0$, the values of the solution and its derivative are prescribed. It is natural to ask the following questions. Is there a solution to this problem? If there is, is the solution unique, and how far can it be continued?

Theorem 2.4.1 *Assume that the functions $p(t)$, $g(t)$ and $f(t)$ are continuous on some interval (a, b) that includes t_0. Then the problem (4.1) has a solution, and only one solution. This solution can be continued to the left and to the right of the initial point t_0, so long as t remains in (a, b).*

If the functions $p(t)$, $g(t)$ and $f(t)$ are continuous for all t, then the solution can be continued for all t, $-\infty < t < \infty$. This is a stronger conclusion than what we had for first order equations (where blow up in finite time was possible). Why? Because the equation here is linear. Linearity pays!

Corollary 2.4.1 *Let $z(t)$ be a solution of (4.1) with the same initial data as $y(t)$: $z(t_0) = \alpha$ and $z'(t_0) = \beta$. Then $z(t) = y(t)$ for all t.*

Let us now study the homogeneous equation (for $y = y(t)$)

$$y'' + p(t)y' + g(t)y = 0\,, \tag{4.2}$$

with given coefficient functions $p(t)$ and $g(t)$. Although this equation looks relatively simple, its analytical solution is totally out of reach, in general. (One has to either solve it numerically, or use infinite series.) In this section we study some theoretical aspects. In particular, we shall prove that a linear combination of two solutions, which are not constant multiples of one another, gives the general solution (a fact that we intuitively used for equations with constant coefficients). The equation (4.2) always has a solution $y(t) = 0$ for all t, called the *trivial solution*. We shall study primarily *non-trivial solutions*.

We shall need a concept of the *Wronskian determinant* of two functions $y_1(t)$ and $y_2(t)$, or the *Wronskian*, for short:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)\,.$$

(Named in honor of Polish mathematician J.M. Wronski, 1776-1853.) Sometimes the Wronskian is written as $W(y_1, y_2)(t)$ to stress its dependence on $y_1(t)$ and $y_2(t)$. For example,

$$W(\cos 2t, \sin 2t)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2\,.$$

Given the Wronskian and one of the functions, one can determine the other one.

Example If $f(t) = t$, and $W(f, g)(t) = t^2e^t$, find $g(t)$.

Solution: Here $f'(t) = 1$, and so

$$W(f, g)(t) = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) = t^2e^t\,.$$

This is a linear first order equation for $g(t)$. We solve it as usual, obtaining

$$g(t) = te^t + ct\,.$$

If $g(t) = cf(t)$, with some constant c, we compute that $W(f, g)(t) = 0$, for all t. The converse statement is not true. For example, the functions $f(t) = t^2$ and

$$g(t) = \begin{cases} t^2 & \text{if } t \ge 0 \\ -t^2 & \text{if } t < 0 \end{cases}$$

are not constant multiples of one another, but $W(f,g)(t) = 0$. This is seen by computing the Wronskian separately in case $t \geq 0$, and for $t < 0$. (Observe that $g(t)$ is a differentiable function, with $g'(0) = 0$.)

Theorem 2.4.2 *Let $y_1(t)$ and $y_2(t)$ be two solutions of (4.2), and $W(t)$ is their Wronskian. Then*

$$W(t) = ce^{-\int p(t)\,dt}. \tag{4.3}$$

where c is some constant.

This is a remarkable fact! Even though we do not know $y_1(t)$ and $y_2(t)$, we can compute their Wronskian.

Proof: Differentiate the Wronskian $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$:

$$W' = y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2\,.$$

Because y_1 is a solution of (4.2), we have $y_1'' + p(t)y_1' + g(t)y_1 = 0$, or $y_1'' = -p(t)y_1' - g(t)y_1$, and similarly $y_2'' = -p(t)y_2' - g(t)y_2$. With these formulas, we continue

$$\begin{aligned} W' &= y_1\left(-p(t)y_2' - g(t)y_2\right) - \left(-p(t)y_1' - g(t)y_1\right)y_2 \\ &= -p(t)\left(y_1y_2' - y_1'y_2\right) = -p(t)W\,. \end{aligned}$$

We obtained a linear first order equation for $W(t)$, $W' = -p(t)W$. Solving it, gives (4.3). ♢

Corollary 2.4.2 *We see from (4.3) that either $W(t) = 0$ for all t, when $c = 0$, or else $W(t)$ is never zero, in case $c \neq 0$.*

Theorem 2.4.3 *Let $y_1(t)$ and $y_2(t)$ be two non-trivial solutions of (4.2), and $W(t)$ is their Wronskian. Then $W(t) = 0$ for all t, if and only if $y_1(t)$ and $y_2(t)$ are constant multiples of each other.*

We just saw that if two functions are constant multiples of each other, then their Wronskian is zero, while the converse statement is not true, in general. But if these functions happen to be solutions of (4.2), then the converse statement is true.

Proof: Assume that the Wronskian of two solutions $y_1(t)$ and $y_2(t)$ is zero. In particular it is zero at any point t_0, so that

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0\,.$$

When a 2×2 determinant is zero, its columns are proportional. Let us assume the second column is equal to γ times the first one, where γ is some number, so that $y_2(t_0) = \gamma y_1(t_0)$ and $y_2'(t_0) = \gamma y_1'(t_0)$. We may assume that $\gamma \neq 0$, because otherwise we would have $y_2(t_0) = y_2'(t_0) = 0$, and then $y_2(t)$ would be the trivial solution, contrary to our assumptions. Consider the function $z(t) = y_2(t)/\gamma$. This function is a solution of the homogeneous equation (4.2), and it has initial values $z(t_0) = y_1(t_0)$ and $z'(t_0) = y_1'(t_0)$, the same as $y_1(t)$. By Corollary 2.4.1, it follows that $z(t) = y_1(t)$, so that $y_2(t) = \gamma y_1(t)$, for all t. ♢

Definition We say that two solutions $y_1(t)$ and $y_2(t)$ of (4.2) form a *fundamental set*, if for any other solution $z(t)$, we can find two constants c_1^0 and c_2^0, so that $z(t) = c_1^0 y_1(t) + c_2^0 y_2(t)$. In other words, the linear combination $c_1 y_1(t) + c_2 y_2(t)$ gives us *all solutions of* (4.2).

Theorem 2.4.4 *Let $y_1(t)$ and $y_2(t)$ be two solutions of (4.2), that are not constant multiples of one another. Then they form a fundamental set.*

Proof: Let $y(t)$ be any solution of the equation (4.2). Let us try to find the constants c_1 and c_2, so that $z(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfies the same initial conditions as $y(t)$, so that

$$
\begin{aligned}
z(t_0) &= c_1 y_1(t_0) + c_2 y_2(t_0) = y(t_0) \\
z'(t_0) &= c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'(t_0)\,.
\end{aligned}
\tag{4.4}
$$

This is a system of two linear equations to find c_1 and c_2. The determinant of this system is just the Wronskian of $y_1(t)$ and $y_2(t)$, evaluated at t_0. This determinant is not zero, because $y_1(t)$ and $y_2(t)$ are not constant multiples of one another. (This determinant is $W(t_0)$. If $W(t_0) = 0$, then $W(t) = 0$ for all t, by the Corollary 2.4.2, and then by the Theorem 2.4.3, $y_1(t)$ and $y_2(t)$ would have to be constant multiples of one another, contrary to our assumption.) It follows that the 2×2 system (4.4) has a unique solution $c_1 = c_1^0$, $c_2 = c_2^0$. The function $z(t) = c_1^0 y_1(t) + c_2^0 y_2(t)$ is then a solution of the same equation (4.2), satisfying the same initial conditions, as does $y(t)$. By the Corollary 2.4.1, $y(t) = z(t)$ for all t. So that any solution $y(t)$ is a particular case of the general solution $c_1 y_1(t) + c_2 y_2(t)$. ♢

Finally, we mention that two functions are called *linearly independent*, if they are not constant multiples of one another. So that two solutions $y_1(t)$ and $y_2(t)$ form a fundamental set, if and only if they are linearly independent.

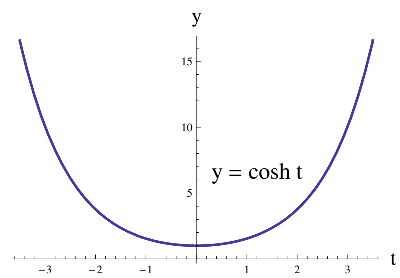


Figure 2.1: The cosine hyperbolic function

2.5 Some Applications of the Theory

We shall give some practical applications of the theory from the last section. But first, we recall the functions $\sinh t$ and $\cosh t$.

2.5.1 The Hyperbolic Sine and Cosine Functions

One defines

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \text{and} \quad \sinh t = \frac{e^t - e^{-t}}{2}\,.$$

In particular, $\cosh 0 = 1$, $\sinh 0 = 0$. Observe that $\cosh t$ is an even function, while $\sinh t$ is odd. Compute:

$$\frac{d}{dt}\cosh t = \sinh t \quad \text{and} \quad \frac{d}{dt}\sinh t = \cosh t\,.$$

These formulas are similar to those for cosine and sine. By squaring out, one sees that

$$\cosh^2 t - \sinh^2 t = 1, \quad \text{for all } t.$$

(There are other similar formulas.) We see that the derivatives, and the algebraic properties of the new functions are similar to those for cosine and sine. However, the graphs of $\sinh t$ and $\cosh t$ look totally different: they are not periodic, and they are unbounded, see Figures 2.1 and 2.2.

2.5.2 Different Ways to Write the General Solution

For the equation

$$y'' - a^2 y = 0 \tag{5.1}$$

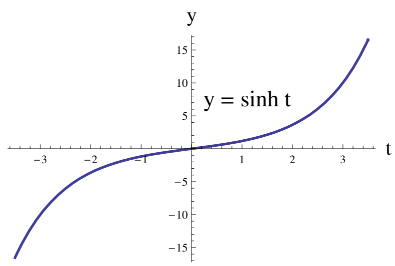


Figure 2.2: The sine hyperbolic function

you remember that the functions e^{-at} and e^{at} form a fundamental set, and $y(t) = c_1 e^{-at} + c_2 e^{at}$ is the general solution. But $y = \sinh at$ is also a solution, because $y' = a \cosh at$ and $y'' = a^2 \sinh at = a^2 y$. Similarly, $\cosh at$ is a solution. It is not a constant multiple of $\sinh at$, so that together they form another fundamental set, and we have another form of the general solution of (5.1)

$$y = c_1 \cosh at + c_2 \sinh at \,.$$

This is not a "new" general solution, as it can be reduced to the old one, by expressing $\cosh at$ and $\sinh at$ through the exponentials. However, the new form is useful.

Example 1 Solve: $y'' - 4y = 0$, $y(0) = 0$, $y'(0) = -5$.

Write the general solution as $y(t) = c_1 \cosh 2t + c_2 \sinh 2t$. Using that $\cosh 0 = 1$ and $\sinh 0 = 0$, gives $y(0) = c_1 = 0$. With $c_1 = 0$, we update the solution: $y(t) = c_2 \sinh 2t$. Compute $y'(t) = 2c_2 \cosh 2t$, and $y'(0) = 2c_2 = -5$, giving $c_2 = -\frac{5}{2}$. Answer: $y(t) = -\frac{5}{2} \sinh 2t$.

Yet another form of the general solution of (5.1) is

$$y(t) = c_1 e^{-a(t-t_0)} + c_2 e^{a(t-t_0)},$$

where t_0 is any number. (Both functions $e^{-a(t-t_0)}$ and $e^{a(t-t_0)}$ are solutions of (5.1).)

Example 2 Solve: $y'' - 9y = 0$, $y(2) = -1$, $y'(2) = 9$.

We select $t_0 = 2$, writing the general solution as

$$y(t) = c_1 e^{-3(t-2)} + c_2 e^{3(t-2)} \,.$$

Compute $y'(t) = -3c_1e^{-3(t-2)} + 3c_2e^{3(t-2)}$, and use the initial conditions

$$\begin{aligned} c_1 + c_2 &= -1 \\ -3c_1 + 3c_2 &= 9\,. \end{aligned}$$

Calculate $c_1 = -2$, and $c_2 = 1$. Answer: $y(t) = -2e^{-3(t-2)} + e^{3(t-2)}$.

One can write general solutions, *centered at* t_0, for other simple equations as well. For the equation

$$y'' + a^2y = 0$$

the functions $\cos a(t-t_0)$ and $\sin a(t-t_0)$ are both solutions, for any value of t_0, and they form a fundamental set (because they are not constant multiples of one another). We can then write the general solution as

$$y = c_1 \cos a(t - t_0) + c_2 \sin a(t - t_0)\,.$$

Example 3 Solve: $y'' + 4y = 0$, $y(\frac{\pi}{5}) = 2$, $y'(\frac{\pi}{5}) = -6$.

Write the general solution as

$$y = c_1 \cos 2(t - \frac{\pi}{5}) + c_2 \sin 2(t - \frac{\pi}{5})\,.$$

Using the initial conditions, we quickly compute $c_1 = 2$, and $c_2 = -3$.
Answer: $y = 2\cos 2(t - \dfrac{\pi}{5}) - 3\sin 2(t - \dfrac{\pi}{5})$.

2.5.3 Finding the Second Solution

Next we solve *Legendre's equation* (for $-1 < t < 1$)

$$(1 - t^2)y'' - 2ty' + 2y = 0\,.$$

It has a solution $y = t$. (Lucky!) We need to find another solution in the fundamental set. Divide this equation by $1-t^2$, to put it into the form (4.2) from the previous section:

$$y'' - \frac{2t}{1-t^2}\,y' + \frac{2}{1-t^2}\,y = 0\,.$$

Denote another solution in the fundamental set by $y(t)$. By the Theorem 2.4.2, we can calculate the Wronskian of the two solutions:

$$W(t, y) = \begin{vmatrix} t & y(t) \\ 1 & y'(t) \end{vmatrix} = ce^{\int \frac{2t}{1-t^2}\,dt}\,.$$

Set here $c = 1$, because we need just one solution, which is not a constant multiple of the solution $y = t$. Then

$$ty' - y = e^{\int \frac{2t}{1-t^2}\,dt} = e^{-\ln(1-t^2)} = \frac{1}{1-t^2}\,.$$

This is a linear equation, which is solved as usual:

$$y' - \frac{1}{t}y = \frac{1}{t(1-t^2)}\,, \quad \mu(t) = e^{-\int 1/t\,dt} = e^{-\ln t} = \frac{1}{t}\,,$$

$$\frac{d}{dt}\left[\frac{1}{t}y\right] = \frac{1}{t^2(1-t^2)}\,, \quad \frac{1}{t}y = \int \frac{1}{t^2(1-t^2)}\,dt\,.$$

The last integral was calculated above, by the guess-and-check method, so that

$$y = t\int \frac{1}{t^2\,(1-t^2)}\,dt = t\left[-\frac{1}{t} - \frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(1+t)\right] = -1+\frac{1}{2}t\ln\frac{1+t}{1-t}\,.$$

Again, we took the constant of integration $c = 0$, because we need just one solution, which is not a constant multiple of the solution $y = t$. Answer: $y(t) = c_1 t + c_2\left(-1 + \frac{1}{2}t\ln\frac{1+t}{1-t}\right)$.

2.5.4 Problems

I. 1. Find the Wronskians of the following functions.

(i) $f(t) = e^{3t}$, $g(t) = e^{-\frac{1}{2}t}$. Answer. $-\frac{7}{2}e^{\frac{5}{2}t}$.

(ii) $f(t) = e^{2t}$, $g(t) = te^{2t}$. Answer. e^{4t}.

(iii) $f(t) = e^t\cos 3t$, $g(t) = e^t\sin 3t$. Answer. $3e^{2t}$.

(iv) $f(t) = \cosh 4t$, $g(t) = \sinh 4t$. Answer. 4.

2. If $f(t) = t^2$, and the Wronskian $W(f,g)(t) = t^5e^t$, find $g(t)$.
Answer. $g(t) = t^3e^t - t^2e^t + ct^2$.

3. If $f(t) = e^{-t}$, and the Wronskian $W(f,g)(t) = t$, find $g(t)$ given that $g(0) = 0$. Answer. $g(t) = \frac{te^t}{2} + \frac{e^{-t}}{4} - \frac{e^t}{4}$.

4. Assume that $f(t) > 0$, $g(t) > 0$ and $W(f,g)(t) = 0$ for all t. Show that $g(t) = cf(t)$, for some constant c.

Hint: Express $\dfrac{g'}{g} = \dfrac{f'}{f}$, then integrate.

5. Let $y_1(t)$ and $y_2(t)$ be any two solutions of

$$y'' - t^2 y = 0 \,.$$

Show that $W(y_1(t), y_2(t))(t) = \textit{constant}$.

II. Express the solution, by using the hyperbolic sine and cosine functions.

1. $y'' - 4y = 0$, $y(0) = 0$, $y'(0) = -\frac{1}{3}$. Answer. $y = -\dfrac{1}{6} \sinh 2t$.

2. $y'' - 9y = 0$, $y(0) = 2$, $y'(0) = 0$. Answer. $y = 2\cosh 3t$.

3. $y'' - y = 0$, $y(0) = -3$, $y'(0) = 5$. Answer. $y = -3\cosh t + 5\sinh t$.

III. Solve the problem, by using the general solution centered at the initial point.

1. $y'' + y = 0$, $y(\pi/8) = 0$, $y'(\pi/8) = 3$. Answer. $y = 3\sin(t - \pi/8)$.

2. $y'' + 4y = 0$, $y(\pi/4) = 0$, $y'(\pi/4) = 4$.

Answer. $y = 2\sin 2(t - \pi/4) = 2\sin(2t - \pi/2) = -2\cos 2t$.

3. $y'' - 2y' - 3y = 0$, $y(1) = 1$, $y'(1) = 7$. Answer. $y = 2e^{3(t-1)} - e^{-(t-1)}$.

4. $y'' - 9y = 0$, $y(2) = -1$, $y'(2) = 15$.

Answer. $y = -\cosh 3(t - 2) + 5\sinh 3(t - 2)$.

IV. For the following equations one solution is given. Using Wronskians, find the second solution, and the general solution.

1. $y'' - 2y' + y = 0$, $y_1(t) = e^t$.

2. $t^2 y'' - 2ty' + 2y = 0$, $y_1(t) = t$.

Answer. $y = c_1 t + c_2 t^2$.

3. $(1 + t^2)y'' - 2ty' + 2y = 0$, $y_1(t) = t$.

Answer. $y = c_1 t + c_2(t^2 - 1)$.

4. $(t - 2)y'' - ty' + 2y = 0$, $y_1(t) = e^t$.

Answer. $y = c_1 e^t + c_2 \left(-t^2 + 2t - 2\right)$.

5. $ty'' + 2y' + ty = 0$, $y_1(t) = \frac{\sin t}{t}$. Answer. $y = c_1 \dfrac{\sin t}{t} + c_2 \dfrac{\cos t}{t}$.

6. $ty'' - (t+1)y' - (2t-2)y = 0$.
Hint: Search for $y_1(t)$ in the form $y = e^{at}$.
Answer. $y = c_1 e^{2t} + c_2(3t+1)e^{-t}$.

V. 1. Show that the functions $y_1 = t$, and $y_2 = \sin t$, cannot be both solutions of

$$y'' + p(t)y' + g(t)y = 0\,,$$

no matter what $p(t)$ and $g(t)$ are.

Hint: Consider $W(y_1, y_2)(0)$, then use the Corollary 2.4.2.

2. Assume that the functions $y_1 = 1$, and $y_2 = \cos t$, are both solutions of some equation of the form

$$y'' + p(t)y' + g(t)y = 0\,.$$

What is the equation?

Hint: Use the Theorem 2.4.2 to determine $p(t)$, then argue that $g(t)$ must be zero. Answer. The equation is

$$y'' - \cot t y' = 0\,.$$

3. Let us return to Legendre's equation

$$(1-t^2)y'' - 2ty' + 2y = 0\,, \quad -1 < t < 1\,,$$

for which one solution, $y_1 = t$, is known. Show that the substitution $y = tv$ produces an equation for $v(t)$, which can be reduced to a separable first order equation by letting $v' = z$. Solve this equation, to produce the second solution y_2 in the fundamental set. This technique is known as *reduction of order*.

2.6 Non-homogeneous Equations

This section deals with *non-homogeneous equations*

$$y'' + p(t)y' + g(t)y = f(t)\,. \tag{6.1}$$

Here the coefficient functions $p(t)$ and $g(t)$, and the *non-zero* function $f(t)$ are given. The *corresponding homogeneous equation* is

$$y'' + p(t)y' + g(t)y = 0\,. \tag{6.2}$$

Assume that the general solution of the corresponding homogeneous equation is known: $y(t) = c_1y_1(t) + c_2y_2(t)$. Our goal is to find the general solution of the non-homogeneous equation. Suppose that we can find somehow a particular solution $Y(t)$ of the non-homogeneous equation, so that

$$Y'' + p(t)Y' + g(t)Y = f(t)\,. \tag{6.3}$$

From the equation (6.1) subtract the equation (6.3):

$$(y - Y)'' + p(t)(y - Y)' + g(t)(y - Y) = 0\,.$$

We see that the function $v = y - Y$ is a solution the homogeneous equation (6.2), and then $v(t) = c_1y_1(t) + c_2y_2(t)$, for some constants c_1 and c_2. We express $y = Y + v$, giving

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t)\,.$$

In words: *the general solution of the non-homogeneous equation is equal to the sum of any particular solution of the non-homogeneous equation, and the general solution of the corresponding homogeneous equation.*

Finding a particular solution $Y(t)$ is the subject of this section (and the following two sections). We now study the *method of undetermined coefficients.*

Example 1 Find the general solution of

$$y'' + 9y = -4\cos 2t\,.$$

The general solution of the corresponding homogeneous equation

$$y'' + 9y = 0$$

is $y(t) = c_1 \cos 3t + c_2 \sin 3t$. We look for a particular solution in the form $Y(t) = A\cos 2t$. Substitute this in, then simplify:

$$-4A\cos 2t + 9A\cos 2t = -4\cos 2t,$$

$$5A\cos 2t = -4\cos 2t\,,$$

giving $A = -\frac{4}{5}$, and $Y(t) = -\frac{4}{5}\cos 2t$. Answer: $y(t) = -\dfrac{4}{5}\cos 2t + c_1\cos 3t + c_2\sin 3t$.

This was an easy example, because the y' term was missing. If y' term is present, we need to look for $Y(t)$ in the form $Y(t) = A\cos 2t + B\sin 2t$.

Prescription 1 If the right side of the equation (6.1) has the form $a \cos \alpha t + b \sin \alpha t$, with constants a, b and α, then look for a particular solution in the form $Y(t) = A \cos \alpha t + B \sin \alpha t$. More generally, if the right side of the equation has the form $(at + b) \cos \alpha t + (ct + d) \sin \alpha t$, involving linear polynomials, then look for a particular solution in the form $Y(t) = (At + B) \cos \alpha t + (Ct + D) \sin \alpha t$. Even more generally, if the polynomials are of higher power, we make the corresponding adjustments.

Example 2 Solve $y'' - y' - 2y = -4 \cos 2t + 8 \sin 2t$.

We look for a particular solution $Y(t)$ in the form $y(t) = A \cos 2t + B \sin 2t$. Substitute $y(t)$ into the equation, then combine the like terms:

$$
\begin{aligned}
-4A \cos 2t - 4B \sin 2t - (-2A \sin 2t + 2B \cos 2t) - 2\,(A \cos 2t + B \sin 2t) \\
= -4 \cos 2t + 8 \sin 2t\,,
\end{aligned}
$$

$$
(-6A - 2B) \cos 2t + (2A - 6B) \sin 2t = -4 \cos 2t + 8 \sin 2t\,.
$$

Equating the corresponding coefficients

$$
\begin{aligned}
-6A - 2B &= -4 \\
2A - 6B &= 8\,.
\end{aligned}
$$

Solving this system, gives $A = 1$ and $B = -1$, so that $Y(t) = \cos 2t - \sin 2t$. The general solution of the corresponding homogeneous equation

$$
y'' - y' - 2y = 0
$$

is $y = c_1 e^{-t} + c_2 e^{2t}$. Answer: $y(t) = \cos 2t - \sin 2t + c_1 e^{-t} + c_2 e^{2t}$.

Example 3 Solve $y'' + 2y' + y = t - 1$.

On the right we see a linear polynomial. We look for particular solution in the form $Y(t) = At + B$. Substituting this in, gives

$$
2A + At + B = t - 1\,.
$$

Equating the corresponding coefficients, we get $A = 1$, and $2A + B = -1$, so that $B = -3$. Then, $Y(t) = t - 3$. The general solution of the corresponding homogeneous equation

$$
y'' + 2y' + y = 0
$$

is $y = c_1 e^{-t} + c_2 t e^{-t}$. Answer: $y(t) = t - 3 + c_1 e^{-t} + c_2 t e^{-t}$.

Example 4 Solve $y'' + y' - 2y = t^2$.

On the right we have a quadratic polynomial (two of its coefficients happened to be zero). We look for a particular solution as a quadratic $Y(t) = At^2 + Bt + C$. Substituting this in, gives

$$2A + 2At + B - 2(At^2 + Bt + C) = t^2 \,.$$

Equating the coefficients in t^2, t, and the constant terms, gives

$$\begin{aligned} -2A &= 1 \\ 2A - 2B &= 0 \\ 2A + B - 2C &= 0 \,. \end{aligned}$$

From the first equation, $A = -\frac{1}{2}$, from the second one, $B = -\frac{1}{2}$, and from the third, $C = A + \frac{1}{2}B = -\frac{3}{4}$. So that $Y(t) = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}$. The general solution of the corresponding homogeneous equation is $y(t) = c_1 e^{-2t} + c_2 e^t$.

Answer: $y(t) = -\dfrac{1}{2}t^2 - \dfrac{1}{2}t - \dfrac{3}{4} + c_1 e^{-2t} + c_2 e^t$.

The last two examples lead to the following prescription.

Prescription 2 If the right hand side of the equation (6.1) is a polynomial of degree n: $a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$, look for a particular solution as a polynomial of degree n: $A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n$, with the coefficients to be determined.

And on to the final possibility.

Prescription 3 If the right side of the equation (6.1) is a polynomial of degree n, times an exponential : $\left(a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n\right) e^{\alpha t}$, look for a particular solution as a polynomial of degree n times the same exponential: $\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_n\right) e^{\alpha t}$, with the coefficients to be determined.

Example 5 Solve $y'' + y = te^{-2t}$.

We look for a particular solution in the form $Y(t) = (At+B)e^{-2t}$. Compute $Y'(t) = Ae^{-2t} - 2(At+B)e^{-2t}$, $Y''(t) = -4Ae^{-2t} + 4(At+B)e^{-2t}$. Substitute $Y(t)$ into the equation:

$$4Ate^{-2t} - 4Ae^{-2t} + 4Be^{-2t} + Ate^{-2t} + Be^{-2t} = te^{-2t} \,.$$

Divide by e^{-2t}, then equate the coefficients in t, and the constant terms

$$\begin{aligned} 5A &= 1 \\ -4A + 5B &= 0 \,, \end{aligned}$$

which gives $A = 1/5$ and $B = 4/25$, so that $Y(t) = (\frac{1}{5}t + \frac{4}{25})e^{-2t}$. Answer: $y(t) = (\dfrac{1}{5}t + \dfrac{4}{25})e^{-2t} + c_1 \cos t + c_2 \sin t$.

Example 6 Solve $y'' - 4y = t^2 + 3e^t, \quad y(0) = 0,\ y'(0) = 2$.

One can find $Y(t)$ as a sum of two pieces, $Y(t) = Y_1(t) + Y_2(t)$, where $Y_1(t)$ is any particular solution of

$$y'' - 4y = t^2 \,,$$

and $Y_2(t)$ is any particular solution of

$$y'' - 4y = 3e^t \,.$$

(Indeed, adding the identities $Y_1'' - 4Y_1 = t^2$ and $Y_2'' - 4Y_2 = 3e^t$, gives $Y'' - 4Y = t^2 + 3e^t$.) Using our prescriptions, $Y_1(t) = -\frac{1}{4}t^2 - \frac{1}{8}$, and $Y_2(t) = -e^t$. The general solution is $y(t) = -\frac{1}{4}t^2 - \frac{1}{8} - e^t + c_1 \cosh 2t + c_2 \sinh 2t$. Calculate: $c_1 = 9/8$, and $c_2 = 3/2$.
Answer: $y(t) = -\dfrac{1}{4}t^2 - \dfrac{1}{8} - e^t + \dfrac{9}{8}\cosh 2t + \dfrac{3}{2}\sinh 2t$.

2.7 More on Guessing of $Y(t)$

The prescriptions from the previous section do not always work. In this section we sketch a "fix". A more general method for finding $Y(t)$ will be developed in the next section.

Example 1 Solve $y'' + y = \sin t$.

We try $Y(t) = A \sin t + B \cos t$, according to the Prescription 1. Substituting $Y(t)$ into the equation gives

$$0 = \sin t \,,$$

which is impossible. Why did we "strike out"? Because $A \sin t$ and $B \cos t$ are solutions of the corresponding homogeneous equation. Let us multiply the initial guess by t, and try $Y = At \sin t + Bt \cos t$. Calculate $Y' = A \sin t + At \cos t + B \cos t - Bt \sin t$, and $Y'' = 2A \cos t - At \sin t - 2B \sin t - Bt \cos t$. Substitute Y into our equation, and simplify:

$$2A \cos t - At \sin t - 2B \sin t - Bt \cos t + At \sin t + Bt \cos t = \sin t \,,$$

$$2A \cos t - 2B \sin t = \sin t \,.$$

We conclude that $A = 0$, and $B = -1/2$, so that $Y = -\frac{1}{2}t \cos t$.
Answer: $y = -\frac{1}{2}t \cos t + c_1 \sin t + c_2 \cos t$.

This example prompts us to change the strategy. We now *begin by solving the corresponding homogeneous equation*. The *Prescriptions* from the previous section are now the *Initial Guesses* for the particular solution. We now describe the complete strategy, which is justified in the book of W.E. Boyce and R.C. DiPrima.

If any of the functions, appearing in the Initial Guess, is a solution of the corresponding homogeneous equation, multiply the entire Initial Guess by t, and look at the new functions. If some of them are still solutions of the corresponding homogeneous equation, multiply the entire Initial Guess by t^2. This is guaranteed to work. (Of course, if none of the functions appearing in the Initial Guess is a solution of the corresponding homogeneous equation, then the Initial Guess works.)

In the preceding example, the Initial Guess involved the functions $\sin t$ and $\cos t$, both solutions of the corresponding homogeneous equation. After we multiplied the Initial Guess by t, the new functions $t \sin t$ and $t \cos t$ are not solutions of the corresponding homogeneous equation, and the new guess worked.

Example 2 Solve $y'' + 4y' = 2t - 5$.

The fundamental set of the corresponding homogeneous equation

$$y'' + 4y' = 0$$

consists of the functions $y_1(t) = 1$, and $y_2(t) = e^{-4t}$. The Initial Guess, according to the Prescription 2, $Y(t) = At + B$, is a linear combination of the functions t and 1, and the second of these functions is a solution of the corresponding homogeneous equation. We multiply the Initial Guess by t, obtaining $Y(t) = t(At + B) = At^2 + Bt$. This is a linear combination of t^2 and t, both of which are not solutions of the corresponding homogeneous equation. Substituting $Y(t)$ into the equation, gives

$$2A + 4(2At + B) = 2t - 5\,.$$

Equating the coefficients in t, and the constant terms, we have

$$\begin{aligned} 8A &= 2 \\ 2A + 4B &= -5\,, \end{aligned}$$

giving $A = 1/4$, and $B = -11/8$. The particular solution is $Y(t) = \frac{t^2}{4} - \frac{11}{8}t$.

Answer: $y(t) = \dfrac{t^2}{4} - \dfrac{11}{8}t + c_1 + c_2 e^{-4t}$.

Example 3 Solve $y'' + 2y' + y = te^{-t}$.

The fundamental set of the corresponding homogeneous equation consists of the functions $y_1(t) = e^{-t}$, and $y_2(t) = te^{-t}$. The Initial Guess, according to the Prescription 3, $(At + B)e^{-t} = Ate^{-t} + Be^{-t}$, is a linear combination of the same two functions. We multiply the Initial Guess by t: $t(At+B)e^{-t} = At^2e^{-t}+Bte^{-t}$. The new guess is a linear combination of the functions t^2e^{-t} and te^{-t}. The first of these functions is not a solution of the corresponding homogeneous equation, but the second one is. Therefore, we multiply the Initial Guess by t^2: $Y = t^2(At + B)e^{-t} = At^3e^{-t} + Bt^2e^{-t}$. It is convenient to write $Y = (At^3 + Bt^2)e^{-t}$, and we substitute this in:

$$(6At + 2B)e^{-t} - 2(3At^2 + 2Bt)e^{-t} + (At^3 + Bt^2)e^{-t}$$
$$+2\left[(3At^2 + 2Bt)e^{-t} - (At^3 + Bt^2)e^{-t}\right] + (At^3 + Bt^2)e^{-t} = te^{-t}\,.$$

Divide both sides by e^{-t}, and simplify:

$$6At + 2B = t\,.$$

It follows that

$$\begin{aligned} 6A &= 1 \\ 2B &= 0\,, \end{aligned}$$

giving $A = 1/6$, and $B = 0$. Answer: $y(t) = \dfrac{1}{6}t^3e^{-t} + c_1e^{-t} + c_2te^{-t}$.

2.8 The Method of Variation of Parameters

We now present a more general way to find a particular solution of the non-homogeneous equation

$$y'' + p(t)y' + g(t)y = f(t)\,. \tag{8.1}$$

Let us assume that $y_1(t)$ and $y_2(t)$ form a fundamental solution set for the corresponding homogeneous equation

$$y'' + p(t)y' + g(t)y = 0\,. \tag{8.2}$$

(So that $c_1y_1(t) + c_2y_2(t)$ gives the general solution of (8.2).) We look for a particular solution of (8.1) in the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)\,, \tag{8.3}$$

with some functions $u_1(t)$ and $u_2(t)$, that shall be chosen to satisfy the following two equations

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= f(t)\,. \end{aligned} \tag{8.4}$$

We have a system of two linear equations to find $u_1'(t)$ and $u_2'(t)$. Its determinant

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is the Wronskian of $y_1(t)$ and $y_2(t)$. By the Theorem 2.4.3, $W(t) \neq 0$ for all t, because $y_1(t)$ and $y_2(t)$ form a fundamental solution set. By Cramer's rule (or by elimination), the solution of (8.4) is

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} \tag{8.5}$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)}\,.$$

The functions $u_1(t)$ and $u_2(t)$ are then computed by integration. We shall show that $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ is a particular solution of the non-homogeneous equation (8.1). Let us compute the derivatives of $Y(t)$, in order to substitute $Y(t)$ into the equation (8.1). Obtain:

$$Y'(t) = u_1'(t)y_1(t)+u_2'(t)y_2(t)+u_1(t)y_1'(t)+u_2(t)y_2'(t) = u_1(t)y_1'(t)+u_2(t)y_2'(t).$$

Here the first two terms have disappeared (they add up to zero), thanks to the first equation in (8.4). Next:

$$\begin{aligned} Y''(t) &= u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + u_1(t)y_1''(t) + u_2(t)y_2''(t) \\ &= f(t) + u_1(t)y_1''(t) + u_2(t)y_2''(t)\,, \end{aligned}$$

by using the second equation in (8.4). Then

$$\begin{aligned} Y'' + pY' + gY &= f(t) + u_1y_1'' + u_2y_2'' + p(u_1y_1' + u_2y_2') + g(u_1y_1 + u_2y_2) \\ &= f(t) + u_1(y_1'' + py_1' + gy_1) + u_2(y_2'' + py_2' + gy_2) \\ &= f(t)\,, \end{aligned}$$

which proves that $Y(t)$ is a particular solution of (8.1). (Both brackets are zero, because $y_1(t)$ and $y_2(t)$ are solutions of the corresponding homogeneous equation (8.2).)

In practice, one begins by writing down the formulas (8.5).

Example 1 Find the general solution of

$$y'' + y = \tan t\,.$$

The fundamental set of the corresponding homogeneous equation

$$y'' + y = 0$$

consists of $y_1 = \sin t$ and $y_2 = \cos t$. Their Wronskian

$$W(t) = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1\,,$$

and the formulas (8.5) give

$$\begin{aligned} u_1'(t) &= \tan t \cos t = \sin t \\ u_2'(t) &= -\tan t \sin t = -\frac{\sin^2 t}{\cos t}\,. \end{aligned}$$

Integrating, $u_1(t) = -\cos t$. We set the constant of integration to zero, because we only need one particular solution. Integrating the second formula,

$$\begin{aligned} u_2(t) &= -\int \frac{\sin^2 t}{\cos t}\,dt = -\int \frac{1-\cos^2 t}{\cos t}\,dt = \int (-\sec t + \cos t)\,dt \\ &= -\ln|\sec t + \tan t| + \sin t\,. \end{aligned}$$

We have a particular solution ($Y = u_1 y_1 + u_2 y_2$)

$$Y(t) = -\cos t \sin t + (-\ln|\sec t + \tan t| + \sin t)\cos t = -\cos t\, \ln|\sec t + \tan t|\,.$$

Answer: $y(t) = -\cos t\, \ln|\sec t + \tan t| + c_1 \sin t + c_2 \cos t$.

Example 2 Let us revisit the equation

$$y'' + 2y' + y = te^{-t}\,,$$

for which we needed to use a modified prescription in the preceding section. The fundamental set of the corresponding homogeneous equation

$$y'' + 2y' + y = 0$$

consists of $y_1 = e^{-t}$ and $y_2 = te^{-t}$, and their Wronskian is $W(y_1, y_2)(t) = e^{-2t}$. Then by the formulas (8.5), $u_1' = -t^2$, giving $u_1 = -\frac{t^3}{3}$, and $u_2' = t$, giving $u_2 = \frac{t^2}{2}$. Obtain $Y = u_1 y_1 + u_2 y_2 = \frac{1}{6}t^3 e^{-t}$. Answer: $y(t) = \dfrac{1}{6}t^3 e^{-t} + c_1 e^{-t} + c_2 t e^{-t}$.

2.9 The Convolution Integral

This section introduces the convolution integral, which allows quick computation of a particular solution $Y(t)$, in case of constant coefficients.

2.9.1 Differentiation of Integrals

If $g(t,s)$ is a continuously differentiable function of two variables, then the integral $\int_a^b g(t,s)\,ds$ depends on a parameter t (s is a dummy variable). This integral is differentiated as follows

$$\frac{d}{dt}\int_a^b g(t,s)\,ds = \int_a^b g_t(t,s)\,ds\,,$$

where $g_t(t,s)$ denotes the partial derivative in t. To differentiate the integral $\int_a^t g(s)\,ds$, one uses the fundamental theorem of calculus:

$$\frac{d}{dt}\int_a^t g(s)\,ds = g(t)\,.$$

The integral $\int_a^t g(t,s)\,ds$ depends on a parameter t, and it also has t as its upper limit. It is known from calculus that

$$\frac{d}{dt}\int_a^t g(t,s)\,ds = \int_a^t g_t(t,s)\,ds + g(t,t)\,,$$

so that, in effect, we combine the previous two formulas. Let now $z(t)$ and $f(t)$ be some functions, then the last formula gives

$$\frac{d}{dt}\int_a^t z(t-s)f(s)\,ds = \int_a^t z'(t-s)f(s)\,ds + z(0)f(t)\,. \tag{9.1}$$

2.9.2 Yet Another Way to Compute a Particular Solution

We consider the non-homogeneous equation

$$y'' + py' + gy = f(t)\,, \tag{9.2}$$

where p and g are given numbers, and $f(t)$ is a given function. Let $z(t)$ denote the solution of the corresponding homogeneous equation, satisfying

$$z'' + pz' + gz = 0,\;\; z(0) = 0,\; z'(0) = 1\,. \tag{9.3}$$

Then we can write a particular solution of (9.2) as a *convolution integral*

$$Y(t) = \int_0^t z(t-s) f(s)\, ds\,. \tag{9.4}$$

To justify this formula, we compute the derivatives of $Y(t)$, by using the formula (9.1), and the initial conditions $z(0) = 0$ and $z'(0) = 1$:

$$Y'(t) = \int_0^t z'(t-s) f(s)\, ds + z(0) f(t) = \int_0^t z'(t-s) f(s)\, ds\,,$$

$$Y''(t) = \int_0^t z''(t-s) f(s)\, ds + z'(0) f(t) = \int_0^t z''(t-s) f(s)\, ds + f(t)\,.$$

Then

$$\begin{gathered} Y''(t) + pY'(t) + gY(t) \\ = \int_0^t \left[z''(t-s) + pz'(t-s) + gz(t-s) \right] f(s)\, ds + f(t) = f(t)\,. \end{gathered}$$

Here the integral is zero, because $z(t)$ satisfies the homogeneous equation in (9.3), with *constant coefficients* p and g, at all values of its argument t, including $t - s$.

Example Let us now revisit the equation

$$y'' + y = \tan t\,.$$

Solving

$$z'' + z = 0, \quad z(0) = 0, \; z'(0) = 1\,,$$

gives $z(t) = \sin t$. Then

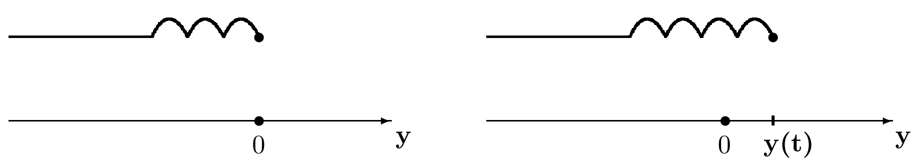
$$Y(t) = \int_0^t \sin(t-s) \tan s\, ds\,.$$

Writing $\sin(t-s) = \sin t \cos s - \cos t \sin s$, and integrating, it is easy to obtain the solution we had before.

2.10 Applications of Second Order Equations

One of the main applications of differential equations is to model mechanical and electrical oscillations. This section is mostly devoted to oscillations of springs, like the springs used in our cars.

2.10.1 Vibrating Spring



Spring at rest Extended spring

If a spring is either extended or compressed, it will oscillate around its equilibrium position. We direct the y-axis along the spring, with the origin chosen at the equilibrium position. Let $y = y(t)$ denote the displacement of a spring from its natural position. Its motion is governed by *Newton's second law*

$$ma = f\,.$$

The acceleration $a = y''(t)$. We assume that the only force f, acting on the spring, is its own restoring force, which by *Hooke's law* is $f = -ky$, for small displacements. Here the physical constant $k > 0$ describes the *stiffness* (or the *hardness*) of the spring. Then

$$my'' = -ky\,.$$

Divide both sides by the mass m of the spring, and denote $k/m = \omega^2$ (so that $\omega = \sqrt{k/m}$), obtaining

$$y'' + \omega^2 y = 0\,.$$

The general solution, $y(t) = c_1 \cos\omega t + c_2 \sin\omega t$, gives us the *harmonic motion* of the spring.

To understand the solution better, let us write $y(t)$ as

$$\begin{aligned} y(t) &= \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega t\right) \\ &= A\left(\frac{c_1}{A}\cos\omega t + \frac{c_2}{A}\sin\omega t\right)\,, \end{aligned}$$

where we denoted $A = \sqrt{c_1^2 + c_2^2}$. Observe that $(\frac{c_1}{A})^2 + (\frac{c_2}{A})^2 = 1$, which means that we can find an angle δ, such that $\cos\delta = \frac{c_1}{A}$, and $\sin\delta = \frac{c_2}{A}$. Then our solution takes the form

$$y(t) = A\left(\cos\omega t \cos\delta + \sin\omega t \sin\delta\right) = A\cos(\omega t - \delta)\,.$$

We conclude that any harmonic motion is just a shifted cosine curve of amplitude $A = \sqrt{c_1^2 + c_2^2}$, and of period $\dfrac{2\pi}{\omega}$. The larger ω is, the smaller is the period, and the oscillations are more frequent. So that ω gives us the *frequency of oscillations*, called the *natural frequency of the spring*. The constants c_1 and c_2 can be computed, once the initial displacement $y(0)$, and the initial velocity $y'(0)$ are prescribed.

Example 1 Solving the initial value problem

$$y'' + 4y = 0, \;\; y(0) = 3, \; y'(0) = -8$$

one gets $y(t) = 3\cos 2t - 4\sin 2t$. This solution is a periodic function, with the amplitude 5, the frequency 2, and the period π.

The equation

$$y'' + \omega^2 y = f(t) \tag{10.1}$$

models the case when an *external force*, with acceleration equal to $f(t)$, is applied to the spring. Indeed, the corresponding equation of motion is now

$$my'' = -ky + mf(t)\,,$$

from which we get (10.1), dividing by m.

Let us consider the case of a *periodic forcing term*

$$y'' + \omega^2 y = a\sin\nu t\,, \tag{10.2}$$

where $a > 0$ is the amplitude of the external force, and ν is the *forcing frequency*. If $\nu \neq \omega$, we look for a particular solution of this non-homogeneous equation in the form $Y(t) = A\sin\nu t$. Substituting this in, gives $A = \dfrac{a}{\omega^2 - \nu^2}$. The general solution of (10.2), which is $y(t) = \dfrac{a}{\omega^2 - \nu^2}\sin\nu t + c_1\cos\omega t + c_2\sin\omega t$, is a superposition (sum) of the harmonic motion, and the *response term* $(\dfrac{a}{\omega^2 - \nu^2}\sin\nu t)$ to the external force. We see that the solution is still bounded, although not periodic anymore, for general ν and ω, as a sum of functions of different periods $\frac{2\pi}{\nu}$ and $\frac{2\pi}{\omega}$ (such functions are called *quasiperiodic*).

A very important case is when $\nu = \omega$, so that the forcing frequency is the same as the natural frequency. Then a particular solution has the form $Y(t) = At\sin\nu t + Bt\cos\nu t$, so that solutions become unbounded, as time t increases. This is the case of *resonance*, when a bounded external force

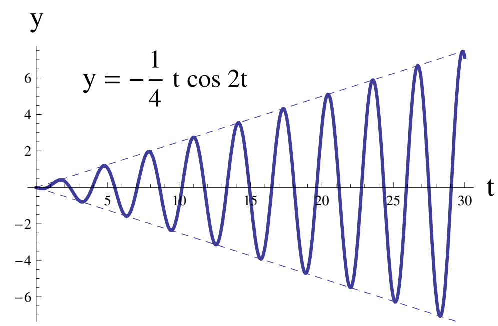


Figure 2.3: The graph of the secular term $y = -\frac{1}{4}t\cos 2t$, oscillating between the lines $y = -\frac{1}{4}t$, and $y = \frac{1}{4}t$

produces unbounded response. Large displacements will break the spring. Resonance is a serious engineering concern.

Example 2 $y'' + 4y = \sin 2t$, $y(0) = 0$, $y'(0) = 1$.

Both the natural and forcing frequencies are equal to 2. The fundamental set of the corresponding homogeneous equation consists of $\sin 2t$ and $\cos 2t$. We search for a particular solution in the form $Y(t) = At\sin 2t + Bt\cos 2t$, corresponding to a modified prescription (alternatively, one can use the variation of parameters method). As before, we compute $Y(t) = -\frac{1}{4}t\cos 2t$. Then the general solution is $y(t) = -\frac{1}{4}t\cos 2t + c_1\cos 2t + c_2\sin 2t$. Using the initial conditions, calculate $c_1 = 0$ and $c_2 = \frac{5}{8}$, so that $y(t) = -\frac{1}{4}t\cos 2t + \frac{5}{8}\sin 2t$. The term $-\frac{1}{4}t\cos 2t$ introduces oscillations, with the amplitude $\frac{1}{4}t$ increasing without bound, as time $t \to \infty$. (It is customary to call such unbounded term a *secular term*, which seems to imply that the harmonic terms are divine.)

2.10.2 Problems

I. Solve the following non-homogeneous equations.

1. $2y'' - 3y' + y = 2\sin t$. Answer. $y = c_1 e^{t/2} + c_2 e^t + \dfrac{3}{5}\cos t - \dfrac{1}{5}\sin t$.

2. $y'' + 4y' + 5y = 2\cos 2t - 3\sin 2t$.

Answer. $y = \dfrac{2}{5}\cos 2t + \dfrac{1}{5}\sin 2t + c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$.

3. $y'' - y' = 5\sin 2t$. Answer. $y = c_1 + c_2 e^t + \dfrac{1}{2}\cos 2t - \sin 2t$.

4. $y'' + 9y = 2\cos \nu t$, $\nu \neq 3$ is a constant.

Answer. $y = \dfrac{2}{9-\nu^2}\cos \nu t + c_1 \cos 3t + c_2 \sin 3t$.

5. $y'' + 2y' + y = 2t\cos t$.

Hint: By Prescription 1, look for a particular solution in the form $y = (At+B)\cos t + (Ct+D)\sin t$. Answer. $y = \cos t + (t-1)\sin t + c_1 e^{-t} + c_2 te^{-t}$.

6. $y'' - 2y' + y = t + 2$. Answer. $y = t + 4 + c_1 e^t + c_2 e^t t$.

7. $y'' + 4y = t^2 - 3t + 1$. Answer. $y = \dfrac{1}{8}\left(2t^2 - 6t + 1\right) + c_1 \cos 2t + c_2 \sin 2t$.

8. $y'' - 9y = e^{5t}$. Answer. $y = \dfrac{e^{5t}}{16} + c_1 e^{-3t} + c_2 e^{3t}$.

9. $y'' - 4y = te^{3t}$, $y(0) = 0$, $y'(0) = 1$.

Answer. $y = \left(\dfrac{1}{5}t - \dfrac{6}{25}\right)e^{3t} - \dfrac{13}{50}e^{-2t} + \dfrac{1}{2}e^{2t}$.

10. $2y'' + y' - y = (5t^2 + t - 1)e^{-2t}$.

Answer. $y = \dfrac{1}{5}e^{-2t}\left(5t^2 + 15t + 16\right) + c_1 e^{t/2} + c_2 e^{-t}$.

11. $4y'' + 8y' + 5y = 5t - \sin t$.

Answer. $y = t - \dfrac{8}{5} + \dfrac{8}{65}\cos t - \dfrac{1}{65}\sin t + c_1 e^{-t}\cos\dfrac{t}{2} + c_2 e^{-t}\sin\dfrac{t}{2}$.

12. $y'' + y = 2e^{4t} + t^2$.

Answer. $y = \dfrac{2}{17}e^{4t} + t^2 - 2 + c_1 \cos t + c_2 \sin t$.

13. $y'' - y' = 2\sin t - \cos 2t$.

Answer. $y = \cos t - \sin t + \dfrac{1}{5}\cos 2t + \dfrac{1}{10}\sin 2t + c_1 + c_2 e^t$.

14. For the first order equation

$$y' - x^2 y = 2x - x^4$$

the integrating factor $\mu = e^{-\frac{x^3}{3}}$ leads to an intractable integral. Instead, look for a particular solution as a quadratic polynomial, and add to it the general solution of the corresponding homogeneous equation.

Answer. $y = x^2 + ce^{\frac{x^3}{3}}$.

II. Solve the non-homogeneous equations (using the modified prescriptions).

1. $y'' + y = 2\cos t$. Answer. $y = t\sin t + c_1 \cos t + c_2 \sin t$.

2. $y'' + y' - 6y = -e^{2t}$. Answer. $y = -\dfrac{1}{5}e^{2t}t + c_1 e^{-3t} + c_2 e^{2t}$.

3. $y'' + 2y' + y = 2e^{-t}$. Answer. $y = t^2 e^{-t} + c_1 e^{-t} + c_2 te^{-t}$.

4. $y'' - 2y' + y = te^t$. Answer. $y = \dfrac{t^3 e^t}{6} + c_1 e^t + c_2 te^t$.

5. $y'' - 4y' = 2 - \cos t$. Answer. $y = -\dfrac{t}{2} + \dfrac{\cos t}{17} + \dfrac{4\sin t}{17} + c_1 e^{4t} + c_2$.

6. $2y'' - y' - y = 3e^t$. Answer. $y = te^t + c_1 e^{-\frac{1}{2}t} + c_2 e^t$.

III. Write down the form in which one should look for a particular solution, but DO NOT compute the coefficients.

1. $y'' + y = 2\sin 2t - \cos 3t - 5t^2 e^{3t} + 4t$.

Answer. $A\cos 2t + B\sin 2t + C\cos 3t + D\sin 3t + \left(Et^2 + Ft + G\right)e^{3t} + Ht + I$.

2. $y'' + y = 4\cos t - \cos 5t + 8$.

Answer. $t\,(A\cos t + B\sin t) + C\cos 5t + D\sin 5t + E$.

3. $y'' - 4y' + 4y = 3te^{2t} + \sin 4t - t^2$.

Answer. $t^2\,(At + B)\,e^{2t} + C\cos 4t + D\sin 4t + Et^2 + Ft + G$.

IV. Find a particular solution, by using the method of variation of parameters, and then write down the general solution.

1. $y'' + y' - 6y = 5e^{2t}$. Answer. $y = te^{2t} + c_1 e^{-3t} + c_2 e^{2t}$.

2. $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$.

Answer. $y = c_1 e^t + c_2 te^t - \dfrac{1}{2}e^t \ln(1+t^2) + te^t \tan^{-1} t$.

3. $y'' + y = \sin t$. Answer. $y = -\dfrac{t}{2}\cos t + c_1 \cos t + c_2 \sin t$.

4. $y'' + 9y = -2e^{3t}$.

Hint: Similar integrals were considered in Section 1.1. For this equation it is easier to use the Prescription 3 from Section 2.6.

Answer. $y = -\frac{e^{3t}}{9} + c_1 \cos 3t + c_2 \sin 3t$.

5. $y'' + 2y' + y = \frac{e^{-t}}{t}$. Answer. $y = -te^{-t} + te^{-t} \ln t + c_1 e^{-t} + c_2 te^{-t}$.

6. $y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}$. Answer. $y = -e^{-2t}(1 + \ln t) + c_1 e^{-2t} + c_2 te^{-2t}$.

7. $y'' - 4y = 8e^{2t}$. Answer. $y = 2te^{2t} + c_1 e^{-2t} + c_2 e^{2t}$.

8. $y'' + y = \sec t$. Answer. $y = c_1 \cos t + c_2 \sin t + \cos t\, \ln|\cos t| + t \sin t$.

9. $y'' + 3y' = 6t$. Answer. $y = t^2 - \frac{2}{3}t + c_1 e^{-3t} + c_2$.

10. $y'' - y' - 2y = e^{-t}$. $y(0) = 1$, $y'(0) = 0$.

Answer. $y = \frac{4}{9}e^{2t} + \frac{1}{9}e^{-t}(3t - 5)$.

11. $y'' + 4y = \sin 2t$, $y(0) = 0$, $y'(0) = 1$.

Answer. $y = -\frac{1}{4}t \cos 2t + \frac{5}{8} \sin 2t$.

12. $2y'' + 2y' - 4y = e^{-2t}$.

Hint: Put this equation into the right form for the variation of parameters formula (8.5).

Answer. $y = -\frac{1}{6}te^{-2t} + c_1 e^{-2t} + c_2 e^t$.

13. $4y'' + 4y' + y = 3te^t$. Answer. $y = \frac{1}{9}e^t(3t - 4) + c_1 e^{-\frac{t}{2}} + c_2 te^{-\frac{t}{2}}$.

V. Verify that the functions $y_1(t)$ and $y_2(t)$ form a fundamental solution set for the corresponding homogeneous equation, and then use variation of parameters to find the general solution.

1. $t^2 y'' - 2y = t^3 - 1$. $y_1(t) = t^2$, $y_2(t) = t^{-1}$.

Hint: Begin by putting this equation into the right form to use (8.5).

Answer. $y = \frac{1}{2} + \frac{1}{4}t^3 + c_1 t^2 + c_2 \frac{1}{t}$.

2. $ty'' - (1 + t)y' + y = t^2 e^{3t}$. $y_1(t) = t + 1$, $y_2(t) = e^t$.

Answer. $y = \frac{1}{12}e^{3t}(2t-1) + c_1(t+1) + c_2e^t$.

3. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2}$. (Non-homogeneous *Bessel's equation*.)
$y_1(x) = x^{-1/2}\cos x$, $y_2(x) = x^{-1/2}\sin x$.
Answer. $y = x^{-1/2}\left(1 + c_1\cos x + c_2\sin x\right)$.

4*. $(3t^3+t)y'' + 2y' - 6ty = 4 - 12t^2$. $\quad y_1(t) = \frac{1}{t}$, $\; y_2(t) = t^2+1$.
Hint: Use *Mathematica* to compute the integrals.
Answer. $y = 2t + c_1\frac{1}{t} + c_2(t^2+1)$.

VI.
1. Use the convolution integral, to solve

$$y'' + y = t^2, \;\; y(0) = 0, \; y'(0) = 1\,.$$

Answer. $y = t^2 - 2 + 2\cos t + \sin t$.

2. (i) Show that $y(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)\,ds$ gives a solution of the n-th order equation

$$y^{(n)} = f(t)\,.$$

(This formula lets you compute n consecutive antiderivatives at once.)
Hint: Use the formula (9.1).

(ii) Solve the following *integral equation*

$$y(t) + \int_0^t (t-s)\,y(s)\,ds = t\,.$$

Hint: Differentiate the equation twice, and also evaluate $y(0)$, $y'(0)$.

Answer. $y = \sin t$.

3*. For the equation

$$u'' + (1 + f(t))\,u = 0$$

assume that $|f(t)| \le \frac{c}{t^{1+\alpha}}$, with positive constants α and c, for $t \ge 1$.

(i) Show that all solutions are bounded as $t \to \infty$.

Hint: Consider the "energy" of the solution $E(t) = \frac{1}{2}u'^2 + \frac{1}{2}u^2$. Then

$$E' = -f(t)uu' \le |f(t)||u'u| \le |f(t)|\left(\frac{1}{2}u'^2 + \frac{1}{2}u^2\right) \le \frac{c}{t^{1+\alpha}}E\,.$$

(ii) Show that this equation has two solutions such that for $t \to \infty$

$$u_1(t) = \cos t + O\left(\frac{1}{t^\alpha}\right), \quad u_2(t) = \sin t + O\left(\frac{1}{t^\alpha}\right).$$

(The "big O" $O\left(\frac{1}{t^\alpha}\right)$ denotes any function, whose absolute value is bounded by $\frac{const}{t^\alpha}$, as $t \to \infty$.)

Hint: Take $f(t)u$ to the right hand side, and treat it as a known function. Then for any $1 < t < a$

$$u(t) = \cos t + \int_t^a \sin(t-s) f(s) u(s)\, ds\,,$$

gives the unique solution of our equation, satisfying the initial conditions $u(a) = \cos a$, $u'(a) = -\sin a$. This solution is written using an integral involving itself. Since $u(s)$ is bounded, $|\int_t^a \sin(t-s) f(s) u(s)\, ds| \le \frac{c_1}{t^\alpha}$.

4*. For the equation

$$u'' - (1 + f(t))\, u = 0$$

assume that $|f(t)| \le \frac{c}{t^{1+\alpha}}$, with positive constants α and c, for $t \ge 1$. Show that the equation has two solutions such that for $t \to \infty$

$$u_1(t) = e^t \left(1 + O\left(\frac{1}{t^\alpha}\right)\right), \quad u_2(t) = e^{-t}\left(1 + O\left(\frac{1}{t^\alpha}\right)\right).$$

Hint: Expressing a solution as $u(t) = e^t + \int_1^t \sinh(t-s) f(s) u(s)\, ds$, estimate

$$|u(t)| \le e^t + \frac{1}{2} e^t \int_1^t e^{-s} |f(s)|\, |u(s)|\, ds\,.$$

Apply Bellman-Gronwall's lemma to show that $z(t) = e^{-t}|u(t)|$ is bounded, and therefore $|u(t)| \le c e^t$. Then for any $1 < t < a$

$$u(t) = e^t - \int_t^a \sinh(t-s) f(s) u(s)\, ds\,,$$

and estimate the integral as above. (Similar questions are discussed in a nice old book by R. Bellman [2].)

5*. In the equation

$$y'' \pm a^4(x) y = 0$$

make a substitution $y = b(x)u$ to obtain

$$b^2 \left(b^2 u'\right)' + \left(b'' b^3 \pm a^4 b^4\right) u = 0\,.$$

Select $b(x) = \frac{1}{a(x)}$, then make a change of the independent variable $x \to t$, so that $\frac{dx}{dt} = \frac{1}{a^2}$, or $t = \int a^2(x)\, dx$. Show that $u = u(t)$ satisfies

$$u'' + \left(\pm 1 + \frac{1}{a^3}\left(\frac{1}{a}\right)_{xx}\right) u = 0\,.$$

This procedure is known as the *Liouville transformation.* It often happens that $\frac{1}{a^3}\left(\frac{1}{a}\right)_{xx} \to 0$ as $t \to \infty$.

6*. Apply the Liouville transformation to the equation

$$y'' + e^{2x} y = 0\,. \tag{10.3}$$

Hint: Here $a(x) = e^{\frac{1}{2}x}$, $t = e^x$, $\frac{1}{a^3}\left(\frac{1}{a}\right)_{xx} = \frac{1}{4}e^{-2x} = \frac{1}{4t^2}$. Obtain:

$$u'' + \left(1 + \frac{1}{4t^2}\right) u = 0\,.$$

Conclude that the general solution of (10.3) satisfies

$$y = c_1 e^{-x/2}\cos(e^x) + c_2 e^{-x/2}\sin(e^x) + O\left(e^{-\frac{3}{2}x}\right)\,, \quad \text{as } x \to \infty\,.$$

7*. Apply the Liouville transformation to the equation

$$xy'' - y = 0\,.$$

Conclude that the general solution satisfies

$$y = c_1 x^{\frac{1}{4}} e^{2\sqrt{x}} + c_2 x^{\frac{1}{4}} e^{-2\sqrt{x}} + O\left(x^{-\frac{1}{4}} e^{2\sqrt{x}}\right)\,, \quad \text{as } x \to \infty\,.$$

VII.

1. A spring has natural frequency $\omega = 2$. Its initial displacement is -1, and the initial velocity is 2. Find its displacement $y(t)$ at any time t. What is the amplitude of the oscillations?

Answer. $y(t) = \sin 2t - \cos 2t$, $A = \sqrt{2}$.

2. A spring of mass 2 lb is hanging down from the ceiling, and its stiffness constant is $k = 18$. Initially, the spring is pushed up 3 inches, and is given velocity of 2 inch/sec, directed downward. Find the displacement of the spring $y(t)$ at any time t, and the amplitude of oscillations. (Assume that the y axis is directed down from the equilibrium position.)

Answer. $y(t) = \frac{2}{3}\sin 3t - 3\cos 3t$, $A = \frac{\sqrt{85}}{3}$.

3. A spring has natural frequency $\omega = 3$. An outside force, with acceleration $f(t) = 2\cos \nu t$, is applied to the spring. Here ν is a constant, $\nu \neq 3$. Find the displacement of the spring $y(t)$ at any time t. What happens to the amplitude of oscillations in case ν is close to 3?

Answer. $y(t) = \dfrac{2}{9-\nu^2}\cos \nu t + c_1 \cos 3t + c_2 \sin 3t$.

4. Assume that $\nu = 3$ in the preceding problem. Find the displacement of the spring $y(t)$ at any time t. What happens to the spring in the long run?

Answer. $y(t) = \dfrac{1}{3} t \sin 3t + c_1 \cos 3t + c_2 \sin 3t$.

5. Consider *dissipative* (or *damped*) *motion* of a spring

$$y'' + \alpha y' + 9y = 0\,.$$

Write down the solution, assuming that $\alpha < 6$. What is the smallest value of the dissipation constant α, which will prevent the spring from oscillating?

Answer. No oscillations for $\alpha \geq 6$.

6. Consider forced vibrations of a dissipative spring

$$y'' + \alpha y' + 9y = \sin 3t\,.$$

Write down the general solution for
(i) $\alpha = 0$
(ii) $\alpha \neq 0$.
What does friction do to the resonance?

2.10.3 A Meteor Approaching the Earth

Let $r = r(t)$ denote the distance of some meteor from the center of the Earth. The motion of the meteor is governed by Newton's law of gravitation

$$mr'' = -\frac{mMG}{r^2}\,. \tag{10.4}$$

Here m is the mass of the meteor, M denotes the mass of the Earth, and G is the universal gravitational constant. Let a be the radius of the Earth. If an object is sitting on Earth's surface, then $r = a$, and the acceleration $r'' = -g$, the *gravity of Earth*, so that from (10.4)

$$g = \frac{MG}{a^2}\,.$$

Then $MG = ga^2$, and we can rewrite (10.4) as

$$r'' = -g\frac{a^2}{r^2}\,. \tag{10.5}$$

We could solve this equation by letting $r' = v(r)$, because the independent variable t is missing. Instead, to obtain the solution in a more instructive way, let us multiply both sides of the equation by r', and write the result as

$$r'r'' + g\frac{a^2}{r^2}r' = 0\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}r'^2 - g\frac{a^2}{r}\right) = 0\,,$$

$$\frac{1}{2}r'^2(t) - g\frac{a^2}{r(t)} = c\,. \tag{10.6}$$

So that the *energy of the meteor*, $E(t) = \frac{1}{2}r'^2(t) - g\frac{a^2}{r(t)}$, remains constant at all time. (That is why the gravitational force field is called *conservative*.) We can now express $r'(t) = -\sqrt{2c + \frac{2ga^2}{r(t)}}$, and calculate the motion of meteor $r(t)$ by separation of variables. However, as we are not riding on the meteor, this seems to be not worth the effort. What really concerns us is the velocity of impact, when the meteor hits the Earth, which is discussed next.

Let us assume that the meteor "begins" its journey with zero velocity $r'(0) = 0$, and at a distance so large that we may assume $r(0) = \infty$. Then the energy of the meteor at time $t = 0$ is zero, $E(0) = 0$. As the energy remains constant at all time, the energy at the time of impact is also zero. At the time of impact, we have $r = a$, and the velocity of impact we denote by v ($r' = v$). Then from (10.6)

$$\frac{1}{2}v^2(t) - g\frac{a^2}{a} = 0\,,$$

and the velocity of impact is

$$v = \sqrt{2ga}\,.$$

Food for thought: the velocity of impact is the same, as it would have been achieved by free fall from height a.

Let us now revisit the harmonic oscillations of a spring:

$$y'' + \omega^2 y = 0\,.$$

Similarly to the meteor case, multiply this equation by y':

$$y'y'' + \omega^2 yy' = 0\,,$$

$$\frac{d}{dt}\left(\frac{1}{2}y'^2 + \frac{1}{2}\omega^2 y^2\right) = 0\,,$$

$$E(t) = \frac{1}{2}y'^2 + \frac{1}{2}\omega^2 y^2 = \textit{constant}\,.$$

With the energy $E(t)$ being conserved, no wonder the motion of the spring was periodic.

2.10.4 Damped Oscillations

We add an extra term to our model of spring motion:

$$my'' = -ky - k_1 y'\,,$$

where k_1 is another positive constant. It represents an additional force, which is directed in the opposite direction, and is proportional to the velocity of motion y'. This can be either air resistance or friction. Denoting $k_1/m = \alpha > 0$, and $k/m = \omega^2$, rewrite the equation as

$$y'' + \alpha y' + \omega^2 y = 0\,. \tag{10.7}$$

Let us see what effect the extra term $\alpha y'$ has on the energy of the spring, $E(t) = \frac{1}{2}y'^2 + \frac{1}{2}\omega^2 y^2$. We differentiate the energy, and express from the equation (10.7), $y'' = -\alpha y' - \omega^2 y$, obtaining

$$E'(t) = y'y'' + \omega^2 yy' = y'(-\alpha y' - \omega^2 y) + \omega^2 yy' = -\alpha y'^2\,.$$

It follows that $E'(t) \le 0$, and the energy decreases. This is an example of a *dissipative motion*. We expect the amplitude of oscillations to decrease with time. We call α the *dissipation (or damping) coefficient*.

To solve the equation (10.7), write down its characteristic equation

$$r^2 + \alpha r + \omega^2 = 0\,.$$

The roots are $r = \dfrac{-\alpha \pm \sqrt{\alpha^2 - 4\omega^2}}{2}$.

There are three cases to consider.

(i) $\alpha^2 - 4\omega^2 < 0$. (The dissipation coefficient α is small.) The roots are complex. If we denote $\alpha^2 - 4\omega^2 = -q^2$, with $q > 0$, the roots are $-\dfrac{\alpha}{2} \pm i\dfrac{q}{2}$. The general solution

$$y(t) = c_1 e^{-\frac{\alpha}{2}t} \sin \frac{q}{2}t + c_2 e^{-\frac{\alpha}{2}t} \cos \frac{q}{2}t$$

exhibits damped oscillations (the amplitude of oscillations tends to zero, as $t \to \infty$).

(ii) $\alpha^2 - 4\omega^2 = 0$. There is a double real root $-\dfrac{\alpha}{2}$. The general solution

$$y(t) = c_1 e^{-\frac{\alpha}{2}t} + c_2 t e^{-\frac{\alpha}{2}t}$$

tends to zero as $t \to \infty$, without oscillating.

(iii) $\alpha^2 - 4\omega^2 > 0$. The roots are real and distinct. If we denote $q = \sqrt{\alpha^2 - 4\omega^2}$, then the roots are $r_1 = \dfrac{-\alpha - q}{2}$, and $r_2 = \dfrac{-\alpha + q}{2}$. Both roots are negative, because $q < \alpha$. The general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

tends to zero as $t \to \infty$, without oscillating. We see that large enough dissipation coefficient α "kills" the oscillations.

2.11 Further Applications

This section begins with forced vibrations in presence of damping. It turns out that any amount of damping "kills" the resonance, and the largest amplitude of oscillations occurs when the forcing frequency ν is a little smaller than the natural frequency ω. Then oscillations of a pendulum, and of two coupled pendulums, are studied.

2.11.1 Forced and Damped Oscillations

It turns out that even a little damping is enough to avoid resonance. Consider the equation

$$y'' + \alpha y' + \omega^2 y = \sin \nu t \tag{11.1}$$

modeling forced vibrations of a spring in the presence of damping. Our theory tells us to look for a particular solution in the form $Y(t) = A_1 \cos \nu t + A_2 \sin \nu t$. Once the constants A_1 and A_2 are determined, we can use trigonometric identities to put this solution into the form

$$Y(t) = A \sin(\nu t - \gamma) \,, \tag{11.2}$$

with the constants $A > 0$ and γ depending on A_1 and A_2. So, let us look for a particular solution directly in the form (11.2). We transform the forcing term as a linear combination of $\sin(\nu t - \gamma)$ and $\cos(\nu t - \gamma)$:

$$\sin \nu t = \sin\left((\nu t - \gamma) + \gamma\right) = \sin(\nu t - \gamma) \cos \gamma + \cos(\nu t - \gamma) \sin \gamma \,.$$

Substitute $Y(t) = A \sin(\nu t - \gamma)$ into the equation (11.1):

$$-A\nu^2 \sin(\nu t - \gamma) + A\alpha\nu \cos(\nu t - \gamma) + A\omega^2 \sin(\nu t - \gamma) =$$
$$\sin(\nu t - \gamma) \cos \gamma + \cos(\nu t - \gamma) \sin \gamma \,.$$

Equating the coefficients in $\sin(\nu t - \gamma)$ and $\cos(\nu t - \gamma)$, gives

$$\begin{aligned} A(\omega^2 - \nu^2) &= \cos \gamma \\ A\alpha\nu &= \sin \gamma \,. \end{aligned} \tag{11.3}$$

Square both of these equations, and add the results

$$A^2(\omega^2 - \nu^2)^2 + A^2 \alpha^2 \nu^2 = 1 \,,$$

which allows us to calculate A:

$$A = \frac{1}{\sqrt{(\omega^2 - \nu^2)^2 + \alpha^2 \nu^2}} \,.$$

To calculate γ, divide the second equation in (11.3) by the first

$$\tan \gamma = \frac{\alpha \nu}{\omega^2 - \nu^2} \,, \quad \text{or} \quad \gamma = \tan^{-1} \frac{\alpha \nu}{\omega^2 - \nu^2} \,.$$

We computed a particular solution

$$Y(t) = \frac{1}{\sqrt{(\omega^2 - \nu^2)^2 + \alpha^2 \nu^2}} \sin(\nu t - \gamma), \quad \text{where } \gamma = \tan^{-1} \frac{\alpha \nu}{\omega^2 - \nu^2} \,.$$

We now make a physically reasonable assumption that the damping coefficient α is small, so that $\alpha^2 - 4\omega^2 < 0$. Then the characteristic equation for the homogeneous equation corresponding to (11.1)

$$r^2 + \alpha r + \omega^2 = 0$$

has a pair of complex roots $-\frac{\alpha}{2} \pm i\beta$, where $\beta = \frac{\sqrt{4\omega^2-\alpha^2}}{2}$. The general solution of (11.1) is then

$$y(t) = c_1 e^{-\frac{\alpha}{2}t}\cos\beta t + c_2 e^{-\frac{\alpha}{2}t}\sin\beta t + \frac{1}{\sqrt{(\omega^2-\nu^2)^2+\alpha^2\nu^2}}\sin(\nu t - \gamma)\,.$$

The first two terms of this solution are called the *transient oscillations*, because they quickly tend to zero, as the time t goes on ("sic transit gloria mundi"). So that the third term, $Y(t)$, describes the oscillations in the long run. We see that oscillations of $Y(t)$ are bounded, no matter what is the frequency ν of the forcing term. The resonance is gone! Moreover, the largest amplitude of $Y(t)$ occurs not at $\nu = \omega$, but at a slightly smaller value of ν. Indeed, the maximal amplitude happens when the quantity in the denominator, $(\omega^2-\nu^2)^2 + \alpha^2\nu^2$, is the smallest. This quantity is a quadratic in ν^2. Its minimum occurs when $\nu^2 = \omega^2 - \frac{\alpha^2}{2}$, or $\nu = \sqrt{\omega^2 - \frac{\alpha^2}{2}}$.

2.11.2 An Example of a Discontinuous Forcing Term

We now consider equations with a jumping force function. A simple function with a jump at some number c, is the *Heaviside step function*

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}\,.$$

Example For $t > 0$, solve the problem

$$y'' + 4y = f(t)$$
$$y(0) = 0, \;\; y'(0) = 3\,,$$

where

$$f(t) = \begin{cases} 0 & \text{if } t < \pi/4 \\ t+1 & \text{if } t \geq \pi/4 \end{cases}\,.$$

Physical interpretation: no external force is applied to the spring before the time $t = \pi/4$, and the force is equal to $t+1$ afterwards. The forcing function can be written as $f(t) = u_{\pi/4}(t)(t+1)$.

The problem naturally breaks down into two parts. When $0 < t < \pi/4$, we are solving

$$y'' + 4y = 0\,.$$

Its general solution is $y(t) = c_1 \cos 2t + c_2 \sin 2t$. Using the initial conditions, calculate $c_1 = 0$, $c_2 = \frac{3}{2}$, so that

$$y(t) = \frac{3}{2} \sin 2t, \quad \text{for } t \le \pi/4\,. \tag{11.4}$$

At later times, when $t \ge \pi/4$, our equation is

$$y'' + 4y = t + 1\,. \tag{11.5}$$

But what are the new initial conditions at the time $t = \pi/4$? Clearly, we can get them from (11.4):

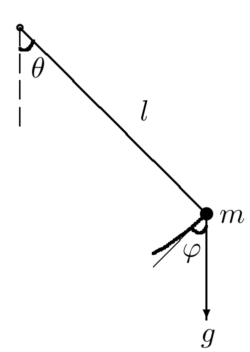
$$y(\pi/4) = \frac{3}{2}, \quad y'(\pi/4) = 0\,. \tag{11.6}$$

The general solution of (11.5) is $y(t) = \frac{1}{4}t + \frac{1}{4} + c_1 \cos 2t + c_2 \sin 2t$. Calculating c_1 and c_2 from the initial conditions in (11.6), gives $y(t) = \frac{1}{4}t + \frac{1}{4} + \frac{1}{8}\cos 2t + (\frac{5}{4} - \frac{\pi}{16})\sin 2t$. Answer:

$$y(t) = \begin{cases} \frac{3}{2} \sin 2t, & \text{if } t < \pi/4 \\ \\ \frac{1}{4}t + \frac{1}{4} + \frac{1}{8}\cos 2t + (\frac{5}{4} - \frac{\pi}{16})\sin 2t, & \text{if } t \ge \pi/4 \end{cases}.$$

Observe that the solution $y(t)$ is continuous at $t = \pi/4$.

2.11.3 Oscillations of a Pendulum



Gravity acting on a pendulum, $\varphi = \frac{\pi}{2} - \theta$

Assume that a small ball of mass m is attached to one end of a rigid rod of length l, while the other end of the rod is attached to the ceiling. Assume also that the mass of the rod itself is so small, that it can be neglected. Clearly, the ball will move on an arch of a circle of radius l. Let $\theta = \theta(t)$ be the angle that the pendulum makes with the vertical line, at the time t. We assume that $\theta > 0$ if the pendulum is to the left of the vertical line, and $\theta < 0$ to the right of the vertical. If the pendulum moves by an angle θ radians, it covers the distance $l\theta = l\theta(t)$. It follows that $l\theta'(t)$ gives its velocity, and $l\theta''(t)$ the acceleration. We assume that the only force acting on the mass is the force of gravity. Only the projection of this force on the tangent line to the circle is active, which is $mg\cos(\frac{\pi}{2} - \theta) = mg\sin\theta$. Newton's second law of motion gives

$$ml\theta''(t) = -mg\sin\theta\,.$$

(Minus, because the force works to decrease the angle θ, when $\theta > 0$, and to increase θ, if $\theta < 0$.) Denoting $g/l = \omega^2$, we obtain the *pendulum equation*

$$\theta''(t) + \omega^2\sin\theta(t) = 0\,.$$

If the oscillation angle $\theta(t)$ is small, then $\sin\theta(t) \approx \theta(t)$, giving us again a harmonic oscillator

$$\theta''(t) + \omega^2\theta(t) = 0\,,$$

this time as a model of small oscillations of a pendulum.

2.11.4 Sympathetic Oscillations

Suppose that we have two pendulums hanging from the ceiling, and they are coupled (connected) through a weightless spring. Let x_1 denote the angle the left pendulum makes with the vertical line. We consider this angle to be positive if the pendulum is to the left of the vertical line, and $x_1 < 0$ if the pendulum is to the right of the vertical line. Let x_2 be the angle the right pendulum makes with the vertical, with the same assumptions on its sign. We assume that x_1 and x_2 are small in absolute value, which means that each pendulum separately can be modeled by a harmonic oscillator.

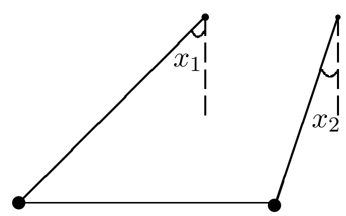
For the *coupled pendulums* the model is

$$\begin{aligned} x_1'' + \omega^2 x_1 &= -k(x_1 - x_2) \\ x_2'' + \omega^2 x_2 &= k(x_1 - x_2)\,, \end{aligned} \tag{11.7}$$

where $k > 0$ is a physical constant, describing the stiffness of the coupling spring. Indeed, if $x_1 > x_2 > 0$, then the coupling spring is extended, so that the spring tries to contract, and in doing so it pulls back the left pendulum, while pulling forward (accelerating) the right pendulum. (Correspondingly, the forcing term is negative in the first equation, and positive in the second one.) In case $0 < x_1 < x_2$, the spring is compressed, and as it tries to expand, it accelerates the first (left) pendulum, and slows down the second (right) pendulum. We shall solve the system (11.7), together with the simple initial conditions

$$x_1(0) = a, \;\; x_1'(0) = 0, \;\; x_2(0) = 0, \;\; x_2'(0) = 0\,, \tag{11.8}$$

which correspond to the first pendulum beginning with a small displacement angle a, and zero initial velocity, while the second pendulum is at rest.



Two pendulums connected by a weightless spring

Add the equations in (11.7), and call $z_1 = x_1 + x_2$. Obtain

$$z_1'' + \omega^2 z_1 = 0, \;\; z_1(0) = a, \;\; z_1'(0) = 0\,.$$

The solution of this initial value problem is $z_1(t) = a\cos\omega t$. Subtracting the second equation in (11.7) from the first one, and calling $z_2 = x_1 - x_2$, gives $z_2'' + \omega^2 z_2 = -2kz_2$, or

$$z_2'' + (\omega^2 + 2k)z_2 = 0, \;\; z_2(0) = a, \;\; z_2'(0) = 0\,.$$

Denoting $\omega^2 + 2k = \omega_1^2$, or $\omega_1 = \sqrt{\omega^2 + 2k}$, we have $z_2(t) = a\cos\omega_1 t$. Clearly, $z_1 + z_2 = 2x_1$. Then

$$x_1 = \frac{z_1 + z_2}{2} = \frac{a\cos\omega t + a\cos\omega_1 t}{2} = a\,\cos\frac{\omega_1 - \omega}{2}t\,\cos\frac{\omega_1 + \omega}{2}t\,, \tag{11.9}$$

using a trigonometric identity on the last step. Similarly,

$$x_2 = \frac{z_1 - z_2}{2} = \frac{a\cos\omega t - a\cos\omega_1 t}{2} = a\,\sin\frac{\omega_1 - \omega}{2}t\,\sin\frac{\omega_1 + \omega}{2}t\,. \tag{11.10}$$

We now analyze the solution, given by the formulas (11.9) and (11.10). If k is small (the coupling is weak), then ω_1 is close to ω, and so their difference $\omega_1 - \omega$ is small. It follows that both $\cos \dfrac{\omega_1 - \omega}{2} t$ and $\sin \dfrac{\omega_1 - \omega}{2} t$ change very slowly with time t. Rewrite the solution as

$$x_1 = A \cos \frac{\omega_1 + \omega}{2} t, \quad \text{and} \quad x_2 = B \sin \frac{\omega_1 + \omega}{2} t \,,$$

where we regard $A = a \cos \dfrac{\omega_1 - \omega}{2} t$, and $B = a \sin \dfrac{\omega_1 - \omega}{2} t$, as slowly varying amplitudes. We interpret this by saying that the pendulums oscillate with the frequency $\dfrac{\omega_1 + \omega}{2}$, and with slowly varying amplitudes A and B. (The amplitudes A and B are also periodic. Oscillations with periodically varying amplitudes are known as *beats*, see Figure 2.4.)

At times t, when $\cos \dfrac{\omega_1 - \omega}{2} t$ is zero, and the first pendulum is at rest ($x_1 = 0$), the amplitude of the second pendulum satisfies $|\sin \dfrac{\omega_1 - \omega}{2} t| = 1$, obtaining its largest possible absolute value. There is a complete exchange of energy: when one of the pendulums is doing the maximal work, the other one is resting. We see "sympathy" between the pendulums. Observe also that $A^2 + B^2 = a^2$. This means that the point $(x_1(t), x_2(t))$ lies on the circle of radius a in the (x_1, x_2) plane, for all t.

Example Using *Mathematica*, we solved a particular case of (11.7)

$$\begin{gathered} x_1'' + x_1 = -0.1(x_1 - x_2) \\ x_2'' + x_2 = 0.1(x_1 - x_2) \\ x_1(0) = 2 \,, \; x_1'(0) = 0 \,, \; x_2(0) = x_2'(0) = 0 \,. \end{gathered}$$

The graphs of $x_1(t)$ and $x_2(t)$ in Figure 2.4 both exhibit beats. Observe that maximal amplitude of each of these functions occurs at times when the amplitude of the other function is zero.

2.12 Oscillations of a Spring Subject to a Periodic Force

This section develops the Fourier series, one of the most important concepts of mathematics. Application is made to periodic vibrations of a spring.

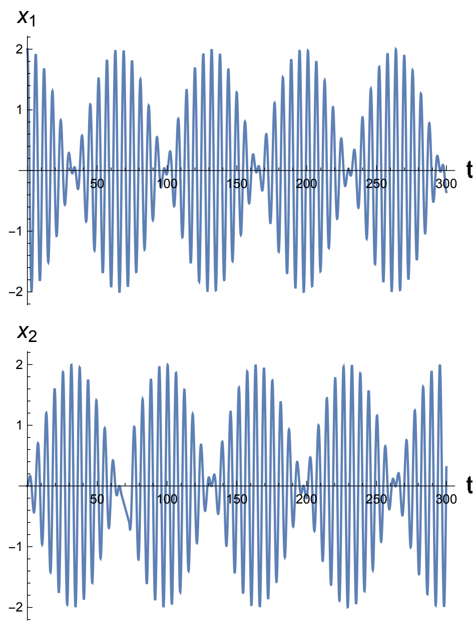


Figure 2.4: Oscillations of coupled pendulums: beats

2.12.1 The Fourier Series

For vectors in three dimensions, one of the central notions is that of the *scalar product* (also known as the “inner product”, or the “dot product”). Namely, if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then their scalar product is

$$(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 \,.$$

Scalar product can be used to compute the length of a vector $||x|| = \sqrt{(x,x)}$, and the angle θ between the vectors x and y

$$\cos\theta = \frac{(x,y)}{||x||\,||y||} \,.$$

In particular, the vectors x and y are orthogonal (perpendicular) if $(x, y) = 0$. If i, j and k are the unit coordinate vectors, then $(x, i) = x_1$, $(x, j) = x_2$, and $(x, k) = x_3$. Writing $x = x_1 i + x_2 j + x_3 k$, we express

$$x = (x, i)i + (x, j)j + (x, k)k \,. \tag{12.1}$$

This formula gives probably the simplest example of the *Fourier Series*.

We shall now consider functions $f(t)$ that are periodic, with period 2π. Such functions are determined by their values on any interval of length 2π. So let us consider them on the interval $(-\pi, \pi)$. Given two functions $f(t)$ and $g(t)$, we define their scalar product as

$$(f, g) = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt\,.$$

We call the functions *orthogonal* if $(f, g) = 0$. For example, $(\sin t, \cos t) = \int_{-\pi}^{\pi} \sin t\, \cos t\, dt = 0$, so that $\sin t$ and $\cos t$ are orthogonal. (Observe that the orthogonality of these functions has nothing to do with the angle at which their graphs intersect.) The notion of scalar product allows us to define the *norm of a function*

$$||f|| = \sqrt{(f, f)} = \sqrt{\int_{-\pi}^{\pi} f^2(t)\, dt}\,.$$

For example,

$$||\sin t|| = \sqrt{\int_{-\pi}^{\pi} \sin^2 t\, dt} = \sqrt{\int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\, dt} = \sqrt{\pi}\,.$$

Similarly, for any positive integer n, $||\sin nt|| = \sqrt{\pi}$, $||\cos nt|| = \sqrt{\pi}$, and $||1|| = \sqrt{2\pi}$.

We now consider an infinite set of functions

$$1,\ \cos t,\ \cos 2t,\ \ldots, \cos nt,\ \ldots, \sin t,\ \sin 2t,\ \ldots, \sin nt,\ \ldots\,.$$

They are all mutually orthogonal. This is because

$$(1, \cos nt) = \int_{-\pi}^{\pi} \cos nt\, dt = 0\,,$$

$$(1, \sin nt) = \int_{-\pi}^{\pi} \sin nt\, dt = 0\,,$$

$$(\cos nt, \cos mt) = \int_{-\pi}^{\pi} \cos nt\, \cos mt\, dt = 0, \quad \text{for all } n \neq m\,,$$

$$(\sin nt, \sin mt) = \int_{-\pi}^{\pi} \sin nt\, \sin mt\, dt = 0, \quad \text{for all } n \neq m\,,$$

$$(\sin nt, \cos mt) = \int_{-\pi}^{\pi} \sin nt\, \cos mt\, dt = 0, \quad \text{for any } n \text{ and } m\,.$$

The last three integrals are computed by using trigonometric identities. If we divide these functions by their norms, we shall obtain an *orthonormal set of functions*

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \ldots, \frac{\cos nt}{\sqrt{\pi}}, \ldots, \frac{\sin t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \ldots, \frac{\sin nt}{\sqrt{\pi}}, \ldots,$$

which is similar to the coordinate vectors i, j and k. It is known that these functions form a *complete set*, so that "any" function $f(t)$ can be represented as their linear combination. Similarly to the formula for vectors (12.1), we decompose an arbitrary function $f(t)$ as

$$f(t) = \alpha_0 \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(\alpha_n \frac{\cos nt}{\sqrt{\pi}} + \beta_n \frac{\sin nt}{\sqrt{\pi}} \right) ,$$

where

$$\alpha_0 = (f(t), \frac{1}{\sqrt{2\pi}}) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, dt \,,$$

$$\alpha_n = (f(t), \frac{\cos nt}{\sqrt{\pi}}) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(t) \, \cos nt \, dt \,,$$

$$\beta_n = (f(t), \frac{\sin nt}{\sqrt{\pi}}) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(t) \, \sin nt \, dt \,.$$

It is customary to denote $a_0 = \alpha_0/\sqrt{2\pi}$, $a_n = \alpha_n/\sqrt{\pi}$, and $b_n = \beta_n/\sqrt{\pi}$, so that the Fourier Series takes the final form

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt \right) ,$$

with the coefficients given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt \,,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, \cos nt \, dt \,,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, \sin nt \, dt \,.$$

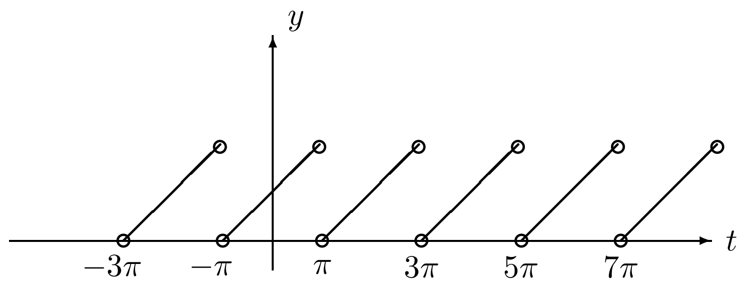
The Pythagorean theorem takes the form

$$||f||^2 = \alpha_0^2 + \sum_{n=1}^{\infty} \left(\alpha_n^2 + \beta_n^2 \right) ,$$

or

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\,,$$

which is known as *Parseval's identity.*



The saw-tooth function

Example Let $f(t)$ be a function of period 2π, which is equal to $t+\pi$ on the interval $(-\pi,\pi)$. This is the *saw-tooth function.* It is not defined at the points $n\pi$ and $-n\pi$, with n odd, but this does not affect the integrals that we need to compute. Compute

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(t+\pi)\,dt = \frac{1}{4\pi}(t+\pi)^2\Big|_{-\pi}^{\pi} = \pi\,.$$

Integrating by parts (or using guess-and-check)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(t+\pi)\,\cos nt\,dt = \left[\frac{1}{\pi}(t+\pi)\,\frac{\sin nt}{n}+\frac{\cos nt}{n^2\pi}\right]\Big|_{-\pi}^{\pi} = 0\,,$$

because $\sin n\pi = 0$, and cosine is an even function. Similarly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(t+\pi)\,\sin nt\,dt = \left[\frac{1}{\pi}(t+\pi)(-\frac{\cos nt}{n})+\frac{\sin nt}{n^2\pi}\right]\Big|_{-\pi}^{\pi}$$
$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}\,.$$

(Observe that $\cos n\pi$ is equal to 1 for even n, and to -1 for odd n, which may be combined as $\cos n\pi = (-1)^n$.) The Fourier series for the function $f(t)$ is then

$$f(t) = \pi + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nt\,,$$

which is valid on $(-\infty, \infty)$ (with the exception of points $n\pi$ and $-n\pi$, with n odd). Restricting to the interval $(-\pi, \pi)$, gives

$$t + \pi = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nt, \quad \text{for } -\pi < t < \pi\,.$$

It might look that we did not accomplish much by expressing a simple function $t + \pi$ through an infinite series. However, one can now express solutions of differential equations through Fourier series.

2.12.2 Vibrations of a Spring Subject to a Periodic Force

Consider the model

$$y'' + \omega^2 y = f(t)\,,$$

where $y = y(t)$ is the displacement of a spring, $\omega > 0$ is a constant (the natural frequency), and $f(t)$ is a given function of period 2π, the acceleration of an external force. This equation also models oscillations in electrical circuits. Expressing $f(t)$ by its Fourier series, rewrite this model as

$$y'' + \omega^2 y = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right)\,.$$

Let us assume that $\omega \neq n$, for any integer n (to avoid resonance). According to our theory, we look for a particular solution in the form $Y(t) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt\right)$. Substituting this in, we find

$$Y(t) = \frac{a_0}{\omega^2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{\omega^2 - n^2} \cos nt + \frac{b_n}{\omega^2 - n^2} \sin nt \right)\,.$$

The general solution is then

$$y(t) = \frac{a_0}{\omega^2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{\omega^2 - n^2} \cos nt + \frac{b_n}{\omega^2 - n^2} \sin nt \right) + c_1 \cos \omega t + c_2 \sin \omega t\,.$$

We see that the coefficients in the m-th harmonics (in $\cos mt$ and $\sin mt$) are large, provided that the natural frequency ω is selected to be close to m. That is basically what happens, when one is turning the tuning knob of a radio set. (The knob controls ω, while your favorite station broadcasts at a frequency m, so that its signal has the form $f(t) = a_m \cos mt + b_m \sin mt$.)

2.13 Euler's Equation

This section covers an important class of equations with variable coefficients. The understanding of Euler's equations will play a crucial role when infinite series are used in Chapter 3 to solve differential equations.

Preliminaries

What is the meaning of $3^{\sqrt{2}}$? Or, more generally, what is the definition of the function t^r, where r is any real number? Here it is: $t^r = e^{\ln t^r} = e^{r \ln t}$. We see that the function t^r is defined only for $t > 0$. The function $|t|^r$ is defined for all $t \neq 0$, but what is the derivative of this function?

More generally, if $f(t)$ is a differentiable function, the function $f(|t|)$ is differentiable for all $t \neq 0$ ($|t|$ is not differentiable at $t = 0$). Let us define the following *step function*

$$\operatorname{sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}.$$

Observe that

$$\frac{d}{dt}|t| = \operatorname{sign}(t)\,, \quad \text{for all } t \neq 0\,,$$

as follows by considering separately the cases $t > 0$, and $t < 0$.

The chain rule gives

$$\frac{d}{dt} f(|t|) = f'(|t|) \operatorname{sign}(t)\,, \quad \text{for all } t \neq 0\,. \tag{13.1}$$

In particular, $\frac{d}{dt}|t|^r = r|t|^{r-1}\operatorname{sign}(t)$, for all $t \neq 0$. Also

$$\frac{d}{dt} \ln|t| = \frac{\operatorname{sign}(t)}{|t|} = \frac{1}{t}\,, \quad \text{for all } t \neq 0\,. \tag{13.2}$$

The Important Class of Equations

Euler's equation has the form (here $y = y(t)$)

$$at^2y'' + bty' + cy = 0\,, \tag{13.3}$$

where a, b and c are given numbers. Assume first that $t > 0$. We look for a solution in the form $y = t^r$, with the constant r to be determined. Substituting this in, gives

$$at^2r(r-1)t^{r-2} + btrt^{r-1} + ct^r = 0\,.$$

Dividing by a positive quantity t^r

$$ar(r-1)+br+c=0 \tag{13.4}$$

gives us a quadratic equation, called the *characteristic equation*. There are three possibilities with respect to its roots, which we consider next.

Case 1 There are two real and distinct roots $r_1 \neq r_2$. Then t^{r_1} and t^{r_2} are two solutions, which are not constant multiples of each other, and the general solution (valid for $t>0$) is

$$y(t)=c_1t^{r_1}+c_2t^{r_2}\,.$$

If r_1 is either an integer, or a fraction with an odd denominator, then t^{r_1} is also defined for $t<0$. If the same is true for r_2, then the above general solution is valid for all $t \neq 0$. For other r_1 and r_2, this solution is not even defined for $t<0$.

We claim that $y(t)=|t|^{r_1}$ gives a solution of Euler's equation, which is valid for all $t \neq 0$. Indeed, calculate $y'(t)=r_1|t|^{r_1-1}\text{sign}(t)$, $y''(t)=r_1(r_1-1)|t|^{r_1-2}\left(\text{sign}(t)\right)^2=r_1(r_1-1)|t|^{r_1-2}$, and then substituting $y(t)$ into Euler's equation (13.3) gives

$$\begin{aligned} &at^2\,r_1(r_1-1)|t|^{r_1-2}+bt\,r_1|t|^{r_1-1}\text{sign}(t)+c|t|^{r_1} \\ &=|t|^{r_1}\left(ar_1(r_1-1)+br_1+c\right)=0\,, \end{aligned}$$

because $t\,\text{sign}(t)=|t|$, and r_1 is a root of characteristic equation. So that

$$y(t)=c_1|t|^{r_1}+c_2|t|^{r_2}$$

gives a general solution valid for all $t \neq 0$.

Example 1 Solve $\quad 2t^2y''+ty'-3y=0.$

The characteristic equation

$$2r(r-1)+r-3=0$$

has roots $r_1=-1$, and $r_2=\frac{3}{2}$. The general solution $y(t)=c_1t^{-1}+c_2t^{\frac{3}{2}}$ is valid only for $t>0$, while $y(t)=c_1|t|^{-1}+c_2|t|^{\frac{3}{2}}$ is valid for $t \neq 0$.

Example 2 Solve $\quad t^2y''+2ty'-2y=0.$

The characteristic equation

$$r(r-1)+2r-2=0$$

has roots $r_1 = -2$, and $r_2 = 1$. The general solution $y(t) = c_1 t^{-2} + c_2 t$ is valid not just for $t > 0$, but for all $t \neq 0$. Another general solution valid for all $t \neq 0$ is $y(t) = c_1|t|^{-2} + c_2|t| = c_1 t^{-2} + c_2|t|$. This is a truly different function! Why such an unexpected complexity? If one divides this equation by t^2, then the functions $p(t) = 2/t$ and $g(t) = -2/t^2$ from our general theory, are both discontinuous at $t = 0$. We have a *singularity* at $t = 0$, and, in general, the solution $y(t)$ is not even defined at $t = 0$ (as we see in this example), and that is the reason for the complexity. However, when solving initial value problems, it does not matter which form of the general solution one uses. For example, if we prescribe some initial conditions at $t = -1$, then both forms of the general solution can be continued only on the interval $(-\infty, 0)$, and on that interval both forms are equivalent.

We now turn to the cases of equal roots, and of complex roots, for the characteristic equation. One could proceed similarly to the linear equations with constant coefficients. Instead, to understand what lies behind the nice properties of Euler's equation, we make a change of independent variables from t to a new variable s, by letting $t = e^s$, or $s = \ln t$. By the chain rule

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{dy}{ds}\frac{1}{t}.$$

Using the product rule, and then the chain rule,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{ds}\right)\frac{1}{t} - \frac{dy}{ds}\frac{1}{t^2} = \frac{d^2y}{ds^2}\frac{ds}{dt}\frac{1}{t} - \frac{dy}{ds}\frac{1}{t^2} = \frac{d^2y}{ds^2}\frac{1}{t^2} - \frac{dy}{ds}\frac{1}{t^2}.$$

Then Euler's equation (13.3) becomes

$$a\frac{d^2y}{ds^2} - a\frac{dy}{ds} + b\frac{dy}{ds} + cy = 0.$$

This is a linear equations with constant coefficients! It can be solved for any a, b and c. Let us use primes again to denote the derivatives in s in this equation. Then it becomes

$$ay'' + (b-a)y' + cy = 0. \tag{13.5}$$

Its characteristic equation

$$ar^2 + (b-a)r + c = 0 \tag{13.6}$$

is exactly the same as (13.4).

We now return to Euler's equation, and its characteristic equation (13.4).

Case 2 r_1 is a double root of the characteristic equation (13.4), i.e., r_1 is a double root of (13.6). Then $y = c_1 e^{r_1 s} + c_2 s e^{r_1 s}$ is the general solution of (13.5). Returning to the original variable t, by substituting $s = \ln t$, and simplifying (using that $e^{r_1 \ln t} = t^{r_1}$), obtain

$$y(t) = c_1 t^{r_1} + c_2 t^{r_1} \ln t\,.$$

This general solution of Euler's equation is valid for $t > 0$. More generally, it is straightforward to verify that

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|$$

gives us the general solution of Euler's equation, valid for all $t \neq 0$.

Case 3 $p \pm iq$ are complex roots of the characteristic equation (13.4). Then $y = c_1 e^{ps} \cos qs + c_2 s e^{ps} \sin qs$ is the general solution of (13.5). Returning to the original variable t, by substituting $s = \ln t$, we get the general solution of Euler's equation

$$y(t) = c_1 t^p \cos(q \ln t) + c_2 t^p \sin(q \ln t)\,,$$

valid for $t > 0$. One verifies that replacing t by $|t|$, gives the general solution of Euler's equation, valid for all $t \neq 0$:

$$y(t) = c_1 |t|^p \cos(q \ln |t|) + c_2 |t|^p \sin(q \ln |t|)\,.$$

Example 3 Solve $t^2 y'' - 3ty' + 4y = 0$, $t > 0$.

The characteristic equation

$$r(r-1) - 3r + 4 = 0$$

has a double root $r = 2$. The general solution: $y = c_1 t^2 + c_2 t^2 \ln t$.

Example 4 Solve $t^2 y'' - 3ty' + 4y = 0, \quad y(1) = 4,\ y'(1) = 7$.

Using the general solution from the preceding example, calculate $c_1 = 4$ and $c_2 = -1$. Answer: $y = 4t^2 - t^2 \ln t$.

Example 5 Solve $t^2 y'' + ty' + 4y = 0, \quad y(-1) = 0,\ y'(-1) = 3$.

The characteristic equation

$$r(r-1) + r + 4 = 0$$

has a pair of complex roots $\pm 2i$. Here $p = 0$, $q = 2$, and the general solution, valid for both positive and negative t, is

$$y(t) = c_1 \cos(2 \ln |t|) + c_2 \sin(2 \ln |t|)\,.$$

From the first initial condition, $y(-1) = c_1 = 0$, so that $y(t) = c_2 \sin(2 \ln |t|)$. Using the chain rule and the formula (13.2)

$$y'(t) = c_2 \cos(2 \ln |t|)\, \frac{2}{t}\,,$$

and then $y'(-1) = -2c_2 = 3$, giving $c_2 = -3/2$. Answer: $y(t) = -\dfrac{3}{2} \sin(2 \ln |t|)$.

Example 6 Solve $t^2 y'' - 3ty' + 4y = t - 2$, $t > 0$.

This is a non-homogeneous equation. Look for a particular solution in the form $Y = At + B$, and obtain $Y = t - \dfrac{1}{2}$. The fundamental solution set of the corresponding homogeneous equation is given by t^2 and $t^2 \ln t$, as we saw in Example 3 above. The general solution is $y = t - \dfrac{1}{2} + c_1 t^2 + c_2 t^2 \ln t$.

2.14 Linear Equations of Order Higher Than Two

Differential equations of order higher than two occur frequently in applications, for example when modeling vibrations of a beam.

2.14.1 The Polar Form of Complex Numbers

For a complex number $x + iy$, one can use the point (x, y) to represent it. This turns the usual plane into the *complex plane*. The point (x, y) can also be identified by its polar coordinates (r, θ). We shall always take $r > 0$. Then

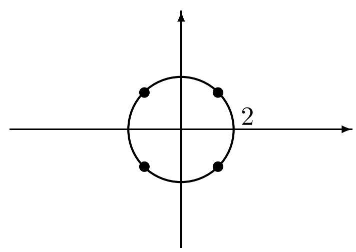
$$z = x + iy = r \cos\theta + ir \sin\theta = r(\cos\theta + i \sin\theta)$$

gives us the polar form of a complex number z. Using Euler's formula, we can also write $z = re^{i\theta}$. For example, $-2i = 2e^{i\frac{3\pi}{2}}$, because the point $(0, -2)$ has polar coordinates $(2, \frac{3\pi}{2})$. Similarly, $1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$, and $-1 = e^{i\pi}$ (real numbers are just particular cases of complex ones).

There are infinitely many ways to represent a complex number using polar coordinates $z = re^{i(\theta + 2\pi m)}$, where m is any integer (positive or negative). Let n be a positive integer. We now compute the n-th root(s) of z:

$$z^{1/n} = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2\pi m}{n})}, \quad m = 0, 1, \ldots, n-1\,. \tag{14.1}$$

Here $r^{1/n}$ is the positive n-th root of the positive number r. (The "high school" n-th root.) Clearly, $\left(z^{1/n}\right)^n = z$. When m varies from 0 to $n-1$, we get different values, and then the roots repeat themselves. There are n complex n-th roots of any complex number (and in particular, of any real number). All of these roots lie on a circle of radius $r^{1/n}$ around the origin, and the difference in the polar angles between any two neighbors is $2\pi/n$.



The four complex fourth roots of -16, on the circle of radius 2

Example 1 Solve the equation: $z^4 + 16 = 0$.

We need the four complex roots of $-16 = 16e^{i(\pi+2\pi m)}$. The formula (14.1) gives

$$(-16)^{(1/4)} = 2e^{i(\frac{\pi}{4}+\frac{\pi m}{2})}, \quad m = 0, 1, 2, 3 .$$

When $m = 0$, the root is $2e^{i\frac{\pi}{4}} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2} + i\sqrt{2}$. The other roots, $2e^{i\frac{3\pi}{4}}$, $2e^{i\frac{5\pi}{4}}$ and $2e^{i\frac{7\pi}{4}}$, are computed similarly. They come in two complex conjugate pairs: $\sqrt{2} \pm i\sqrt{2}$ and $-\sqrt{2} \pm i\sqrt{2}$. In the complex plane, they all lie on the circle of radius 2, and the difference in the polar angles between any two neighbors is $\pi/2$.

Example 2 Solve the equation: $r^3 + 8 = 0$.

We need the three complex cube roots of -8. One of them is $r_1 = -2 = 2e^{i\pi}$, and the other two lie on the circle of radius 2, at an angle $2\pi/3$ away, so that $r_2 = 2e^{i\pi/3} = 1 + \sqrt{3}i$, and $r_3 = 2e^{-i\pi/3} = 1 - \sqrt{3}i$. (Alternatively, the root $r = -2$ is easy to guess. Then factor $r^3 + 8 = (r+2)(r^2 - 2r + 4)$, and set the second factor to zero, to find the other two roots.)

2.14.2 Linear Homogeneous Equations

Let us consider fourth order equations

$$a_0 y'''' + a_1 y''' + a_2 y'' + a_3 y' + a_4 y = 0 , \tag{14.2}$$

with given numbers a_0, a_1, a_2, a_3, and a_4. Again, we search for a solution in the form $y(t) = e^{rt}$, with a constant r to be determined. Substituting this in, and dividing by the positive exponent e^{rt}, we obtain the characteristic equation

$$a_0 r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 = 0\,. \tag{14.3}$$

If one has an equation of order n

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0\,, \tag{14.4}$$

with constant coefficients, then the corresponding characteristic equation is

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0\,. \tag{14.5}$$

The fundamental theorem of algebra says that any polynomial of degree n has n roots in the complex plane, counted according to their multiplicity (double root is counted as two roots, and so on). The characteristic equation (14.3) has four roots.

The theory is similar to the second order case. We need four *different solutions* of (14.2), so that every solution is *not a linear combination of the other three* (for the equation (14.4) we need n different solutions). Every root of the characteristic equation must "pull its weight". If the root is simple, it brings in one solution, if it is repeated twice, then two solutions. (Three solutions, if the root is repeated three times, and so on.) The following cases may occur for the n-th order equation (14.4).

Case 1 r_1 is a simple real root. Then it brings $e^{r_1 t}$ into the fundamental set.

Case 2 r_1 is a real root repeated s times. Then it brings the following s solutions into the fundamental set: $e^{r_1 t}$, $te^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}$.

Case 3 $p+iq$ and $p-iq$ are simple complex roots. They contribute: $e^{pt}\cos qt$ and $e^{pt}\sin qt$ into the fundamental set.

Case 4 $p+iq$ and $p-iq$ are repeated s times each. They bring the following $2s$ solutions into the fundamental set: $e^{pt}\cos qt$ and $e^{pt}\sin qt$, $te^{pt}\cos qt$ and $te^{pt}\sin qt, \ldots,$ $t^{s-1}e^{pt}\cos qt$ and $t^{s-1}e^{pt}\sin qt$.

The cases 1 and 3 are justified as for the second order equations. The other two cases are discussed in the Problems.

Example 1 Solve $\;y'''' - y = 0$.

The characteristic equation is

$$r^4 - 1 = 0\,.$$

We solve it by factoring

$$(r-1)(r+1)(r^2+1) = 0\,.$$

The roots are -1, 1, $-i$, i. The general solution: $y(t) = c_1 e^{-t} + c_2 e^t + c_3 \cos t + c_4 \sin t$.

Example 2 Solve $\quad y''' - 3y'' + 3y' - y = 0$.

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0\,.$$

This is a cubic equation. You probably did not study how to solve it by Cardano's formula. Fortunately, you must remember that the quantity on the left is an exact cube:

$$(r-1)^3 = 0\,.$$

The root $r = 1$ is repeated 3 times. The general solution: $y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$.

Let us suppose that you did not know the formula for cube of a difference. Then one can guess that $r = 1$ is a root. This means that the cubic polynomial can be factored, with one factor being $r - 1$. The other factor is then found by the long division. The other factor is a quadratic polynomial, and its roots are easy to find.

Example 3 Solve $\quad y''' - y'' + 3y' + 5y = 0$.

The characteristic equation is

$$r^3 - r^2 + 3r + 5 = 0\,.$$

We need to guess a root. The procedure for guessing a root (for textbook examples) is a simple one: try $r = 0$, $r = \pm 1$, $r = \pm 2$, and then give up. One sees that $r = -1$ is a root, $r_1 = -1$. It follows that the first factor is $r + 1$, and the second factor is found by the long division:

$$(r+1)(r^2 - 2r + 5) = 0\,.$$

The roots of the quadratic are $r_2 = 1 - 2i$, and $r_3 = 1 + 2i$. The general solution: $y(t) = c_1 e^{-t} + c_2 e^t \cos 2t + c_3 e^t \sin 2t$.

Example 4 Solve $y'''' + 2y'' + y = 0$.

The characteristic equation is

$$r^4 + 2r^2 + 1 = 0\,.$$

It can be solved by factoring

$$(r^2 + 1)^2 = 0\,.$$

(Or one could set $z = r^2$, and obtain a quadratic equation for z.) The roots are $-i$, i, each repeated twice. The general solution: $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$.

Example 5 Solve $y'''' + 16y = 0$.

The characteristic equation is

$$r^4 + 16 = 0\,.$$

Its solutions are the four complex roots of -16, computed earlier: $\sqrt{2} \pm i\sqrt{2}$, and $-\sqrt{2} \pm i\sqrt{2}$. The general solution:

$$y(t) = c_1 e^{\sqrt{2}\,t} \cos(\sqrt{2}\,t) + c_2 e^{\sqrt{2}\,t} \sin(\sqrt{2}\,t) + c_3 e^{-\sqrt{2}\,t} \cos(\sqrt{2}\,t) + c_4 e^{-\sqrt{2}\,t} \sin(\sqrt{2}\,t)\,.$$

Example 6 Solve $y^{(5)} + 9y''' = 0$.

The characteristic equation is

$$r^5 + 9r^3 = 0\,.$$

Factoring $r^3(r^2 + 9) = 0$, we see that the roots are: 0, 0, 0, $-3i$, $3i$. The general solution: $y(t) = c_1 + c_2 t + c_3 t^2 + c_4 \cos 3t + c_5 \sin 3t$.

2.14.3 Non-Homogeneous Equations

The theory is parallel to the second order case. Again, a particular solution is needed, to which we add the general solution of the corresponding homogeneous equation.

Example Solve $y^{(5)} + 9y''' = 3t - \sin 2t$.

General solution of the corresponding homogeneous equation was found in the Example 6. A particular solution is produced in the form $Y(t) = Y_1(t) + Y_2(t)$, where $Y_1(t)$ is a particular solution of

$$y^{(5)} + 9y''' = 3t\,,$$

and $Y_2(t)$ is a particular solution of

$$y^{(5)} + 9y''' = -\sin 2t\,.$$

We guess that $Y_1(t) = At^4$, and compute $A = \frac{1}{72}$, and that $Y_2(t) = B\cos 2t$, which gives $B = -\frac{1}{40}$. So that $Y(t) = \frac{1}{72}t^4 - \frac{1}{40}\cos 2t$.

Answer: $y(t) = \frac{1}{72}t^4 - \frac{1}{40}\cos 2t + c_1 + c_2 t + c_3 t^2 + c_4 \cos 3t + c_5 \sin 3t$.

2.14.4 Problems

I. Solve the non-homogeneous equations with discontinuous forcing function.

1. $y'' + 9y = f(t)$, where $f(t) = 0$ for $0 < t < \pi$, and $f(t) = t$ for $t > \pi$, $y(0) = 0$, $y'(0) = -2$. Answer:

$$y(t) = \begin{cases} -\frac{2}{3}\sin 3t, & \text{if } t \le \pi \\ \\ \frac{1}{9}t + \frac{\pi}{9}\cos 3t - \frac{17}{27}\sin 3t, & \text{if } t > \pi \end{cases}.$$

2. $y'' + y = f(t)$, where $f(t) = 0$ for $0 < t < \pi$, and $f(t) = t$ for $t > \pi$, $y(0) = 2$, $y'(0) = 0$.

II. Find the general solution, valid for $t > 0$.

1. $t^2y'' - 2ty' + 2y = 0$. Answer. $y = c_1 t + c_2 t^2$.
2. $t^2y'' + ty' + 4y = 0$. Answer. $y = c_1\cos(2\ln t) + c_2\sin(2\ln t)$.
3. $t^2y'' + 5ty' + 4y = 0$. Answer. $y = c_1 t^{-2} + c_2 t^{-2}\ln t$.
4. $t^2y'' + 5ty' + 5y = 0$. Answer. $y = c_1 t^{-2}\cos(\ln t) + c_2 t^{-2}\sin(\ln t)$.
5. $t^2y'' - 3ty' = 0$. Answer. $y = c_1 + c_2 t^4$.
6. $y'' + \dfrac{1}{4}t^{-2}y = 0$. Answer. $y = c_1\sqrt{t} + c_2\sqrt{t}\ln t$.
7. $2t^2y'' + 5ty' + y = 0$. Answer. $y = c_1 t^{-\frac{1}{2}} + c_2 t^{-1}$.
8. $9t^2y'' - 3ty' + 4y = 0$. Answer. $y = c_1 t^{2/3} + c_2 t^{2/3}\ln t$.
9. $4x^2y''(x) + 4xy'(x) + y(x) = 0$, $x > 0$.

Answer. $y = c_1\cos\left(\dfrac{1}{2}\ln x\right) + c_2\sin\left(\dfrac{1}{2}\ln x\right)$.

10. Find the general solution of

$$y'' + \frac{3}{t}y' + \frac{5}{t^2}y = \frac{1}{t^3}\,, \quad t > 0\,.$$

Hint: Look for a particular solution in the form $y = \frac{A}{t}$.

Answer. $y = \frac{1}{4t} + c_1 \frac{\cos(2\ln t)}{t} + c_2 \frac{\sin(2\ln t)}{t}$.

11. Use variation of parameters to find the general solution of

$$y'' + \frac{3}{t}y' + \frac{5}{t^2}y = \frac{\ln t}{t^3}\,, \quad t > 0\,.$$

Answer. $y = \frac{\ln t}{4t} + c_1 \frac{\cos(2\ln t)}{t} + c_2 \frac{\sin(2\ln t)}{t}$.

12. Find the general solution of

$$t^3 y''' + t^2 y'' - 2ty' + 2y = 0\,.$$

Hint: Look for a solution in the form $y = t^r$.

Answer. $y = c_1 \frac{1}{t} + c_2 t + c_3 t^2$.

III. Find the general solution, valid for all $t \neq 0$.

1. $t^2 y'' + ty' + 4y = 0$. Answer. $y = c_1 \cos(2\ln|t|) + c_2 \sin(2\ln|t|)$.

2. $2t^2 y'' - ty' + y = 0$. Answer. $y = c_1\sqrt{|t|} + c_2|t|$. ($y = c_1\sqrt{|t|} + c_2 t$ is also a correct answer.)

3. $4t^2 y'' - 4ty' + 13y = 0$. Answer. $y = c_1|t|\cos\left(\frac{3}{2}\ln|t|\right) + c_2|t|\sin\left(\frac{3}{2}\ln|t|\right)$.

4. $9t^2 y'' + 3ty' + y = 0$. Answer. $y = c_1|t|^{1/3} + c_2|t|^{1/3}\ln|t|$.

5. $2ty'' + y' = 0$. Answer. $y = c_1 + c_2\sqrt{|t|}$.

6. $2t^2 y'' - ty' + y = t^2 - 3$.

Hint: Look for a particular solution as $Y = At^2 + Bt + C$.

Answer. $y = \frac{1}{3}t^2 - 3 + c_1\sqrt{|t|} + c_2|t|$.

7. $2t^2 y'' - ty' + y = t^3$. Hint: Look for a particular solution as $Y = At^3$.

Answer. $y = \frac{1}{10}t^3 + c_1\sqrt{|t|} + c_2|t|$.

8. $2(t+1)^2y'' - 3(t+1)y' + 2y = 0,\ t \neq -1$.

Hint: Look for a solution in the form $y = (t+1)^r$.

Answer. $c_1\sqrt{|t+1|} + c_2(t+1)^2$.

9. Solve the following *integro-differential equation*

$$4y'(t) + \int_0^t \frac{y(s)}{(s+1)^2}\, ds = 0\,,\quad t > -1\,.$$

Hint: Differentiate the equation, and observe that $y'(0) = 0$.

Answer. $y = c\left[2\,(t+1)^{1/2} - (t+1)^{1/2}\ln\,(t+1)\right]$.

IV. Solve the following initial value problems.

1. $t^2y'' - 2ty' + 2y = 0,\ y(1) = 2,\ y'(1) = 5$. Answer. $y = -t + 3t^2$.

2. $t^2y'' - 3ty' + 4y = 0,\ y(-1) = 1,\ y'(-1) = 2$. Answer. $y = t^2 - 4t^2\ln|t|$.

3. $t^2y'' + 3ty' - 3y = 0,\ y(-1) = 1,\ y'(-1) = 2$. Answer. $y = -\frac{3}{4}t^{-3} - \frac{1}{4}t$.

4. $t^2y'' - ty' + 5y = 0,\ y(1) = 0,\ y'(1) = 2$. Answer. $y = t\sin(2\ln t)$.

5. $t^2y'' + ty' + 4y = 0,\ y(-1) = 0,\ y'(-1) = 4$. Answer. $y = -2\sin\,(2\ln|t|)$.

6. $6t^2y'' + ty' + y = 0,\ y(2) = 0,\ y'(2) = 1$. Answer. $y = 12\left[\left(\frac{t}{2}\right)^{1/2} - \left(\frac{t}{2}\right)^{1/3}\right]$.

7. $ty'' + y' = 0,\ y(-3) = 0,\ y'(-3) = 1$. Answer. $y = 3\ln 3 - 3\ln|t|$.

8. $2t^2y'' - ty' + y = 0,\ y(-1) = 0,\ y'(-1) = \frac{1}{2}$. Answer. $y = t + \sqrt{|t|}$.

V. Solve the polynomial equations.

1. $r^3 - 1 = 0$. Answer. The roots are $1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$.

2. $r^3 + 27 = 0$. Answer. $-3,\ \frac{3}{2} - \frac{3\sqrt{3}}{2}i,\ \frac{3}{2} + \frac{3\sqrt{3}}{2}i$.

3. $r^4 - 16 = 0$. Answer. ± 2 and $\pm 2i$.

4. $r^3 - 3r^2 + r + 1 = 0$. Answer. $1,\ 1-\sqrt{2},\ 1+\sqrt{2}$.

5. $2r^3 - 5r^2 + 4r - 1 = 0$. Answer. $\frac{1}{2},\ 1,\ 1$.

6. $r^3 + 2r^2 + r + 2 = 0$. Answer. $-2,\ -i,\ i$.

7. $3r^4 + 5r^3 + r^2 - r = 0$. Answer. $0,\ \frac{1}{3},\ -1,\ -1$.

8. $r^4 + 1 = 0$. Answer. $e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}},\ e^{i\frac{5\pi}{4}},\ e^{i\frac{7\pi}{4}}$.

9. $r^4 + 4 = 0$. Answer. $1+i,\ 1-i,\ -1+i,\ -1-i$.

10. $r^4 + 8r^2 + 16 = 0$. Answer. $2i$ and $-2i$ are both double roots.

11. $r^4 + 5r^2 + 4 = 0$. Answer. $\pm i$ and $\pm 2i$.

12. $r^6 + r^4 + 4r^2 + 4 = 0$.

Hint: Write the equation as $r^2(r^4+4) + r^4 + 4 = 0$.

Answer. $\pm i,\ 1 \pm i,\ -1 \pm i$.

VI. Find the general solution.

1. $y''' - y = 0$. Answer. $y = c_1 e^t + c_2 e^{-t/2} \cos \frac{\sqrt{3}}{2} t + c_3 e^{-t/2} \sin \frac{\sqrt{3}}{2} t$.

2. $y''' - 5y'' + 8y' - 4y = 0$. Answer. $y = c_1 e^t + c_2 e^{2t} + c_3 t e^{2t}$.

3. $y''' - 3y'' + y' + y = 0$. Answer. $y = c_1 e^t + c_2 e^{(1-\sqrt{2})t} + c_3 e^{(1+\sqrt{2})t}$.

4. $y''' - 3y'' + y' - 3y = 0$. Answer. $y = c_1 e^{3t} + c_2 \cos t + c_3 \sin t$.

5. $y^{(4)} - 8y'' + 16y = 0$. Answer. $y = c_1 e^{-2t} + c_2 e^{2t} + c_3 t e^{-2t} + c_4 t e^{2t}$.

6. $y^{(4)} + 8y'' + 16y = 0$.

Answer. $y = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t$.

7. $y^{(4)} + y = 0$.

Answer. $y = c_1 e^{\frac{t}{\sqrt{2}}} \cos \frac{t}{\sqrt{2}} + c_2 e^{\frac{t}{\sqrt{2}}} \sin \frac{t}{\sqrt{2}} + c_3 e^{-\frac{t}{\sqrt{2}}} \cos \frac{t}{\sqrt{2}} + c_4 e^{-\frac{t}{\sqrt{2}}} \sin \frac{t}{\sqrt{2}}$.

8. $y''' - y = t^2$. Answer. $y = -t^2 + c_1 e^t + c_2 e^{-t/2} \cos \frac{\sqrt{3}}{2} t + c_3 e^{-t/2} \sin \frac{\sqrt{3}}{2} t$.

9. $y^{(6)} - y'' = 0$. Answer. $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^t + c_5 \cos t + c_6 \sin t$.

10. $2y''' - 5y'' + 4y' - y = 0$. Answer. $y = c_1 e^{\frac{1}{2}t} + c_2 e^t + c_3 t e^t$.

11. $y^{(5)} - 3y'''' + 3y''' - 3y'' + 2y' = 0$.

Answer. $y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 \cos t + c_5 \sin t$.

12. $y^{(8)} - y^{(6)} = \sin t$.

Answer. $y = \dfrac{1}{2} \sin t + c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 t^5 + c_7 e^{-t} + c_8 e^t$.

13. $y'''' + 4y = 4t^2 - 1$.

Answer. $y = t^2 - \dfrac{1}{4} + c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$.

14. $y'''' - 2y''' - 8y' + 16y = 27e^{-t}$.

Answer. $y = e^{-t} + c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t} \cos \sqrt{3}t + c_4 e^{-t} \sin \sqrt{3}t$.

VII. Solve the following initial value problems.

1. $y''' + 4y' = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 2$.

Answer. $y = \frac{3}{2} - \frac{1}{2} \cos 2t - \frac{1}{2} \sin 2t$.

2. $y'''' + 4y = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 2$, $y'''(0) = 3$.

Answer. $y = -\frac{1}{8} e^t (\cos t - 5 \sin t) + \frac{3}{8} e^{-t} (3 \cos t - \sin t)$.

3. $y''' + 8y = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -2$.

Answer. $y = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t \cos \sqrt{3}\, t$.

4. $y^{(5)} + y^{(4)} = 1$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 1$, $y'''(0) = -1$, $y''''(0) = 2$.

Answer. $y = \dfrac{t^4}{24} + e^{-t}$.

5. $y'''' + y''' + y'' + y' = 0$, $y(0) = 0$, $y'(0) = 3$, $y''(0) = -1$, $y'''(0) = -1$.

Answer. $y = 1 - e^{-t} + 2 \sin t$.

6. $y'''' - 3y'' - 4y = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 4$, $y'''(0) = 1$.

Answer. $y = \cosh 2t - \sin t$.

7. $y^{(5)} - y' = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, $y''''(0) = 1$.

Hint: Write the general solution as $y = c_1 + c_2 \cos t + c_3 \sin t + c_4 \cosh t + c_5 \sinh t$. Answer. $y = 1 + \cosh t$.

VIII.

1. Write the equation (14.4) in the *operator form*

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \,. \tag{14.6}$$

Here $L[y]$ is a function of a function $y(t)$, or an *operator*.

(i) Show that

$$L[e^{rt}] = e^{rt} \left(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \right) \,. \tag{14.7}$$

(ii) Assume that r_1 is a real root of the characteristic equation (14.5), which is repeated s times, so that

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = a_0 (r - r_1)^s q(r) \,,$$

where $q(r)$ is a polynomial of degree $n - s$, with $q(r_1) \neq 0$. Differentiate the equation (14.7) in r, and set $r = r_1$, to show that $te^{r_1 t}$ is a solution of (14.6). Show that $e^{r_1 t}$, $te^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}$ are solutions of (14.6).

(iii) Assume that $p + iq$ and $p - iq$ are roots of the characteristic equation (14.5), each repeated s times. By above, $z_1 = t^k e^{(p+iq)t}$ and $z_2 = t^k e^{(p-iq)t}$ are solutions of (14.6), for $k = 0, 1, \ldots, s-1$. By considering $\frac{z_1 + z_2}{2}$ and $\frac{z_1 - z_2}{2i}$, justify that these roots bring the following $2s$ solutions into the fundamental set: $e^{pt} \cos qt$ and $e^{pt} \sin qt$, $te^{pt} \cos qt$ and $te^{pt} \sin qt, \ldots$, $t^{s-1} e^{pt} \cos qt$ and $t^{s-1} e^{pt} \sin qt$.

2. Find the linear homogeneous differential equation of the lowest possible order, which has the following functions as its solutions: 1, e^{-2t}, $\sin t$.

Answer. $y'''' + 2y''' + y'' + 2y' = 0$.

3. Find the general solution of

$$(t+1)^2 y'' - 4(t+1) y' + 6y = 0 \,.$$

Hint: Look for the solution in the form $y = (t+1)^r$.

Answer. $y = c_1 (t+1)^2 + c_2 (t+1)^3$.

4. Find the general solution of

$$ty''' + y'' = 1 \,.$$

Answer. $y = \dfrac{t^2}{2} + c_1 (t \ln t - 1) + c_2 t + c_3$.

5. Solve the following nonlinear equation ($y = y(t)$)

$$2y'y''' - 3y''^2 = 0\,.$$

(This equation is connected to the *Schwarzian derivative*, defined as $S(y(t)) = \frac{y'''(t)}{y'(t)} - \frac{3}{2}\left(\frac{y''(t)}{y'(t)}\right)^2$.)

Hint: Write this equation as

$$\frac{y'''}{y''} = \frac{3}{2}\frac{y''}{y'}\,,$$

then integrate, to get

$$y'' = c_1 {y'}^{(3/2)}\,.$$

Let $y' = v$, and obtain a first order equation.

Answer. $y = \dfrac{1}{c_1 t + c_2} + c_3$, and also $y = c_4 t + c_5$. (Equivalently, the answer can be written as $y = \dfrac{c_1 t + c_2}{c_3 t + c_4}$.)

6. Consider the nonlinear equation

$$y''(t) + a(t) y^3(t) = 0, \quad 0 \le t < \infty\,,$$

where $a(t)$ is a continuously differentiable function, and $a(t) \ge a_0 > 0$ for all $t \ge 0$, a_0 is a constant. Assume that either $a'(t) > 0$, or $a'(t) < 0$ for all $t \in [0, \infty)$. Show that any solution remains bounded on $[0, \infty)$.

Hint: Consider the "energy" function $E(t) = \frac{1}{2}{y'}^2(t) + \frac{1}{4}a(t)y^4(t)$. Using the equation, $E'(t) = \frac{1}{4}a'(t)y^4(t)$. In case $a'(t) < 0$ for all t, the energy is decreasing, and so $y(t)$ is bounded. In case $a'(t) > 0$ for all t, we have $E'(t) = \frac{1}{4}a'(t)y^4(t) \le a'(t)\frac{E(t)}{a(t)}$, or $\frac{E'(t)}{E(t)} \le \frac{a'(t)}{a(t)}$. Integrating this over $(0, t)$, we get $E(t) \le a(t)\frac{E(0)}{a(0)}$, which implies that $\frac{1}{4}y^4(t) \le \frac{E(0)}{a(0)} = \textit{constant}$.

7. Find the *homoclinic solutions* of (a is a given number, $u = u(x)$)

$$u'' - a^2 u + 2u^3 = 0, \quad -\infty < x < \infty, \quad u(-\infty) = u'(-\infty) = u(\infty) = u'(\infty) = 0\,.$$

Hint: Multiply the equation by u', and integrate:

$${u'}^2 - a^2 u^2 + u^4 = \textit{constant} = 0.$$

Solve this equation for u', to obtain a first order separable equation.

Answer. $u(x) = \dfrac{a}{\cosh a(x-c)}$, for any number c.

8. (i) Solve the nonlinear equation ($y = y(t)$)

$$y'' - \frac{1}{y^3} = 0\,, \;\; y(0) = q\,, \; y'(0) = p\,,$$

with the given numbers $q \neq 0$ and p.

Hint: Multiply the equation by y' to get

$$\frac{d}{dt}\left(y'^2 + y^{-2}\right) = 0\,.$$

Integration gives

$$y'^2 + y^{-2} = p^2 + \frac{1}{q^2}\,. \tag{14.8}$$

Now multiply the equation by y:

$$yy'' - y^{-2} = 0\,. \tag{14.9}$$

Using (14.8),

$$yy'' = \frac{1}{2}\left(y^2\right)'' - y'^2 = \frac{1}{2}\left(y^2\right)'' + y^{-2} - p^2 - \frac{1}{q^2}$$

Setting $v = y^2$, obtain from (14.9)

$$v'' = 2\left(p^2 + \frac{1}{q^2}\right)\,, \;\; v(0) = q^2\,, \; v'(0) = 2pq\,.$$

Answer. $y = \pm\sqrt{\left(p^2 + \dfrac{1}{q^2}\right)t^2 + 2pqt + q^2}$, with "plus" if $q > 0$, and "minus" if $q < 0$.

(ii) Solve *Pinney's equation* ($a > 0$ is a constant)

$$y'' + a^2 y - \frac{1}{y^3} = 0\,, \;\; y(0) = q \neq 0\,, \; y'(0) = p\,.$$

Hint: Proceed similarly, and show that $v = y^2$ satisfies

$$v'' + 4a^2 v = 2\left(p^2 + a^2 q^2 + \frac{1}{q^2}\right)\,, \;\; v(0) = q^2\,, \; v'(0) = 2pq\,.$$

Answer. $y = \dfrac{\sqrt{(a^2q^4 - p^2q^2 - 1)\cos(2at) + 2apq^3\sin(2at) + p^2q^2 + a^2q^4 + 1}}{\sqrt{2}aq}$.

(iii) Let $u(x)$ and $v(x)$ be the solutions of the linear equation

$$y'' + a(x)y = 0\,, \tag{14.10}$$

for which $u(x_0) = q$, $u'(x_0) = p$, and $v(x_0) = 0$, $v'(x_0) = \frac{1}{q}$. Here $a(x)$ is a given function, $q \neq 0$, p and x_0 are given numbers. Use the Theorem 2.4.2 to show that their Wronskian $W(x) = W(u,v)(x)$ satisfies

$$W(x) = u'(x)v(x) - u(x)v'(x) = 1\,, \quad \text{for all } x\,.$$

(iv) Consider *Pinney's equation* (more general than the one in part (ii))

$$y'' + a(x)y + \frac{c}{y^3} = 0\,, \quad y(x_0) = q \neq 0\,,\ y'(x_0) = p\,,$$

with a given function $a(x)$ and a constant $c \neq 0$. Show that its solution is

$$y(x) = \pm\sqrt{u^2(x) - cv^2(x)}\,,$$

where one takes "plus" if $q > 0$, and "minus" if $q < 0$.

Hint: Substituting $y = \sqrt{u^2(x) - cv^2(x)}$ into Pinney's equation, and using that $u'' = -a(x)u$ and $v'' = -a(x)v$, obtain

$$y'' + a(x)y + \frac{c}{y^3} = -c\,\frac{\left[u'(x)v(x) - u(x)v'(x)\right]^2 - 1}{\left[u^2(x) - cv^2(x)\right]^{\frac{3}{2}}} = 0\,.$$

2.15 Oscillation and Comparison Theorems

The equation

$$y'' + n^2y = 0$$

has a solution $y(t) = \sin nt$. The larger is n, the more roots this solution has, and so it oscillates faster. In 1836, J.C.F. Sturm discovered the following theorem.

Theorem 2.15.1 *(The Sturm Comparison Theorem.) Let $y(t)$ and $v(t)$ be respectively non-trivial solutions of the following equations*

$$y'' + b(t)y = 0\,, \tag{15.1}$$

$$v'' + b_1(t)v = 0\,. \tag{15.2}$$

Assume that the given continuous functions $b(t)$, and $b_1(t)$ satisfy

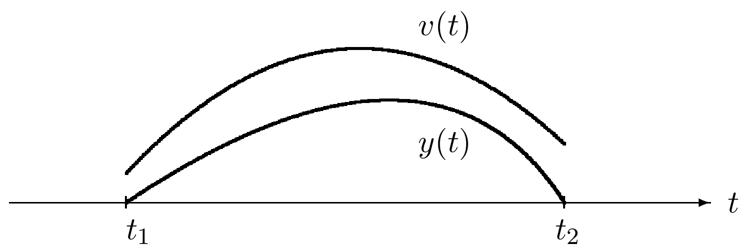
$$b_1(t) \geq b(t) \ \text{ for all } t. \tag{15.3}$$

In case $b_1(t) = b(t)$ on some interval (t_1, t_2), assume additionally that $y(t)$ and $v(t)$ are not constant multiples of one another on (t_1, t_2). Then $v(t)$ has a root between any two consecutive roots of $y(t)$.

Proof: Let $t_1 < t_2$ be two consecutive roots of $y(t)$,

$$y(t_1) = y(t_2) = 0\,. \tag{15.4}$$

We may assume that $y(t) > 0$ on (t_1, t_2) (in case $y(t) < 0$ on (t_1, t_2), we may consider $-y(t)$, which is also a solution of (15.1)). Assume, *contrary to what we want to prove*, that $v(t)$ has no roots on (t_1, t_2). We may assume that $v(t) > 0$ on (t_1, t_2) (by considering $-v(t)$, in case $v(t) < 0$).



The functions $y(t)$ and $v(t)$

Multiply the equation (15.2) by $y(t)$, and subtract from that the equation (15.1), multiplied by $v(t)$. The result may be written as

$$(v'y - vy')' + (b_1 - b)yv = 0\,.$$

Integrating this over (t_1, t_2), and using (15.4) gives

$$-v(t_2)y'(t_2) + v(t_1)y'(t_1) + \int_{t_1}^{t_2} \left[b_1(t) - b(t)\right] y(t)v(t)\,dt = 0\,. \tag{15.5}$$

All three terms on the left are non-negative. If $b_1(t) > b(t)$ on some subinterval of (t_1, t_2), then the third term is strictly positive, and we have a contradiction.

Now consider the remaining case when $b_1(t) = b(t)$ for all $t \in (t_1, t_2)$, so that the equations (15.1) and (15.2) coincide on (t_1, t_2), and $v(t)$ is a solution of (15.1). We claim that $v(t)$ cannot vanish at t_1 and at t_2, so that $v(t_1) > 0$, and $v(t_2) > 0$. Indeed, in case $v(t_1) = 0$, we consider the function $z(t) = \frac{y'(t_1)}{v'(t_1)} v(t)$. This function is a solution of (15.1), and $z(t_1) = y(t_1) = 0$, $z'(t_1) = y'(t_1)$, so that by the uniqueness of solutions for initial value problems, $z(t) = y(t)$ for all $t \in (t_1, t_2)$, and then $y(t)$ and $v(t)$ are constant multiples of one another on (t_1, t_2), which is not allowed. It follows that $v(t_1) > 0$, and similarly we prove that $v(t_2) > 0$. Clearly, $y'(t_1) \geq 0$ and $y'(t_2) \leq 0$. The uniqueness Theorem 2.4.1 for initial value problems implies that $y'(t_1) > 0$, and $y'(t_2) < 0$ (otherwise, if say $y'(t_1) = 0$, then $y(t) = 0$ for all t, by Theorem 2.4.1). Then the first two terms in (15.5) are strictly positive, and we have a contradiction in (15.5). $\diamondsuit$

In case $y(t)$ and $v(t)$ are two solutions of the same equation (15.1), which are not constant multiples of one another, the theorem implies that *their roots interlace*, which means that between any two roots of one of the solutions there is a root of the other one.

By a similar argument, one proves the following version of the Sturm comparison theorem, involving a *differential inequality.*

Lemma 2.15.1 *Assume that the functions $u(t)$ and $v(t)$ are twice continuously differentiable, and they satisfy*

$$\begin{aligned} & v'' + q(t)v = 0, \quad v(a) = 0\,, \\ & u'' + q(t)u \geq 0, \quad u(a) = 0, \quad u'(a) > 0\,, \end{aligned} \tag{15.6}$$

on some interval (a, b) (with a given continuous function $q(t)$). Then $v(t)$ oscillates faster than $u(t)$, provided that both functions are positive. Namely, (i) if $v(t) > 0$ on (a, b), then $u(t) > 0$ on (a, b).

(ii) If, on the other hand, $u(t) > 0$ on (a, b) and $u(b) = 0$, then $v(t)$ must vanish on $(a, b]$.

Proof: As in Theorem 2.15.1, obtain

$$(v'u - vu')' \leq 0, \quad \text{for } x \in (a, b)\,. \tag{15.7}$$

Let us prove (i). Assume that $v(t) > 0$ on (a, b), and *contrary to what we want to prove*, $u(\xi) = 0$ at some $\xi \in (a, b)$.

Case 1. The inequality in (15.6) is strict on some sub-interval of (a, ξ). The the same is then true for the inequality (15.7). Integrating (15.7) over (a, ξ), obtain

$$-v(\xi)u'(\xi) < 0\,,$$

which is a contradiction (because $v(\xi) > 0$, $u'(\xi) \le 0$).

Case 2. Assume that $u'' + q(t)u = 0$ on (a, ξ). Then $u(t)$ and $v(t)$ are solutions of the same equation on (a, ξ), and $u(a) = v(a) = 0$. It follows that $u(t)$ and $v(t)$ are constant multiples of one another, but $u(\xi) = 0$, while $v(\xi) > 0$, a contradiction, proving the first part of the lemma.

The second statement of the lemma is proved similarly. ♢

We shall need the following formula from calculus, discovered by an Italian mathematician Mauro Picone in 1909.

Lemma 2.15.2 *(Picone's Identity) Assume that the functions $a(t)$ and $a_1(t)$ are differentiable, the functions $u(t)$ and $v(t)$ are twice differentiable, and $v(t) > 0$ for all t. Then*

$$\left[\frac{u}{v}\left(vau' - ua_1v'\right)\right]' = u(au')' - \frac{u^2}{v}(a_1v')' + (a - a_1)u'^2 + a_1\left(u' - \frac{u}{v}v'\right)^2 .$$

Proof: The derivative on the left is equal to

$$\frac{u'v - uv'}{v^2}\left(vau' - ua_1v'\right) + \frac{u}{v}\left(v(au')' - u(a_1v')'\right) + (a - a_1)\frac{u}{v}u'v' \equiv A + B + C\,.$$

The middle term B is equal to the first two terms on the right in Picone's identity. It remains to prove that $A + C$ is equal to the sum of the last two terms on the right in Picone's identity. We expand A and C, and after a cancellation, obtain

$$A + C = (a - a_1)u'^2 + a_1\left[u'^2 - 2\frac{u}{v}u'v' + v'^2\left(\frac{u}{v}\right)^2\right] = (a - a_1)u'^2 + a_1\left(u' - \frac{u}{v}v'\right)^2,$$

completing the proof. ♢

We now turn to the general second order equations

$$p(t)u'' + q(t)u' + r(t)u = 0\,. \tag{15.8}$$

Assume that the functions $p(t)$, $q(t)$ and $r(t)$ are differentiable, with $p(t) > 0$ for all t. We divide this equation by $p(t)$

$$u'' + \frac{q(t)}{p(t)}u' + \frac{r(t)}{p(t)}u = 0\,,$$

and then multiply by the integrating factor $a(t) = e^{\int \frac{q(t)}{p(t)}\,dt}$. Denoting $b(t) = a(t)\dfrac{r(t)}{p(t)}$, we arrive at

$$(a(t)u')' + b(t)u = 0\,, \tag{15.9}$$

which is known as the *self-adjoint form* of (15.8).

With the help of his clever identity, M. Picone was able to give the following generalization of the Sturm comparison theorem.

Theorem 2.15.2 *Let $u(t)$ and $v(t)$ be respectively solutions of (15.9), and*

$$(a_1(t)v')' + b_1(t)v = 0\,.$$

Assume that the differentiable functions $a(t)$, $a_1(t)$, and the continuous functions $b(t)$, $b_1(t)$ satisfy

$$b_1(t) \geq b(t), \quad \text{and } \ 0 < a_1(t) \leq a(t) \quad \text{for all } t.$$

In case $a_1(t) = a(t)$ and $b_1(t) = b(t)$ on some interval (t_1, t_2), assume additionally that $u(t)$ and $v(t)$ are not constant multiples of one another on (t_1, t_2). Then $v(t)$ has a root between any two consecutive roots of $u(t)$.

Proof: The proof is similar to that of the Sturm comparison theorem. Let $t_1 < t_2$ be two consecutive roots of $u(t)$,

$$u(t_1) = u(t_2) = 0\,. \tag{15.10}$$

Again, we may assume that $u(t) > 0$ on (t_1, t_2). Assume, *contrary to what we want to prove*, that $v(t)$ has no roots on (t_1, t_2). We may assume that $v(t) > 0$ on (t_1, t_2). Apply Picone's identity to $u(t)$ and $v(t)$. Expressing from the corresponding equations, $(a(t)u')' = -b(t)u$ and $(a_1(t)v')' = -b_1(t)v$, we rewrite Picone's identity as

$$\left[\frac{u}{v}\left(vau' - ua_1v'\right)\right]' = (b_1 - b)u^2 + (a - a_1){u'}^2 + a_1\left(u' - \frac{u}{v}v'\right)^2\,.$$

Integrating this over (t_1, t_2), and using (15.10), gives

$$0 = \int_{t_1}^{t_2} (b_1 - b)u^2 \, dt + \int_{t_1}^{t_2} (a - a_1){u'}^2 \, dt + \int_{t_1}^{t_2} a_1 \left(u' - \frac{u}{v} v' \right)^2 \, dt \, .$$

(In case $v(t_1) = 0$, we have $v'(t_1) > 0$, by the uniqueness of solutions for initial value problems. Then $\lim_{t \to t_1} \frac{u^2}{v} = \lim_{t \to t_1} \frac{2uu'}{v'} = 0$. Similarly, in case $v(t_2) = 0$, the upper limit vanishes for the integral on the left.) The integrals on the right are non-negative. We obtain an immediate contradiction, unless $a_1(t) = a(t)$ and $b_1(t) = b(t)$ for all $t \in (t_1, t_2)$. In such a case, we must also have $u' - \frac{u}{v} v' = 0$ on (t_1, t_2) (so that all three integrals vanish). But then $\frac{u'}{u} = \frac{v'}{v}$, and integrating we see that $u(t)$ and $v(t)$ are constant multiples of one another, contradicting our assumption. ♢

Our next goal is the famous *Lyapunov inequality*. It will follow from the next lemma. Define $q^+(t) = \max(q(t), 0)$, the *positive part* of the function $q(t)$. Similarly, one defines $q^-(t) = \min(q(t), 0)$, the *negative part* of the function $q(t)$. Clearly, $q(t) = q^+(t) + q^-(t)$.

Lemma 2.15.3 *Assume that $u(t)$ is twice continuously differentiable, and it satisfies the following conditions on some interval $(0, b)$ (here $q(t)$ is a given continuous function)*

$$u'' + q(t)u = 0, \quad u(0) = u(b) = 0, \quad u(t) > 0 \ \textit{ on } (0, b) \, . \tag{15.11}$$

Then

$$\int_0^b q^+(t) \, dt > \frac{4}{b} \, .$$

Proof: From (15.11) it follows that

$$u'' + q^+(t)u = -q^-(t)u \geq 0 \, .$$

Let $v(t)$ be the solution of (c is any positive number)

$$v'' + q^+(t)v = 0, \quad v(0) = 0, \quad v'(0) = c > 0 \, . \tag{15.12}$$

By Lemma 2.15.1, $v(t)$ must vanish on $(0, b]$. Let $t_2 \in (0, b]$ be the first root of $v(t)$, so that $v(t) > 0$ on $(0, t_2)$. (In case $q^-(t) \equiv 0$, we have $t_2 = b$, because $v(t)$ is a constant multiple of $u(t)$.) Integrating (15.12) (treating $q^+(t)v$ as a known quantity),

$$v(t) = ct - \int_0^t (t - s)q^+(s)v(s) \, ds, \quad \text{for } t \in [0, t_2] \, . \tag{15.13}$$

From $v(t_2) = 0$, it follows that $c = \dfrac{1}{t_2}\displaystyle\int_0^{t_2} (t_2 - s)q^+(s)v(s)\, ds$. Substituting this back into (15.13), we express

$$v(t) = \frac{t}{t_2}\int_0^{t_2} (t_2 - s)q^+(s)v(s)\, ds - \int_0^t (t - s)q^+(s)v(s)\, ds\,, \quad \text{for } t \in [0, t_2]\,.$$

Breaking the first integral, $\int_0^{t_2} = \int_0^t + \int_t^{t_2}$, we continue:

$$\begin{aligned} t_2 v(t) &= \int_0^t \left[t(t_2 - s) - t_2(t - s)\right] q^+(s)v(s)\, ds + t\int_t^{t_2} (t_2 - s)q^+(s)v(s)\, ds \\ &= (t_2 - t)\int_0^t sq^+(s)v(s)\, ds + t\int_t^{t_2} (t_2 - s)q^+(s)v(s)\, ds\,. \end{aligned}$$

Let t_0 be the point of maximum of $v(t)$ on $(0, t_2)$, $v(t_0) > 0$. Estimate $v(s) < v(t_0)$ in both integrals on the right, then evaluate the last formula at t_0, and cancel $v(t_0)$. Obtain:

$$\begin{aligned} t_2 &< (t_2 - t_0)\int_0^{t_0} sq^+(s)\, ds + t_0\int_{t_0}^{t_2} (t_2 - s)q^+(s)\, ds \\ &< \int_0^{t_0} (t_2 - s)sq^+(s)\, ds + \int_{t_0}^{t_2} s(t_2 - s)q^+(s)\, ds = \int_0^{t_2} (t_2 - s)sq^+(s)\, ds\,. \end{aligned}$$

Dividing by t_2, gives

$$1 < \int_0^{t_2} (1 - \frac{s}{t_2})sq^+(s)\, ds \le \int_0^b (1 - \frac{s}{b})sq^+(s)\, ds < \int_0^b \frac{b}{4}q^+(s)\, ds\,,$$

which implies our inequality. (On the last step we estimated the function $(1 - \dfrac{s}{b})s$ by its maximum value of $\dfrac{b}{4}$.) ♢

Theorem 2.15.3 *(Lyapunov's inequality) If a non-trivial solution of the equation*

$$u'' + q(t)u = 0$$

has two roots on an interval $[a, b]$*, then*

$$\int_a^b q^+(t)\, dt > \frac{4}{b - a}\,.$$

Proof: Let t_1 and t_2 be two consecutive roots of $u(t)$, $a \le t_1 < t_2 \le b$. We may assume that $u(t) > 0$ on (t_1, t_2), and use the above lemma

$$\int_a^b q^+(t)\,dt \ge \int_{t_1}^{t_2} q^+(t)\,dt > \frac{4}{t_2 - t_1} \ge \frac{4}{b-a}\,.$$

(We may declare the point t_1 to be the origin, to use the above lemma.) $\diamondsuit$

Remarkably, the constant 4 appears in another well-known and useful inequality.

Theorem 2.15.4 *(Hardy's inequality) Assume that $f(x)$ is a continuously differentiable function on $[0,b]$, where $b > 0$ is arbitrary, and $f(0) = 0$. Then*

$$\int_0^b \frac{f^2(x)}{x^2}\,dx \le 4\int_0^b f'^2(x)\,dx\,.$$

Proof: Write

$$f'^2 = \left[\tfrac{1}{2}x^{-1}f + x^{1/2}\left(x^{-1/2}f\right)'\right]^2$$
$$= \tfrac{1}{4}x^{-2}f^2 + \left(x^{-1/2}f\right)\left(x^{-1/2}f\right)' + \left(x^{1/2}\left(x^{-1/2}f\right)'\right)^2.$$

Integrating both sides, and dropping a non-negative term on the right

$$\int_0^b f'^2(x)\,dx \ge \int_0^b \frac{1}{4}x^{-2}f^2\,dx + \int_0^b \left(x^{-1/2}f\right)\left(x^{-1/2}f\right)'\,dx \ge \frac{1}{4}\int_0^b x^{-2}f^2\,dx\,,$$

because $\int_0^b \left(x^{-1/2}f\right)\left(x^{-1/2}f\right)'\,dx = \frac{1}{2}x^{-1}f^2(x)\big|_0^b = \frac{1}{2}b^{-1}f^2(b) \ge 0$. (Observe that $\lim_{x\to 0} x^{-1}f^2(x) = \lim_{x\to 0}\frac{f(x)}{x}\cdot\lim_{x\to 0} f(x) = f'(0)\cdot 0 = 0$.) $\diamondsuit$

Chapter 3

Using Infinite Series to Solve Differential Equations

"Most" differential equations cannot be solved by a formula. One traditional approach involves using infinite series to approximate solutions near some point a. (Another possibility is to use numerical methods, which is discussed in Chapters 1 and 9.) We begin with the case when the point a is *regular*, and it is possible to compute all derivatives of solutions at $x = a$, and then write down the corresponding Taylor's series. Turning to singular a, we distinguish the easier case when a is a simple root of the leading coefficient (we call such equations *mildly singular*). Then we show that the case when a is a double root of the leading coefficient can often be reduced to a mildly singular case, by a change of variables.

3.1 Series Solution Near a Regular Point

3.1.1 Maclauren and Taylor Series

Infinitely differentiable functions can often be represented by a series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots . \tag{1.1}$$

Letting $x = 0$, we see that $a_0 = f(0)$. Differentiating (1.1), and then letting $x = 0$, shows that $a_1 = f'(0)$. Differentiating (1.1) twice, and then letting $x = 0$, gives $a_2 = \dfrac{f''(0)}{2}$. Continuing this way, we see that $a_n = \dfrac{f^{(n)}(0)}{n!}$, giving us the Maclauren series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \,.$$

It is known that for each $f(x)$ there is a number R, so that the Maclauren series converges for x inside the interval $(-R, R)$, and diverges outside of this interval, when $|x| > R$. We call R *the radius of convergence*. For some $f(x)$, we have $R = \infty$ (for example, for $\sin x$, $\cos x$, e^x), while for some series $R = 0$, and in general $0 \le R \le \infty$.

Computing the Maclauren series for some specific functions, gives:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \,,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,,$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \,.$$

The last series, called *the geometric series*, converges on the interval $(-1, 1)$, so that $R = 1$.

Maclauren's series gives an approximation of $f(x)$, for x close to zero. For example, $\sin x \approx x$ gives a reasonably good approximation for $|x|$ small. If we add one more term of the Maclauren series: $\sin x \approx x - \frac{x^3}{6}$, then, say on the interval $(-1, 1)$, we get an excellent approximation, see Figure 3.1.

If one needs the Maclauren series for $\sin x^2$, one begins with a series for $\sin x$, and then replaces each x by x^2, obtaining

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \,.$$

One can split Maclauren's series into a sum of series with either even or odd powers:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1} \,.$$

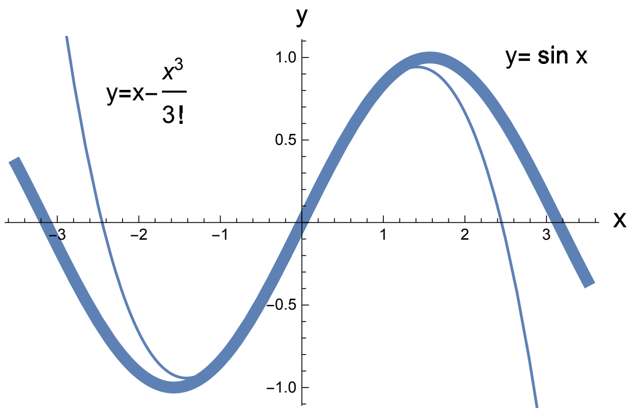


Figure 3.1: The approximation of $y = \sin x$ by $y = x - \dfrac{x^3}{6}$ near $x = 0$

In the following series only the odd powers have non-zero coefficients

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} x^n &= \frac{2}{1}x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1} = 2\sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1} \,.\end{aligned}$$

All of the series above were *centered at* 0. One can replace zero by any number a, obtaining Taylor's series

$$\begin{aligned}f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \\ = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \,.\end{aligned}$$

It converges on some interval $(a-R, a+R)$, centered at a. The radius of convergence satisfies $0 \le R \le \infty$, as before. Taylor's series allows us to approximate $f(x)$, for x close to a. For example, one can usually expect (but this is not always true) that $f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$, for x close to 2, say for $1.8 < x < 2.2$.

Clearly, $\sum\limits_{n=1}^{\infty} a_n x^n = \sum\limits_{m=1}^{\infty} a_m x^m$, so that the index n can be regarded as a "dummy" index of summation. It is often desirable to put a series like

$\sum_{n=1}^{\infty} \frac{n}{n+4} x^{n+1}$ into the form $\sum a_n x^n$. We set $n+1=m$, or $n=m-1$, and get

$$\sum_{n=1}^{\infty} \frac{n}{n+4} x^{n+1} = \sum_{m=2}^{\infty} \frac{m-1}{m+3} x^m = \sum_{n=2}^{\infty} \frac{n-1}{n+3} x^n \, .$$

The same result can be accomplished in one step, by the *shift of the index of summation*: $n \to n-1$, or replacing each occurrence of n by $n-1$, and incrementing by 1 the limit(s) of summation.

3.1.2 A Toy Problem

Let us begin with the equation (here $y = y(x)$)

$$y'' + y = 0 \, ,$$

for which we already know the general solution. Let us denote by $y_1(x)$ the solution of the initial value problem

$$y'' + y = 0, \;\; y(0) = 1, \; y'(0) = 0 \, . \tag{1.2}$$

By $y_2(x)$ we denote the solution of the same equation, together with the initial conditions $y(0) = 0$, $y'(0) = 1$. Clearly, $y_1(x)$ and $y_2(x)$ are not constant multiples of each other. Therefore, they form a fundamental set, giving us the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Let us now compute $y_1(x)$, the solution of (1.2). From the initial conditions, we already know the first two terms of its Maclauren series

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2} x^2 + \cdots + \frac{y^{(n)}(0)}{n!} x^n + \cdots \, .$$

To get more terms, we need to compute the derivatives of $y(x)$ at zero. From the equation (1.2), $y''(0) = -y(0) = -1$. We now differentiate the equation (1.2), getting $y''' + y' = 0$, and then set $x = 0$ to obtain

$$y'''(0) = -y'(0) = 0 \, .$$

Differentiating again, gives $y'''' + y'' = 0$, and setting $x = 0$,

$$y''''(0) = -y''(0) = 1 \, .$$

On the next step:

$$y^{(5)}(0) = -y'''(0) = 0 \, .$$

We see that all derivatives of odd order vanish at $x = 0$, while the derivatives of even order alternate between 1 and -1. The Maclauren series is then

$$y_1(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \cos x\,.$$

Similarly, we compute the series representation for $y_2(x)$:

$$y_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sin x\,.$$

We shall solve the equations with variable coefficients

$$P(x)y'' + Q(x)y' + R(x)y = 0\,, \tag{1.3}$$

where continuous functions $P(x)$, $Q(x)$ and $R(x)$ are given. We shall always denote by $y_1(x)$ the solution of (1.3) satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$, and by $y_2(x)$ the solution of (1.3) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$. If one needs to solve (1.3), together with the given initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta\,,$$

then the solution is

$$y(x) = \alpha y_1(x) + \beta y_2(x)\,.$$

Indeed, $y(0) = \alpha y_1(0) + \beta y_2(0) = \alpha$, and $y'(0) = \alpha y_1'(0) + \beta y_2'(0) = \beta$.

3.1.3 Using Series When Other Methods Fail

Let us try to find the general solution of the equation

$$y'' + xy' + 2y = 0\,.$$

This equation has variable coefficients, and none of the previously considered methods will apply here. Our goal is to use the Maclauren series $\sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}\, x^n$ to approximate solutions near $x = 0$.

We shall derive a formula for $y^{(n)}(0)$, and use it to calculate the solutions $y_1(x)$ and $y_2(x)$, defined in the preceding subsection. From the equation we express $y''(0) = -2y(0)$. Differentiate the equation

$$y''' + xy'' + 3y' = 0\,,$$

which gives $y'''(0) = -3y'(0)$. Differentiate the last equation again

$$y'''' + xy''' + 4y'' = 0\,,$$

and get: $y''''(0) = -4y''(0)$. In general, we get a *recurrence relation*

$$y^{(n)}(0) = -ny^{(n-2)}(0), \quad n = 2, 3, \ldots . \tag{1.4}$$

By a convention, $y^{(0)}(x) = y(x)$, so that $y^{(0)}(0) = y(0)$.

Let us begin with the computation of $y_2(x)$, for which we use the initial conditions $y(0) = 0$, $y'(0) = 1$. Then, using the recurrence relation (1.4), obtain

$$\begin{aligned} y''(0) &= -2y(0) = 0\,, \\ y'''(0) &= -3y'(0) = -3 \cdot 1\,, \\ y''''(0) &= -4y''(0) = 0\,. \end{aligned}$$

It is clear that all derivatives of even order are zero at $x = 0$. Let us continue with the derivatives of odd order:

$$\begin{aligned} y^{(5)}(0) &= -5y'''(0) = (-1)^2\, 5 \cdot 3 \cdot 1\,, \\ y^{(7)}(0) &= -7y'''(0) = (-1)^3\, 7 \cdot 5 \cdot 3 \cdot 1\,. \end{aligned}$$

And in general,

$$y^{(2n+1)}(0) = (-1)^n\, (2n+1) \cdot (2n-1) \cdots 3 \cdot 1\,.$$

Then the Maclauren series for $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n = x + \sum_{n=1}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} \\ &= x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1) \cdot (2n-1) \cdots 3 \cdot 1}{(2n+1)!} x^{2n+1} \\ &= x + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n \cdot (2n-2) \cdots 4 \cdot 2}\, x^{2n+1}\,. \end{aligned}$$

One can also write this solution as $y_2(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!}\, x^{2n+1}$.

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$, $y'(0) = 0$. Similarly to the above, we see from the recurrence relation that all derivatives of odd order vanish at $x = 0$, while the even ones satisfy

$$y^{(2n)}(0) = (-1)^n\, 2n \cdot (2n-2) \cdots 4 \cdot 2\,, \quad \text{for } n = 1, 2, 3, \ldots .$$

This leads to

$$y_1(x) = 1+\sum_{n=1}^{\infty}(-1)^n \frac{y^{(2n)}(0)}{(2n)!}\, x^{2n} = 1+\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}\, x^{2n} \,.$$

The general solution:

$$y(x) = c_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n-3)\cdots 3\cdot 1}x^{2n}\right)+c_2\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!}\,x^{2n+1}\,.$$

We shall need a formula for repeated differentiation of a product of two functions. Starting with the product rule $(fg)' = f'g + fg'$, express

$$(fg)'' = f''g + 2f'g' + fg'' \,,$$
$$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg''' \,,$$

and in general, for the n-th derivative:

$$(1.5) \qquad (fg)^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2}f^{(n-2)}g'' + \cdots$$
$$+\frac{n(n-1)}{2}f''g^{(n-2)} + nf'g^{(n-1)} + fg^{(n)} \,.$$

(Convention: $f^{(0)} = f$.) This formula is similar to the binomial formula for the expansion of $(x+y)^n$. Using the summation notation, we can write it as

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}\, g^{(k)} \,,$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are the *binomial coefficients*.

The formula (1.5) simplifies considerably in case $f(x) = x$, or if $f(x) = x^2$:

$$(xg)^{(n)} = ng^{(n-1)} + xg^{(n)} \,,$$
$$(x^2g)^{(n)} = n(n-1)g^{(n-2)} + 2nxg^{(n-1)} + x^2g^{(n)} \,.$$

We shall use Taylor's series centered at $x = a$, $y(x) = \sum_{n=0}^{\infty}\frac{y^{(n)}(a)}{n!}(x-a)^n$, to solve linear second order equations with variable coefficients

$$P(x)y'' + Q(x)y' + R(x)y = 0 \,,$$

where the functions $P(x)$, $Q(x)$ and $R(x)$ are given. A number a is called a *regular point* if $P(a) \neq 0$. If $P(a) = 0$, then $x = a$ is called a *singular point*. If a is a regular point, we can compute $y^{(n)}(a)$ from the equation, as in the examples above. If the point a is singular, it is not even possible to compute $y''(a)$ from the equation.

Example 1 $(2 + x^2)y'' - xy' + 4y = 0$.

For this equation, any a is a regular point. Let us find the general solution as an infinite series, centered at $a = 0$, the Maclauren series for $y(x)$. We differentiate both sides of this equation n times. When we use the formula (1.5) to differentiate the first term, only the last three terms are non-zero, because the derivatives of $2+x^2$, of order three and higher, are zero. Obtain

$$\left[(2 + x^2)y''\right]^{(n)} = \frac{n(n-1)}{2} 2y^{(n)} + n(2x)y^{(n+1)} + (2 + x^2)y^{(n+2)} .$$

When we differentiate n times xy', only the last two terms survive, giving

$$\left[xy'\right]^{(n)} = ny^{(n)} + xy^{(n+1)} .$$

It follows that n differentiations of our equation produce

$$\frac{n(n-1)}{2} 2y^{(n)} + n(2x)y^{(n+1)} + (2 + x^2)y^{(n+2)} - ny^{(n)} - xy^{(n+1)} + 4y^{(n)} = 0 .$$

Set here $x = 0$. Several terms vanish. Combining the like terms, we get

$$2y^{(n+2)}(0) + \left(n^2 - 2n + 4\right) y^{(n)}(0) = 0 ,$$

which gives us the recurrence relation

$$y^{(n+2)}(0) = -\frac{(n^2 - 2n + 4)}{2}\, y^{(n)}(0) .$$

This relation is too involved to get a general formula for $y^{(n)}(0)$ as a function of n. However, it can be used to crank out the derivatives at zero, as many as you wish.

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$. Setting $n = 0$ in the recurrence relation, obtain

$$y''(0) = -2y(0) = -2 .$$

When $n = 2$,

$$y''''(0) = -2y''(0) = 4\,.$$

Using these derivatives in the Maclauren series, gives

$$y_1(x) = 1 - x^2 + \frac{1}{6}x^4 + \cdots\,.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero. When $n = 1$, we get

$$y'''(0) = -\frac{3}{2}y'(0) = -\frac{3}{2}\,.$$

Setting $n = 3$, obtain

$$y^{(5)}(0) = -\frac{7}{2}y'''(0) = \frac{21}{4}\,.$$

Using these derivatives in the Maclauren series, we conclude

$$y_2(x) = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots\,.$$

The general solution:

$$\begin{aligned} y(x) = c_1y_1(x) + c_2y_2(x) = c_1 \left(1 - x^2 + \frac{1}{6}x^4 + \cdots\right) \\ +c_2 \left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots\right)\,. \end{aligned}$$

Suppose that we wish to solve the above equation, together with the initial conditions: $y(0) = -2$, $y'(0) = 3$. Then $y(0) = c_1y_1(0) + c_2y_2(0) = c_1 = -2$, and $y'(0) = c_1y_1'(0) + c_2y_2'(0) = c_2 = 3$. It follows that $y(x) = -2y_1(x) + 3y_2(x)$. If one needs to approximate $y(x)$ near $x = 0$, say on the interval $(-0.3, 0.3)$, then

$$y(x) \approx -2\left(1 - x^2 + \frac{1}{6}x^4\right) + 3\left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5\right)$$

will provide an excellent approximation.

Example 2 $\quad y'' - xy' + y = 0$, $a = 0$.

Differentiating this equation n times gives

$$y^{(n+2)}(x) - ny^{(n)}(x) - xy^{(n+1)}(x) + y^{(n)}(x) = 0\,.$$

Setting here $x = 0$, we obtain the recurrence relation

$$y^{(n+2)}(0) = (n-1)y^{(n)}(0)\,.$$

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$, $y'(0) = 0$. Then all derivatives of odd order vanish, $y^{(2n+1)}(0) = 0$, as follows by repeated application of the recurrence relation. We now compute the derivatives of even order. Setting $n = 0$ in the recurrence relation gives: $y''(0) = -y(0) = -1$. When $n = 2$, we get $y''''(0) = y''(0) = -1$, and then $y^{(6)}(0) = 3y''''(0) = -1 \cdot 3$, at $n = 4$. We continue, $y^{(8)}(0) = 5y^{(6)}(0) = -1 \cdot 3 \cdot 5$, and in general

$$y^{(2n)}(0) = -1 \cdot 3 \cdot 5 \cdots (2n-3)\,.$$

Then

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{(2n)!} x^{2n} = 1 - \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdots 2n(2n-1)} x^{2n}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{1}{2^n\, n!(2n-1)} x^{2n} = -\sum_{n=0}^{\infty} \frac{1}{2^n\, n!(2n-1)} x^{2n}\,.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$, $y'(0) = 1$. Then all derivatives of even order vanish, $y^{(2n)}(0) = 0$, as follows by repeated application of the recurrence relation. Setting $n = 1$ in the recurrence relation gives: $y'''(0) = 0$, and so all derivatives of odd order vanish as well. We conclude that $y_2(x) = x$. The general solution is

$$y(x) = c_1 \sum_{n=0}^{\infty} \frac{1}{2^n\, n!(2n-1)} x^{2n} + c_2 x\,.$$

Example 3 Approximate the general solution of *Airy's equation*

$$y'' - xy = 0\,,$$

near $x = 1$. This equation was encountered in 1838 by G.B. Airy, in his study of optics.

We need to compute Taylor's series about the regular point $a = 1$, which is $y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n$. From the equation,

$$y''(1) = y(1)\,.$$

To get higher derivatives, we differentiate our equation n times, and then set $x = 1$, to get the recurrence relation

$$y^{(n+2)}(x) - ny^{(n-1)}(x) - xy^{(n)}(x) = 0\,,$$

$$y^{(n+2)}(1) = ny^{(n-1)}(1) + y^{(n)}(1), \quad n = 1, 2, \ldots. \tag{1.6}$$

To compute $y_1(x)$, we use the initial conditions $y(1) = 1$, $y'(1) = 0$. Then

$$y''(1) = y(1) = 1\,.$$

Setting $n = 1$ in the recurrence relation (1.6), gives

$$y^{(3)}(1) = y(1) + y'(1) = 1\,.$$

When $n = 2$,

$$y^{(4)}(1) = 2y'(1) + y''(1) = 1\,.$$

Then, for $n = 3$,

$$y^{(5)}(1) = 3y''(1) + y'''(1) = 4\,.$$

Obtain

$$y_1(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots.$$

Again, it does not seem possible to get a general formula for the coefficients.

To compute $y_2(x)$, we use the initial conditions $y(1) = 0$, $y'(1) = 1$. Then $y''(1) = y(1) = 0$. Setting $n = 1$ in the recurrence relation (1.6), gives $y^{(3)}(1) = y(1) + y'(1) = 1$. When $n = 2$, $y^{(4)}(1) = 2y'(1) + y''(1) = 2$. Then, for $n = 3$, $y^{(5)}(1) = 3y''(1) + y'''(1) = 1$. Obtain

$$y_2(x) = x - 1 + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots.$$

The general solution is, of course, $y(x) = c_1y_1(x) + c_2y_2(x)$.

3.2 Solution Near a Mildly Singular Point

We consider again the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0\,,$$

with given functions $P(x)$, $Q(x)$ and $R(x)$ that are continuous near a point a, at which we wish to compute solution as a series $y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(x-a)^n$.

If $P(a) = 0$, we have a problem: one cannot compute $y''(a)$ from the equation (and the same problem occurs for higher derivatives). However, if a is a simple root of $P(x)$, it turns out that one can still use series to produce a solution. Namely, we assume that $P(x) = (x-a)P_1(x)$, with $P_1(a) \neq 0$. We call $x = a$ a *mildly singular point*. Dividing the equation by $P_1(x)$, and calling $q(x) = \frac{Q(x)}{P_1(x)}$, $r(x) = \frac{R(x)}{P_1(x)}$, we put it into the form

$$(x-a)y'' + q(x)y' + r(x)y = 0\,.$$

The functions $q(x)$ and $r(x)$ are continuous near a. In case $a = 0$, the equation becomes

$$xy'' + q(x)y' + r(x)y = 0\,. \tag{2.1}$$

For this equation we cannot expect to obtain two linearly independent solutions, by prescribing $y_1(0) = 1$, $y_1'(0) = 0$, and $y_2(0) = 0$, $y_2'(0) = 1$, the way we did before. This equation is singular at $x = 0$ (the functions $\frac{q(x)}{x}$ and $\frac{r(x)}{x}$ are discontinuous at $x = 0$, and so the existence and uniqueness Theorem 2.4.1 from Chapter 2 does not apply).

Example 1 Let us try to solve:

$$xy'' - y' = 0, \;\; y(0) = 0, \, y'(0) = 1\,.$$

Multiplying through by x, we obtain Euler's equation with the general solution $y(x) = c_1x^2 + c_2$. Then $y'(x) = 2c_1x$, and $y'(0) = 0 \neq 1$. This initial value problem has no solution. However, if we change the initial conditions, and consider the problem

$$xy'' - y' = 0, \;\; y(0) = 1, \, y'(0) = 0\,,$$

then there are infinitely many solutions $y = 1 + c_1x^2$.

We therefore lower our expectations, and we shall be satisfied to compute just one series solution of (2.1). It turns out that in most cases it is possible to calculate a series solution of the form $y(x) = \sum_{n=0}^{\infty} a_nx^n$, starting with $a_0 = 1$, which corresponds to $y(0) = 1$. (If $a_0 \neq 1$, then $\frac{1}{a_0}y(x)$ is another solution of (2.1), which begins with 1. So that we shall always assume that $a_0 = 1$. The possibility of $a_0 = 0$ is considered later.)

Example 2 Find a series solution, centered at $a = 0$, of

$$xy'' + 3y' - 2y = 0\,.$$

It is convenient to multiply this equation by x:

$$x^2y'' + 3xy' - 2xy = 0\,. \tag{2.2}$$

Look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\,,$$

with $a_0 = 1$. Calculate

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots\,,$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + \cdots\,.$$

Observe that each differentiation "kills" a term. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation (2.2), gives

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0\,. \tag{2.3}$$

The third series is not "lined up" with the other two. We therefore shift the index of summation, replacing n by $n-1$ in that series, obtaining

$$\sum_{n=0}^{\infty} 2a_n x^{n+1} = \sum_{n=1}^{\infty} 2a_{n-1}x^n\,.$$

Then (2.3) becomes

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0\,. \tag{2.4}$$

We shall use the following fact: if $\sum_{n=1}^{\infty} b_n x^n = 0$ for all x, then $b_n = 0$ for all $n = 1, 2, \ldots$. Our goal is to combine the three series in (2.4) into a single one, so that we can set all of the resulting coefficients to zero. The x term is present in the second and the third series, but not in the first. However, we can start the first series at $n = 1$, because at $n = 1$ the coefficient is zero. So that (2.4) becomes

$$\sum_{n=1}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0\,.$$

Now for all $n \geq 1$, the x^n term is present in all three series, so that we can combine these series into one series. We therefore just set the sum of the coefficients to zero

$$a_n n(n-1) + 3a_n n - 2a_{n-1} = 0\,.$$

Solve for a_n, to get the recurrence relation

$$a_n = \frac{2}{n(n+2)} a_{n-1}\,, \quad n \geq 1\,.$$

Starting with $a_0 = 1$, compute $a_1 = \dfrac{2}{1 \cdot 3}$, then

$$a_2 = \frac{2}{2 \cdot 4}\, a_1 = \frac{2^2}{(1 \cdot 2)(3 \cdot 4)} = \frac{2^3}{2!\, 4!}\,,$$

$$a_3 = \frac{2}{3 \cdot 5}\, a_2 = \frac{2^4}{3!\, 5!}\,,$$

and, in general, $a_n = \dfrac{2^{n+1}}{n!\,(n+2)!}$.

Answer: $y(x) = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n!\,(n+2)!}\, x^n = \sum_{n=0}^{\infty} \frac{2^{n+1}}{n!\,(n+2)!}\, x^n$.

Example 3 Find a series solution, centered at $a = 0$, of

$$xy'' - 3y' - 2y = 0\,.$$

This equation is a small modification of the preceding one, so that we can quickly derive the recurrence relation:

$$a_n = \frac{2}{n(n-4)} a_{n-1}\,, \quad n \geq 1\,. \tag{2.5}$$

If we start with $a_0 = 1$, and proceed as before, then at $n = 4$ the denominator is zero, and the computation stops! To avoid the trouble at $n = 4$, we look for the solution in the form $y = \displaystyle\sum_{n=4}^{\infty} a_n x^n$. Substituting this series into the equation

$$x^2 y'' - 3xy' - 2xy = 0$$

(which is the original equation, multiplied by x), gives

$$\sum_{n=4}^{\infty} a_n n(n-1) x^n - \sum_{n=4}^{\infty} 3a_n n x^n - \sum_{n=4}^{\infty} 2a_n x^{n+1} = 0\,.$$

The coefficient in x^4, which is $a_4(4 \cdot 3 - 3 \cdot 4)$, is zero for any choice of a_4. We can then begin the first two series at $n = 5$:

$$\sum_{n=5}^{\infty} a_n n(n-1)x^n - \sum_{n=5}^{\infty} 3a_n n x^n - \sum_{n=4}^{\infty} 2a_n x^{n+1} = 0\,. \tag{2.6}$$

Shifting $n \to n-1$ in the last series in (2.6), we see that the recurrence relation (2.5) holds for $n \geq 5$. We choose $a_4 = 1$, and use the recurrence relation (2.5) to calculate a_5, a_6, etc.

Compute: $a_5 = \dfrac{2}{5 \cdot 1} a_4 = \dfrac{2}{5 \cdot 1}$, $a_6 = \dfrac{2}{6 \cdot 2} a_5 = \dfrac{2^2}{6 \cdot 5 \cdot 2 \cdot 1} = 24\,\dfrac{2^2}{6!2!}$, and in general, $a_n = 24\,\dfrac{2^{n-4}}{n!(n-4)!}$.

Answer: $y(x) = x^4 + 24 \displaystyle\sum_{n=5}^{\infty} \frac{2^{n-4}}{n!(n-4)!}\, x^n = 24 \sum_{n=4}^{\infty} \frac{2^{n-4}}{n!(n-4)!}\, x^n$.

Our experience with the previous two problems can be summarized as follows (convergence of the series is proved in more advanced books).

Theorem 3.2.1 *Consider the equation (2.1). Assume that the functions $q(x)$ and $r(x)$ have convergent Maclauren series expansions on some interval $(-\delta, \delta)$. If $q(0)$ is not a non-positive integer ($q(0)$ is not equal to $0, -1, -2, \ldots$), one can find a series solution of the form $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, starting with $a_0 = 1$. In case $q(0) = -k$, where k is a non-negative integer, one can find a series solution of the form $y(x) = \displaystyle\sum_{n=k+1}^{\infty} a_n x^n$, starting with $a_{k+1} = 1$. In both cases, the series for $y(x)$ is convergent on $(-\delta, \delta)$.*

We now turn to one of the most important examples of this chapter.

Example 4 $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$.

This is the *Bessel equation*, which is of great importance in mathematical physics! It depends on a real parameter ν. It is also called *Bessel's equation of order ν*. Its solutions are called *Bessel's functions of order ν*. We see that $a = 0$ is not a mildly singular point, for $\nu \neq 0$. (Zero is a double root of x^2.) But in case $\nu = 0$, one can cancel x, putting Bessel's equation of order zero into the form

$$x y'' + y' + x y = 0\,, \tag{2.7}$$

so that $a = 0$ is a mildly singular point. We shall find the solution of this equation, as a series centered at $a = 0$, the Maclauren series for $y(x)$.

We put the equation (2.7) back into the form

$$x^2y'' + xy' + x^2y = 0\,,$$

and look for solution in the form $y = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$. Substitute this in:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0\,.$$

In the last series we replace n by $n-2$:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0\,.$$

None of the three series has a constant term. The x term is present only in the second series. Its coefficient is a_1, and so

$$a_1 = 0\,.$$

The terms involving x^n, starting with $n = 2$, are present in all series, so that (after combining the series)

$$a_n n(n-1) + a_n n + a_{n-2} = 0\,,$$

giving

$$a_n = -\frac{1}{n^2}\,a_{n-2}\,.$$

This recurrence relation tells us that all odd coefficients are zero, $a_{2n+1} = 0$. Starting with $a_0 = 1$, compute $a_2 = -\frac{1}{2^2}\,a_0 = -\frac{1}{2^2}$, $a_4 = -\frac{1}{4^2}\,a_2 = (-1)^2\frac{1}{2^2\,4^2}$, $a_6 = -\frac{1}{6^2}\,a_4 = (-1)^3\frac{1}{2^2\,4^2\,6^2}$, and in general,

$$a_{2n} = (-1)^n \frac{1}{2^2\,4^2\,6^2 \cdots (2n)^2} = (-1)^n \frac{1}{(2\cdot 4\cdot 6 \cdots 2n)^2} = (-1)^n \frac{1}{2^{2n}(n!)^2}\,.$$

We then have

$$y(x) = 1 + \sum_{n=1}^{\infty} a_{2n}x^{2n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^{2n}(n!)^2} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(n!)^2} x^{2n}\,.$$

We obtained *Bessel's function of order zero of the first kind*, with the customary notation: $J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}\,(n!)^2}\,x^{2n}$.

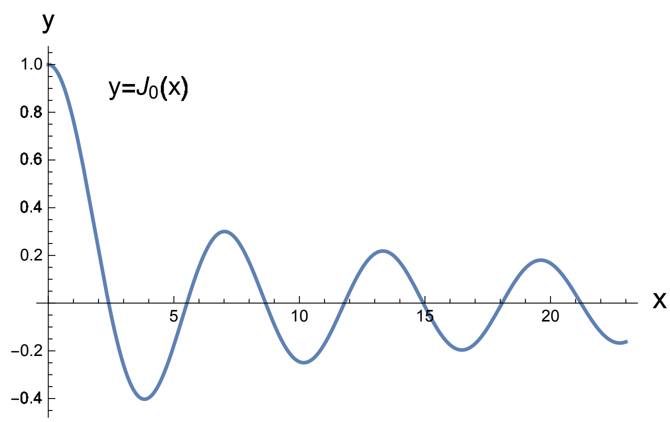


Figure 3.2: The graph of Bessel's function $J_0(x)$

3.2.1* Derivation of $J_0(x)$ by Differentiation of the Equation

It turns out that one can obtain $J_0(x)$ by differentiating the equation, and using the initial conditions $y(0) = 1$ and $y'(0) = 0$, even though the point $a = 0$ is singular.

Differentiate the equation (2.7) n times, and then set $x = 0$:

$$ny^{(n+1)} + xy^{(n+2)} + y^{(n+1)} + ny^{(n-1)} + xy^{(n)} = 0\,;$$

$$ny^{(n+1)}(0) + y^{(n+1)}(0) + ny^{(n-1)}(0) = 0\,.$$

(It is not always true that $xy^{(n+2)} \to 0$ as $x \to 0$. However, in case of the initial conditions $y(0) = 1$, $y'(0) = 0$ that is true, as was justified in author's paper [16].) We get the recurrence relation

$$y^{(n+1)}(0) = -\frac{n}{n+1} y^{(n-1)}(0)\,.$$

We use the initial conditions $y(0) = 1$, $y'(0) = 0$. Then all derivatives of odd order vanish, while

$$y^{(2n)}(0) = -\frac{2n-1}{2n} y^{(2n-2)}(0) = (-1)^2 \frac{2n-1}{2n} \frac{2n-3}{2n-4} y^{(2n-4)}(0) = \ldots$$

$$= (-1)^n \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n(2n-2)\cdots 2} y(0) = (-1)^n \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n\, n!}\,.$$

Then

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(2n)}(0)}{(2n)!}x^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n}\,(n!)^2}\,x^{2n}\,.$$

We obtained again Bessel's function of order zero of the first kind, $J_0(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n}\,(n!)^2}\,x^{2n}$.

In case of the initial conditions $y(0)=0$ and $y'(0)=1$, there is no solution of Bessel's equation (2.7) (the recurrence relation above is not valid, because the relation $xy^{(n+2)}\to 0$ as $x\to 0$ is not true in this case). In fact, the second solution of Bessel's equation cannot possibly be continuously differentiable at $x=0$. Indeed, by the Theorem 2.4.2 from Chapter 2, the Wronskian of any two solutions is equal to $ce^{-\int\frac{1}{x}\,dx}=\frac{c}{x}$, so that $W(y_1,y_2)=y_1y_2'-y_1'y_2=\frac{c}{x}$. The solution $J_0(x)$ satisfies $J_0(0)=1$, $J_0'(0)=0$. Therefore, the other solution, or its derivative, must be discontinuous at $x=0$. It turns out that the other solution, called *Bessel's function of the second type*, and denoted $Y_0(x)$, has a term involving $\ln x$ in its series representation, see the book of J. Bowman [3] for a concise introduction.

Bessel's function of order ν of the first kind is denoted by $J_\nu(x)$. Similarly to $J_0(x)$, the function $J_\nu(x)$ is continuous at $x=0$, and it has an infinite sequence of roots, tending to infinity, see [3].

3.3 Moderately Singular Equations

This section deals only with series centered at zero, so that $a=0$, and the general case is similar. We consider the equation

$$x^2y''+xp(x)y'+q(x)y=0\,, \tag{3.1}$$

where the given functions $p(x)$ and $q(x)$ are assumed to be infinitely differentiable functions, that can be represented by their Maclauren series

$$\begin{aligned} p(x)&=p(0)+p'(0)x+\tfrac{1}{2}p''(0)x^2+\cdots \\ q(x)&=q(0)+q'(0)x+\tfrac{1}{2}q''(0)x^2+\cdots\,. \end{aligned} \tag{3.2}$$

(Observe the special form of the coefficient function in front of y'.) If it so happens that $q(0)=0$, then $q(x)$ has a factor of x, and one can divide the equation (3.1) by x, to obtain a mildly singular equation. So that the difference with the preceding section is that we now allow the case of

$q(0) \neq 0$. Observe also that in case $p(x)$ and $q(x)$ are constants, the equation (3.1) is Euler's equation, that was studied in Section 2.13. This connection with Euler's equation is the "guiding light" of the theory that follows.

We change to a new unknown function $v(x)$, by letting $y(x) = x^r v(x)$, with a constant r to be specified. With $y' = rx^{r-1}v + x^r v'$, and $y'' = r(r-1)x^{r-2}v + 2rx^{r-1}v' + x^r v''$, we substitute $y(x)$ into (3.1), obtaining

$$x^{r+2}v'' + x^{r+1}v'\,(2r + p(x)) + x^r v\,[r(r-1) + rp(x) + q(x)] = 0\,. \tag{3.3}$$

Now choose r to satisfy the following *characteristic equation*

$$r(r-1) + rp(0) + q(0) = 0\,. \tag{3.4}$$

(We shall only consider the case when this quadratic equation has two real and distinct roots.) In view of (3.2), the quantity in the square bracket in (3.3) then becomes

$$rp'(0)x + r\frac{1}{2}p''(0)x^2 + \cdots + q'(0)x + \frac{1}{2}q''(0)x^2 + \cdots\,,$$

so that it has a factor of x. We take this factor out, and divide the equation (3.3) by x^{r+1}, obtaining a mildly singular equation. We conclude that the substitution $y(x) = x^r v(x)$, with r being a root of (3.4), produces a mildly singular equation for $v(x)$, that was analyzed in the preceding section.

If r_1 and r_2 are real roots of the characteristic equation (3.4), then we get solutions of (3.1) in the form $y_1 = x^{r_1}v_1(x)$ and $y_2 = x^{r_2}v_2(x)$, where $v_1(x)$ and $v_2(x)$ are solutions of the corresponding mildly singular equations. This approach will produce a fundamental set of (3.1), except if r_1 and r_2 are either the same or differ by an integer. In those cases $y_1(x)$ and $y_2(x)$ may coincide, and one needs a different method to construct another solution, see the book of W.E. Boyce and R.C. DiPrima [4].

Example 1 Solve $2x^2y'' - xy' + (1+x)y = 0$.

To put this equation into the right form (3.1), divide it by 2:

$$x^2y'' - \frac{1}{2}xy' + (\frac{1}{2} + \frac{1}{2}x)y = 0\,. \tag{3.5}$$

Here $p(x) = -\frac{1}{2}$ and $q(x) = \frac{1}{2} + \frac{1}{2}x$. The characteristic equation is then

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0\,.$$

Its roots are $r = \frac{1}{2}$, and $r = 1$.

The case $r = \frac{1}{2}$. We know that the substitution $y = x^{\frac{1}{2}} v$ will produce a mildly singular equation for $v(x)$. Substituting this y into our equation (3.5) (or using (3.3)), gives

$$x^{5/2}v'' + \frac{1}{2}x^{3/2}v' + \frac{1}{2}x^{3/2}v = 0\,.$$

Dividing by $x^{3/2}$, produces a mildly singular equation

$$xv'' + \frac{1}{2}v' + \frac{1}{2}v = 0\,.$$

Multiply this equation by $2x$, for convenience,

$$2x^2v'' + xv' + xv = 0\,, \tag{3.6}$$

and look for a solution in the form $v(x) = \sum_{n=0}^{\infty} a_n x^n$. Substituting $v(x)$ into (3.6), gives

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,. \tag{3.7}$$

To line up the powers, shift $n \to n-1$ in the last series. The first series we may begin at $n = 1$, instead of $n = 2$, because its coefficient at $n = 1$ is zero. Then (3.7) becomes

$$\sum_{n=1}^{\infty} 2a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

Combine these series into a single series, and set its coefficients to zero

$$2a_n n(n-1) + a_n n + a_{n-1} = 0\,,$$

which gives us the recurrence relation

$$a_n = -\frac{1}{n(2n-1)}\, a_{n-1}\,.$$

Starting with $a_0 = 1$, compute $a_1 = -\frac{1}{1 \cdot 1}$, $a_2 = -\frac{1}{2 \cdot 3}\, a_1 = (-1)^2 \frac{1}{(1 \cdot 2)\;(1 \cdot 3)}$, $a_3 = -\frac{1}{3 \cdot 5}\, a_2 = (-1)^3 \frac{1}{(1 \cdot 2 \cdot 3)\;(1 \cdot 3 \cdot 5)}$, and in general

$$a_n = (-1)^n \frac{1}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}\,.$$

We obtained the first solution:

$$y_1(x) = x^{1/2}v(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{1}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}\, x^n\right].$$

The case $r = 1$. Set $y = xv$. Substituting this y into (3.5), and simplifying

$$x^3v'' + \frac{3}{2}x^2v' + \frac{1}{2}x^2v = 0\,.$$

Dividing by x^2, gives a mildly singular equation

$$xv'' + \frac{3}{2}v' + \frac{1}{2}v = 0\,.$$

Multiply this equation by $2x$, for convenience,

$$2x^2v'' + 3xv' + xv = 0\,, \tag{3.8}$$

and look for a solution in the form $v = \sum_{n=0}^{\infty} a_n x^n$. Substituting $v(x)$ into (3.8), obtain

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

We start the first series at $n = 1$, and make a shift $n \to n-1$ in the third series:

$$\sum_{n=1}^{\infty} 2a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

Setting the coefficient of x^n to zero,

$$2a_n n(n-1) + 3a_n n + a_{n-1} = 0\,,$$

gives us the recurrence relation

$$a_n = -\frac{1}{n(2n+1)}\, a_{n-1}\,.$$

Starting with $a_0 = 1$, compute $a_1 = -\dfrac{1}{1 \cdot 3}$, $a_2 = -\dfrac{1}{2 \cdot 5}\, a_1 = (-1)^2 \dfrac{1}{(1 \cdot 2)\ (1 \cdot 3 \cdot 5)}$, $a_3 = -\dfrac{1}{3 \cdot 7}\, a_2 = (-1)^3 \dfrac{1}{(1 \cdot 2 \cdot 3)\ (1 \cdot 3 \cdot 5 \cdot 7)}$, and in general

$$a_n = (-1)^n \frac{1}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)}\,.$$

The second solution is then

$$y_2(x) = x\left[1+\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!\,\cdot 1\cdot 3\cdot 5\cdots(2n+1)}\,x^n\right].$$

The general solution is, of course, $y(x) = c_1 y_1 + c_2 y_2$.

Example 2 Solve $\quad x^2y'' + xy' + (x^2 - \dfrac{1}{9})y = 0.$

This is Bessel's equation of order $\frac{1}{3}$. Here $p(x) = 1$, and $q(x) = x^2 - \frac{1}{9}$. The characteristic equation

$$r(r-1) + r - \frac{1}{9} = 0$$

has roots $r = -\frac{1}{3}$, and $r = \frac{1}{3}$.

The case $r = -\frac{1}{3}$. Set $y = x^{-\frac{1}{3}}v$. Compute $y' = -\frac{1}{3}x^{-\frac{4}{3}}v + x^{-\frac{1}{3}}v'$, $y'' = \frac{4}{9}x^{-\frac{7}{3}}v - \frac{2}{3}x^{-\frac{4}{3}}v' + x^{-\frac{1}{3}}v''$. Substituting this y in and simplifying, produces a mildly singular equation

$$xv'' + \frac{1}{3}v' + xv = 0\,.$$

Multiply this equation by $3x$

$$3x^2v'' + xv' + 3x^2v = 0\,, \tag{3.9}$$

and look for a solution in the form $v = \sum_{n=0}^{\infty} a_n x^n$. Substituting this series into (3.9), gives

$$\sum_{n=2}^{\infty} 3a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 3a_n x^{n+2} = 0\,.$$

We shift $n \to n-2$ in the last series:

$$\sum_{n=2}^{\infty} 3a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} 3a_{n-2} x^{n} = 0\,.$$

The x term is present only in the second series. Its coefficient must be zero, so that

$$a_1 = 0\,. \tag{3.10}$$

The term x^n, with $n \geq 2$, is present in all three series. Setting its coefficient to zero

$$3a_n n(n-1) + a_n n + 3a_{n-2} = 0\,,$$

gives us the recurrence relation

$$a_n = -\frac{3}{n(3n-2)}\, a_{n-2}\,.$$

All odd coefficients are zero (because of (3.10)), while for the even ones our recurrence relation gives

$$a_{2n} = -\frac{3}{2n(6n-2)}\, a_{2n-2}\,.$$

Starting with $a_0 = 1$, compute $a_2 = -\frac{3}{2\cdot 4}$, $a_4 = (-1)^2 \frac{3^2}{(2\cdot 4)\ (4\cdot 10)}$, $a_6 = (-1)^3 \frac{3^2}{(2\cdot 4\cdot 6)\ (4\cdot 10\cdot 16)}$, and in general,

$$a_{2n} = (-1)^n \frac{3^n}{(2\cdot 4\cdots 2n)\,(4\cdot 10\cdots (6n-2))}\,.$$

The first solution is then

$$y_1(x) = x^{-1/3}\left[1+\sum_{n=1}^{\infty}(-1)^n \frac{3^n}{(2\cdot 4\cdots 2n)\,(4\cdot 10\cdots (6n-2))}x^{2n}\right].$$

The case $r = \frac{1}{3}$. Set $y = x^{\frac{1}{3}}v$. Compute $y' = \frac{1}{3}x^{-\frac{2}{3}}v + x^{\frac{1}{3}}v'$, $y'' = -\frac{2}{9}x^{-\frac{5}{3}}v + \frac{2}{3}x^{-\frac{2}{3}}v' + x^{\frac{1}{3}}v''$. Substituting this y into our equation and simplifying, produces a mildly singular equation

$$xv'' + \frac{5}{3}v' + xv = 0\,.$$

Multiply the last equation by $3x$

$$3x^2v'' + 5xv' + 3x^2v = 0\,,$$

and look for a solution in the form $v = \sum_{n=0}^{\infty} a_n x^n$. Substituting this in, we conclude again that $a_1 = 0$, and that the following recurrence relation holds

$$a_n = -\frac{3}{n(3n+2)}\, a_{n-2}\,.$$

It follows that all odd coefficients are zero, while the even ones satisfy

$$a_{2n} = -\frac{3}{2n(6n+2)}\, a_{2n-2} = -\frac{1}{2n(2n+2/3)}\, a_{2n-2}\,.$$

We then derive the second solution

$$y_2(x) = x^{1/3}\left[1+\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2\cdot 4\cdots 2n)\,((2+2/3)\cdot(4+2/3)\cdots(2n+2/3))}x^{2n}\right].$$

3.3.1 Problems

I. Find the Maclauren series of the following functions, and state their radius of convergence

1. $\sin x^2$. 2. $\dfrac{1}{1+x^2}$. 3. xe^{-x^3}.

II. 1. Find the Taylor series of $f(x)$ centered at a.

(i) $f(x) = \sin x$, $a = \frac{\pi}{2}$. (ii) $f(x) = e^x$, $a = 1$. (iii) $f(x) = \dfrac{1}{x}$, $a = 1$.

2. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n^2}\, x^n = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}\, x^{2n}$.

3. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^3}\, x^n = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}\, x^{2n-1}$.

Hint: $\cos n\pi = (-1)^n$.

4. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{n+3}{n!\,(n+1)}\, x^{n+1} = \sum_{n=1}^{\infty} \frac{n+2}{(n-1)!\,n}\, x^n$.

5. Show that $\displaystyle\sum_{n=0}^{\infty} a_n\, x^{n+2} = \sum_{n=2}^{\infty} a_{n-2}\, x^n$.

6. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{n+3}{n!\,(n+1)}\, x^{n+2} = \sum_{n=2}^{\infty} \frac{n+1}{(n-2)!\,(n-1)}\, x^n$.

7. Expand the n-th derivative: $\left[(x^2+x)g(x)\right]^{(n)}$.

Answer. $n(n-1)g^{(n-2)}(x) + n(2x+1)g^{(n-1)}(x) + (x^2+x)g^{(n)}(x)$.

8. Find the n-th derivative: $\left[(x^2+x)e^{2x}\right]^{(n)}$.

Answer. $2^{n-2}e^{2x}\left[n(n-1) + 2n(2x+1) + 4(x^2+x)\right]$.

9. Expand the n-th derivative: $\left[xy'\right]^{(n)}$.

Answer. $ny^{(n)} + xy^{(n+1)}$.

10. Expand the n-th derivative: $\left[\left(x^2+1\right)y''\right]^{(n)}$.

Answer. $n(n-1)y^{(n)} + 2nxy^{(n+1)} + \left(x^2+1\right)y^{(n+2)}$.

11. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Show that
$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$, and $y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$.

12. Let $y(x) = \sum_{n=1}^{\infty} \frac{1}{n!(n-1)!} x^n$. Show that
$y''(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)!(n-2)!} x^{n-2}$, and $xy''(x) = \sum_{n=1}^{\infty} \frac{1}{n!(n-1)!} x^n$.
Conclude that $y(x)$ is a solution of

$$xy'' - y = 0\,.$$

Can you solve this equation by another method? Hint: Probably not.

III. Find the general solution, using power series centered at a (find the recurrence relation, and two linearly independent solutions).

1. $y'' - xy' - y = 0$, $a = 0$.

Answer. $y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$, $y_2(x) = \sum_{n=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}$.

2. $y'' - xy' + 2y = 0$, $a = 0$.

Answer. $y_1(x) = 1 - x^2$, $y_2(x) = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \cdots$.

3. $(x^2+1)y'' + xy' + y = 0$, $a = 0$.

Answer. The recurrence relation: $y^{(n+2)}(0) = -(n^2+1)y^{(n)}(0)$.
$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \cdots$, $\quad y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{6}x^5 - \cdots$.

4. $(x^2+1)y'' + 3xy' + y = 0$, $a = 0$.

Answer. The recurrence relation: $y^{(n+2)}(0) = -(n+1)^2 y^{(n)}(0)$.
$y_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} x^{2n}$,
$y_2(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}$.

5. $(x^2+1)y'' - xy' + y = 0$, $a = 0$.

Answer. The recurrence relation: $y^{(n+2)}(0) = -(n-1)^2 y^{(n)}(0)$.

$y_1(x)=1+\sum_{n=1}^{\infty}(-1)^n\frac{1^2\cdot 3^2\cdots(2n-3)^2}{(2n)!}\,x^{2n}$, $y_2(x)=x$.

6. $y''-xy=0$, $a=2$.

Answer. $y_1(x)=1+(x-2)^2+\frac{1}{3}(x-2)^3+\frac{1}{6}(x-2)^4+\frac{1}{15}(x-2)^5\cdots$,
$y_2(x)=(x-2)+\frac{1}{3}(x-2)^3+\frac{1}{12}(x-2)^4+\frac{1}{30}(x-2)^5+\cdots$.

7. $y''-xy'-y=0$, $a=1$.

Answer. $y_1(x)=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots$,
$y_2(x)=(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots$.

8. $y''+(x+2)y'+y=0$, $a=-2$.

Answer. The recurrence relation: $y^{(n+2)}(-2)=-(n+1)y^{(n)}(-2)$. $y_1(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!}\,(x+2)^{2n}$, $y_2(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{1\cdot 3\cdot 5\cdots(2n+1)}\,(x+2)^{2n+1}$.

9. $y''+(x+1)y'-y=0$.

Answer. The recurrence relation: $y^{(n+2)}(-1)=-(n-1)y^{(n)}(-1)$.
$y_1(x)=1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^n n!(2n-1)}\,(x+1)^{2n}$, $y_2(x)=x+1$.

10. $xy''+y=0$, $a=-2$.

Answer. The recurrence relation: $y^{(n+2)}(-2)=\frac{1}{2}\left[ny^{(n+1)}(-2)+y^{(n)}(-2)\right]$.
$y_1(x)=1+\frac{1}{4}(x+2)^2+\frac{1}{24}(x+2)^3+\cdots$,

$y_2(x)=x+2+\frac{1}{12}(x+2)^3+\frac{1}{48}(x+2)^4+\cdots$.

11. $y''+x^2y=0$, $a=1$.

Hint: Before using the recurrence relation

$$y^{(n+2)}(1)=-n(n-1)y^{(n-2)}(1)-2ny^{(n-1)}(1)-y^{(n)}(1)\,,$$

calculate from the equation $y''(1)=-y(1)$, and $y'''(1)=-2y(1)-y'(1)$.

Answer. $y_1(x)=1-\frac{1}{2}(x-1)^2-\frac{1}{3}(x-1)^3-\frac{1}{24}(x-1)^4+\cdots$,
$y_2(x)=x-1-\frac{1}{6}(x-1)^3-\frac{1}{6}(x-1)^4+\cdots$.

IV. 1. Find the solution of the initial value problem, using power series centered at 0

$$y'' - xy' + 2y = 0, \quad y(0) = 1, \, y'(0) = 2\,.$$

Answer. $y = 1 + 2x - x^2 - \dfrac{1}{3}x^3 + \cdots$.

2. Find the solution of the initial value problem, using power series centered at 2

$$y'' - 2xy = 0, \quad y(2) = 1, \, y'(2) = 0\,.$$

Answer. $y = 1 + 2(x-2)^2 + \dfrac{1}{3}(x-2)^3 + \dfrac{2}{3}(x-2)^4 + \cdots$.

3. Find the solution of the initial value problem, using power series centered at -1

$$y'' + xy = 0, \quad y(-1) = 2, \, y'(-1) = -3\,.$$

Answer. $y = 2 - 3(x+1) + (x+1)^2 - \dfrac{5}{6}(x+1)^3 + \cdots$.

4. Find the solution of the initial value problem, using power series centered at 0

$$(1+x^2)y'' - 2xy' + 2y = 0\,, \quad y(0) = 1, \;\; y'(0) = -2\,.$$

Hint: Differentiate the equation to conclude that $y'''(x) = 0$ for all x, so that $y(x)$ is a quadratic polynomial. Answer. $y = 1 - 2x - x^2$.

V. Find one series solution of the following mildly singular equations, centered at $a = 0$

1. $2xy'' + y' + xy = 0$.

Answer. $y = 1 - \dfrac{x^2}{2 \cdot 3} + \dfrac{x^4}{2 \cdot 4 \cdot 3 \cdot 7} - \dfrac{x^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\, 3 \cdot 7 \cdot 11 \cdots (4n-1)}\,.$$

2. $xy'' + y' - y = 0$.

Answer. $y = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$.

3. $xy'' + 2y' + y = 0$.

Answer. $y = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\, x^n$.

4. $xy'' + y' - 2xy = 0$.

Answer. $y = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n (n!)^2} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{2^n (n!)^2} x^{2n}$.

5. $xy'' + y' + xy = 0$.

Answer. $y = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n} (n!)^2} x^{2n}$.

VI. 1. Find one series solution in the form $y = \sum_{n=5}^{\infty} a_n x^n$ of the following mildly singular equation, centered at $a = 0$

$$xy'' - 4y' + y = 0\,.$$

Answer. $y = x^5 + 120 \sum_{n=6}^{\infty} \frac{(-1)^{n-5}}{n!(n-5)!} x^n = 120 \sum_{n=5}^{\infty} \frac{(-1)^{n-5}}{n!(n-5)!} x^n$.

2. Find one series solution of the following mildly singular equation, centered at $a = 0$

$$xy'' - 2y' - 2y = 0\,.$$

Hint: Look for a solution in the form $y = \sum_{n=3}^{\infty} a_n x^n$, starting with $a_3 = 1$.

Answer. $y = x^3 + 6 \sum_{n=4}^{\infty} \frac{2^{n-3}}{n!(n-3)!} x^n = 6 \sum_{n=3}^{\infty} \frac{2^{n-3}}{n!(n-3)!} x^n$.

3. Find one series solution of the following mildly singular equation, centered at $a = 0$

$$xy'' + y = 0\,.$$

Hint: Look for a solution in the form $y = \sum_{n=1}^{\infty} a_n x^n$, starting with $a_1 = 1$.

Answer. $y = x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n!(n-1)!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!(n-1)!} x^n$.

4. Recall that Bessel's function $J_0(x)$ is a solution of (for $x > 0$)

$$xy'' + y' + xy = 0\,.$$

Show that the "energy" $E(x) = y'^2(x) + y^2(x)$ is a decreasing function. Conclude that each maximum value of $J_0(x)$ is greater than the absolute value of the minimum value that follows it, which in turn is larger than the next maximum value, and so on (see the graph of $J_0(x)$).

5. Show that the absolute value of the slope of the tangent line decreases at each consecutive root of $J_0(x)$.

Hint: Use the energy function $E(x)$ from the preceding problem.

VII. We assume that $a = 0$ for all problems of this set.

1. Verify that the Bessel equation of order $1/2$

$$x^2y'' + xy' + (x^2 - 1/4)y = 0$$

has a moderate singularity at zero. Write down the characteristic equation, and find its roots.

(i) Corresponding to the root $r = 1/2$, perform a change of variables $y = x^{1/2}v$, and obtain a mildly singular equation for $v(x)$. Solve that equation, to obtain one of the solutions of the Bessel equation.

Answer. $y = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] = x^{-1/2}\left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = x^{-1/2}\sin x$.

(ii) Corresponding to the root $r = -1/2$, perform a change of variables $y = x^{-1/2}v$, and obtain a mildly singular equation for $v(x)$. Solve that equation, to obtain the second solution of the Bessel equation.

Answer. $y = x^{-1/2}\cos x$.

(iii) Find the general solution.

Answer. $y = c_1 x^{-1/2}\sin x + c_2 x^{-1/2}\cos x$.

2. Find the fundamental solution set of the Bessel equation of order $3/2$

$$x^2y'' + xy' + (x^2 - 9/4)y = 0\,.$$

Answer. $y_1 = x^{\frac{3}{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\, 2^{2n}\,(1+\frac{3}{2})(2+\frac{3}{2})\cdots(n+\frac{3}{2})}\right]$,

$y_2 = x^{-\frac{3}{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{2n}}{n!\, 2^{2n}\,(1-\frac{3}{2})(2-\frac{3}{2})\cdots(n-\frac{3}{2})}\right]$.

3. Find the fundamental solution set of

$$2x^2y'' + 3xy' - (1+x)y = 0\,.$$

Answer. $y_1 = x^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,2^n\,(1+\frac{3}{2})(2+\frac{3}{2})\cdots(n+\frac{3}{2})}\right]$,
$y_2 = x^{-1}\left[1 - x - \sum_{n=2}^{\infty} \frac{x^n}{n!\,1\cdot 3\cdots(2n-3)}\right]$.

4. Find the fundamental solution set of

$$9x^2y'' + (2-x^2)y = 0\,.$$

Answer. $y_1 = x^{\frac{1}{3}}\left(1 + \frac{x^2}{5\cdot 6} + \frac{x^4}{5\cdot 6\cdot 11\cdot 12} + \cdots\right)$,
$y_2 = x^{\frac{2}{3}}\left(1 + \frac{x^2}{6\cdot 7} + \frac{x^4}{6\cdot 7\cdot 12\cdot 13} + \cdots\right)$.

5. Find the fundamental solution set of

$$9x^2y'' + (2+x)y = 0\,.$$

6. Find the fundamental solution set of

$$2x^2y'' + 3xy' - (x^2+1)y = 0\,.$$

Answer. $y_1 = x^{\frac{1}{2}}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \cdots\right)$, $y_2 = x^{-1}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \cdots\right)$.

Chapter 4

The Laplace Transform

The method of Laplace Transform is prominent in engineering, and in fact it was developed by an English electrical engineer - Oliver Heaviside (1850-1925). We present this method in great detail, show its many uses, and make an application to the historic tautochrone problem. The chapter concludes with a brief presentation of distribution theory.

4.1 The Laplace Transform And Its Inverse

4.1.1 Review of Improper Integrals

The mechanics of computing the integrals, involving infinite limits, is similar to that for integrals with finite end-points. For example,

$$\int_0^\infty e^{-2t}\,dt = -\frac{1}{2}e^{-2t}\,\Big|_0^\infty = \frac{1}{2}\,.$$

Here we did not set the upper limit $t=\infty$, but rather computed the limit as $t \to \infty$ (the limit is zero). This is an example of a *convergent integral.* On the other hand, the integral

$$\int_1^\infty \frac{1}{t}\,dt = \ln t\,\Big|_1^\infty$$

is *divergent*, because $\ln t$ has an infinite limit as $t \to \infty$. When computing improper integrals, we use the same techniques of integration, in essentially the same way. For example

$$\int_0^\infty te^{-2t}\,dt = \left[-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right]\Big|_0^\infty = \frac{1}{4}\,.$$

Here the antiderivative is computed by the guess-and-check method (or by integration by parts). The limit at infinity is computed by L'Hospital's rule to be zero.

4.1.2 The Laplace Transform

Let the function $f(t)$ be defined on the interval $[0, \infty)$. Let $s > 0$ be a positive parameter. We define the Laplace transform of $f(t)$ as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}(f(t)),$$

provided that this integral converges. It is customary to use the corresponding capital letters to denote the Laplace transform (so that the Laplace transform of $g(t)$ is denoted by $G(s)$, of $h(t)$ by $H(s)$, etc.). We also use the *operator notation* for the Laplace transform: $\mathcal{L}(f(t))$.

We now build up a collection of Laplace transforms.

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{e^{-st}}{s} \Big|_0^\infty = \frac{1}{s},$$

$$\mathcal{L}(t) = \int_0^\infty e^{-st} t\, dt = \left[-\frac{e^{-st} t}{s} - \frac{e^{-st}}{s^2} \right] \Big|_0^\infty = \frac{1}{s^2}.$$

Using integration by parts (n is a positive integer)

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n\, dt = -\frac{e^{-st} t^n}{s} \Big|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1}\, dt = \frac{n}{s} \mathcal{L}(t^{n-1}).$$

(Here $\lim_{t\to\infty} e^{-st} t^n = \lim_{t\to\infty} \frac{t^n}{e^{st}} = 0$, after n applications of L'Hospital's rule.) With this recurrence relation, we now compute $\mathcal{L}(t^2) = \frac{2}{s}\mathcal{L}(t) = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$, $\mathcal{L}(t^3) = \frac{3}{s}\mathcal{L}(t^2) = \frac{3!}{s^4}$, and in general

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

The next class of functions are the exponentials e^{at}, where a is some number:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty = \frac{1}{s-a}, \quad \text{provided that } s > a.$$

Here we had to assume that $s > a$, to obtain a convergent integral.

Next we observe that for any constants c_1 and c_2

$$\mathcal{L}(c_1 f(t) + c_2 g(t)) = c_1 F(s) + c_2 G(s), \tag{1.1}$$

because a similar property holds for integrals (and the Laplace transform is an integral). This formula expands considerably the set of functions for which one can write down the Laplace transform. For example, with $a > 0$,

$$\mathcal{L}(\cosh at) = \mathcal{L}(\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}) = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}, \quad \text{for} \;\; s > a\,.$$

Similarly,

$$\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}, \quad \text{for} \;\; s > a\,.$$

The formula (1.1) holds with an arbitrary number of terms, and it allows to compute the Laplace transform of any polynomial. For example,

$$\mathcal{L}\left(2t^5 - 3t^2 + 5\right) = 2\mathcal{L}(t^5) - 3\mathcal{L}(t^2) + 5\mathcal{L}(1) = \frac{240}{s^6} - \frac{6}{s^3} + \frac{5}{s}\,.$$

Compute

$$\mathcal{L}(\cos at) = \int_0^\infty e^{-st}\cos at\, dt = \frac{e^{-st}(a\sin at - s\cos at)}{s^2+a^2}\Big|_0^\infty = \frac{s}{s^2+a^2}\,.$$

(One guesses that the antiderivative of $e^{-st}\cos at$ is of the form $Ae^{-st}\cos at + Be^{-st}\sin at$, and then evaluates the constants A and B by differentiation.) Similarly,

$$\mathcal{L}(\sin at) = \int_0^\infty e^{-st}\sin at\, dt = -\frac{e^{-st}(s\sin at + a\cos at)}{s^2+a^2}\Big|_0^\infty = \frac{a}{s^2+a^2}\,.$$

For example,

$$\mathcal{L}\left(\cos^2 3t\right) = \mathcal{L}\left(\frac{1}{2} + \frac{1}{2}\cos 6t\right) = \frac{1}{2s} + \frac{s}{2(s^2+36)}\,.$$

If c is some number, then

$$\mathcal{L}(e^{ct}f(t)) = \int_0^\infty e^{-st}e^{ct}f(t)\, dt = \int_0^\infty e^{-(s-c)t}f(t)\, dt = F(s-c)\,.$$

We derived the *shift formula*:

$$\mathcal{L}(e^{ct}f(t)) = F(s-c)\,. \tag{1.2}$$

For example,

$$\mathcal{L}\left(e^{5t}\sin 3t\right) = \frac{3}{(s-5)^2+9}\,.$$

(Start with $\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$, and then perform the shift $s \to s-5$, to account for the extra exponential factor e^{5t}.) Another example:

$$\mathcal{L}\left(e^{-2t}\cosh 3t\right) = \frac{s+2}{(s+2)^2-9}\,.$$

In the last example $c=-2$, so that $s-c=s+2$. Similarly,

$$\mathcal{L}\left(e^{t}t^5\right) = \frac{5!}{(s-1)^6}\,.$$

4.1.3 The Inverse Laplace Transform

This is just going from $F(s)$ back to $f(t)$. We denote it by $\mathcal{L}^{-1}(F(s)) = f(t)$. We have

$$\mathcal{L}^{-1}\left(c_1F(s)+c_2G(s)\right) = c_1f(t)+c_2g(t)\,,$$

corresponding to the formula (1.1), read backward. Each of the formulas for the Laplace Transform leads to the corresponding formula for its inverse:

$$\mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}\,,$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at\,,$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at\,,$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}\,,$$

and so on. For example,

$$\mathcal{L}^{-1}\left(\frac{1}{4s^2+1}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s^2+\frac{1}{4}}\right) = \frac{1}{2}\sin\frac{t}{2}\,.$$

To compute $\mathcal{L}^{-1}$, one often uses partial fractions, as well as the inverse of the shift formula (1.2)

$$\mathcal{L}^{-1}(F(s-c)) = e^{ct}f(t)\,, \tag{1.3}$$

which is also called the *shift formula*.

Example 1 Find $\mathcal{L}^{-1}\left(\frac{3s-5}{s^2+4}\right)$.

Breaking this fraction into a difference of two fractions, obtain

$$\mathcal{L}^{-1}\left(\frac{3s-5}{s^2+4}\right) = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = 3\cos 2t - \frac{5}{2}\sin 2t\,.$$

Example 2 Find $\mathcal{L}^{-1}\left(\frac{2}{(s-5)^4}\right)$.

We recognize that a shift by 5 is performed in the function $\frac{2}{s^4}$. Begin by inverting this function, $\mathcal{L}^{-1}\left(\frac{2}{s^4}\right) = \frac{t^3}{3}$, and then account for the shift, according to the shift formula (1.3):

$$\mathcal{L}^{-1}\left(\frac{2}{(s-5)^4}\right) = e^{5t}\frac{t^3}{3}\,.$$

Example 3 Find $\mathcal{L}^{-1}\left(\frac{s+7}{s^2-s-6}\right)$.

Factor the denominator, and use partial fractions

$$\frac{s+7}{s^2-s-6} = \frac{s+7}{(s-3)(s+2)} = \frac{2}{s-3} - \frac{1}{s+2}\,,$$

which gives

$$\mathcal{L}^{-1}\left(\frac{s+7}{s^2-s-6}\right) = 2e^{3t} - e^{-2t}\,.$$

The method of partial fractions is reviewed in the Appendix.

Example 4 Find $\mathcal{L}^{-1}\left(\frac{s^3+2s^2-s+12}{s^4+10s^2+9}\right)$.

Again, we factor the denominator, and use partial fractions

$$\frac{s^3+2s^2-s+12}{s^4+10s^2+9} = \frac{s^3+2s^2-s+12}{(s^2+1)(s^2+9)} = \frac{-\frac{1}{4}s+\frac{5}{4}}{s^2+1} + \frac{\frac{5}{4}s+\frac{3}{4}}{s^2+9}\,,$$

which leads to

$$\mathcal{L}^{-1}\left(\frac{s^3+2s^2-s+12}{s^4+10s^2+9}\right) = \frac{1}{4}(-\cos t + 5\sin t + 5\cos 3t + \sin 3t)\,.$$

Example 5 Find $\mathcal{L}^{-1}\left(\dfrac{2s-1}{s^2+2s+5}\right)$.

One cannot factor the denominator, so we complete the square

$$\frac{2s-1}{s^2+2s+5} = \frac{2s-1}{(s+1)^2+4} = \frac{2(s+1)-3}{(s+1)^2+4},$$

and then adjust the numerator, so that it involves the same shift as in the denominator. Without the shift, we have the function $\dfrac{2s-3}{s^2+4}$, with the inverse Laplace transform equal to $2\cos 2t - \frac{3}{2}\sin 2t$. By the shift formula (1.3), obtain

$$\mathcal{L}^{-1}\left(\frac{2s-1}{s^2+2s+5}\right) = 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t\,.$$

4.2 Solving The Initial Value Problems

Integrating by parts,

$$\mathcal{L}(y'(t)) = \int_0^\infty e^{-st}y'(t)\,dt = \left[e^{-st}y(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st}y(t)\,dt\,.$$

Let us assume that $y(t)$ does not grow too fast, as $t \to \infty$, so that $|y(t)| \le be^{at}$, for some positive constants a and b. If we now require that $s > a$, then the limit as $t \to \infty$ is zero, while the lower limit gives $-y(0)$. We conclude

$$\mathcal{L}(y'(t)) = -y(0) + sY(s)\,. \tag{2.1}$$

This formula shows that the Laplace transform of the derivative of $y(t)$ is obtained from the Laplace transform of $y(t)$ by a simple *algebraic operation*. To compute the Laplace transform of $y''(t)$, we use the formula (2.1) twice

$$\mathcal{L}(y''(t)) = \mathcal{L}((y'(t))') = -y'(0) + s\mathcal{L}(y'(t)) = -y'(0) - sy(0) + s^2Y(s)\,. \tag{2.2}$$

In general,

$$\mathcal{L}(y^{(n)}(t)) = -y^{(n-1)}(0) - sy^{(n-2)}(0) - \cdots - s^{n-1}y(0) + s^nY(s)\,. \tag{2.3}$$

Example 1 Solve $y'' + 3y' + 2y = 0, \quad y(0) = -1,\ y'(0) = 4$.

Apply the Laplace transform to both sides of the equation. Using the linearity of the Laplace transform (the formula (1.1)), and that $\mathcal{L}(0) = 0$, obtain

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = 0\,.$$

By the formulas (2.1), (2.2), and our initial conditions ($\mathcal{L}\left(y(t)\right) = Y(s)$):

$$-y'(0) - sy(0) + s^2Y(s) + 3\left(-y(0) + sY(s)\right) + 2Y(s) = 0\,,$$

$$-4 + s + s^2Y(s) + 3\left(1 + sY(s)\right) + 2Y(s) = 0\,,$$

$$\left(s^2 + 3s + 2\right)Y(s) + s - 1 = 0\,.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1-s}{s^2+3s+2}\,.$$

To get the solution, it remains to find the inverse Laplace transform $y(t) = \mathcal{L}^{-1}(Y(s))$. We factor the denominator, and use partial fractions

$$\frac{1-s}{s^2+3s+2} = \frac{1-s}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{3}{s+2}\,.$$

Answer: $y(t) = 2e^{-t} - 3e^{-2t}$.

Of course, this problem could also be solved without using the Laplace transform. The Laplace transform gives an alternative solution method, it is more convenient for discontinuous forcing functions, and in addition, it provides a tool that can be used in more involved situations, for example, to solve partial differential equations.

Example 2 Solve $y'' - 4y' + 5y = 0, \quad y(0) = 1,\, y'(0) = -2$.

Apply the Laplace transform to both sides of the equation. Using the initial conditions, obtain

$$2 - s + s^2Y(s) - 4\left(-1 + sY(s)\right) + 5Y(s) = 0\,,$$

$$Y(s) = \frac{s-6}{s^2-4s+5}\,.$$

To invert the Laplace transform, we complete the square in the denominator, and then produce the same shift in the numerator:

$$Y(s) = \frac{s-6}{s^2-4s+5} = \frac{(s-2)-4}{(s-2)^2+1}\,.$$

Using the shift formula, leads to the answer: $y(t) = e^{2t}\cos t - 4e^{2t}\sin t$.

Example 3 Solve

$$y'' + \omega^2 y = 5\cos 2t, \quad \omega \neq 2,$$
$$y(0) = 1, \quad y'(0) = 0\,.$$

This problem models a spring, with the natural frequency ω, subjected to an external force of frequency 2. Applying the Laplace transform to both sides of the equation, and using the initial conditions, we get

$$-s+s^2Y(s)+\omega^2Y(s)=\frac{5s}{s^2+4}\,,$$

$$Y(s)=\frac{5s}{(s^2+4)(s^2+\omega^2)}+\frac{s}{s^2+\omega^2}\,.$$

The second term is easy to invert. To find the inverse Laplace transform of the first term, we use the guess-and-check method (or partial fractions)

$$\frac{s}{(s^2+4)(s^2+\omega^2)}=\frac{1}{\omega^2-4}\left[\frac{s}{s^2+4}-\frac{s}{s^2+\omega^2}\right]\,.$$

Answer: $y(t)=\dfrac{5}{\omega^2-4}\left(\cos 2t-\cos\omega t\right)+\cos\omega t$.

When ω approaches 2, the amplitude of the oscillations becomes large.

To treat the case of resonance, when $\omega=2$, we need one more formula. Differentiate in s both sides of the formula

$$F(s)=\int_0^\infty e^{-st}f(t)\,dt\,,$$

to obtain

$$F'(s)=-\int_0^\infty e^{-st}tf(t)\,dt=-\mathcal{L}(tf(t))\,,$$

or

$$\mathcal{L}(tf(t))=-F'(s)\,.$$

For example,

$$\mathcal{L}(t\sin 2t)=-\frac{d}{ds}\,\mathcal{L}(\sin 2t)=-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)=\frac{4s}{(s^2+4)^2}\,. \tag{2.4}$$

Example 4 Solve (a case of resonance)

$$y''+4y=5\cos 2t,$$
$$y(0)=0,\;\; y'(0)=0\,.$$

Using the Laplace transform, obtain

$$s^2Y(s)+4Y(s)=\frac{5s}{s^2+4}\,,$$

$$Y(s) = \frac{5s}{(s^2+4)^2}\,.$$

Then, using (2.4), $y(t) = \dfrac{5}{4} t \sin 2t$. We see that the amplitude of oscillations (which is equal to $\frac{5}{4}t$) tends to infinity with time.

Example 5 Solve the initial value problem for the fourth order equation

$$y'''' - y = 0,$$
$$y(0) = 1,\;\; y'(0) = 0,\;\; y''(0) = 1,\;\; y'''(0) = 0\,.$$

Applying the Laplace transform, using the formula (2.3), and our initial conditions, obtain

$$-y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + s^4Y(s) - Y(s) = 0\,,$$

$$(s^4-1)Y(s) = s^3 + s\,,$$

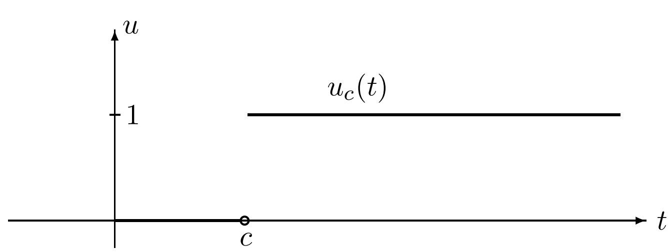
$$Y(s) = \frac{s^3+s}{s^4-1} = \frac{s(s^2+1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1}\,.$$

We conclude that $y(t) = \cosh t$.

4.2.1 Step Functions

Sometimes an external force acts only over some time interval. One uses *step functions* to model such forces. The basic step function is the *Heaviside function* $u_c(t)$, defined for any positive constant c by

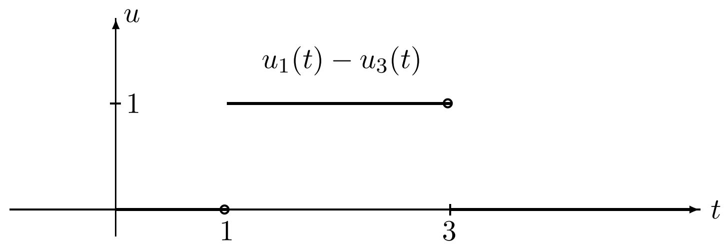
$$u_c(t) = \begin{cases} 0 & \text{if } 0 \le t < c \\ 1 & \text{if } t \ge c \end{cases}\,.$$



The Heaviside step function $u_c(t)$

(Oliver Heaviside, 1850 - 1925, was a self-taught English electrical engineer.) Using $u_c(t)$, we can build up other step functions. For example, the function $u_1(t) - u_3(t)$ is equal to 1 for $1 \le t < 3$, and is zero otherwise. Indeed

$$u_1(t) - u_3(t) = \begin{cases} 0 - 0 = 0\,, & \text{if } 0 \le t < 1 \\ 1 - 0 = 1\,, & \text{if } 1 \le t < 3 \\ 1 - 1 = 0\,, & \text{if } t \ge 3 \end{cases} .$$



The graph of $u_1(t) - u_3(t)$

The function $[u_1(t) - u_3(t)]\, t^2$ models a force that is equal to t^2 for $1 \le t < 3$, and is zero for other t.

Compute:

$$\mathcal{L}(u_c(t)) = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \frac{e^{-cs}}{s}\,.$$

Correspondingly,

$$\mathcal{L}^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t)\,.$$

For example, if $f(t)$ is equal to 3 for $2 \le t < 7$, and is equal to zero for all other $t \ge 0$, then $f(t) = 3\,[u_2(t) - u_7(t)]$, and

$$\mathcal{L}(f(t)) = 3\mathcal{L}(u_2(t)) - 3\mathcal{L}(u_7(t)) = 3\frac{e^{-2s}}{s} - 3\frac{e^{-7s}}{s}\,.$$

We compute the Laplace transform of the following "shifted" function, which "begins" at $t = c$ (it is zero for $0 < t < c$):

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} f(t-c)\, dt\,.$$

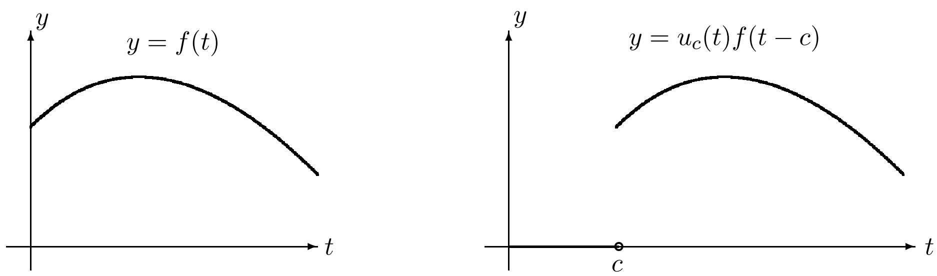
In the last integral we change the variable $t \to z$, by setting $t - c = z$. Then $dt = dz$, and the integral becomes

$$\int_0^\infty e^{-s(c+z)} f(z)\, dz = e^{-cs} \int_0^\infty e^{-sz} f(z)\, dz = e^{-cs} F(s)\,.$$

The result is another pair of *shift formulas*:

$$\mathcal{L}\left(u_c(t)f(t-c)\right) = e^{-cs}F(s)\,, \tag{2.5}$$

$$\mathcal{L}^{-1}\left(e^{-cs}F(s)\right) = u_c(t)f(t-c)\,. \tag{2.6}$$



The function $y = u_c(t)f(t-c)$

For example,

$$\mathcal{L}\left(u_1(t)(t-1)\right) = e^{-s}\mathcal{L}(t) = \frac{e^{-s}}{s^2}\,.$$

Using that $\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$, obtain

$$\mathcal{L}^{-1}\left(e^{-\pi s}\frac{1}{s^2+1}\right) = u_\pi(t)\sin(t-\pi) = -u_\pi(t)\sin t\,.$$

Example 1 Solve

$$y'' + 9y = u_2(t) - u_4(t), \quad y(0) = 1, \;\; y'(0) = 0\,.$$

Here the forcing term is equal to 1, for $2 \le t < 4$, and is zero for other t. Taking the Laplace transform, then solving for $Y(s)$, we have

$$s^2Y(s) - s + 9Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\,,$$

$$Y(s) = \frac{s}{s^2+9} + e^{-2s}\frac{1}{s(s^2+9)} - e^{-4s}\frac{1}{s(s^2+9)}\,.$$

Using the guess-and-check method (or partial fractions)

$$\frac{1}{s(s^2+9)} = \frac{1}{9}\left[\frac{1}{s} - \frac{s}{s^2+9}\right]\,,$$

and therefore

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+9)}\right) = \frac{1}{9} - \frac{1}{9}\cos 3t\,.$$

Using (2.6), we conclude

$$y(t) = \cos 3t + u_2(t)\left[\frac{1}{9} - \frac{1}{9}\cos 3(t-2)\right] - u_4(t)\left[\frac{1}{9} - \frac{1}{9}\cos 3(t-4)\right].$$

Observe that the solution undergoes jumps in its behavior at $t = 2$, and at $t = 4$, which corresponds to the force being switched on at $t = 2$, and switched off at $t = 4$.

Example 2 Solve

$$y'' + 4y = g(t), \quad y(0) = 0, \;\; y'(0) = 0\,,$$

where $g(t)$ is the *ramp function*:

$$g(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}.$$

Express

$$g(t) = t\,(1 - u_1(t)) + u_1(t) = t - u_1(t)(t-1)\,,$$

so that by the shift formula (2.5) its Laplace transform is

$$G(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}\,.$$

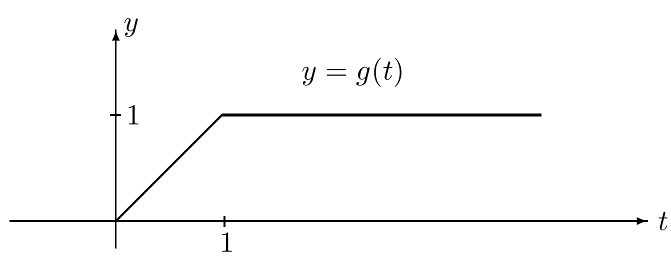
Take the Laplace transform of the equation:

$$s^2 Y(s) + 4Y(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}\,,$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)} = \frac{1/4}{s^2} - \frac{1/4}{s^2+4} - e^{-s}\left[\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right].$$

Using the shift formula (2.6), we conclude that

$$y(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t - u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right].$$



The ramp function $y = g(t)$

4.3 The Delta Function and Impulse Forces

Imagine a rod so thin that we may consider it to be one dimensional and so long that we assume it to extend for $-\infty < t < \infty$, along the t axis. Assume that the function $\rho(t)$ gives the density of the rod (weight per unit length). If we subdivide the interval $(-N, N)$, for some $N > 0$, using the points t_1, $t_2, \ldots, t_n$, at a distance $\Delta t = \frac{2N}{n}$ apart, then the weight of the piece i can be approximated by $\rho(t_i)\Delta t$, and $\displaystyle\sum_{i=1}^{n} \rho(t_i)\Delta t$ gives an approximation of the total weight. Passing to the limit, letting $\Delta t \to 0$, and $N \to \infty$, we get the exact value of the weight:

$$w = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \rho(t_i)\Delta t = \int_{-\infty}^{\infty} \rho(t)\, dt\,.$$

Assume now that the rod is moved to a new position in the (t, y) plane, with each point $(t, 0)$ moved to a point $(t, f(t))$, where $f(t)$ is a given function. What is the work needed for this move? For the piece i, the work is approximated by $f(t_i)\rho(t_i)\Delta t$. The total work is then

$$\text{Work} = \int_{-\infty}^{\infty} \rho(t) f(t)\, dt\,.$$

Assume now that the rod has unit weight, $w = 1$, and the entire weight is pushed into a single point $t = 0$. The resulting distribution of weight is called the *delta distribution* or the *delta function*, and is denoted $\delta(t)$. In view of the discussion above, it has the following properties:

(i) $\delta(t) = 0, \quad \text{for } t \neq 0,$

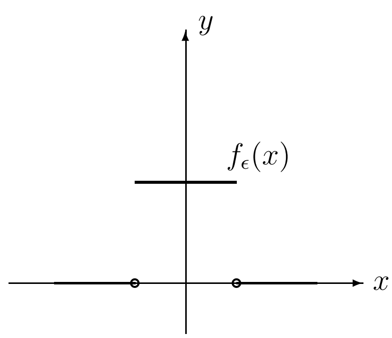
(ii) $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$ (unit weight),

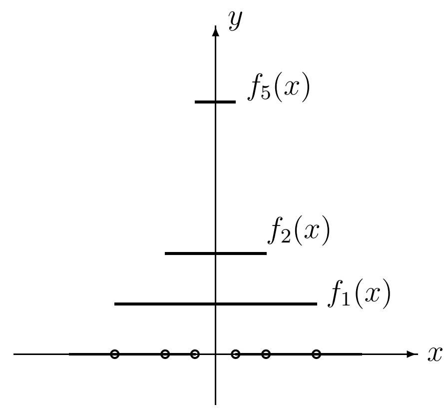
(iii) $\displaystyle\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).$

The last formula holds, because work is expended only to move the weight 1 at $t = 0$, the distance of $f(0)$. Observe that $\delta(t)$ is not a usual function, like the ones studied in calculus. (If a usual function is equal to zero, except at one point, its integral is zero, over any interval.) One can think of $\delta(t)$ as the limit of the following sequence of functions (a *delta sequence*)

$$f_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } -\epsilon \le t \le \epsilon \\ \\ 0 & \text{for other } t \end{cases}\,,$$

as $\epsilon \to 0$. (Observe that $\int_{-\infty}^{\infty} f_\epsilon(t)\,dt = 1$.)





The step function $f_\epsilon(x)$ A delta sequence

For any number t_0, the function $\delta(t-t_0)$ gives a translation of the delta function, with the unit weight concentrated at $t = t_0$. Correspondingly, its properties are

(i) $\delta(t-t_0) = 0$, for $t \neq t_0$,

(ii) $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$,

(iii) $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$.

Using the properties **(i)** and **(iii)**, we compute the Laplace transform, for any $t_0 \geq 0$,

$$\mathcal{L}\left(\delta(t-t_0)\right) = \int_0^\infty \delta(t-t_0)e^{-st}\,dt = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-st}\,dt = e^{-st_0}\,.$$

In particular,

$$\mathcal{L}\left(\delta(t)\right) = 1\,.$$

Correspondingly,

$$\mathcal{L}^{-1}\left(e^{-st_0}\right) = \delta(t-t_0), \text{ and } \mathcal{L}^{-1}(1) = \delta(t)\,.$$

For example,

$$\mathcal{L}^{-1}\left(\frac{s+1}{s+3}\right) = \mathcal{L}^{-1}\left(1 - \frac{2}{s+3}\right) = \delta(t) - 2e^{-3t}\,.$$

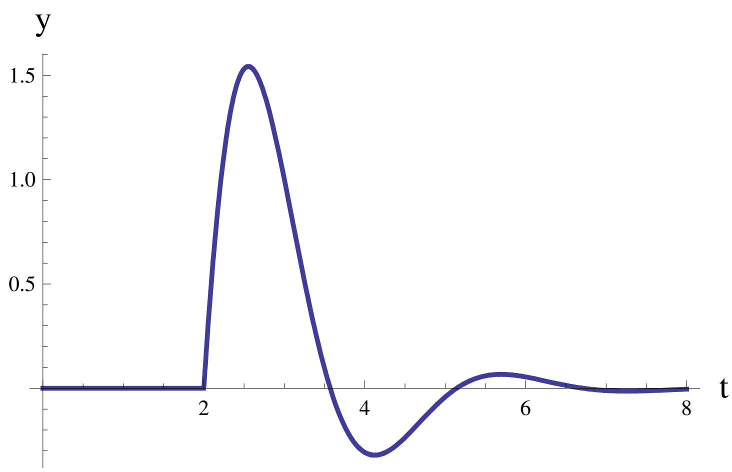


Figure 4.1: Spring's response to an impulse force

Other physical quantities may be concentrated at a single point. In the following example we consider forced vibrations of a spring, with the external force concentrated at $t = 2$. We say that an external *impulse force* is applied at $t = 2$.

Example Solve the initial value problem

$$y'' + 2y' + 5y = 6\,\delta(t-2), \quad y(0) = 0, \quad y'(0) = 0\,.$$

Applying the Laplace transform, then solving for $Y(s)$ and completing the square, obtain

$$(s^2 + 2s + 5)Y(s) = 6e^{-2s}\,,$$

$$Y(s) = \frac{6e^{-2s}}{s^2 + 2s + 5} = e^{-2s}\frac{6}{(s+1)^2 + 4}\,.$$

By the shift formula $\mathcal{L}^{-1}\left(\dfrac{1}{(s+1)^2+4}\right) = \dfrac{1}{2}e^{-t}\sin 2t$, and using the second shift formula (2.6), we conclude that

$$y(t) = 3u_2(t)e^{-(t-2)}\sin 2(t-2)\,.$$

Before the time $t = 2$, the external force is zero. Coupled with zero initial conditions, this leaves the spring at rest for $t \le 2$. The impulse force at $t = 2$ sets the spring in motion, but the vibrations quickly die down, because of the heavy damping; see the Figure 4.1 for the graph of $y(t)$.

4.4 Convolution and the Tautochrone Curve

The problem

$$y'' + y = 0, \quad y(0) = 0, \ y'(0) = 1$$

has solution $y = \sin t$. If we now add a forcing term $g(t)$, and consider

$$y'' + y = g(t), \quad y(0) = 0, \ y'(0) = 0\,,$$

then the solution is

$$y(t) = \int_0^t \sin(t-v)g(v)\,dv, \tag{4.1}$$

as we saw in the section on convolution integrals. Motivated by this formula, we now define the concept of *convolution of two functions* $f(t)$ and $g(t)$

$$f * g = \int_0^t f(t-v)g(v)\,dv\,.$$

The result is a function of t, which is also denoted as $(f*g)(t)$. The formula (4.1) can now be written as

$$y(t) = \sin t * g(t)\,.$$

Another example of convolution:

$$t * t^2 = \int_0^t (t-v)v^2\,dv = t\int_0^t v^2\,dv - \int_0^t v^3\,dv = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}\,.$$

If you compute $t^2 * t$, the answer is the same. More generally, for any two continuous functions $f(t)$ and $g(t)$

$$g * f = f * g\,.$$

Indeed, making a change of variables $v \to u$, by letting $u = t - v$, we express

$$\begin{aligned} g * f &= \int_0^t g(t-v)f(v)\,dv = -\int_t^0 g(u)f(t-u)\,du \\ &= \int_0^t f(t-u)g(u)\,du = f * g\,. \end{aligned}$$

It turns out that the Laplace transform of a convolution is equal to the product of the Laplace transforms:

$$\mathcal{L}\left(f*g\right) = F(s)G(s)\,.$$

Indeed,

$$\mathcal{L}\left(f * g\right) = \int_0^\infty e^{-st} \int_0^t f(t-v)g(v)\,dv\,dt = \iint_D e^{-st} f(t-v)g(v)\,dv\,dt\,,$$

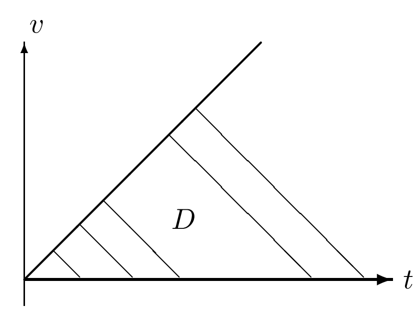
where the double integral on the right hand side is taken over the region D of the tv-plane, which is an infinite wedge $0 < v < t$ in the first quadrant. We now evaluate this double integral by using the reverse order of repeated integrations:

$$\iint_D e^{-st} f(t-v)g(v)\,dv\,dt = \int_0^\infty g(v) \left(\int_v^\infty e^{-st} f(t-v)\,dt \right) dv\,. \tag{4.2}$$

For the integral in the brackets, we make a change of variables $t \to u$, by letting $u = t - v$,

$$\int_v^\infty e^{-st} f(t-v)\,dt = \int_0^\infty e^{-s(v+u)} f(u)\,du = e^{-sv} F(s)\,,$$

and then the right hand side of (4.2) is equal to $F(s)G(s)$.



The infinite wedge D

We conclude a useful formula

$$\mathcal{L}^{-1}\left(F(s)G(s)\right) = (f * g)(t)\,.$$

For example,

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+4)^2}\right) = \cos 2t * \cos 2t = \int_0^t \cos 2(t-v)\cos 2v\,dv\,.$$

Using that

$$\cos 2(t-v) = \cos 2t \cos 2v + \sin 2t \sin 2v\,,$$

we conclude

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+4)^2}\right) = \cos 2t \int_0^t \cos^2 2v\, dv + \sin 2t \int_0^t \sin 2v \cos 2v\, dv$$
$$= \frac{1}{2} t \cos 2t + \frac{1}{4} \sin 2t\,.$$

Example Consider the vibrations of a *spring at resonance*

$$y'' + y = -3\cos t, \quad y(0) = 0, \;\; y'(0) = 0\,.$$

Taking the Laplace transform, compute

$$Y(s) = -3\,\frac{s}{(s^2+1)^2}\,.$$

Writing $Y(s) = -3\,\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$, we invert it as

$$y(t) = -3 \sin t * \cos t = -\frac{3}{2} t \sin t\,,$$

because

$$\sin t * \cos t = \int_0^t \sin(t-v)\cos v\, dv = \int_0^t [\sin t \cos v - \cos t \sin v] \cos v\, dv$$
$$= \sin t \int_0^t \cos^2 v\, dv - \cos t \int_0^t \sin v \cos v\, dv$$
$$= \frac{1}{2} t \sin t + \frac{1}{4} \sin 2t \sin t - \frac{1}{2} \cos t \sin^2 t = \frac{1}{2} t \sin t\,.$$

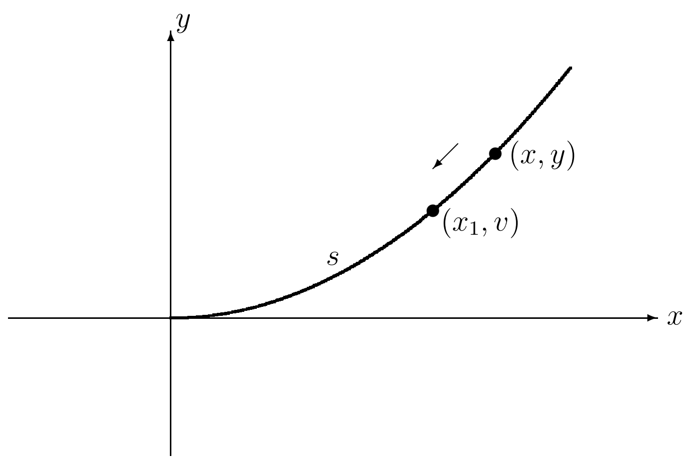
We see again that the amplitude of oscillations, which is $\frac{1}{2}t$, tends to infinity with time t.

The Tautochrone curve

Assume that we have a curve through the origin in the first quadrant of the xy-plane, and a particle slides down this curve, under the influence of the force of gravity. The initial velocity at the starting point is assumed to be zero. We wish to find the curve so that the time T it takes to reach the bottom at $(0,0)$ is the same, for any starting point (x,y). This historic curve, called the *tautochrone* (which means loosely "the same time" in Latin), was found by Christian Huygens in 1673. He was motivated by the construction of a clock pendulum with the period independent of its amplitude.

Let (x_1, v) be any intermediate position of the particle, $v < y$. Let $s = f(v)$ be the length of the curve from $(0,0)$ to (x_1, v). Of course, the length s depends also on the time t, and $\frac{ds}{dt}$ gives the speed of the particle. The kinetic energy of the particle at (x_1, v) is due to the decrease of its potential energy (m is the mass of the particle):

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mg(y-v)\,.$$



The Tautochrone curve

By the chain rule, $\frac{ds}{dt} = \frac{ds}{dv}\frac{dv}{dt} = f'(v)\frac{dv}{dt}$, so that

$$\frac{1}{2}\left(f'(v)\frac{dv}{dt}\right)^2 = g(y-v)\,,$$

$$f'(v)\frac{dv}{dt} = -\sqrt{2g}\sqrt{y-v}\,.$$

(Minus, because the function $v(t)$ is decreasing, while $f'(v) > 0$.) We separate the variables, and integrate

$$\int_0^y \frac{f'(v)}{\sqrt{y-v}}\,dv = \int_0^T \sqrt{2g}\,dt = \sqrt{2g}\,T\,. \tag{4.3}$$

(Over the time interval $(0, T)$, the particle descends from $v = y$ to $v = 0$.) To find the function f', we need to solve the *integral equation* (4.3), which may be written as

$$y^{-1/2} * f'(y) = \sqrt{2g}\,T\,. \tag{4.4}$$

Recall that in Problems we had the formula $\mathcal{L}\left(t^{-\frac{1}{2}}\right) = \sqrt{\frac{\pi}{s}}$, or in terms of the variable y

$$\mathcal{L}\left(y^{-\frac{1}{2}}\right) = \sqrt{\frac{\pi}{s}}\,. \tag{4.5}$$

Now apply the Laplace transform to the equation (4.4), and get

$$\sqrt{\frac{\pi}{s}}\mathcal{L}\left(f'(y)\right) = \sqrt{2g}\,T\,\frac{1}{s}\,.$$

Solving for $\mathcal{L}\left(f'(y)\right)$, gives

$$\mathcal{L}\left(f'(y)\right) = \frac{T}{\pi}\sqrt{2g}\,\sqrt{\frac{\pi}{s}} = \sqrt{a}\,\sqrt{\frac{\pi}{s}}\,,$$

where we denoted $a = \frac{T^2}{\pi^2}2g$. Using (4.5) again

$$f'(y) = \sqrt{a}y^{-1/2}\,. \tag{4.6}$$

We have $ds = \sqrt{dx^2 + dy^2}$, and so $f'(y) = \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$. Use this expression in (4.6):

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{a}\frac{1}{\sqrt{y}}\,. \tag{4.7}$$

This is a first order differential equation. One could solve it for $\frac{dx}{dy}$, and then separate the variables. But it appears easier to use the parametric integration technique. To do that, we solve this equation for y

$$y = \frac{a}{1 + \left(\frac{dx}{dy}\right)^2}\,, \tag{4.8}$$

and set

$$\frac{dx}{dy} = \frac{1 + \cos\theta}{\sin\theta}\,, \tag{4.9}$$

where θ is a parameter. Using (4.9) in (4.8), express

$$y = a\frac{\sin^2\theta}{\sin^2\theta + (1+\cos\theta)^2} = a\frac{\sin^2\theta}{2+2\cos\theta} = \frac{a}{2}\frac{1-\cos^2\theta}{1+\cos\theta} = \frac{a}{2}(1-\cos\theta)\,.$$

It follows that $dy = \frac{a}{2}\sin\theta\,d\theta$. Then from (4.9), we get

$$dx = \frac{1+\cos\theta}{\sin\theta}dy = \frac{a}{2}(1+\cos\theta)\,d\theta\,.$$

Compute x by integration, obtaining

$$x = \frac{a}{2}(\theta+\sin\theta), \quad y = \frac{a}{2}(1-\cos\theta)\,,$$

which is a parametric representation of the tautochrone. The name of this curve is *cycloid*, and it appears in many other applications.

4.5 Distributions

A function $f(t)$ converts a number t into a number $f(t)$. (Sometimes we do not use all real t, but restrict $f(t)$ to a smaller domain.) *Functionals* convert functions into numbers. We shall allow only "nice" functions to be plugged into functionals.

A function $\varphi(t)$, defined on the interval $(-\infty,\infty)$, is said to be *of compact support* if it is equal to zero outside of some bounded interval (a,b). Functions that are infinitely differentiable on $(-\infty,\infty)$, and of compact support, are called *test functions*. We shall reserve writing $\varphi(t)$ exclusively for test functions.

Definition *Distribution* is a linear functional on test functions. Notation: (f,φ). (A test function $\varphi(t)$ goes in, the number (f,φ) is the output. One usually also assumes (f,φ) to be continuous in φ, but we do not need that in a brief presentation.)

Example 1 Let $f(t)$ be any continuous function. Define the distribution

$$(f,\varphi) = \int_{-\infty}^{\infty} f(t)\varphi(t)\,dt\,. \tag{5.10}$$

Convergence is not a problem here, because the integrand vanishes outside of some bounded interval. This functional is linear, because

$$(f, c_1\varphi_1 + c_2\varphi_2) = \int_{-\infty}^{\infty} f(t)\left(c_1\varphi_1 + c_2\varphi_2\right)\,dt$$

$$= c_1 \int_{-\infty}^{\infty} f(t)\varphi_1 \, dt + c_2 \int_{-\infty}^{\infty} f(t)\varphi_2 \, dt = c_1(f, \varphi_1) + c_2(f, \varphi_2) \, ,$$

for any two constants c_1 and c_2, and any two test functions $\varphi_1(t)$ and $\varphi_2(t)$. This way "usual" functions can be viewed as distributions. The formula (5.10) lets us consider $f(t)$ *in the sense of distributions*.

Example 2 *The Delta distribution.* Define

$$(\delta(t), \varphi) = \varphi(0) \, .$$

(Compare this with (5.10), and the intuitive formula $\int_{-\infty}^{\infty} \delta(t)\varphi(t) \, dt = \varphi(0)$ from Section 4.3.) We see that in the realm of distributions the delta function $\delta(t)$ sits next to usual functions, as an equal member of the club.

Assume that $f(t)$ is a differentiable function. Viewing $f'(t)$ as a distribution, we have

$$(f', \varphi) = \int_{-\infty}^{\infty} f'(t)\varphi(t) \, dt = -\int_{-\infty}^{\infty} f(t)\varphi'(t) \, dt = -(f, \varphi') \, ,$$

using integration by parts (recall that $\varphi(t)$ is zero outside of some bounded interval). Motivated by this formula, we now define the derivative of *any distribution* f:

$$(f', \varphi) = -(f, \varphi') \, .$$

In particular,

$$(\delta', \varphi) = -(\delta, \varphi') = -\varphi'(0) \, .$$

(So that δ' is another distribution. We know how it acts on test functions.) Similarly, $(\delta'', \varphi) = -(\delta', \varphi') = \varphi''(0)$, and in general

$$(\delta^{(n)}, \varphi) = (-1)^n \varphi^{(n)}(0) \, .$$

We see that all distributions are infinitely differentiable! In particular, all continuous functions are infinitely differentiable, if we view them as distributions.

Example 3 The Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } -\infty \le t < 0 \\ 1 & \text{if } t \ge 0 \end{cases}$$

has a jump at $t=0$. Clearly, $H(t)$ is not differentiable at $t=0$. But, in the sense of distributions, we claim that

$$H'(t) = \delta(t)\,.$$

Indeed,

$$(H'(t),\varphi) = -(H(t),\varphi') = -\int_0^\infty \varphi'(t)\,dt = \varphi(0) = (\delta(t),\varphi)\,.$$

Example 4 The function $|t|$ is not differentiable at $t=0$. But, in the sense of distributions,

$$|t|' = 2H(t) - 1\,.$$

Indeed,

$$(|t|',\varphi) = -(|t|,\varphi') = -\int_{-\infty}^0 (-t)\varphi'(t)\,dt - \int_0^\infty t\varphi'(t)\,dt\,.$$

Integrating by parts in both integrals, we continue

$$(|t|',\varphi) = -\int_{-\infty}^0 \varphi(t)\,dt + \int_0^\infty \varphi(t)\,dt = (2H(t)-1,\varphi)\,.$$

4.5.1 Problems

I. Find the Laplace transform of the following functions.

1. $5+2t^3-e^{-4t}$. Answer. $\dfrac{5}{s}+\dfrac{12}{s^4}-\dfrac{1}{s+4}$.

2. $2\sin 3t - t^3$. Answer. $\dfrac{6}{s^2+9}-\dfrac{6}{s^4}$.

3. $\cosh 2t - e^{4t}$. Answer. $\dfrac{s}{s^2-4}-\dfrac{1}{s-4}$.

4. $e^{2(t-1)}$. Answer. $\dfrac{1}{e^2(s-2)}$.

5. $e^{2t}\cos 3t$. Answer. $\dfrac{s-2}{(s-2)^2+9}$.

6. $\dfrac{t^3-3t}{t}$. Answer. $\dfrac{2}{s^3}-\dfrac{3}{s}$.

7. $e^{-3t}t^4$. Answer. $\dfrac{24}{(s+3)^5}$

8. $\sin^2 2t$. Answer. $\dfrac{1}{2s} - \dfrac{s}{2(s^2+16)}$.

9. $\sin 2t \cos 2t$. Answer. $\dfrac{2}{s^2+16}$.

10. $\sinh t \cosh t$. Answer. $\dfrac{1}{s^2-4}$.

11. $|t-2|$. Answer. $\dfrac{2e^{-2s}+2s-1}{s^2}$.

Hint: Split the integral into two pieces.

12. $f(t) = t$ for $1 < t < 3$, and $f(t) = 0$ for all other $t > 0$.

Answer. $F(s) = \dfrac{e^{-s}(s+1)}{s^2} - \dfrac{e^{-3s}(3s+1)}{s^2}$.

II. Find the inverse Laplace transform of the following functions.

1. $\dfrac{1}{s^2+4} - \dfrac{2}{s^3}$. Answer. $\dfrac{1}{2}\sin 2t - t^2$.

2. $\dfrac{s}{s^2-9} - \dfrac{2}{s+3}$. Answer. $\cosh 3t - 2e^{-3t}$.

3. $\dfrac{1}{s^2+s}$. Answer. $1 - e^{-t}$.

4. $\dfrac{1}{s^2-3s}$. Answer. $\dfrac{1}{3}e^{3t} - \dfrac{1}{3}$.

5. $\dfrac{1}{s^3-7s+6}$. Answer. $\dfrac{1}{5}e^{2t} - \dfrac{1}{4}e^{t} + \dfrac{1}{20}e^{-3t}$.

6. $\dfrac{1}{s^3+s}$. Answer. $1 - \cos t$.

7. $\dfrac{1}{(s^2+1)(s^2+4)}$. Answer. $\dfrac{1}{3}\sin t - \dfrac{1}{6}\sin 2t$.

8. $\dfrac{s}{s^4+5s^2+4}$. Answer. $\dfrac{1}{3}\cos t - \dfrac{1}{3}\cos 2t$.

9. $\dfrac{1}{s^2+2s+10}$. Answer. $\dfrac{1}{3}e^{-t}\sin 3t$.

10. $\dfrac{1}{s^2+s-2}$. Answer. $\dfrac{1}{3}e^{t} - \dfrac{1}{3}e^{-2t}$.

11. $\dfrac{s}{s^2+s+1}$. Answer. $e^{-\frac{1}{2}t}\left[\cos\dfrac{\sqrt{3}}{2}t - \dfrac{1}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}t\right]$.

12. $\dfrac{s-1}{s^2-s-2}$. Answer. $\dfrac{1}{3}e^{2t}+\dfrac{2}{3}e^{-t}$.

13. $\dfrac{s+3}{4s^2+1}$. Answer. $\dfrac{1}{4}\cos\dfrac{t}{2}+\dfrac{3}{2}\sin\dfrac{t}{2}$.

14. $\dfrac{s}{4s^2-4s+5}$. Answer. $e^{\frac{1}{2}t}\left(\dfrac{1}{4}\cos t+\dfrac{1}{8}\sin t\right)$.

15. $\dfrac{s+2}{s^3-3s^2+2s}$. Answer. $1-3e^t+2e^{2t}$.

16. $\dfrac{s^3-s}{s^4+5s^2+4}$. Answer. $-\dfrac{2}{3}\cos t+\dfrac{5}{3}\cos 2t$.

17. $\dfrac{s^2+2}{s^3-2s^2+2s}$. Answer. $1+2e^t\sin t$.

18*. (i) Consider $\dfrac{p(s)}{q(s)}$, where $q(s)=(s-s_1)(s-s_2)\cdots(s-s_n)$ with some numbers $s_1,s_2,\ldots,s_n$, and $p(s)$ is a polynomial of degree less than n. By the method of partial fractions

$$\frac{p(s)}{q(s)}=\frac{a_1}{s-s_1}+\frac{a_2}{s-s_2}+\cdots+\frac{a_n}{s-s_n}\,, \tag{5.11}$$

for some numbers $a_1,a_2,\ldots,a_n$. Show that $a_1=\frac{p(s_1)}{q'(s_1)}$, and derive similar formulas for the other a_i's.

Hint: Multiply (5.11) by $s-s_1$, take the limit as $s\to s_1$, and use L'Hospital's rule.

(ii) Show that

$$\mathcal{L}^{-1}\left(\frac{p(s)}{q(s)}\right)=\sum_{i=1}^{n}\frac{p(s_i)}{q'(s_i)}\,e^{s_it}\,.$$

(iii) Calculate $\mathcal{L}^{-1}\left(\dfrac{s^2+5}{(s-1)(s-2)(s-3)}\right)$. Answer. $y=3e^t-9e^{2t}+7e^{3t}$.

III. Using the Laplace transform, solve the following initial value problems.

1. $y''+3y'+2y=0$, $y(0)=-1$, $y'(0)=2$. Answer. $y=-e^{-2t}$.

2. $y''+2y'+5y=0$, $y(0)=1$, $y'(0)=-2$.

Answer. $\frac{1}{2}e^{-t}\left(2\cos 2t-\sin 2t\right)$.

3. $y''+y=\sin 2t$, $y(0)=0$, $y'(0)=1$. Answer. $y=\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$.

4. $y''+2y'+2y=e^t$, $y(0)=0$, $y'(0)=1$.

Answer. $y=\frac{1}{5}e^t-\frac{1}{5}e^{-t}\left(\cos t-3\sin t\right)$.

5. $y''''-y=0$, $y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y'''(0)=0$.

Answer. $y=\frac{1}{2}\sin t+\frac{1}{2}\sinh t$.

6. $y''''-16y=0$, $y(0)=0$, $y'(0)=2$, $y''(0)=0$, $y'''(0)=8$.

Answer. $y=\sinh 2t$.

7. $y'''+3y''+2y'=0$, $y(0)=0$, $y'(0)=0$, $y''(0)=-1$.

Answer. $y=-\frac{1}{2}-\frac{e^{-2t}}{2}+e^{-t}$.

8. $y'''+3y''+3y'+y=e^{-t}$, $y(0)=0$, $y'(0)=0$, $y''(0)=0$.

Answer. $y=\frac{t^3e^{-t}}{6}$.

IV.

1. (a) Let $s>0$. Show that

$$\int_{-\infty}^{\infty} e^{-sx^2}\,dx=\sqrt{\frac{\pi}{s}}\,.$$

Hint: Denote $I=\int_{-\infty}^{\infty} e^{-sx^2}\,dx$. Then

$$I^2=\int_{-\infty}^{\infty} e^{-sx^2}\,dx\int_{-\infty}^{\infty} e^{-sy^2}\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s(x^2+y^2)}\,dA\,.$$

This is a double integral over the entire xy-plane. Evaluate it by using the polar coordinates, to obtain $I^2=\frac{\pi}{s}$.

(b) Show that $\mathcal{L}\left(t^{-\frac{1}{2}}\right)=\sqrt{\frac{\pi}{s}}$.

Hint: $\mathcal{L}\left(t^{-\frac{1}{2}}\right)=\int_0^{\infty} t^{-\frac{1}{2}}e^{-st}\,dt$. Make a change of variables $t\to x$, by letting $x=t^{\frac{1}{2}}$. Then $\mathcal{L}\left(t^{-\frac{1}{2}}\right)=2\int_0^{\infty} e^{-sx^2}\,dx=\int_{-\infty}^{\infty} e^{-sx^2}\,dx=\sqrt{\frac{\pi}{s}}$.

2. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Hint: Consider $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{x}\,dx$, and calculate its Laplace transform $F(s) = \frac{\pi}{2}\frac{1}{s}$.

3. Solve the following system of differential equations

$$\frac{dx}{dt} = 2x - y, \quad x(0) = 4$$

$$\frac{dy}{dt} = -x + 2y, \quad y(0) = -2\,.$$

Answer. $x(t) = e^t + 3e^{3t}$, $y(t) = e^t - 3e^{3t}$.

4. Solve the following non-homogeneous system of differential equations

$$x' = 2x - 3y + t, \quad x(0) = 0$$
$$y' = -2x + y, \quad y(0) = 1\,.$$

Answer. $x(t) = e^{-t} - \dfrac{9}{16}e^{4t} + \dfrac{1}{16}(-7+4t)$, $y(t) = e^{-t} + \dfrac{3}{8}e^{4t} + \dfrac{1}{8}(-3+4t)$.

V.

1. A function $f(t)$ is equal to 1 for $1 \le t < 5$, and is equal to 0 for all other $t \ge 0$. Represent $f(t)$ as a difference of two step functions, and find its Laplace transform.

Answer. $f(t) = u_1(t) - u_5(t)$, $F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-5s}}{s}$.

2. A function $g(t)$ is equal to 1 for $0 \le t < 5$, and is equal to -2 for $t \ge 5$. Represent $g(t)$ using step functions, and find its Laplace transform.

Answer. $g(t) = 1 - 3u_5(t)$, $G(s) = \dfrac{1}{s} - 3\dfrac{e^{-5s}}{s}$.

3. A function $h(t)$ is equal to -2 for $0 \le t < 3$, to 4 for $3 \le t < 7$, and to zero for $t \ge 7$. Represent $h(t)$ using step functions, and find its Laplace transform.

Answer. $h(t) = -2 + 6u_3(t) - 4u_7(t)$, $H(s) = -\dfrac{2}{s} + 6\dfrac{e^{-3s}}{s} - 4\dfrac{e^{-7s}}{s}$.

4. A function $k(t)$ is equal to t for $0 \le t < 4$, and to 4 for $t \ge 4$. Represent $k(t)$ using step functions, and find its Laplace transform.

Answer. $k(t) = t\,(1 - u_4(t)) + 4u_4(t) = t - u_4(t)\,(t-4)$, $K(s) = \dfrac{1}{s^2} - \dfrac{e^{-4s}}{s^2}$, by using the second shift formula (2.5).

5. Find the Laplace transform of $t^2 - 2u_4(t)$. Answer. $F(s) = \dfrac{2}{s^3} - 2\dfrac{e^{-4s}}{s}$.

6. Sketch the graph of the function $u_2(t) - 2u_3(t) + 4u_6(t)$, and find its Laplace transform.

7. Find the inverse Laplace transform of $\dfrac{1}{s^2}\left(2e^{-s} - 3e^{-4s}\right)$.

Answer. $2u_1(t)(t-1) - 3u_4(t)(t-4)$.

8. Find the inverse Laplace transform of $e^{-2s}\dfrac{3s-1}{s^2+4}$.

Answer. $u_2(t)\left[3\cos 2(t-2) - \dfrac{1}{2}\sin 2(t-2)\right]$.

9. Find the inverse Laplace transform of $e^{-s}\dfrac{1}{s^2+s-6}$.

Answer. $u_1(t)\left(\dfrac{1}{5}e^{2t-2} - \dfrac{1}{5}e^{-3t+3}\right)$.

10. Find the inverse Laplace transform of $e^{-\frac{\pi}{2}s}\dfrac{1}{s^2+2s+5}$, and simplify the answer.

Answer. $-\dfrac{1}{2}u_{\pi/2}(t)e^{-t+\pi/2}\sin 2t$.

11. Solve

$$y'' + y = 4u_1(t) - u_5(t), \quad y(0) = 2, \; y'(0) = 0.$$

Answer. $2\cos t + 4u_1(t)\,[1 - \cos(t-1)] - u_5(t)\,[1 - \cos(t-5)]$.

12. Solve

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \; y'(0) = -1.$$

Answer. $y(t) = e^{-2t} - e^{-t} + u_2(t)\left[\dfrac{1}{2} - e^{-(t-2)} + \dfrac{1}{2}e^{-2(t-2)}\right]$.

13. Solve

$$y'' + 4y = u_\pi(t)\,\sin t, \quad y(0) = -1, \; y'(0) = 1.$$

Hint: Write $\sin t = -\sin(t-\pi)$, and use the shift formula (2.5).

Answer. $y(t) = -\cos 2t + \dfrac{1}{2}\sin 2t + u_\pi(t)\left(\dfrac{1}{3}\sin t + \dfrac{1}{6}\sin 2t\right)$.

14. Solve

$$y'' + y = g(t), \quad y(0) = 0, \; y'(0) = 0\,,$$

where

$$g(t) = \begin{cases} t & \text{if } 0 \le t < \pi \\ \pi & \text{if } t \ge \pi \end{cases} .$$

Answer. $y(t) = t - \sin t - u_\pi(t)\,(t - \pi + \sin t)$.

VI.

1. Show that $\mathcal{L}\left(u_c'(t)\right) = \mathcal{L}\left(\delta(t-c)\right)$, $c > 0$.

This formula shows that $u_c'(t) = \delta(t-c)$.

2. Find the Laplace transform of $\delta(t-4) - 2u_4(t)$.

3. Find the inverse Laplace transform of $\dfrac{s+1}{s+3}$. Answer. $\delta(t) - 2e^{-3t}$.

4. Find the inverse Laplace transform of $\dfrac{s^2+1}{s^2+2s+2}$.

Answer. $\delta(t) - e^{-t}(2\cos t - \sin t)$.

5. Solve

$$y'' + y = \delta(t-\pi), \; y(0) = 0, \; y'(0) = 2.$$

Answer. $y(t) = 2\sin t - u_\pi(t)\sin t$.

6. Solve

$$y'' + 2y' + 10y = \delta(t-\pi), \; y(0) = 0, \; y'(0) = 0.$$

Answer. $y(t) = -\dfrac{1}{3}u_\pi(t)e^{-t+\pi}\sin 3t$.

7. Solve

$$4y'' + y = \delta(t), \; y(0) = 0, \; y'(0) = 0.$$

Answer. $y(t) = \dfrac{1}{2}\sin\dfrac{1}{2}t$.

8. Solve

$$4y'' + 4y' + 5y = \delta(t-2\pi), \; y(0) = 0, \; y'(0) = 1.$$

Answer. $y(t) = e^{-\frac{1}{2}t}\sin t + \dfrac{1}{4}u_{2\pi}(t)\,e^{-\frac{1}{2}(t-2\pi)}\sin t$.

9. Show that $\mathcal{L}\left(\delta(t-t_0)f(t)\right) = e^{-st_0}f(t_0)$.

10. Solve

$$y'' + 4y = \delta(t - \frac{\pi}{3})\cos t, \; y(0) = 0, \; y'(0) = 0.$$

Answer. $y(t) = \frac{1}{4} u_{\pi/3}(t) \sin 2(t - \frac{\pi}{3})$.

VII.

1. Show that $\sin t * 1 = 1 - \cos t$. (Observe that $\sin t * 1 \neq \sin t$.)

2. Show that $f(t) * \delta(t) = f(t)$, for any $f(t)$.

(So that the delta function plays the role of *unity* for convolution.)

3. Find the convolution $t * t$. Answer. $\frac{t^3}{6}$.

4. Find the convolution $t * \sin at$. Answer. $\frac{at - \sin at}{a^2}$.

5. Find the convolution $\cos t * \cos t$.

Hint: $\cos(t - v) = \cos t \cos v + \sin t \sin v$. Answer. $\frac{1}{2} t \cos t + \frac{1}{2} \sin t$.

6. Using convolutions, find the inverse Laplace transform of the following functions

(a) $\frac{1}{s^3(s^2+1)}$. Answer. $\frac{t^2}{2} * \sin t = \frac{t^2}{2} + \cos t - 1$.

(b) $\frac{s}{(s+1)(s^2+9)}$. Answer. $-\frac{e^{-t}}{10} + \frac{1}{10} \cos 3t + \frac{3}{10} \sin 3t$.

(c) $\frac{s}{(s^2+1)^2}$. Answer. $\frac{1}{2} t \sin t$.

(d) $\frac{1}{(s^2+9)^2}$. Answer. $\frac{1}{54} \sin 3t - \frac{3}{54} t \cos 3t$.

7. Solve the following initial value problem at resonance

$$y'' + 9y = \cos 3t, \quad y(0) = 0, \; y'(0) = 0.$$

Answer. $y(t) = \frac{1}{6} t \sin 3t$.

8. Solve the initial value problem with a given forcing term $g(t)$

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0.$$

Answer. $y(t) = \frac{1}{2} \int_0^t \sin 2(t - v) g(v) \, dv$.

9. Find $f(t)$ given that

$$\mathcal{L}\left(t * f(t)\right) = \frac{1}{s^4(s^2+1)}\,.$$

Answer. $t - \sin t$.

10. Find the Laplace transform of $\displaystyle\int_0^t e^{-(t-v)} \cos v \, dv$.

Answer. $\mathcal{L}\left(e^{-t}\right)\mathcal{L}\left(\cos t\right) = \dfrac{s}{(s+1)(s^2+1)}$.

11. Solve the following Volterra's integral equation

$$y(t) + \int_0^t (t-v)y(v)\,dv = \cos 2t\,.$$

Answer. $y(t) = -\dfrac{1}{3}\cos t + \dfrac{4}{3}\cos 2t$.

12. By using the Laplace transform, calculate $t * t * t$. Answer. $\dfrac{t^5}{5!}$.

VIII.

1. Find the second derivative of $|t|$ in the sense of distributions.

Answer. $2\delta(t)$.

2. Find $f(t)$, such that (n is a positive integer)

$$f^{(n)}(t) = \delta(t)\,.$$

Answer. $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t^{n-1}}{(n-1)!} & \text{if } t \geq 0 \end{cases}$.

3. Let $f(t) = \begin{cases} t^2 & \text{if } t < 0 \\ t^2 + 5 & \text{if } t \ \geq 0 \end{cases}$.

Show that $f'(t) = 2t + 5\delta(t)$.

Hint: $f(t) = t^2 + 5H(t)$.

4. Consider a family of functions $f_\epsilon(t) = \dfrac{\epsilon}{\pi\left(t^2+\epsilon^2\right)}$, $\epsilon > 0$. Show that in the sense of distributions

$$\lim_{\epsilon \to 0} f_\epsilon(t) = \delta(t)\,.$$

One refers to $f_\epsilon(t)$ as a *delta sequence*. (Other delta sequences can be found in the book of M. Renardy and R.C. Rogers [25].)

Hint: $(f_\epsilon, \varphi) = \int_{-\infty}^{\infty} \frac{\epsilon \varphi(t)}{\pi\,(t^2+\epsilon^2)}\,dt = \int_{-\infty}^{\infty} \frac{\varphi(\epsilon z)}{\pi\,(z^2+1)}\,dz$

$$\to \varphi(0) \int_{-\infty}^{\infty} \frac{1}{\pi\,(z^2+1)}\,dz = \varphi(0) = (\delta, \varphi)\,.$$

5. (i) The function $K(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$ is known as the *heat kernel* ($k > 0$ is a constant). Show that in the sense of distributions

$$\lim_{t \to 0} K(x,t) = \delta(x)\,.$$

(ii) Conclude that

$$\lim_{t \to 0} K(x,t) * f(x) = \lim_{t \to 0} \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy = f(x)\,.$$

(iii) Show that the function $K(x,t)$ satisfies the *heat equation*

$$K_t = k K_{xx}\,.$$

(iv) Conclude that the function $u(x,t) = K(x,t) * f(x)$ satisfies

$$u_t = k u_{xx}\,, \quad u(x,0) = f(x)\,.$$

Chapter 5

Linear Systems of Differential Equations

We begin this chapter by solving systems of linear differential equations with constant coefficients, using the eigenvalues and eigenvectors of the corresponding coefficient matrices. Then we study the long term properties of these systems, and the notion of the exponential of a matrix. We develop the Floquet theory for systems with periodic coefficients, and make an application to Massera's theorem. We classify the pictures at the origin for 2×2 systems, and discuss the controllability and observability of linear systems.

5.1 The Case of Distinct Eigenvalues

The case when the coefficient matrix has distinct eigenvalues turns out to be the easiest. We begin by recalling the basic notions of matrix theory.

5.1.1 Review of Vectors and Matrices

Recall that given two vectors

$$C_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ C_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

we can add them as $C_1 + C_2 = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$, or multiply by a constant x: $xC_1 = \begin{bmatrix} xa_1 \\ xa_2 \\ xa_3 \end{bmatrix}$. More generally, we can compute the *linear combination*

$$x_1 C_1 + x_2 C_2 = \begin{bmatrix} x_1 a_1 + x_2 b_1 \\ x_1 a_2 + x_2 b_2 \\ x_1 a_3 + x_2 b_3 \end{bmatrix},$$

for any two constants x_1 and x_2.

We shall be dealing only with the square matrices, like the following 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{1.1}$$

We shall view A as a row of column vectors $A = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$, where

$$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \quad C_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}.$$

The product of a matrix A and of a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is defined as the vector

$$Ax = C_1 x_1 + C_2 x_2 + C_3 x_3 .$$

(This definition is equivalent to the more traditional one, that you might have seen before.) We get

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}.$$

Two vectors C_1 and C_2 are called *linearly dependent* if one of them is a constant multiple of the other, so that $C_2 = aC_1$, for some number a. (The zero vector is linearly dependent with any other vector.) Linearly dependent vectors C_1 and C_2 go along the same line. If the vectors C_1 and C_2 do not go along the same line, they are *linearly independent*. Three vectors C_1, C_2 and C_3 are called *linearly dependent* if one of them is a linear combination of the others, e.g., if $C_3 = aC_1 + bC_2$, for some numbers a, b. This means that C_3 lies in the plane determined by C_1 and C_2, so that all three vectors lie in the same plane. If C_1, C_2 and C_3 do not lie in the same plane, they are *linearly independent*.

A system of 3 equations with 3 unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\tag{1.2}
$$

can be written in the matrix form

$$Ax = b\,,$$

where A is the 3×3 matrix above, and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is the given vector of the right hand sides. Recall that the system (1.2) has a unique solution for any vector b if and only if the columns of the matrix A are linearly independent. (In that case, the determinant $|A| \neq 0$, and the inverse matrix A^{-1} exists.)

5.1.2 Linear First Order Systems with Constant Coefficients

We wish to find the functions $x_1(t)$, $x_2(t)$ and $x_3(t)$ that solve the following system of equations, with given constant coefficients $a_{11}, \ldots, a_{33}$,

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3\,,
\end{aligned}
\tag{1.3}
$$

subject to the given initial conditions

$$x_1(t_0) = \alpha, \;\; x_2(t_0) = \beta, \;\; x_3(t_0) = \gamma\,.$$

We may write this system using matrix notation:

$$x' = Ax, \;\; x(t_0) = x_0\,, \tag{1.4}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ is the unknown vector function, A is the 3×3 matrix (1.1) of the coefficients, and $x_0 = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is the vector of initial conditions. Indeed, on the left in (1.3) we have components of the vector $x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix}$, while on the right we see the components of the vector Ax.

Let us observe that given two vector functions $y(t)$ and $z(t)$, which are solutions of the system $x' = Ax$, their *linear combination* $c_1y(t) + c_2z(t)$ is also a solution of the same system, for any constants c_1 and c_2. Our system of differential equations is *linear*, because it involves only linear combinations of the unknown functions.

We now search for a solution of (1.4) in the form

$$x(t) = e^{\lambda t}\xi\,, \tag{1.5}$$

with a constant λ, and a vector ξ, with entries independent of t. Substituting this into (1.4), we have

$$\lambda e^{\lambda t}\xi = A\left(e^{\lambda t}\xi\right),$$

giving

$$A\xi = \lambda\xi\,.$$

So that if λ is an eigenvalue of A, and ξ the corresponding eigenvector, then (1.5) gives us a solution of the problem (1.4). Observe that the same is true for any square $n \times n$ matrix A. Let λ_1, λ_2 and λ_3 be the eigenvalues of our 3×3 matrix A. There are several cases to consider.

Case 1 The eigenvalues of A are real and distinct. It is known from matrix theory that the corresponding eigenvectors ξ_1, ξ_2 and ξ_3 are then linearly independent. We know that $e^{\lambda_1 t}\xi_1$, $e^{\lambda_2 t}\xi_2$ and $e^{\lambda_3 t}\xi_3$ are solutions of our system (1.4), so that their linear combination

$$x(t) = c_1e^{\lambda_1 t}\xi_1 + c_2e^{\lambda_2 t}\xi_2 + c_3e^{\lambda_3 t}\xi_3 \tag{1.6}$$

also solves the system (1.4). We claim that (1.6) gives the general solution of our system, meaning that it is possible to determine the constants c_1, c_2, and c_3 to satisfy any initial conditions:

$$x(t_0) = c_1 e^{\lambda_1 t_0}\xi_1 + c_2 e^{\lambda_2 t_0}\xi_2 + c_3 e^{\lambda_3 t_0}\xi_3 = x_0 \,. \tag{1.7}$$

This is a system of three linear equations with three unknowns c_1, c_2, and c_3. The matrix of this system is non-singular, because its columns, $e^{\lambda_1 t_0}\xi_1$, $e^{\lambda_2 t_0}\xi_2$ and $e^{\lambda_3 t_0}\xi_3$, are linearly independent (observe that these columns are constant multiples of the linearly independent vectors ξ_1, ξ_2 and ξ_3). Therefore, we can find a unique solution triple $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3$ of the system (1.7). Then $x(t) = \bar{c}_1 e^{\lambda_1 t}\xi_1 + \bar{c}_2 e^{\lambda_2 t}\xi_2 + \bar{c}_3 e^{\lambda_3 t}\xi_3$ is the desired solution of our initial value problem (1.4).

Example 1 Solve the system

$$x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x, \;\; x(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

This is a 2×2 system, so that there are only two terms in (1.6). We compute the eigenvalues $\lambda_1 = 1$, and the corresponding eigenvector $\xi_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\lambda_2 = 3$, with the corresponding eigenvector $\xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

or in components

$$\begin{aligned} x_1(t) &= -c_1 e^t + c_2 e^{3t} \\ x_2(t) &= c_1 e^t + c_2 e^{3t} \,. \end{aligned}$$

Turning to the initial conditions,

$$\begin{aligned} x_1(0) &= -c_1 + c_2 = -1 \\ x_2(0) &= c_1 + c_2 = 2 \,. \end{aligned}$$

Calculate $c_1 = 3/2$, $c_2 = 1/2$. Answer:

$$\begin{aligned} x_1(t) &= -\tfrac{3}{2} e^t + \tfrac{1}{2} e^{3t} \\ x_2(t) &= \tfrac{3}{2} e^t + \tfrac{1}{2} e^{3t} \,. \end{aligned}$$

Case 2 The eigenvalue λ_1 is double, $\lambda_2 = \lambda_1$, $\lambda_3 \neq \lambda_1$, however, λ_1 has two linearly independent eigenvectors ξ_1 and ξ_2. Let ξ_3 denote again an eigenvector corresponding to λ_3. This vector cannot lie in the plane spanned by ξ_1 and ξ_2, and then the vectors ξ_1, ξ_2 and ξ_3 are linearly independent. The general solution is given again by the formula (1.6), with λ_2 replaced by λ_1:

$$x(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_1 t} \xi_2 + c_3 e^{\lambda_3 t} \xi_3 \,.$$

To satisfy the initial conditions, we get a linear system for c_1, c_2 and c_3

$$c_1 e^{\lambda_1 t_0} \xi_1 + c_2 e^{\lambda_1 t_0} \xi_2 + c_3 e^{\lambda_3 t_0} \xi_3 = x_0 \,,$$

which has a unique solution, because its matrix has linearly independent columns. (Linearly independent eigenvectors is the key here!)

Example 2 Solve the system

$$x' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} x \,, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} .$$

Expanding the determinant of $A - \lambda I$ in the first row, we write the characteristic equation as

$$(2-\lambda)\left[(2-\lambda)^2 - 1\right] - (2-\lambda-1) + 1 + \lambda - 2 = 0 \,,$$

$$(2-\lambda)(1-\lambda)(3-\lambda) - 2(1-\lambda) = 0 \,,$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda) - 2\right] = 0 \,.$$

The roots are: $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 4$. We calculate that the double eigenvalue $\lambda_1 = 1$ has two linearly independent eigenvectors $\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\xi_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. The other eigenvalue $\lambda_3 = 4$ has the corresponding eigenvector $\xi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

Or, in components,

$$\begin{aligned} x_1(t) &= -c_1e^t + c_3e^{4t} \\ x_2(t) &= -c_2e^t + c_3e^{4t} \\ x_3(t) &= c_1e^t + c_2e^t + c_3e^{4t}\,. \end{aligned}$$

Using the initial conditions, calculate $c_1 = -2$, $c_2 = -1$, and $c_3 = -1$. Answer:

$$\begin{aligned} x_1(t) &= 2e^t - e^{4t} \\ x_2(t) &= e^t - e^{4t} \\ x_3(t) &= -3e^t - e^{4t}\,. \end{aligned}$$

Proceeding similarly, we can solve the initial value problem (1.4) for any $n \times n$ matrix A, provided that all of its eigenvalues are real, and it has a *full set of n linearly independent eigenvectors*. Recall that if an $n \times n$ matrix A is symmetric ($a_{ij} = a_{ji}$, for all i and j), then all of its eigenvalues are real. The eigenvalues of a symmetric matrix may be repeated, but there is always a full set of n linearly independent eigenvectors. So that one can solve the initial value problem (1.4) for any system with a symmetric matrix.

Case 3 The eigenvalue λ_1 has multiplicity two (λ_1 is a double root of the characteristic equation, $\lambda_2 = \lambda_1$), $\lambda_3 \neq \lambda_1$, but λ_1 has only one linearly independent eigenvector ξ. The eigenvalue λ_1 brings in only one solution $e^{\lambda_1 t}\xi$. By analogy with the second order equations, we try $te^{\lambda_1 t}\xi$ for the second solution. However, this vector function is a scalar multiple of the first solution, so that it is linearly dependent with it, at any $t = t_0$. We modify our guess:

$$x(t) = te^{\lambda_1 t}\xi + e^{\lambda_1 t}\eta\,, \tag{1.8}$$

and choose a vector η, to obtain a second linearly independent solution. Substituting (1.8) into our system (1.4), and using that $A\xi = \lambda_1\xi$, obtain

$$e^{\lambda_1 t}\xi + \lambda_1 te^{\lambda_1 t}\xi + \lambda_1 e^{\lambda_1 t}\eta = \lambda_1 te^{\lambda_1 t}\xi + e^{\lambda_1 t}A\eta.$$

Cancelling a pair of terms, and dividing by $e^{\lambda_1 t}$, we simplify this to

$$(A - \lambda_1 I)\eta = \xi\,. \tag{1.9}$$

Even though the matrix $A - \lambda_1 I$ is singular (its determinant is zero), it can be shown (using the Jordan normal form) that the linear system (1.9) always has a solution η, called *generalized eigenvector*. We see from (1.9) that η

is not a multiple of ξ. Using this η in (1.8), gives us the second linearly independent solution, corresponding to $\lambda = \lambda_1$. (Observe that $c\,\eta$ is *not* a generalized eigenvector for any constant $c \neq 1$, unlike the usual eigenvectors. If η is a generalized eigenvector, then so is $\eta + c\xi$, for any constant c.)

Example 3 Solve the system

$$x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x \,.$$

This matrix has a double eigenvalue $\lambda_1 = \lambda_2 = 2$, and only one linearly independent eigenvector $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We have one solution: $x_1(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The system (1.9) to determine the vector $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ takes the form

$$\begin{aligned} -\eta_1 - \eta_2 &= 1 \\ \eta_1 + \eta_2 &= -1 \,. \end{aligned}$$

Discard the second equation, because it is a multiple of the first. The first equation has infinitely many solutions. But all we need is just one solution, that is not a multiple of ξ. So we set $\eta_2 = 0$, which gives $\eta_1 = -1$. We computed the second linearly independent solution:

$$x_2(t) = te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 \\ 0 \end{bmatrix} .$$

The general solution is then

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) .$$

5.2 A Pair of Complex Conjugate Eigenvalues

5.2.1 Complex Valued and Real Valued Solutions

Recall that one differentiates complex valued functions much the same way, as the real ones. For example,

$$\frac{d}{dt} e^{it} = ie^{it} \,,$$

where $i = \sqrt{-1}$ is treated the same way as any other constant. Any complex valued function $f(t)$ can be written in the form $f(t) = u(t)+iv(t)$, where $u(t)$ and $v(t)$ are real valued functions. It follows by the definition of derivative that $f'(t) = u'(t) + iv'(t)$. For example, using Euler's formula,

$$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i \sin t) = -\sin t + i \cos t = i(\cos t + i \sin t) = ie^{it} \,.$$

Any complex valued vector function $x(t)$ can also be written as $x(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real valued vector functions. Again, we have $x'(t) = u'(t) + iv'(t)$.

If $x(t) = u(t) + iv(t)$ is a complex valued solution of our system (1.4), then

$$u'(t) + iv'(t) = A\,(u(t) + iv(t)) \;.$$

Equating the real and imaginary parts, we see that both $u(t)$ and $v(t)$ are real valued solutions of our system (1.4).

5.2.2 The General Solution

Assume that the matrix A has a pair of complex conjugate eigenvalues $p+iq$ and $p-iq$. They need to contribute two linearly independent solutions. The eigenvector corresponding to $p + iq$ is complex valued, which we may write as $\xi + i\eta$, where ξ and η are real valued vectors. Then $x(t) = e^{(p+iq)t}(\xi + i\eta)$ is a solution of our system. To get two real valued solutions, we take the real and the imaginary parts of this solution. Obtain:

$$\begin{aligned} x(t) &= e^{pt}(\cos qt + i \sin qt)(\xi + i\eta) \\ &= e^{pt}(\cos qt\ \xi - \sin qt\ \eta) + ie^{pt}(\sin qt\ \xi + \cos qt\ \eta) \,. \end{aligned}$$

So that

$$\begin{aligned} u(t) &= e^{pt}(\cos qt\ \xi - \sin qt\ \eta) \,, \\ v(t) &= e^{pt}(\sin qt\ \xi + \cos qt\ \eta) \end{aligned}$$

give us two real valued solutions. In case of a 2×2 matrix (when there are no other eigenvalues), the general solution is

$$x(t) = c_1 u(t) + c_2 v(t) \,. \tag{2.1}$$

(If one uses the other eigenvalue $p - iq$, and the corresponding eigenvector, the answer is the same.) We show in the exercises that one can choose the constants c_1 and c_2 to satisfy any initial condition $x(t_0) = x_0$.

Example 1 Solve the system

$$x' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} x, \;\; x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Calculate the eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. The eigenvector corresponding to λ_1 is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. So that we have a complex valued solution $e^{(1+2i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}$. Using Euler's formula, rewrite it as

$$e^t(\cos 2t + i \sin 2t) \begin{bmatrix} i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + ie^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

The real and imaginary parts give us two linearly independent solutions, so that the general solution is

$$x(t) = c_1 e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

In components

$$\begin{aligned} x_1(t) &= -c_1 e^t \sin 2t + c_2 e^t \cos 2t \\ x_2(t) &= c_1 e^t \cos 2t + c_2 e^t \sin 2t\,. \end{aligned}$$

From the initial conditions

$$\begin{aligned} x_1(0) &= c_2 = 2 \\ x_2(0) &= c_1 = 1\,, \end{aligned}$$

so that $c_1 = 1$, and $c_2 = 2$. Answer:

$$\begin{aligned} x_1(t) &= -e^t \sin 2t + 2e^t \cos 2t \\ x_2(t) &= e^t \cos 2t + 2e^t \sin 2t\,. \end{aligned}$$

Example 2 Solve the system

$$x' = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ -2 & 1 & -1 \end{bmatrix} x\,.$$

One calculates the eigenvalue $\lambda_1 = 1$, and the corresponding eigenvector $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. One of the solutions is then $e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. The other two eigenvalues are $\lambda_2 = i$ and $\lambda_3 = -i$. The eigenvector corresponding to $\lambda_2 = i$ is $\begin{bmatrix} -1-i \\ -1-i \\ 1 \end{bmatrix}$, giving us a complex valued solution $e^{it} \begin{bmatrix} -1-i \\ -1-i \\ 1 \end{bmatrix}$. We rewrite it as

$$(\cos t + i \sin t) \begin{bmatrix} -1-i \\ -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos t + \sin t \\ -\cos t + \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} -\cos t - \sin t \\ -\cos t - \sin t \\ \sin t \end{bmatrix}$$

Taking its real and imaginary parts gives us two more linearly independent solutions, so that the general solution is

$$x(t) = c_1 e^t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\cos t + \sin t \\ -\cos t + \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} -\cos t - \sin t \\ -\cos t - \sin t \\ \sin t \end{bmatrix}.$$

Examining the form of the solutions in all of the above cases, we see that *if all eigenvalues of a matrix A are either negative or have negative real parts, then* $\lim_{t\to\infty} x(t) = 0$ *(all components of the vector $x(t)$ tend to zero).*

5.2.3 Non-Homogeneous Systems

Similarly to the case of a single equation, the general solution of a non-homogeneous system

$$x' = Ax + f(t)\,,$$

with a given vector-function $f(t)$, is the sum of any particular solution $Y(t)$ of this system and the general solution of the corresponding homogeneous system

$$x' = Ax\,.$$

Sometimes one can guess the form of $Y(t)$.

Example 3 Solve the system

$$x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}.$$

We look for a particular solution in the form $Y(t) = \begin{bmatrix} Ae^{-t} \\ Be^{-t} \end{bmatrix}$, and calculate $A = -\frac{3}{8}$, $B = \frac{1}{8}$, so that $Y(t) = \begin{bmatrix} -\frac{3}{8}e^{-t} \\ \frac{1}{8}e^{-t} \end{bmatrix}$. The general solution of the corresponding homogeneous system was found in the Example 1.

Answer. $x(t) = \begin{bmatrix} -\frac{3}{8}e^{-t} \\ \frac{1}{8}e^{-t} \end{bmatrix} + c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Later on we shall develop the method of variation of parameters for non-homogeneous systems.

5.3 The Exponential of a Matrix

In matrix notation, a linear system with a square matrix A,

$$x' = Ax, \quad x(0) = x_0 \tag{3.1}$$

looks like a single equation. In case A and x_0 are constants, the solution of (3.1) is

$$x(t) = e^{At} x_0 \,. \tag{3.2}$$

In order to write the solution of our system in the form (3.2), we shall define the notion of the *exponential of a matrix*. First, we define powers of a matrix: $A^2 = A \cdot A$, $A^3 = A^2 \cdot A$, and so on, using repeated matrix multiplications. Starting with the Maclauren series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,,$$

we define (I is the identity matrix)

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \,.$$

So that e^A is the sum of infinitely many matrices, and each entry of e^A is an infinite series. It can be shown that all of these series are convergent for *any matrix* A, so that we can compute e^A for any square matrix A (we shall prove this fact for diagonalizable matrices). If O denotes a square matrix with all entries equal to zero, then $e^O = I$.

We have

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \cdots ,$$

and then

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots = Ae^{At} .$$

We conclude by a direct substitution that the formula (3.2) gives the solution of the initial-value problem (3.1). (Observe that $x(0) = e^O x_0 = x_0$.)

Example 1 Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where a and b are constants. Then $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$, and we have

$$e^A = \begin{bmatrix} 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots & 0 \\ 0 & 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix} .$$

Example 2 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate: $A^2 = -I$, $A^3 = -A$, $A^4 = I$. Then for any constant t

$$e^{At} = \begin{bmatrix} 1 - t^2/2! + t^4/4! + \cdots & -t + t^3/3! - t^5/5! + \cdots \\ t - t^3/3! + t^5/5! + \cdots & 1 - t^2/2! + t^4/4! + \cdots \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} .$$

The last example lets us express the solution of the system

$$\begin{aligned} x_1' &= -x_2, \quad x_1(0) = \alpha \\ x_2' &= x_1, \qquad x_2(0) = \beta \end{aligned} \tag{3.3}$$

in the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} ,$$

which is rotation of the initial vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ by the angle t, counterclockwise. We see that the integral curves of our system are circles in the (x_1, x_2) plane.

This makes sense, because the velocity vector $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ is always perpendicular to the position vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Observe that the system (3.3) is equivalent to the equation

$$x_1'' + x_1 = 0\,,$$

modeling a *harmonic oscillator*.

Recall that we can write a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

as a row of column vectors $A = [C_1\ C_2\ C_3]$, where $C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$, $C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$, and $C_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$. Then the product of the matrix A and of a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is defined as the vector

$$Ax = C_1 x_1 + C_2 x_2 + C_3 x_3\,.$$

If B is another 3×3 matrix, whose columns are the vectors K_1,K_2 and K_3, we define the product of the matrices A and B as follows:

$$AB = A\,[K_1\ K_2\ K_3] = [AK_1\ AK_2\ AK_3]\ .$$

(This definition is equivalent to the traditional one, which is giving the ij-element of the product: $(AB)_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$.) In general, $AB \neq BA$.

Let Λ be a diagonal matrix $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Compute the product

$$A\,\Lambda = \left[A \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ 0 \\ \lambda 3 \end{bmatrix} \right] = [\lambda_1 C_1\ \ \lambda_2 C_2\ \ \lambda_3 C_3]\ .$$

So that multiplying a matrix A from the right by a diagonal matrix Λ, results in the columns of A being multiplied by the corresponding entries of Λ. (Multiplication of a matrix A from the left by a diagonal matrix Λ, results in the rows of A being multiplied by the corresponding entries of Λ.)

Assume now that the matrix A has three linearly independent eigenvectors E_1, E_2, and E_3, so that $AE_1 = \lambda_1 E_1$, $AE_2 = \lambda_2 E_2$, and $AE_3 = \lambda_3 E_3$ (the eigenvalues λ_1, λ_2 and λ_3 are not necessarily different). Form a matrix $S = [E_1 \;\; E_2 \;\; E_3]$. Observe that S has an inverse matrix S^{-1}. Calculate

$$A\,S = [A\,E_1 \;\; A\,E_2 \;\; A\,E_3] = [\lambda_1\,E_1 \;\; \lambda_2\,E_2 \;\; \lambda_3\,E_3] = S\,\Lambda\,.$$

Multiplying both sides from the left by S^{-1}, obtain

$$S^{-1}\,A\,S = \Lambda\,. \tag{3.4}$$

Similarly,

$$A = S\,\Lambda\,S^{-1}\,. \tag{3.5}$$

One refers to the formulas (3.4) and (3.5) as giving the *diagonalization of the matrix A*. We see that any matrix with a *full set* of three linearly independent eigenvectors can be diagonalized. An $n \times n$ matrix A is diagonalizable, if it has a complete set of n linearly independent eigenvectors. In particular, symmetric matrices are diagonalizable.

If A is *diagonalizable*, so that the formula (3.5) holds, then $A^2 = A\,A = S\,\Lambda\,S^{-1}\,S\,\Lambda\,S^{-1} = S\,\Lambda^2\,S^{-1}$, and in general $A^n = S\,\Lambda^n\,S^{-1}$. We then have (for any real scalar t)

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{S\,\Lambda^n t^n\,S^{-1}}{n!} = S\,\sum_{n=0}^{\infty} \frac{\Lambda^n\,t^n}{n!}\,S^{-1} = S\,e^{\Lambda t} S^{-1}\,.$$

The following important theorem holds for any $n \times n$ matrix.

Theorem 5.3.1 *Assume that all eigenvalues of the matrix A are either negative or have negative real parts. Then all solutions of the system*

$$x' = A\,x$$

tend to zero as $t \to \infty$.

We can prove this theorem in the case when A is diagonalizable. Indeed, then we have

$$x(t) = e^{At}x(0) = S\, e^{\Lambda t}\, S^{-1}x(0)\,.$$

The matrix $e^{\Lambda t}$ is a diagonal one, and its entries, $e^{\lambda_1 t}$, $e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$, tend to zero as $t \to \infty$, so that $x(t) \to 0$ as $t \to \infty$.

In general, $e^{A+B} \neq e^A\, e^B$. This is because $AB \neq BA$ for matrices, in general. (One way to show that $e^{x+y} = e^x\, e^y$ is to expand all three exponentials in power series, and show that the series on the left is the same as the one on the right. In the process, we use that $xy = yx$ for numbers.) But $e^{aI+A} = e^{aI}e^A$, because $(aI)A = A(aI)$ (a is any number). For example, if $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, then $A = 2I + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and we have

$$e^{At} = e^{2tI}e^{\begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}} = e^{2tI}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Non-Homogeneous Systems

We shall solve the initial value problem

$$x' = Ax + f(t), \quad x(t_0) = x_0\,, \tag{3.6}$$

where $f(t)$ is a given vector function, and A is a constant square matrix. The solution is

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-As}f(s)\,ds\,. \tag{3.7}$$

How does one think of this formula? In case of one equation (when A is a number) we have an easy linear equation (with the integrating factor $\mu = e^{-At}$), for which (3.7) gives the solution. We use that $\frac{d}{dt}e^{A(t-t_0)} = Ae^{A(t-t_0)}$ to justify this formula for matrices.

Example 3 Solve

$$x' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

By (3.7)

$$x(t) = e^{At}\begin{bmatrix} 0 \\ 3 \end{bmatrix} + e^{At}\int_0^t e^{-As}\begin{bmatrix} 1 \\ s \end{bmatrix} ds\,.$$

We have

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad e^{-As} = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix},$$

$$e^{-As}\begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} \cos s + s\sin s \\ -\sin s + s\cos s \end{bmatrix}.$$

Then

$$\int_0^t e^{-As}\begin{bmatrix} 1 \\ s \end{bmatrix} ds = \begin{bmatrix} \int_0^t (\cos s + s\sin s)\; ds \\ \int_0^t (-\sin s + s\cos s)\; ds \end{bmatrix} = \begin{bmatrix} -t\cos t + 2\sin t \\ -2 + 2\cos t + t\sin t \end{bmatrix};$$

$$e^{At}\int_0^t e^{-As}\begin{bmatrix} 1 \\ s \end{bmatrix} ds = \begin{bmatrix} -t + 2\sin t \\ 2 - 2\cos t \end{bmatrix}.$$

We conclude that

$$x(t) = \begin{bmatrix} -3\sin t \\ 3\cos t \end{bmatrix} + \begin{bmatrix} -t + 2\sin t \\ 2 - 2\cos t \end{bmatrix} = \begin{bmatrix} -t - \sin t \\ 2 + \cos t \end{bmatrix},$$

or, in components, $x_1(t) = -t - \sin t$, $x_2(t) = 2 + \cos t$.

An easier approach for this particular system is to convert it to a single equation

$$x_1'' + x_1 = -t, \;\; x_1(0) = 0, \; x_1'(0) = -2\,,$$

with the solution $x_1(t) = -t - \sin t$, and then $x_2 = -x_1' + 1 = 2 + \cos t$.

5.3.1 Problems

I. Solve the following systems of differential equations.

1. $x' = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} x$. Answer. $x(t) = c_1 e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

2. $x' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} x$. Answer. $x(t) = c_1\begin{bmatrix} \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t + \sin t \\ 2\sin t \end{bmatrix}$.

3. $x' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} x$.

Answer. $x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

4. $x' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} x$. Check your answer by reducing this system to a second order equation for $x_1(t)$.

Answer. $x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.

5. $x' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} x$.

Answer. $x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \left(t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

6. $x' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Answer.

$$x_1(t) = 2e^{2t} - e^{4t}$$
$$x_2(t) = -2e^{2t} - e^{4t}.$$

7. $x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Check your answer by reducing this system to a second order equation for $x_1(t)$. Answer.

$$x_1(t) = \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}$$
$$x_2(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{t}.$$

8. $x' = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Answer.

$$x_1(t) = -3e^{t} - 2e^{2t} + 5e^{3t}$$
$$x_2(t) = -4e^{2t} + 5e^{3t}$$
$$x_3(t) = -3e^{t} + 5e^{3t}.$$

9. $x' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$. Answer.

$$\begin{aligned} x_1(t) &= 2e^{2t} - 4e^{3t} \\ x_2(t) &= 5e^{t} - 4e^{2t} + 4e^{3t} \\ x_3(t) &= -4e^{2t} + 4e^{3t}\,. \end{aligned}$$

10. $x' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$. Answer.

$$\begin{aligned} x_1(t) &= -3e^{t} + e^{4t} \\ x_2(t) &= 2e^{t} + e^{4t} \\ x_3(t) &= e^{t} + e^{4t}\,. \end{aligned}$$

11. $x' = \begin{bmatrix} 1 & 4 & 0 \\ -4 & -7 & 0 \\ 0 & 0 & 5 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$. Answer.

$$\begin{aligned} x_1(t) &= 2e^{-3t}(-1 + 8t) \\ x_2(t) &= -2e^{-3t}(-3 + 8t) \\ x_3(t) &= e^{5t}\,. \end{aligned}$$

12. $x' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Check your answer by reducing this system to a second order equation for $x_1(t)$. Answer.

$$\begin{aligned} x_1(t) &= -2\cos 2t - \sin 2t \\ x_2(t) &= \cos 2t - 2\sin 2t\,. \end{aligned}$$

13. $x' = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Answer.

$$\begin{aligned} x_1(t) &= -e^{3t}\sin 2t \\ x_2(t) &= e^{3t}\cos 2t\,. \end{aligned}$$

14. $x' = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Answer.

$$x_1(t) = -\cos t + 5\sin t$$

$$\begin{aligned} x_2(t) &= -e^t + 2\cos t + 3\sin t \\ x_3(t) &= e^t + \cos t - 5\sin t\,. \end{aligned}$$

15. $x' = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} x, \ \ x(0) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Answer.

$$\begin{aligned} x_1(t) &= \tfrac{1}{5}\left(9\cos 5t - 4\right) \\ x_2(t) &= 3\sin 5t \\ x_3(t) &= \tfrac{3}{5}\left(4\cos 5t + 1\right)\,. \end{aligned}$$

16. Solve the second order system

$$x'' = \begin{bmatrix} 6 & 2 \\ -5 & -1 \end{bmatrix} x\,.$$

Hint: The system

$$x'' = Ax \tag{3.8}$$

has a solution of the form $x = e^{\lambda t}\xi$, provided that λ^2 is an eigenvalue of A, and ξ the corresponding eigenvector. If $\lambda_1 > 0$ and $\lambda_2 > 0$ are the eigenvalues of a 2×2 matrix A with the corresponding eigenvectors ξ_1 and ξ_2, then the general solution of (3.8) is

$$x = c_1 e^{-\sqrt{\lambda_1}\,t}\xi_1 + c_2 e^{\sqrt{\lambda_1}\,t}\xi_1 + c_3 e^{-\sqrt{\lambda_2}\,t}\xi_2 + c_4 e^{\sqrt{\lambda_2}\,t}\xi_2\,.$$

Answer. $x = c_1 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} -2 \\ 5 \end{bmatrix} + c_4 e^{t}\begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

17. (i) Solve the second order system

$$x'' = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} x\,.$$

Hint: If the matrix A has a negative eigenvalue $\lambda = -p^2$, corresponding to an eigenvector ξ, then $x = \cos pt\,\xi$ and $x = \sin pt\,\xi$ are solutions of the system (3.8).

Answer. $x = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \cos 2t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_4 \sin 2t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(ii) Verify the answer by converting this system to a single equation:

$$x_1'''' - 16x_1 = 0\,.$$

18. Solve the non-homogeneous system

$$x' = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ -5e^{2t} \end{bmatrix}.$$

Answer. $x(t) = e^{2t}\begin{bmatrix} -3 \\ 7 \end{bmatrix} + c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

19. Solve the non-homogeneous system

$$x' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Hint: Look for a particular solution in the form $Y(t) = \begin{bmatrix} At + B \\ Ct + D \end{bmatrix}$.

Answer. $x_1 = -\frac{t}{2} + 5e^t \cos t - \frac{1}{2} e^t \sin t - 1$, $x_2 = \frac{1-t}{2} + \frac{1}{2} e^t (\cos t + 10 \sin t)$.

II.

1. Show that all solutions of the system $x' = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} x$, with positive constants a and b, satisfy

$$\lim_{t \to \infty} x_1(t) = 0, \quad \text{and} \quad \lim_{t \to \infty} x_2(t) = 0\,.$$

2. Write the equation (here b and c are constants, $y = y(t)$)

$$y'' + by' + cy = 0$$

as a system of two first order equations, by letting $x_1(t) = y(t)$, $x_2(t) = y'(t)$. Compute the eigenvalues for the matrix of this system. Show that all solutions tend to zero, as $t \to \infty$, provided that b and c are positive constants.

3. Consider the system

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2\,, \end{aligned}$$

with given constants a, b, c and d. Assume that $a + d < 0$ and $ad - bc > 0$. Show that all solutions tend to zero, as $t \to \infty$ ($x_1(t) \to 0$, and $x_2(t) \to 0$, as $t \to \infty$).

Hint: Show that the eigenvalues for the matrix of this system are either negative, or have negative real parts.

4. (i) Let A be a 3×3 constant matrix. Suppose that all solutions of $x' = Ax$ are bounded as $t \to +\infty$, and as $t \to -\infty$. Show that every solution is periodic, and there is a common period for all solutions.

Hint: One of the eigenvalues of A must be zero, and the other two purely imaginary.

(ii) Assume that a constant 3×3 matrix A is *skew-symmetric*, which means that $A^T = -A$. Show that one of the eigenvalues of A is zero, and the other two are purely imaginary.

Hint: Observe that $a_{ji} = -a_{ij}$, $a_{ii} = 0$, and then calculate the characteristic polynomial.

5. Let $x(t)$ and $y(t)$ be two solutions of the system

$$x' = Ax\,,$$

with an $n \times n$ matrix A. Show that $5x(t)$, and $x(t) + y(t)$ are also solutions. Show that the same is true for $c_1 x(t) + c_2 y(t)$, with any numbers c_1 and c_2. Are the above conclusions true if the entries of A depend on t?

6. (i) Suppose that $p + iq$ is an eigenvalue of A, and $\xi + i\eta$ is a corresponding eigenvector. Show that $p - iq$ is also an eigenvalue of A, and $\xi - i\eta$ is a corresponding eigenvector. (A is an $n \times n$ matrix with real entries.)

(ii) Show that ξ and η are linearly independent. (There is no complex number c, such that $\eta = c\,\xi$.)

Hint: Linear dependence of ξ and η would imply linear dependence of the eigenvectors $\xi + i\eta$ and $\xi - i\eta$.

(iii) Show that the formula (2.1) gives the general solution of the 2×2 system $x' = Ax$, so that we can choose c_1 and c_2, with $x(t_0) = x_0$, for any initial condition.

Hint: Decompose x_0 as a linear combination of ξ and η, and then find c_1 and c_2.

7. Consider the following system with a non-constant matrix

$$x' = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix} x\,.$$

Show that the eigenvalues of the matrix are $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$, yet it has an unbounded solution $x(t) = e^{\frac{t}{2}} \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$.

This example shows that for systems with variable coefficients

$$x' = A(t)x$$

the assumption that the eigenvalues of $A(t)$ are either negative or have negative real parts does not imply that all solutions tend to zero (not even if $A(t)$ is a periodic matrix).

Hint: To compute the eigenvalues, calculate the trace and the determinant of $A(t)$.

III.

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$. Show that

$$e^{At} = e^{-3t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} e^{3t}\cos t & -e^{3t}\sin t & 0 \\ e^{3t}\sin t & e^{3t}\cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}.$$

7. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that

$$e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

8. Consider

$$J = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

a *Jordan block*. Show that

$$e^{Jt} = e^{-2t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

9. Show that the series for e^A converges for any diagonalizable matrix A. Hint: If $A = S\Lambda S^{-1}$, then $e^A = Se^{\Lambda}S^{-1}$.

10. Show that for any square matrix A

$$Ae^{At} = e^{At}A\,.$$

Is it true that $A^2e^{At} = e^{At}A^2$?

11. Show that for any square matrix A, and any constants t and s

$$e^{At}e^{As} = e^{As}e^{At} = e^{A(t+s)}\,.$$

12. Two square matrices of the same size, A and B, are said to *commute* if $AB = BA$. Show that then

$$e^{A}e^{B} = e^{A+B}\,.$$

13. Show that for any $n \times n$ matrix A

$$\left(e^{A}\right)^{-1} = e^{-A}\,.$$

14. Show that for any positive integer m

$$\left(e^{A}\right)^{m} = e^{mA}\,.$$

15. Let A be a square matrix. Show that all entries of e^{At} are non-negative for $t \geq 0$ if and only if

$$a_{ij} \geq 0\,, \quad \text{for all } i \neq j\,. \tag{3.9}$$

Hint: For small t, $e^{At} \approx I + At$, so that if all entries of e^{At} are non-negative, then (3.9) holds. The same formula also shows that if (3.9) holds, then all entries of e^{At} are non-negative, if $t > 0$ is small. To see that the same is true for all $t > 0$, write

$$e^{At} = \left(e^{A\frac{t}{m}}\right)^{m}\,,$$

for any integer $m > 0$, and observe that the product of two matrices with all entries non-negative has all entries non-negative.

16. Show that

$$\left(e^{A}\right)^{T} = e^{A^{T}}\,.$$

17. (i) Let λ be an eigenvalue of a square matrix A, corresponding to an eigenvector x. Show that e^{A} has an eigenvalue e^{λ}, corresponding to the same eigenvector x.

Hint: If $Ax = \lambda x$, then

$$e^{A}x = \sum_{k=0}^{\infty} \frac{A^{k}x}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}\,x = e^{\lambda}x\,.$$

(ii) Show that there is no 2×2 matrix A with real entries, such that

$$e^A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

(iii) Show that the determinant $|e^A|$ of the matrix e^A is positive, so that e^A is non-singular.

5.4 Floquet Theory and Massera's Theorem

Logs of Negative Numbers, and Logs of Matrices

We wish to give a meaning to the natural logarithm $\ln(-2)$. We regard -2 as a complex number $-2 + 0\,i$. Writing it in the polar form $-2 = 2e^{i\pi} = 2e^{i(\pi+2\pi m)}$, where m is any integer, suggests that

$$\ln(-2) = \ln 2 + i(\pi + 2\pi m), \quad m = 0, \pm 1, \pm 2, \ldots.$$

If $z = re^{i\theta} \neq 0$ is any complex number, we define

$$\ln z = \ln r + i(\theta + 2\pi m), \quad m = 0, \pm 1, \pm 2, \ldots.$$

Observe that $\ln z$ is a *multi-valued function.*

Given any non-singular square matrix C (so that the determinant $|C| = \det C \neq 0$), it is known that one can find a square matrix B, such that $e^B = C$. It is natural to write: $B = \ln C$. For example,

$$\ln \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}.$$

In case C is a diagonal matrix, just take the logs of the diagonal entries, to compute $\ln C$. If C is diagonalizable, so that $C = S\Lambda S^{-1}$, and the diagonal matrix Λ has non-zero entries, then $\ln C = S \ln \Lambda S^{-1}$. Observe that the entries of $\ln C$ are complex valued, and that $\ln C$ is not unique. For a general matrix C, one needs the *Jordan normal form* to compute $\ln C$, which is outside of the scope of this book.

Linear Dependence and Independence of Vectors

Given n-dimensional vectors $x_1, x_2, \ldots, x_k$, we play the following game: choose the numbers $c_1, c_2, \ldots, c_k$ so that

$$c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0. \quad (4.1)$$

There is an easy way to "win": take all $c_i = 0$. We declare this way to be illegal (like off-side in soccer, or false start in football). If we can choose the constants so that at least one of them is not zero, and (4.1) holds, we call the vectors $x_1, x_2, \ldots, x_k$ *linearly dependent*. Otherwise, if the only way to make (4.1) hold is by "cheating", or by taking $c_1 = c_2 = \cdots = c_k = 0$, then the vectors are called *linearly independent*. Assume that the vectors $x_1, x_2, \ldots, x_k$ are linearly dependent, and say $c_1 \neq 0$. Then from (4.1)

$$x_1 = -\frac{c_2}{c_1}x_2 - \cdots - \frac{c_k}{c_1}x_k \,,$$

so that one of the vectors is a linear combination of the others.

Recall that if the columns of $n \times n$ matrix A are linearly independent, then the determinant $\det A \neq 0$, and the inverse matrix A^{-1} exists. In such a case, the system

$$Ax = b \tag{4.2}$$

is solvable for any vector b ($x = A^{-1}b$). In case $\det A = 0$, the system (4.2) is solvable only for "lucky" b, which we describe next. Consider the system

$$A^T v = 0 \,, \tag{4.3}$$

where A^T is the transpose matrix. In case $\det A = 0$, the system (4.2) is solvable if and only if $(b, v) = 0$, where v is any solution (4.3). (Observe that $\det A^T = \det A = 0$, so that (4.3) has non-zero solutions.) This fact is known as the *Fredholm alternative*. Here $(b, v) = b^T v$ is the scalar (inner) product. (The book by G. Strang [32] has more details.)

Recall also that $(Ax, y) = (x, A^T y)$ for any square matrix A, and vectors x and y.

The Fundamental Solution Matrix

We consider systems of the form (here $x = x(t)$ is an n-dimensional vector)

$$x' = A(t)x \tag{4.4}$$

where the $n \times n$ matrix $A(t)$ depends on t. Assume that the vectors $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent (at all t) solutions of this system. We use these vectors as columns of the matrix

$$X(t) = [x_1(t)\, x_2(t)\, \ldots\, x_n(t)] \,.$$

This $n \times n$ matrix $X(t)$ is called a *fundamental solution matrix* or a *fundamental matrix*, for short. If, moreover, $X(0) = I$ (the identity matrix), we call $X(t)$ the *normalized fundamental matrix*. We claim that the general solution of (4.4) is

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) = X(t)c\,,$$

where c is the column vector $c = [c_1\, c_2\, \ldots\, c_n]^T$, and c_i's are arbitrary constants. Indeed, if $y(t)$ is any solution of (4.4), we choose the vector c_0, so that $X(0)c_0 = y(0)$ (or $c_0 = X^{-1}(0)y(0)$). Then the two solutions of (4.4), $X(t)c_0$ and $y(t)$, have the same initial values at $t = 0$. By the uniqueness of solution theorem (see the Theorem 6.1.1 below), $y(t) = X(t)c_0$.

Let $Y(t)$ be another fundamental matrix $Y(t) = [y_1(t)\, y_2(t)\, \ldots\, y_n(t)]$. Its first column $y_1(t)$ is a solution of (4.4), and so $y_1(t) = X(t)d_1$, where d_1 is a constant n-dimensional vector. Similarly, $y_2(t) = X(t)d_2$, and so on. Form an $n \times n$ matrix $D = [d_1\, d_2\, \ldots\, d_n]$, with constant entries. Then

$$Y(t) = X(t)D\,, \tag{4.5}$$

by the rules of matrix multiplication. Observe that the matrix D is non-singular ($\det(Y(t)) = \det(X(t))\det(D)$, and $\det(Y(t)) \neq 0$).

Observe that any fundamental matrix X satisfies the equation (4.4), so that

$$X' = A(t)X\,. \tag{4.6}$$

Indeed, the first column on the left, which is x_1', is equal to the first column on the right, which is Ax_1, etc.

We now develop the variation of parameters method for the non-homogeneous system

$$x' = A(t)x + f(t)\,. \tag{4.7}$$

Since $X(t)c$ is the general solution of the corresponding homogeneous system (4.4), the general solution of (4.7) has the form

$$x(t) = X(t)c + Y(t)\,,$$

where $Y(t)$ is any particular solution of (4.7). We search for $Y(t)$ in the form $x = X(t)c(t)$, with the appropriate vector-function $c(t)$. Substituting this into (4.7), and using (4.6), we see that $c(t)$ must satisfy

$$X(t)c'(t) = f(t)\,,$$

so that $c(t) = \int_{t_0}^t X^{-1}(s) f(s)\, ds$, with arbitrary number t_0. It follows that the general solution of (4.7) is given by

$$x(t) = X(t)c + X(t) \int_{t_0}^t X^{-1}(s) f(s)\, ds\,. \tag{4.8}$$

In case the matrix A of the system (4.4) has constant entries, one can use $X(t) = e^{At}$ as the fundamental solution matrix, and recover the solution formula (3.7) above.

If $X(t)$ is the normalized fundamental matrix, and $t_0 = 0$, then $c = x(0)$, and (4.8) becomes

$$x(t) = X(t)x(0) + X(t) \int_0^t X^{-1}(s) f(s)\, ds\,. \tag{4.9}$$

Periodic Systems

We now consider $n \times n$ systems with periodic coefficients

$$x' = A(t)\, x, \quad \text{with } \; A(t+p) = A(t)\,. \tag{4.10}$$

We assume that all entries of the $n \times n$ matrix $A(t)$ are functions of the period p. Any solution $x(t)$ of (4.10) satisfies this system at all times t, in particular at the time $t + p$, so that

$$x'(t+p) = A(t+p)\, x(t+p)\,,$$

which implies that

$$x'(t+p) = A(t)\, x(t+p)\,.$$

We conclude that $x(t+p)$ is also a solution of (4.10). Let $X(t)$ be a fundamental matrix of (4.10), then so is $X(t+p)$, and by (4.5)

$$X(t+p) = X(t)D\,, \tag{4.11}$$

with some non-singular $n \times n$ matrix D, with constant entries. Let the matrix B be such that $e^{Bp} = D$, so that $e^{-Bp} = D^{-1}$, and $B = \frac{1}{p} \ln D$ (the entries of B are complex numbers, in general). Define the matrix $P(t) = X(t)e^{-Bt}$. Then $X(t) = P(t)e^{Bt}$. We claim that $P(t+p) = P(t)$. Indeed, using (4.11),

$$P(t+p) = X(t+p)e^{-B(t+p)} = X(t)DE^{-Bp}e^{Bt} = X(t)e^{Bt} = P(t)\,.$$

We have just derived the following *Floquet Theorem*.

Theorem 5.4.1 *Any fundamental matrix of the system (4.10) is of the form $X(t) = P(t)e^{Bt}$, where the matrix $P(t)$ is p-periodic, and B is a constant matrix. (The entries of B and $P(t)$ are complex numbers, in general.)*

The eigenvalues (possibly complex) of the matrix $D = e^{Bp}$ are called the *Floquet multipliers*. Assume that ρ_i is a Floquet multiplier, and c_i is a corresponding eigenvector of e^{Bp}. Consider $x(t) = X(t)c_i = P(t)e^{Bt}c_i$, a solution of our system (4.10). It satisfies

$$x(t+p) = P(t+p)e^{B(t+p)}c_i = P(t)e^{Bt}e^{Bp}c_i = \rho_i P(t)e^{Bt}c_i = \rho_i x(t)\,,$$

so that

$$x(t+p) = \rho_i x(t)\,. \tag{4.12}$$

In particular, the system (4.10) has a periodic solution (satisfying $x(t+p) = x(t)$), exactly when one of the Floquet multipliers is equal to 1. (If one of the Floquet multipliers is equal to -1, then the system (4.10) has a solution of the period $2p$.)

The general solution of the periodic system (4.10) is

$$x(t) = P(t)e^{Bt}c\,. \tag{4.13}$$

The matrix $P(t)$ is periodic, and therefore it is bounded. We see that $x(t) \to 0$, as $t \to \infty$, exactly when the eigenvalues of B are either negative or have negative real parts. The eigenvalues λ_i's of B are called the *characteristic exponents*. The Floquet multipliers ρ_i's are the eigenvalues of e^{Bp}, so that $\rho_i = e^{\lambda_i p}$, or $\lambda_i = \dfrac{1}{p}\ln\rho_i$. It follows that if the (complex) modulus of all Floquet multipliers is < 1, then all characteristic exponents λ_i are either negative or have negative real parts, and then all solutions of the system (4.10) tend to zero, as $t \to \infty$. On the other hand, if some Floquet multiplier ρ_i has complex modulus greater than one, then iterating (4.12) gives $x(t+np) = \rho_i^n x(t)$, for any integer n, concluding that $x(t)$ is an unbounded solution of the system (4.10).

Returning to the formula (4.13), denote $y(t) = e^{Bt}c$. Then

$$y' = By\,. \tag{4.14}$$

It follows that the change of variables $x(t) \to y(t)$, given by

$$x = P(t)y\,,$$

transforms the periodic system (4.10) into the system (4.14) with constant coefficients.

In case $X(t)$ is the normalized fundamental matrix, $D = X(p)$ by (4.11), so that

$$X(t+p) = X(t)X(p)\,,$$

and the matrix $X(p)$ is called the *monodromy matrix*.

Mathieu's and Hill's equations

Mathieu's equation for $y = y(t)$ has the form

$$y'' + (\delta + \epsilon\cos 2t)\,y = 0\,, \tag{4.15}$$

depending on two constant parameters $\delta > 0$ and ϵ. If $\epsilon = 0$, one obtains a harmonic oscillator which models small vibrations of pendulum attached to the ceiling (discussed in Chapter 2). If the support of the pendulum (or the ceiling itself) moves periodically in the vertical direction, one is led to Mathieu's equation (4.15). We shall discuss a more general Hill's equation

$$y'' + a(t)y = 0\,, \tag{4.16}$$

with a given p-periodic function $a(t)$, so that $a(t+p) = a(t)$ for all t. For Mathieu's equation, $p = \pi$.

Let $y_1(t)$ be the solution of (4.16), satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$, and let $y_2(t)$ be the solution of (4.16), with $y(0) = 0$ and $y'(0) = 1$, the *normalized solutions*. Letting $x_1(t) = y(t)$, and $x_2(t) = y'(t)$, one converts Hill's equation (4.16) into a system

$$x' = A(t)x\,, \tag{4.17}$$

with $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and $A(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix}$. Since $A(t+p) = A(t)$, the Floquet theory applies. The matrix $X(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$ gives the normalized fundamental solution matrix of (4.17), and so the Floquet multipliers are the eigenvalues ρ_1 and ρ_2 of the monodromy matrix $D = X(p)$. The sum of the eigenvalues, $\rho_1 + \rho_2 = \operatorname{tr} X(p) = y_1(p) + y_2'(p) \equiv \beta$, a quantity which is easily computed numerically. Since $\operatorname{tr} A(t) = 0$, it follows by Liouville's formula (presented in Problems) that

$$\rho_1\rho_2 = |X(p)| = |X(0)| = |I| = 1\,. \tag{4.18}$$

It follows that ρ_1 and ρ_2 are roots of the quadratic equation

$$\rho^2 - \beta\rho + 1 = 0\,,$$

which are $\rho = \frac{\beta \pm \sqrt{\beta^2-4}}{2}$.

Case 1. Assume that $|\beta| > 2$. The Floquet multipliers are ρ_1 and ρ_2 real and distinct, and by (4.18), one of them is greater than 1 in absolute value. By the Floquet theory, the system (4.17), and hence Hill's equation has unbounded solutions. One says that Hill's equation is *unstable* in this case.

Case 2. Assume that $|\beta| < 2$. The Floquet multipliers ρ_1 and ρ_2 are complex and distinct, and by (4.18), both have complex modulus equal to one, $|\rho_1| = |\rho_2| = 1$. Since $\rho_2 \neq \rho_1$, one can diagonalize $X(p)$:

$$X(p) = S \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} S^{-1}\,,$$

where the entries of S and S^{-1} are complex constants. Any number t can be written in the form $t = t_0 + np$, with some integer n and $t_0 \in [0, p)$. Then iterating the relation $X(t_0 + p) = X(t_0)X(p)$, obtain

$$X(t) = X(t_0 + np) = X(t_0)X(p)^n = X(t_0)S \begin{bmatrix} \rho_1^n & 0 \\ 0 & \rho_2^n \end{bmatrix} S^{-1}\,.$$

Clearly, $|\rho_1^n| = |\rho_2^n| = 1$, for any n. It follows that the fundamental matrix $X(t)$ has bounded entries for all t, and then all solutions of the system (4.17) (which are given by $x(t) = X(t)c$) are bounded. One says that Hill's equation is *stable* in this case.

Case 3. $\beta = 2$. Then $\rho_1 = \rho_2 = 1$. The system (4.17) has a p-periodic solution. (The other solution in the fundamental set is known to be unbounded.)

Case 4. $\beta = -2$. Then $\rho_1 = \rho_2 = -1$. The system (4.17) has a solution of period $2p$. (The other solution in the fundamental set is known to be unbounded.)

Massera's Theorem

We now consider non-homogeneous periodic systems

$$(4.19) \quad x' = A(t)x + f(t), \quad \text{with } A(t+p) = A(t), \;\; f(t+p) = f(t)\,,$$

where $f(t)$ is a given p-periodic vector function. The following spectacular theorem is due to the Uruguayan mathematician J.L. Massera (published in 1950).

Theorem 5.4.2 *Assume that the system (4.19) has a bounded solution, for $t \geq 0$. Then the system (4.19) has a p-periodic solution.*

Observe that this theorem provides a strong conclusion, with the minimum of assumptions.

Proof: Let $z(t)$ be a bounded solution, whose existence is assumed in the statement of the theorem, and let $X(t)$ be the normalized fundamental matrix of $x' = A(t)x$, so that by (4.9)

$$z(t) = X(t)z(0) + X(t)\int_0^t X^{-1}(s)f(s)\,ds\,.$$

In particular,

$$z(p) = X(p)z(0) + b\,,$$

where we denoted $b = X(p)\int_0^p X^{-1}(s)f(s)\,ds$. By the periodicity of our system, $z(t+p)$ is also a solution of (4.19), which is equal to $z(p)$ at $t = 0$. Therefore, using (4.9) again,

$$z(t+p) = X(t)z(p) + X(t)\int_0^t X^{-1}(s)f(s)\,ds\,.$$

Then

$$z(2p) = X(p)z(p) + b = X(p)\left(X(p)z(0) + b\right) + b = X^2(p)z(0) + X(p)b + b\,.$$

By induction, for any integer $m > 0$,

$$z(mp) = X^m(p)z(0) + \sum_{k=0}^{m-1} X^k(p)b\,. \tag{4.20}$$

For any solution of (4.19),

$$x(t) = X(t)x(0) + X(t)\int_0^t X^{-1}(s)f(s)\,ds\,.$$

We obtain a p-periodic solution, with $x(p) = x(0)$, provided that the initial vector $x(0)$ satisfies

$$(I - X(p))\,x(0) = b\,, \tag{4.21}$$

where, as before, $b = X(p) \int_0^p X^{-1}(s) f(s)\, ds$.

Assume, *contrary to what we want to prove*, that the system (4.19) has no p-periodic solutions. Then the system (4.21) has no solutions. This implies that $\det (I - X(p)) = 0$, and then $\det (I - X(p))^T = \det (I - X(p)) = 0$. It follows that the system

$$(I - X(p))^T v = 0 \tag{4.22}$$

has non-trivial solutions, and, by the Fredholm alternative, we can find a non-trivial solution v_0 of (4.22), which satisfies

$$(b, v_0) \neq 0 \,. \tag{4.23}$$

(Otherwise, the system (4.21) would have solutions.) From (4.22), $v_0 = X(p)^T v_0$, then $X(p)^T v_0 = X^2(p)^T v_0$, which gives $v_0 = X^2(p)^T v_0$, and inductively we get

$$v_0 = X^k(p)^T v_0, \quad \text{for all positive integers } k. \tag{4.24}$$

We now take the scalar product of (4.20) with v_0, and use (4.24):

$$(z(mp), v_0) = (X^m(p) z(0), v_0) + \sum_{k=0}^{m-1} (X^k(p) b, v_0)$$

$$= (z(0), X^m(p)^T v_0) + \sum_{k=0}^{m-1} (b, X^k(p)^T v_0) = (z(0), v_0) + m(b, v_0) \to \infty \,,$$

as $m \to \infty$, in view of (4.23). But $z(t)$ is bounded (and so $(z(mp), v_0)$ is bounded). We have a contradiction, which implies that the system (4.19) has a p-periodic solution. ♢

There is a similar Massera's theorem that deals with second order non-linear equations (here $x = x(t)$)

$$x'' + f(t, x) = 0 \,, \tag{4.25}$$

where the function $f(x, t)$ is assumed to be continuous, differentiable in x, and p-periodic in t, so that

$$f(t + p, x) = f(t, x) \,, \quad \text{for all } t \text{ and } x. \tag{4.26}$$

Theorem 5.4.3 *In addition to (4.26), assume that all solutions of (4.25) continue for all $t > 0$, and one of the solutions, $x_0(t)$, is bounded for all $t > 0$ (so that $|x_0(t)| < M$ for some $M > 0$, and all $t > 0$). Then (4.25) has a p-periodic solution.*

For a proof, and an interesting historical discussion, see P. Murthy [20].

5.5 Solutions of Planar Systems Near the Origin

We now describe the solution curves in the x_1x_2-plane, near the origin $(0,0)$, of the system

$$x' = Ax \,, \tag{5.1}$$

with a constant 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

If the eigenvalues of A are real and distinct, $\lambda_1 \neq \lambda_2$, with the corresponding eigenvectors ξ_1 and ξ_2, we know that the general solution is

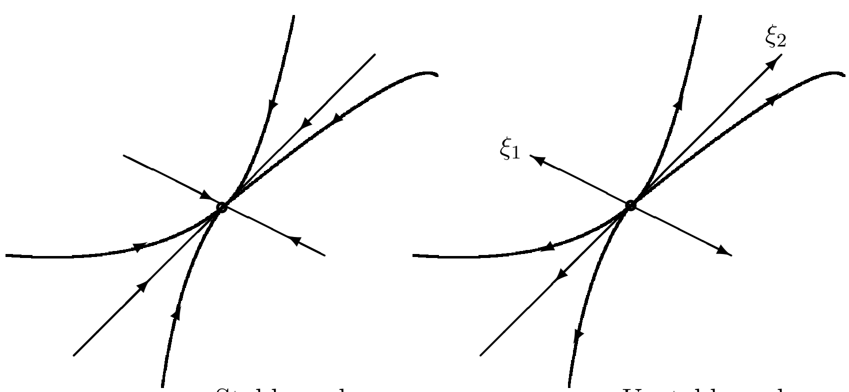
$$x(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2 \,.$$

We study the behavior of solutions as $t \to \pm\infty$, and distinguish between the following cases.

(i) Both eigenvalues are negative, $\lambda_1 < \lambda_2 < 0$. The values of c_1 and c_2 are determined by the initial conditions. If $c_2 = 0$, then $x(t) = c_1 e^{\lambda_1 t} \xi_1$ tends to the origin $(0,0)$ as $t \to \infty$, along the vector ξ_1 (or $-\xi_1$). If $c_2 \neq 0$, then

$$x(t) \approx c_2 e^{\lambda_2 t} \xi_2 \,, \quad \text{for large } t > 0 \,.$$

The solution curves $(x_1(t), x_2(t))$ tend to the origin $(0,0)$ as $t \to \infty$, and they are tangent to the vector ξ_2. The origin is called a *stable node*.



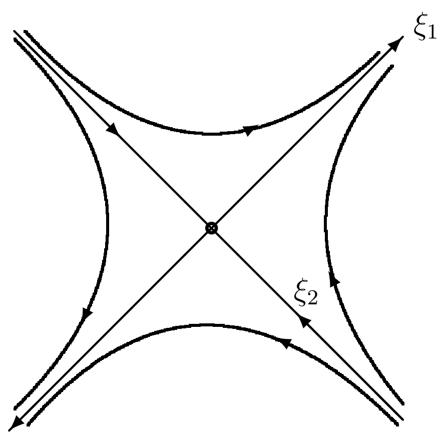
Stable node   Unstable node

(ii) Both eigenvalues are positive, $\lambda_1 > \lambda_2 > 0$. If $c_2 = 0$, then $x(t) = c_1 e^{\lambda_1 t} \xi_1$ tends to the origin $(0,0)$ as $t \to -\infty$, along the vector ξ_1 (or $-\xi_1$). So that solutions *emerge from the origin* along the vectors $\pm\xi_1$. If $c_2 \neq 0$, then

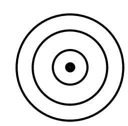
$$x(t) \approx c_2 e^{\lambda_2 t} \xi_2, \quad \text{as } t \to -\infty \,.$$

The solution curves $(x_1(t), x_2(t))$ emerge from the origin $(0,0)$, and they are tangent to the vector ξ_2. The origin is called an *unstable node.*

(iii) The eigenvalues have different sign, $\lambda_1 > 0 > \lambda_2$. In case the initial point lies along the vector ξ_2 (so that $c_1 = 0$), the solution curve $(x_1(t), x_2(t))$ tends to the origin $(0,0)$, as $t \to \infty$. All other solutions (when $c_1 \neq 0$) tend to infinity, and they are tangent to the vector ξ_1, as $t \to \infty$. The origin is called a *saddle* . For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have $\lambda_1 = 1$, $\lambda_2 = -1$, and $x_1 = c_1 e^t$, $x_2 = c_2 e^{-t}$. Express: $x_2 = c_2 \dfrac{1}{e^t} = c_2 \dfrac{1}{x_1/c_1} = \dfrac{c_1 c_2}{x_1}$. Denoting $c = c_1 c_2$, we see that solutions are the hyperbolas $x_2 = \dfrac{c}{x_1}$, which form a saddle near the origin.



Saddle



Center

Turning to the case of complex eigenvalues, we begin with a special matrix $B = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, with the eigenvalues $p \pm iq$, and consider the system

$$y' = By\,. \tag{5.2}$$

Its solutions are

$$y(t) = e^{Bt} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{pt} \begin{bmatrix} \cos qt & \sin qt \\ -\sin qt & \cos qt \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

If $p = 0$, then any initial vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is rotated *clockwise* around the origin (infinitely many times, as $t \to \infty$). We say that the origin is a *center*. If $p < 0$, the solutions spiral into the origin. The origin is called a *stable spiral*. If $p > 0$, we have an *unstable spiral*.

Assume now that A is a 2×2 matrix with complex eigenvalues $p \pm iq$, $q \neq 0$. Let $\xi = u + iv$ be an eigenvector corresponding to $p + iq$, where u and v are real vectors. We have $A(u+iv) = (p+iq)(u+iv) = pu - qv + i(pv+qu)$, and separating the real and imaginary parts

$$Au = pu - qv, \quad \text{and} \quad Av = qu + pv\,. \tag{5.3}$$

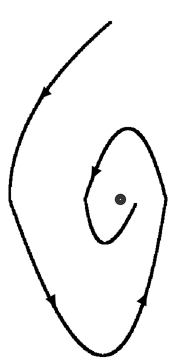
We form a 2×2 matrix $P = [u\ v]$, which has the vectors u and v as its columns. Then, using (5.3)

$$AP = [Au\ Av] = [pu - qv\ \ qu + pv] = [u\ v] \begin{bmatrix} p & q \\ -q & p \end{bmatrix} = PB\,,$$

where B is the special matrix considered above. We now make a change of variables $x = Py$, transforming (5.1) to

$$Py' = APy = PBy\,,$$

and then to the system (5.2), that we have analyzed above (P^{-1}, the inverse of P exists, because the vectors u and v are linearly independent, which is justified in the exercises). We conclude that the origin is a center if $p = 0$, a stable spiral if $p < 0$, and an unstable spiral if $p > 0$. If the determinant $|P| > 0$, the motion on these curves is clockwise, and in case $|P| < 0$, the motion is counterclockwise. (Recall that the solution curves of $y' = By$ move clockwise.)



Stable spiral

Unstable spiral

One often denotes the unknown functions by $x(t)$ and $y(t)$. Then the system (5.1) takes the form

$$\begin{aligned} \tfrac{dx}{dt} &= a_{11}x + a_{12}y \\ \tfrac{dy}{dt} &= a_{21}x + a_{22}y\,. \end{aligned}$$

Dividing the second equation by the first, we can write this system as a single equation

$$\frac{dy}{dx} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}\,,$$

although one can no longer distinguish the direction along the integral curves.

5.5.1 Linearization and the Hartman-Grobman Theorem

We now briefly discuss nonlinear planar systems. Chapter 6 will be entirely devoted to nonlinear systems.

Suppose that a *nonlinear system* (for $x = x(t)$ and $y = y(t)$)

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y)\,, \end{aligned} \tag{5.4}$$

with differentiable functions $f(x,y)$ and $g(x,y)$, has *a rest point* (x_0, y_0), which is defined by

$$f(x_0,y_0) = g(x_0,y_0) = 0\,.$$

By Taylor's formula, we approximate for (x,y) near (x_0,y_0)

$$\begin{aligned} f(x,y) &\approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\,, \\ g(x,y) &\approx g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0)\,. \end{aligned}$$

Letting $x - x_0 = u$ and $y - y_0 = v$, with $x' = u'$ and $y' = v'$, we approximate the system (5.4) by the linear system

$$\begin{aligned} u' &= f_x(x_0,y_0)u + f_y(x_0,y_0)v \\ v' &= g_x(x_0,y_0)u + g_y(x_0,y_0)v \end{aligned} \tag{5.5}$$

This approximation is valid for (u,v) close to $(0,0)$, which corresponds to (x,y) being near (x_0,y_0). One calls (5.5) the *linearized system*. Its matrix

$$A = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$$

is called the *Jacobian matrix*. We analyzed the behavior of linear systems near the rest point $(0,0)$ in the preceding section. The natural question is whether the picture near $(0,0)$ for the linearized system (5.4) remains similar for the nonlinear system (5.4) near (x_0, y_0). The *Hartman-Grobman theorem* says that this is the case if the Jacobian matrix A does not have purely imaginary or zero eigenvalues. So that if the linearized system has a stable or unstable node, or a stable or unstable spiral, or a saddle at $(0,0)$, the picture remains similar for the nonlinear system near (x_0, y_0). On the other hand, in case of a center, the picture may be different.

The Hartman-Grobman theorem also holds for $n \times n$ matrices, and the rest point which does not have purely imaginary or zero eigenvalues is called *hyperbolic*. For the proof, and the precise statement, see the book of M.W. Hirsh and S. Smale [13].

Example 1 The system

$$\begin{aligned} x' &= -y - x\left(x^2+y^2\right) \\ y' &= x - y\left(x^2+y^2\right) \end{aligned} \tag{5.6}$$

has a unique rest point at $x_0 = 0$, $y_0 = 0$. Indeed, to find the rest points we solve

$$\begin{aligned} -y - x\left(x^2+y^2\right) &= 0 \\ x - y\left(x^2+y^2\right) &= 0 \,. \end{aligned}$$

Multiplying the first equation by x, the second one by y, and adding the results gives

$$-\left(x^2+y^2\right)^2 = 0 \,,$$

or $x = y = 0$. The linearized system at the rest point $(0,0)$,

$$\begin{aligned} u' &= -v \\ v' &= u \,, \end{aligned} \tag{5.7}$$

has a center at $(0,0)$, and its trajectories are circles around $(0,0)$. The Hartman-Grobman theorem does not apply. It turns out that the trajectories of (5.6) spiral into $(0,0)$ (so that $(0,0)$ is a stable spiral). Indeed, multiplying the first equation of (5.6) by x, the second one by y, adding the results, and calling $\rho = x^2+y^2$, we see that $\frac{1}{2}\rho' = -\rho^2$. This gives $\rho(t) = \frac{1}{2t+c} \to 0$, as $t \to \infty$.

Example 2 The system

$$x' = -y + xy^2$$
$$y' = x - x^2 y \tag{5.8}$$

has a rest point $(0,0)$. The linearized system at the rest point $(0,0)$ is again given by (5.7), so that $(0,0)$ is a center. We claim that $(0,0)$ is a center for the original system (5.8) too. Indeed, multiplying the first equation in (5.8) by x, the second one by y, and adding the results, we see that $\frac{d}{dt}\left(x^2+y^2\right) = 0$, or

$$x^2 + y^2 = c\,,$$

and all trajectories are circles around the origin.

The system (5.8) has another rest point: $(1,1)$. The Jacobian matrix at $(1,1)$ is $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. It has zero as a double eigenvalue, and so it does not belong to any of the types of the rest points that we considered.

Example 3 The rest points of the system

$$x' = 2x - y + 2$$
$$y' = xy \tag{5.9}$$

are $(0,2)$ and $(-1,0)$. ($xy = 0$ implies that either $x = 0$, or $y = 0$.) The Jacobian matrix at $(0,2)$ is $\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Its eigenvalues are $1 \pm i$, so that the linearized system has an unstable spiral at $(0,0)$. By the Hartman-Grobman theorem, solutions of (5.9) spiral out of $(0,2)$. The Jacobian matrix at $(-1,0)$ is $\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$. Its eigenvalues are 2 and -1. The linearized system has a saddle at $(0,0)$, and by the Hartman-Grobman theorem, the system (5.9) has a saddle at $(-1,0)$.

5.5.2 Phase Plane and the Prüfer Transformation

We saw that some 2×2 systems can be transformed into a single second order equation. Often one transforms the opposite way. For example, in the equation

$$y''(t) - a(t)y(t) = 0 \tag{5.10}$$

we let $y' = v$, then $v' = y'' = a(t)y$, and we convert this equation into an equivalent system, with the variables (y, v) depending on t:

$$
\begin{aligned}
y' &= v \\
v' &= a(t)y\,.
\end{aligned}
\tag{5.11}
$$

One calls the (y, v) plane, or the (y, y') plane, the *phase plane*. Solutions of (5.11) define curves (called *trajectories*) in the (y, y') phase plane.

It is often useful to use polar coordinates in the phase plane, a technique known as the *Prüfer transformation*. We set

$$
\begin{aligned}
y &= r\cos\theta \\
y' &= r\sin\theta\,,
\end{aligned}
\tag{5.12}
$$

with $r = r(t)$, $\theta = \theta(t)$. Using that $\theta(t) = \arctan\frac{y'}{y}$, compute

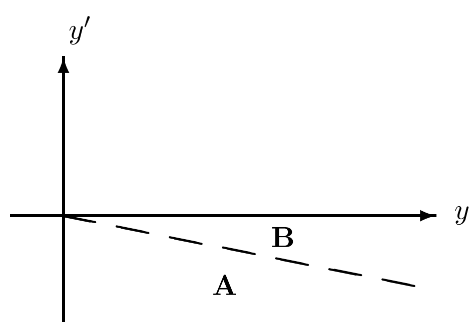
$$\theta' = \frac{1}{1+\left(\frac{y'}{y}\right)^2}\,\frac{y''y - y'^2}{y^2} = \frac{y''y - y'^2}{y^2+y'^2} = \frac{a(t)y^2 - y'^2}{y^2+y'^2}\,.$$

Using (5.12), we have (observe that $y^2 + y'^2 = r^2$)

$$\theta'(t) = a(t)\cos^2\theta(t) - \sin^2\theta(t)\,. \tag{5.13}$$

The point here is that this equation for $\theta(t)$ is *decoupled* from the other polar coordinate $r(t)$ (it does not contain $r(t)$).

To give an application, let us assume that some solution of the equation (5.10) satisfies $y(t) > 0$, $y'(t) < 0$ for $t > t_0 \geq 0$, and $\lim_{t\to\infty} y(t) = 0$. Assume also that $a(t) \geq a_1 > 0$ for $t > t_0$. We shall show that $y(t)$ decays exponentially as $t \to \infty$.



The regions A and B

We are given that the trajectory $(y(t), y'(t))$ lies in the fourth quadrant of the phase plane. Consider the line $\theta = 2\pi - \epsilon$, or

$$y' = \tan(2\pi - \epsilon)\, y = -\tan\epsilon\, y \tag{5.14}$$

in the fourth quadrant of the phase plane (the dashed line in the picture). We claim that for ϵ small, the trajectory $(y(t), y'(t))$ cannot go above this line (or into the region B in the picture). Indeed, assuming the contrary, we can find

$$\theta(t) \in (2\pi - \epsilon, 2\pi)\,, \tag{5.15}$$

with ϵ small, so that $\cos\theta(t) \approx 1$ and $\sin\theta(t) \approx 0$. It follows from (5.13) that

$$\theta'(t) > \frac{1}{2}a_1\,. \tag{5.16}$$

The function $\theta(t)$ is increasing, which implies that (5.16) continues to hold, so long as (5.15) does. Then at some finite time t_1, we have $\theta(t_1) = 2\pi$, which corresponds to $y'(t_1) = 0$, contradicting our assumption that $y'(t) < 0$ for all t. It follows that the trajectory has to stay below the dashed line (or in the region A), so that

$$y'(t) < -\tan\epsilon\, y\,.$$

Integrating this inequality over (t_0, t), we conclude

$$y(t) < y(t_0)e^{-\tan\epsilon\,(t-t_0)}\,,$$

which implies the exponential decay.

5.5.3 Problems

I.

1. Consider a 2×2 system

$$x' = Ax\,. \tag{5.17}$$

Assume that λ_1 is repeated eigenvalue of A ($\lambda_2 = \lambda_1$), which has two linearly independent eigenvectors. Show that the solutions are $x(t) = e^{\lambda_1 t}c$, which are straight lines through the origin in the x_1x_2-plane (c is an arbitrary vector). (If $\lambda_1 < 0$, solutions along all of these lines tend to the origin, as $t \to \infty$, and we say that the origin is a *stable degenerate node*. If $\lambda_1 > 0$, the origin is called an *unstable degenerate node*.)

2. Assume that λ_1 is repeated eigenvalue of A ($\lambda_2 = \lambda_1$), which has only one linearly independent eigenvector ξ. If $\lambda_1 < 0$ show that all solutions of

(5.17) approach the origin in the x_1x_2-plane as $t \to \infty$, and they are tangent to ξ (again one says that the origin is a *stable degenerate node*). If $\lambda_1 > 0$ show that all solutions of (5.17) approach the origin in the x_1x_2-plane as $t \to -\infty$, and they are tangent to ξ (an *unstable degenerate node*).

Hint: Recall that using the generalized eigenvector η, the general solution of (5.17) is

$$x(t) = c_1 e^{\lambda_1 t}\,\xi + c_2 \left(te^{\lambda_1 t}\,\xi + e^{\lambda_1 t}\eta\right).$$

3. Consider the system

$$\frac{dx}{dt} = ax + by \qquad (5.18)$$
$$\frac{dy}{dt} = mx + ny\,,$$

where a, b, m and n are real numbers.

(i) Put this system into the form

$$(mx + ny)\,dx - (ax + by)\,dy = 0\,. \qquad (5.19)$$

Hint: Express $\dfrac{dy}{dx}$.

(ii) Assume that $(0,0)$ is a center for (5.18). Show that the equation (5.19) is exact, and solve it.

Hint: One needs $n = -a$ (and also that b and m have the opposite signs), in order for the matrix of (5.18) to have purely imaginary eigenvalues.

Answer. $mx^2 + nxy - by^2 = c$, a family of closed curves around $(0,0)$.

(iii) Justify that the converse statement is not true.

Hint: For example, if one takes $a = 1$, $b = 1$, $m = 3$, and $n = -1$, then the equation (5.19) is exact, but $(0,0)$ is a saddle.

II. Identify the rest point at the origin $(0,0)$. Sketch the integral curves near the origin, and indicate the direction in which they are traveled for increasing t.

1. $x' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x.$ Answer. Unstable node.

2. $x' = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix} x.$ Answer. Saddle.

3. $x' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x.$ Answer. Center.

4.
$$x' = 2x + 4y$$
$$y' = -5x - 7y\,.$$

Answer. Stable node. (Observe that the variables here are x and y, rather than x_1 and x_2.)

5.
$$\tfrac{dx}{dt} = x - 2y$$
$$\tfrac{dy}{dt} = 4x - 3y\,.$$

Answer. Stable spiral.

6. $\dfrac{dy}{dx} = \dfrac{x-y}{4x+y}.$

Hint: Convert to a system form for $x(t)$ and $y(t)$, with $A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$.

Answer. Unstable spiral.

7. $x' = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix} x.$ Answer. Center.

8. $\dfrac{dy}{dx} = \dfrac{x}{y}.$ Answer. Saddle. Solution: $y^2 - x^2 = c$.

9. $x' = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} x.$

Answer. Stable degenerate node. Solution: $x_1(t) = c_1 e^{-3t}$, $x_2(t) = c_2 e^{-3t}$.

10. $\dfrac{dy}{dx} = \dfrac{y}{x}.$ Answer. Degenerate node.

11. $x' = \begin{bmatrix} 1 & 1 \\ \alpha & 1 \end{bmatrix} x$, α is a constant.

Answer. Saddle for $\alpha > 1$, unstable node for $0 < \alpha < 1$, unstable degenerate node when $\alpha = 0$, unstable spiral if $\alpha < 0$.

III. Find all of the rest points for the following systems, and identify their type, for both the corresponding linearized system and the original system.

1.
$$\begin{aligned} x' &= 2x + y^2 - 1 \\ y' &= 6x - y^2 + 1 \,. \end{aligned}$$

Answer. $(0,1)$ is a saddle, $(0,-1)$ is an unstable spiral for both systems.

2.
$$\begin{aligned} x' &= y - 2 \\ y' &= x^2 - 2y \,. \end{aligned}$$

Answer. $(2,2)$ is a saddle, $(-2,2)$ is a stable spiral for both systems.

3.
$$\begin{aligned} x' &= y - x \\ y' &= (x-2)(y+1) \,. \end{aligned}$$

Answer. $(-1,-1)$ is a stable node, $(2,2)$ is a saddle for both systems.

4.
$$\begin{aligned} x' &= -3y + x\left(x^2+y^2\right) \\ y' &= 3x + y\left(x^2+y^2\right) \,. \end{aligned}$$

Answer. $(0,0)$ is a center for the linearized system, and an unstable spiral for the nonlinear system.

IV. 1. (i) Justify the formula for differentiation of a determinant

$$\frac{d}{dt}\begin{vmatrix} a(t) & b(t) \\ c(t) & d(t) \end{vmatrix} = \begin{vmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{vmatrix} + \begin{vmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{vmatrix} \,.$$

(ii) Consider a system

$$x' = A(t)x \,, \tag{5.20}$$

with a 2×2 matrix $A = [a_{ij}(t)]$. Let $X(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix}$ be its fundamental matrix, so that the vectors $\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}$ and $\begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$ are two linearly independent solutions of (5.20). The determinant $W(t) = |X(t)|$ is called the *Wronskian* of (5.20). Show that for any number t_0

$$W(t) = W(t_0)e^{\int_{t_0}^{t} \operatorname{tr} A(s)\, ds} \,, \tag{5.21}$$

where the trace $\operatorname{tr} A(t) = a_{11}(t) + a_{22}(t)$.

Hint: Calculate

$$W' = \begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}'(t) & x_{22}'(t) \end{vmatrix} . \tag{5.22}$$

Using properties of determinants, calculate

$$\begin{aligned} \begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} &= \begin{vmatrix} a_{11}x_{11} + a_{12}x_{21} & a_{11}x_{12} + a_{12}x_{22} \\ x_{21} & x_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11}x_{11} & a_{11}x_{12} \\ x_{21} & x_{22} \end{vmatrix} = a_{11}W . \end{aligned}$$

Similarly, the second determinant in (5.22) is equal to $a_{22}W$. Then

$$W' = (a_{11} + a_{22})\, W = \mathrm{tr} A\, W .$$

(iii) Show that the formula (5.21), called *Liouville's formula*, holds also in case of $n \times n$ systems (5.20).

2. Consider the system

$$x' = Ax \, , \tag{5.23}$$

where A is an $n \times n$ matrix with constant entries. Show that e^{At} is the normalized fundamental solution matrix.

Hint: Recall that $x(t) = e^{At}c$ gives the general solution. Choosing the first entry of the vector c to be one, and all other entries zero, conclude that the first column of e^{At} is a solution of (5.23).

3. (i) Consider the system

$$x' = A(t)x \, , \tag{5.24}$$

where $A(t)$ is an $n \times n$ matrix with all entries continuous on (t_0, ∞). Derive the following *Ważewski inequality* (for $t > t_0$)

$$||x(t_0)|| e^{\int_{t_0}^t \lambda(s)\, ds} \le ||x(t)|| \le ||x(t_0)|| e^{\int_{t_0}^t \Lambda(s)\, ds} \, ,$$

where $\lambda(t)$ and $\Lambda(t)$ are the smallest and the largest eigenvalues of the matrix $\frac{1}{2}\left(A + A^T\right)$, and $||x(t)||$ is the length of the vector $x(t)$.

Hint: Observe that the matrix $\frac{1}{2}\left(A + A^T\right)$ is symmetric, so that all of its eigenvalues are real. Then integrate the inequality

$$\frac{d}{dt}||x(t)||^2 = \frac{d}{dt}\, x^T x = x^T \left(A + A^T\right) x \le 2\Lambda(t) ||x(t)||^2 .$$

(ii) Let $A(t) = \begin{bmatrix} -e^t & t^3 \\ -t^3 & -3 \end{bmatrix}$. Show that all solutions of (5.24) tend to zero, as $t \to \infty$.

5.6 Controllability and Observability

5.6.1 The Cayley-Hamilton Theorem

Recall that the eigenvalues $\lambda_1 , \lambda_2 , \ldots , \lambda_n$ of an $n \times n$ matrix A are roots of the *characteristic equation*

$$|A - \lambda I| = 0 \, .$$

The determinant $|A - \lambda I|$ is an n-th degree polynomial in λ, called the *characteristic polynomial* :

$$p(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n \, ,$$

with some coefficients $a_0 , a_1 , \ldots , a_n$.

Cayley-Hamilton Theorem Any square matrix A is a root of its own characteristic polynomial, so that

$$p(A) = a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = O \, ,$$

where O is the zero matrix.

Proof: We begin by assuming that the matrix A is diagonalizable, so that $A = S \,\mathrm{diag}\,(\lambda_1 , \lambda_2 , \ldots , \lambda_n)\, S^{-1}$. Here $\mathrm{diag}\,(\lambda_1 , \lambda_2 , \ldots , \lambda_n)$ is the diagonal matrix with entries $\lambda_1 , \lambda_2 , \ldots , \lambda_n$, and S is a non-singular matrix. Recall that $A^k = S \,\mathrm{diag}\left(\lambda_1^k , \lambda_2^k , \ldots , \lambda_n^k\right) S^{-1}$, for any integer k, and then

$$p(A) = S \,\mathrm{diag}\,(p(\lambda_1), p(\lambda_2) , \ldots , p(\lambda_n))\, S^{-1} = SOS^{-1} = O \, .$$

The proof for the general case can be given by a continuity argument, which we sketch next. An arbitrarily small perturbation of any matrix A produces a matrix B with distinct eigenvalues, and hence diagonalizable (over complex numbers). We have $p(B) = O$, by above, while the matrix $p(A)$ is arbitrarily close to $p(B)$, and therefore $p(A) = O$. $\diamondsuit$

We shall use the following corollary of this theorem.

Proposition 5.6.1 *(i) For any integer $s \geq n$, the matrix A^s is a linear combination of $I, A , A^2 , \ldots , A^{n-1}$.*
(ii) The matrix e^A is also a linear combination of $I, A , A^2 , \ldots , A^{n-1}$.

Proof: Performing the long division of polynomials, write

$$\lambda^s = p(\lambda) q(\lambda) + r(\lambda) \, ,$$

where $q(\lambda)$ is a polynomial of degree $s-n$, and $r(\lambda)$ is a polynomial of degree $n-1$, $r(\lambda) = r_0\lambda^{n-1} + \cdots + r_{n-2}\lambda + r_{n-1}$, with some coefficients $r_0, \ldots, r_{n-2}, r_{n-1}$. Then, using the Cayley-Hamilton theorem

$$A^s = p(A)q(A) + r(A) = r(A) = r_0 A^{n-1} + \cdots + r_{n-2}A + r_{n-1}I\,,$$

concluding the proof of the first part. The second part follows from the definition of the exponential $e^A = \sum_{s=0}^{\infty} \frac{A^s}{s!}$. ♢

5.6.2 Controllability of Linear Systems

We consider the linear system

$$x' = Ax + Bu(t)\,. \tag{6.1}$$

Here $x(t) \in R^n$ is the unknown vector function, with the components $x_1(t)$, $x_2(t), \ldots, x_n(t)$, while the vector function $u(t) \in R^m$, $m \geq 1$, is at our disposal, the *control*. The $n \times n$ matrix A and the $n \times m$ matrix B have constant coefficients, and are given. If we regard $u(t)$ as known, then solving the non-homogeneous system (6.1) with the initial condition $x(0) = x_0$ gives

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds\,.$$

One says that *the integral curve $x(t)$ is generated by the control vector $u(t)$, with $x(0) = x_0$.*

Definition We say that *the system (6.1) is controllable* if, given any duration p and two arbitrary points x_0, $x_p \in R^n$, there exists a continuous vector function $\bar{u}(t)$ from $[0,p]$ to R^m, such that the integral curve $\bar{x}(t)$ generated by $\bar{u}(t)$ with $\bar{x}(0) = x_0$ satisfies $\bar{x}(p) = x_p$.

In other words, controllability means that

$$x_p = e^{Ap}x_0 + \int_0^p e^{A(p-s)}B\bar{u}(s)\,ds\,,$$

for some $\bar{u}(t)$, or

$$\int_0^p e^{A(p-t)}B\bar{u}(t)\,dt = x_p - e^{Ap}x_0\,. \tag{6.2}$$

We form *Kalman's controllability matrix*

$$K = \left(B\ AB\ A^2B\ \ldots\ A^{n-1}B\right)\,.$$

(The first m columns of K are those of B, the next m columns are given by AB, and so on.) The type (or size) of K is $n \times nm$. Observe that $\operatorname{rank} K \le n$.

Theorem 5.6.1 *(R. Kalman) The system (6.1) is controllable if and only if*

$$\operatorname{rank} K = n\,.$$

Proof: Define the matrices

$$C(t) = e^{A(p-t)}B\,, \text{ and } G = \int_0^p C(t)\,C^T(t)\,dt\,,$$

where C^T is the transpose of C. The matrix function $C(t)$ is of type $n \times m$, while the constant matrix G is of type $n \times n$.

We claim that controllability of (6.1) is equivalent to G being invertible. Assume that G is invertible. We shall show that the vector

$$\bar{u}(t) = B^T e^{A^T(p-t)}G^{-1}\left(x_p - e^{Ap}x_0\right)$$

gives the desired control. Indeed,

$$\int_0^p e^{A(p-t)}B\bar{u}(t)\,dt = \left(\int_0^p e^{A(p-t)}BB^Te^{A^T(p-t)}\,dt\right)G^{-1}\left(x_p - e^{Ap}x_0\right)$$
$$= GG^{-1}\left(x_p - e^{Ap}x_0\right) = x_p - e^{Ap}x_0\,,$$

and the system (6.1) is controllable by (6.2).

Conversely, assume now that the system (6.1) is controllable. We wish to show that G is invertible. Assume, on the contrary, that G is not invertible. Then its rows are linearly dependent. Therefore there exists a non-zero column vector $v \in R^n$, such that

$$v^TG = 0\,. \tag{6.3}$$

(If $v_1, v_2, \ldots, v_n$ are the entries of v, while $g_1, g_2, \ldots, g_n$ are the rows of G, then (6.3) is equivalent to $v_1g_1 + v_2g_2 + \cdots + v_ng_n = 0$.) Then

$$0 = v^TGv = \int_0^p v^TC(t)C^T(t)v\,dt = \int_0^p ||v^TC(t)||^2\,dt\,.$$

It follows that

$$v^TC(t) = 0\,, \quad \text{for all } t \in [0,p]\,. \tag{6.4}$$

Because the system (6.1) is controllable, for any x_0 and x_p we can find $\bar{u}(t)$ so that (6.2) holds, or

$$\int_0^p C(t)\bar{u}(t)\,dt = x_p - e^{Ap}x_0\,.$$

Choose now $x_0 = 0$, and any x_p such that $v^T x_p \neq 0$. Using (6.4), we have

$$0 = \int_0^p v^T C(t)\bar{u}(t)\,dt = v^T x_p \neq 0\,,$$

a contradiction, proving that G is invertible.

We complete the proof by showing that G being invertible is equivalent to $\operatorname{rank} K = n$. Assume that G is not invertible. Then (6.4) holds for some vector $v \neq 0$, as we saw above. Write

$$(6.5)\quad v^T C(t) = v^T e^{A(p-t)}B = \sum_{i=0}^{\infty} \frac{v^T A^i B}{i!}(p-t)^i = 0\,, \quad \text{for all } t \in [0,p]\,.$$

It follows that $v^T A^i B = 0$ for all $i \geq 0$, which implies that $v^T K = 0$. (Recall that $v^T K$ is a linear combination of the rows of K.) This means that the rows of K are linearly dependent, so that $\operatorname{rank} K < n$. By the logical contraposition, if $\operatorname{rank} K = n$, then G is invertible.

Conversely, assume that $\operatorname{rank} K < n$. Then we can find a non-zero vector $v \in R^n$, such that

$$v^T K = 0\,.$$

By the definition of K this implies that $v^T A^i B = 0$ for $i = 0, 1, \ldots, n-1$. By the Proposition 5.6.1 we conclude that $v^T A^i B = 0$ for all $i \geq 0$. Then by (6.5), $v^T C(t) = 0$. It follows that

$$v^T G = \int_0^p v^T C(t)\,C^T(t)\,dt = 0\,,$$

so that that the rows of G are linearly dependent, and then G is not invertible. Hence, if G is invertible, then $\operatorname{rank} K = n$. $\diamondsuit$

Example 1 The system (we use (x, y) instead of (x_1, x_2))

$$\begin{aligned} x' &= u(t) \\ y' &= x \end{aligned}$$

is controllable. Here $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\operatorname{rank} K = 2$.

Writing this system as a single equation $y'' = u(t)$, we see that the height $y(t)$ of an object, and its velocity $y'(t) = x(t)$, can be jointly controlled with a jet pack (which controls the acceleration function $u(t)$).

Example 2 The system

$$\begin{aligned} x' &= u(t) \\ y' &= -u(t) \end{aligned}$$

is not controllable. Here $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, and $\operatorname{rank} K = 1$.

Writing this system as $(x+y)' = 0$, we conclude that $x(t) + y(t) =$ *constant*, independently of the control $u(t)$. It follows that, for example, the point $(1,1)$ cannot be possibly be steered to $(2,-3)$, since $1+1 \neq 2-3$.

5.6.3 Observability

We now consider a *control-observation process*

$$\begin{aligned} x' &= Ax + Bu \\ v &= Cx\,. \end{aligned} \tag{6.6}$$

Here the first equation corresponds to using the control vector $u(t)$ to steer the solution $x(t)$, as in the preceding section, so that $x \in R^n$, $u \in R^m$, and we assume that the given matrices A of type $n \times n$ and B of type $n \times m$ have constant entries. The second equation involves a $q \times n$ *observation matrix* C with constant entries, and an *observation vector* $v(t) \in R^q$. (If C is the $n \times n$ identity matrix, then $v(t)$ is just $x(t)$.)

Definition We say that *the system (6.6) is observable* if for every time interval (t_0, t_1) the knowledge of the *input-output pair* $(u(t), v(t))$ over (t_0, t_1) uniquely determines the initial state $x(t_0)$.

Define the following $n \times n$ matrix function

$$P(t) = \int_{t_0}^{t} e^{A^T(s-t_0)} C^T C e^{A(s-t_0)}\, ds\,.$$

Lemma 5.6.1 *The system (6.6) is observable if and only if $P(t)$ is invertible for all $t > t_0$.*

Proof: Assume that the system (6.6) is observable, but $P(t_1)$ is singular, at some $t_1 > t_0$, contrary to the statement of the lemma. Then we can find a non-zero vector $x_0 \in R^n$ satisfying $P(t_1)x_0 = 0$. It follows that

$$0 = x_0^T P(t_1) x_0 = \int_{t_0}^{t_1} ||Ce^{A(t-t_0)}x_0||^2 \, ds \,,$$

and then

$$Ce^{A(t-t_0)}x_0 = 0 \,, \quad \text{for all } t \in (t_0, t_1) \,. \tag{6.7}$$

Set $\bar{u}(t) \equiv 0$. Then $\bar{x}(t) = e^{A(t-t_0)}\bar{x}(t_0)$ is the corresponding solution of (6.6), for any initial vector $\bar{x}(t_0)$. We have, in view of (6.7),

$$\bar{v}(t) = Ce^{A(t-t_0)}\bar{x}(t_0) = Ce^{A(t-t_0)}\left(\bar{x}(t_0) + \alpha x_0\right) \,,$$

for any constant α. Hence, the input-output pair $(0, \bar{v}(t))$ does not determine uniquely the initial state at t_0, contrary to the assumption of observability. It follows that $P(t)$ is invertible for all $t > t_0$.

Conversely, assume that $P(t_1)$ is invertible, at some $t_1 > t_0$. Express

$$v(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-s)}Bu(s)\, ds \,.$$

Multiplying both sides by $e^{A^T(t-t_0)}C^T$, and integrating over (t_0, t_1) gives

$$\begin{aligned} P(t_1)x(t_0) = & \textstyle\int_{t_0}^{t_1} e^{A^T(t-t_0)}C^T v(t)\, dt \\ & - \textstyle\int_{t_0}^{t_1} e^{A^T(t-t_0)}C^T \left(C\int_{t_0}^{t} e^{A(t-s)}Bu(s)\, ds\right) dt \,. \end{aligned}$$

Since $P(t_1)$ is invertible, $x(t_0)$ is uniquely determined by the values of $u(t)$ and $v(t)$ over the interval (t_0, t_1), and so (6.6) is observable. $\diamondsuit$

We now consider *Kalman's observability matrix*, of type $qn \times n$,

$$N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} .$$

(Its first q rows are those of C, CA gives the next q rows, and so on.) Clearly, $\operatorname{rank} N \le n$.

Theorem 5.6.2 *(R. Kalman) The system (6.6) is observable if and only if* $\operatorname{rank} N = n$.

Proof: If the rank of N is less than n, its columns are linearly dependent, so that for some non-zero vector $a \in R^n$ we have $Na = 0$, or equivalently

$$Ca = CAa = \cdots = CA^{n-1}a = 0\,.$$

The Proposition 5.6.1 implies that

$$Ce^{A(s-t_0)}a = 0\,, \ \text{ for all } s > t_0\,.$$

Multiplying by $e^{A^T(s-t_0)}C^T$, and integrating over (t_0, t), we conclude that $P(t)a = 0$ for all $t > t_0$, which implies that $P(t)$ is not invertible, and the system (6.6) is not observable by Lemma 5.6.1. By logical contraposition, if the system (6.6) is observable, then $\operatorname{rank} N = n$.

Conversely, assume that $\operatorname{rank} N = n$. Let $x(t_0)$ and $\hat{x}(t_0)$ be two initial states corresponding to the same input-output pair $(u(t), v(t))$. We wish to show that $x(t_0) = \hat{x}(t_0)$, so that the system (6.6) is observable. The difference $z(t) \equiv x(t) - \hat{x}(t)$ satisfies $z' = Az$, and also $Cz = Cx(t) - C\hat{x}(t) = v(t) - v(t) = 0$, so that

$$Ce^{A(t-t_0)}\left[x(t_0) - \hat{x}(t_0)\right] = 0\,, \ \text{ for all } t \geq t_0\,.$$

(Notice that $z(t) = e^{A(t-t_0)}z(t_0)$.) By taking the first $n-1$ derivatives, and setting $t = t_0$, we conclude

$$CA^k\left[x(t_0) - \hat{x}(t_0)\right] = 0\,, \ \ k = 0, 1, \ldots, n-1\,,$$

which is equivalent to $N\left[x(t_0) - \hat{x}(t_0)\right] = 0$. Since N has full rank, its columns are linearly independent, and therefore $x(t_0) - \hat{x}(t_0) = 0$. ♢

Notice that observability does not depend on the matrix B.

Problems

1. Calculate the controllability matrix K, and determine if the system is controllable.

(i)
$$\begin{aligned} x_1' &= x_1 + 2x_2 + u(t) \\ x_2' &= 2x_1 + x_2 + u(t)\,. \end{aligned}$$

Answer. $K = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, $\operatorname{rank} K = 1$, not controllable.

(ii)

$$\begin{aligned} x' &= x_1 + 2x_2 - 2u(t) \\ x_2' &= 2x_1 + x_2 \,. \end{aligned}$$

Answer. $K = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}$, rank $K = 2$, controllable.

2. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Calculate e^A.

Hint: By the Proposition 5.6.1

$$e^A = \alpha A + \beta I \,,$$

for some constants α and β. The eigenvalues of A are -1 and 5, which implies that the eigenvalues of e^A are e^{-1} and e^5, while the eigenvalues of $\alpha A + \beta I$ are $-\alpha + \beta$ and $5\alpha + \beta$. It follows that

$$\begin{aligned} e^{-1} &= -\alpha + \beta \\ e^5 &= 5\alpha + \beta \,. \end{aligned}$$

Solve this system for α and β.

Answer. $\frac{1}{6} \begin{bmatrix} 2e^5 + 4e^{-1} & 2e^5 - 2e^{-1} \\ 4e^5 - 4e^{-1} & 4e^5 + 2e^{-1} \end{bmatrix}$.

Chapter 6

Non-Linear Systems

We begin this chapter with applications to ecology of two species, including both competing species and predator-prey interactions, and to epidemiological modeling. Then we study some important general aspects of non-linear systems, including Lyapunov's stability and limit cycles. Finally, we include an in-depth presentation of periodic ecological models.

6.1 The Predator-Prey Interaction

In 1925, Vito Volterra's future son-in-law, biologist Umberto D'Ancona, told him of the following puzzle. During the World War I, when ocean fishing almost ceased, the ratio of predators (like sharks) to prey (like tuna) had increased. Why did sharks benefit more from the decreased fishing? (While the object of fishing is tuna, sharks are also caught in the nets.)

The Lotka-Volterra Equations

Let $x(t)$ and $y(t)$ give respectively the numbers of prey (tuna) and predators (sharks), as functions of time t. Let us assume that in the absence of sharks, tuna would obey the *Malthusian model*

$$x'(t) = ax(t),$$

with some growth rate $a > 0$. (It would grow exponentially, $x(t) = x(0)e^{at}$.) In the absence of tuna, we assume that the number of sharks would decrease exponentially, and satisfy

$$y'(t) = -cy(t),$$

with some $c > 0$, because its other prey is less nourishing. Clearly, the presence of sharks will decrease the rate of growth of tuna, while tuna is good for sharks. The model is:

$$\begin{aligned} x'(t) &= a\,x(t) - b\,x(t)\,y(t) \\ y'(t) &= -c\,y(t) + d\,x(t)\,y(t)\,, \end{aligned} \tag{1.1}$$

with two more given positive constants b and d. The $x(t)y(t)$ term is proportional to the number of encounters between sharks and tuna. These encounters decrease the growth rate of tuna, and increase the growth rate of sharks. Notice that both equations are nonlinear, and we are interested in solutions with $x(t) > 0$, and $y(t) > 0$. The system (1.1) represents the famous *Lotka-Volterra model*. Alfred J. Lotka was an American mathematician, who developed similar ideas at about the same time as V. Volterra.

A fascinating story of Vito Volterra's life and work, and of life in Italy in the first half of the 20-th Century, is told in a very nice book of Judith R. Goodstein [14].

Analysis of the Model

Remember the energy being constant for a vibrating spring? We have something similar here. It turns out that any solution $(x(t), y(t))$ of (1.1) satisfies

$$a \ln y(t) - b\,y(t) + c \ln x(t) - d\,x(t) = C = constant\,, \tag{1.2}$$

for all time t. To justify that, let us introduce the function $F(x, y) = a \ln y - b\,y + c \ln x - d\,x$. We wish to show that $F(x(t), y(t)) = constant$. Using the chain rule, and expressing the derivatives from the equations (1.1), we have

$$\frac{d}{dt}F(x(t), y(t)) = F_x x' + F_y y' = c\frac{x'(t)}{x(t)} - dx'(t) + a\frac{y'(t)}{y(t)} - by'(t)$$

$$= c(a - by(t)) - d(ax(t) - bx(t)y(t)) + a(-c + dx(t)) - b(-cy(t) + dx(t)y(t))$$

$$= 0\,,$$

proving that $F(x(t), y(t))$ does not change with time t.

We assume that the initial numbers of both sharks and tuna are given:

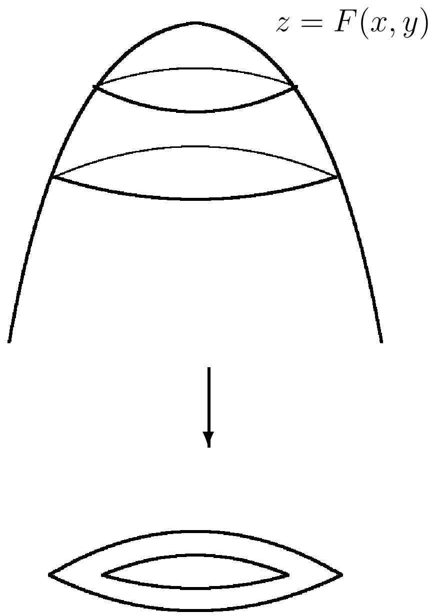
$$x(0) = x_0 > 0, \quad y(0) = y_0 > 0\,. \tag{1.3}$$

The Lotka-Volterra system, together with the initial conditions (1.3), determines both populations $(x(t), y(t))$ at all time t, by the existence and uniqueness Theorem 6.1.1 below. Letting $t = 0$ in (1.2), we calculate the value of C

$$C_0 = a \ln y_0 - b\, y_0 + c \ln x_0 - d\, x_0 \,. \tag{1.4}$$

In the xy-plane, the solution $(x(t), y(t))$ defines a parametric curve, with time t being the parameter. The same curve is described by the implicit relation

$$a \ln y - b\, y + c \ln x - d\, x = C_0 \,. \tag{1.5}$$



The level lines of $z = F(x, y)$

This curve is just a level curve of the function $F(x, y) = a \ln y - b\, y + c \ln x - d\, x$, introduced earlier ($F(x, y) = C_0$). How does the graph of $z = F(x, y)$ look? Like a mountain with a single peak, because $F(x, y)$ is a sum of a function of y, $a \ln y - by$, and of a function of x, $c \ln x - dx$, and both of these functions are concave (down). It is clear that all level lines of $F(x, y)$ are closed curves. Following these closed curves in Figure 6.1, one can see how dramatically the relative fortunes of sharks and tuna change, just as *a result of their interaction*, and not because of any external influences.

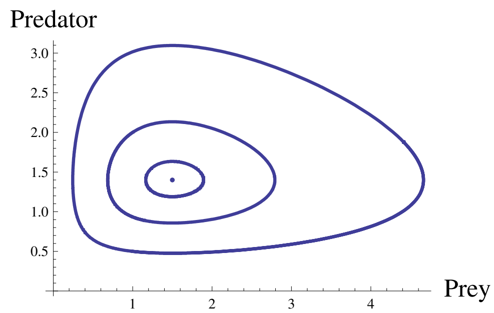


Figure 6.1: The integral curves of the Lotka-Volterra system

Properties of the Solutions

In Figure 6.1 we present a picture of three integral curves, computed by *Mathematica* in the case when $a = 0.7$, $b = 0.5$, $c = 0.3$ and $d = 0.2$. All solutions are closed curves, and there is a dot in the middle (corresponding to a rest point at $(1.5, 1.4)$).

When $x_0 = c/d$, and $y_0 = a/b$, or when the starting point is $(c/d, a/b)$, we calculate from the Lotka-Volterra equations that $x'(0) = 0$, and $y'(0) = 0$. The solution is then $x(t) = c/d$, and $y(t) = a/b$ for all t, as follows by the existence and uniqueness Theorem 6.1.1, which is reviewed at the end of this section. The point $(c/d, a/b)$ is called a *rest point*. (In the above example, the coordinates of the rest point were $x_0 = c/d = 1.5$, and $y_0 = a/b = 1.4$.) All other solutions $(x(t), y(t))$ are periodic, because they represent closed curves. For each trajectory, there is a number T, a period, so that $x(t+T) = x(t)$ and $y(t+T) = y(t)$. This period changes from curve to curve, and it is larger the further the solution curve is from the rest point. (This monotonicity property of the period was proved only around 1985 by Franz Rothe [26], and Jorg Waldvogel [34].) The motion along the integral curves is counterclockwise (at the points due east of the rest point, one has $x' = 0$, and $y' > 0$).

Divide the first of the Lotka-Volterra equations by the solution $x(t)$, and then integrate over its period T:

$$\frac{x'(t)}{x(t)} = a - by(t) \,,$$

$$\int_0^T \frac{x'(t)}{x(t)} \, dt = aT - b \int_0^T y(t) \, dt \,.$$

But $\int_0^T \frac{x'(t)}{x(t)}\,dt = \ln x(t)\,\big|_0^T = 0$, because $x(T) = x(0)$, by periodicity. It follows that

$$\frac{1}{T}\int_0^T y(t)\,dt = a/b\,.$$

Similarly, we derive

$$\frac{1}{T}\int_0^T x(t)\,dt = c/d\,.$$

We have a remarkable fact: the averages of both $x(t)$ and $y(t)$ are the same for all solutions. Moreover, these averages are equal to the coordinates of the rest point.

The Effect of Fishing

Extensive fishing decreases the growth rate of both tuna and sharks. The new model is

$$\begin{aligned} x'(t) &= (a-\alpha)x(t) - b\,x(t)\,y(t) \\ y'(t) &= -(c+\beta)y(t) + d\,x(t)\,y(t)\,, \end{aligned}$$

where α and β are two more given positive constants, related to the intensity of fishing. (There are other ways to model fishing.) As before, we compute the average numbers of both tuna and sharks

$$\frac{1}{T}\int_0^T x(t)\,dt = (c+\beta)/d, \quad \frac{1}{T}\int_0^T y(t)\,dt = (a-\alpha)/b\,.$$

We see an increase for the average number of tuna, and a decrease for the sharks, as a result of moderate amount of fishing (assuming that $\alpha < a$). Conversely, decreased fishing increases the average number of sharks, giving us an explanation of U. D'Ancona's data. This result is known as *Volterra's principle*. It applies also to insecticide treatments. If such a treatment destroys both the pests and their predators, it may be counter-productive, and produce *an increase of the number of pests!*

Biologists have questioned the validity of both the Lotka-Volterra model, and of the way we have accounted for fishing (perhaps, they cannot accept the idea of two simple differential equations ruling the oceans). In fact, it is more common to model fishing, using the system

$$\begin{aligned} x'(t) &= ax(t) - b\,x(t)\,y(t) - h_1(t) \\ y'(t) &= -cy(t) + d\,x(t)\,y(t) - h_2(t)\,, \end{aligned}$$

with some given positive functions $h_1(t)$ and $h_2(t)$.

The Existence and Uniqueness Theorem for Systems

Similarly to the case of one equation, we can expect (under some mild conditions) that an initial value problem for a system of differential equations has a solution, and exactly one solution.

Theorem 6.1.1 *Consider an initial value problem for the system*

$$x'(t) = f(t, x(t), y(t))\,, \quad x(t_0) = x_0$$
$$y'(t) = g(t, x(t), y(t))\,, \quad y(t_0) = y_0\,,$$

with given numbers x_0 and y_0. Assume that the functions $f(t,x,y)$, $g(t,x,y)$, and their partial derivatives f_x, f_y, g_x and g_y are continuous in some three-dimensional region containing the point (t_0, x_0, y_0). Then this system has a unique solution $(x(t), y(t))$, defined on some interval $|t - t_0| < h$.

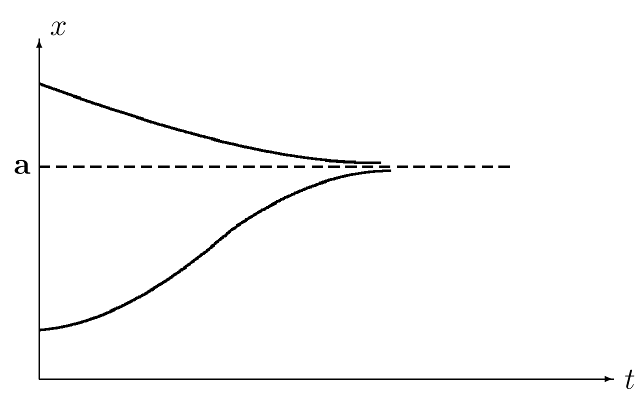
The statement is similar in case of n equations

$$x_i' = f_i(t, x_1, x_2, \ldots, x_n)\,, \quad x_i(t_0) = x_i^0\,, \qquad i = 1, 2, \ldots, n\,,$$

where x_i^0 are given numbers. If the functions f_i, and all of their partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous in some $n+1$-dimensional region containing the initial point $\left(t_0, x_1^0, x_2^0, \ldots, x_n^0\right)$, then this system has a unique solution $(x_1(t), x_2(t), \ldots, x_n(t))$, defined on some interval $|t - t_0| < h$.

One can find the proof in the book of D.W. Jordan and P. Smith [15].

6.2 Competing Species



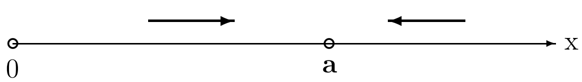
Solution curves for the logistic model

When studying the logistic population model (here $x = x(t) > 0$ gives the number of rabbits)

$$x' = x(a - x)\,, \tag{2.1}$$

we were able to analyze the behavior of solutions, even without solving this equation. Indeed, the quadratic polynomial $x(a - x)$ is positive for $0 < x < a$, and then $x'(t) > 0$, so that $x(t)$ is an increasing function. The same quadratic is negative for $x > a$, and so $x(t)$ is decreasing in this range. If $x(0) = a$, then $x(t) = a$, for all t, by the uniqueness part of the existence and uniqueness Theorem 1.8.1. So that the solution curves look as in the figure above. Here a is the *carrying capacity*, which is the maximum sustainable number of rabbits. (In Chapter 2 the logistic model had the form $x' = ax - bx^2$. Letting here $x = \frac{1}{b}z$, gives $z' = az - z^2$, which is the equation (2.1).)

If $x(0)$ is small, the solution grows exponentially at first, and then the rate of growth gets smaller and smaller, which is a typical *logistic curve*. The point $x = a$ is a *rest point*, and we can describe the situation by the following one-dimensional picture (the arrows indicate that solutions tend to a):



The logistic equation has a stable rest point $x = a$, and an unstable one $x = 0$

The rest point $x = a$ is called *stable rest point*, because solutions of the equation (2.1), with $x(0)$ close to a, tend to a as $t \to \infty$ ($\lim_{t\to\infty} x(t) = a$). The point $x = 0$ is an *unstable rest point*, because solutions of (2.1), starting near $x = 0$, tend away from it.

In general, the point x_0 is called a *rest point* of the equation ($x = x(t)$)

$$x' = g(x)\,, \tag{2.2}$$

with a differentiable function $g(x)$, if $g(x_0) = 0$. It is *stable* (or attractive) if $g'(x_0) < 0$. The solutions of (2.2), with $x(0)$ near x_0, tend to x_0 as $t \to \infty$. The rest point x_0 is *unstable* (or repellent) if $g'(x_0) > 0$. The solutions of (2.2) move away from it. In case $g'(x_0) = 0$, further analysis is needed.

Suppose now that we also have a population $y(t)$ of deer, modeled by a logistic equation

$$y' = y(d - y)\,,$$

with the carrying capacity $d > 0$. Suppose that rabbits and deer live on the same tropical island, and they *compete* (for food, water and hiding places). The *Lotka-Volterra model* of their interaction is

$$\begin{aligned} x' &= x\,(a - x - by) \\ y' &= y\,(d - cx - y)\;. \end{aligned} \tag{2.3}$$

The first equation tells us that the presence of deer effectively decreases the carrying capacity of rabbits (making it $a - by$), and the positive constant b quantifies this influence. Similarly, the positive constant c measures how detrimental the presence of rabbits is for deer. What predictions will follow from the model (2.3)? And how do we analyze this model, since solving the nonlinear system (2.3) is not possible? We shall consider the system (2.3) together with the initial conditions

$$x(0) = \alpha, \quad y(0) = \beta\,. \tag{2.4}$$

The given initial numbers of the animals, $\alpha > 0$ and $\beta > 0$, determine the initial point (α, β) in the xy-plane. We claim that any solution of (2.3), (2.4) satisfies $x(t) > 0$ and $y(t) > 0$ for all $t > 0$. Indeed, write the first equation in (2.3) as

$$x' = A(t)x\,, \quad \text{where } A(t) = a - x(t) - by(t)\,.$$

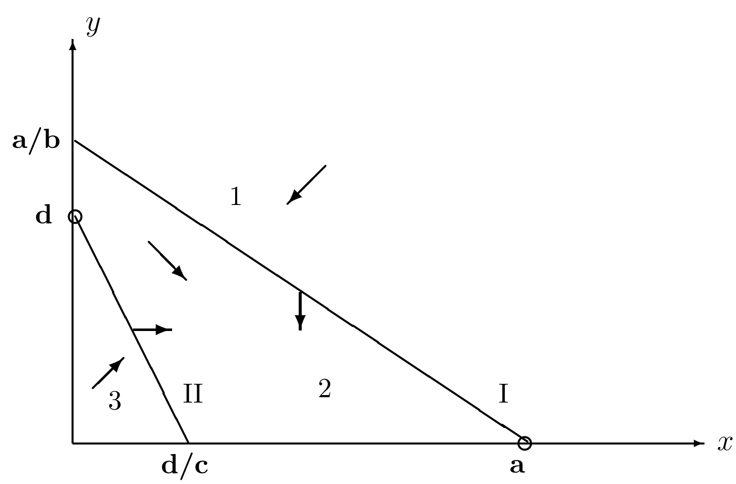
Then $x(t) = \alpha\, e^{\int_0^t A(s)\,ds} > 0$. One shows similarly that $y(t) > 0$. We shall consider the system (2.3) only in the first quadrant of the xy-plane, where $x > 0$ and $y > 0$.

Similarly to the case of a single logistic equation, we look at the points where the right hand sides of (2.3) are zero:

$$\begin{aligned} a - x - by &= 0 \\ d - cx - y &= 0\,. \end{aligned} \tag{2.5}$$

These equations give us two straight lines, called the *null-clines of the system* (2.3). Both of these lines have negative slopes in the (x, y) plane. In the first quadrant these lines may intersect either once, or not at all, depending on the values of a, b, c and d, and that will determine the long turn predictions. The point of intersection of null-clines is a rest point. It follows that there is at most one rest point in the first quadrant.

We shall denote by I the null-cline $a - x - by = 0$. Above this straight line, we have $a - x - by < 0$, which implies that $x'(t) < 0$, and the motion is to the left in the first quadrant. Below the null-cline I, the motion is to the right. We denote by II the null-cline $d - cx - y = 0$. Above II, $y'(t) < 0$, and the motion is down, while below II the point $(x(t), y(t))$ moves up. (For example, if a point lies above both I and II, the motion is to the left and down, in the "southwest" direction. If a point is above I but below II, the motion is northwest, etc.) The system (2.3) has the *trivial solution* $(0,0)$ ($x = 0$, and $y = 0$), and two *semi-trivial solutions* $(a,0)$ and $(0,d)$. The solution $(a,0)$ corresponds to deer becoming extinct, while $(0,d)$ means there are no rabbits. Observe that the semi-trivial solution $(a,0)$ is the x-intercept of the null-cline I, while the second semi-trivial solution $(0,d)$ is the y-intercept of the null-cline II. The behavior of solutions, in the long turn, will depend on whether the null-clines intersect in the first quadrant or not. We consider the following cases.



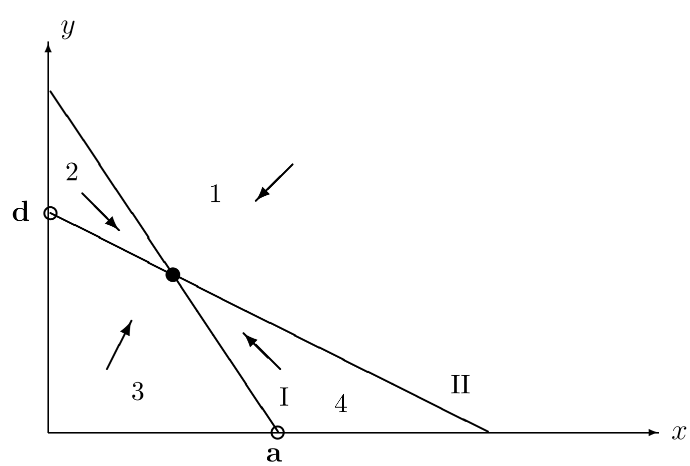
Non-intersecting null-clines, with the semitrivial solutions circled

Case 1: The null-clines do not intersect (in the first quadrant). Assume first that the null-cline I lies above of II, so that $a/b > d$ and $a > d/c$. The null-clines divide the first quadrant into three regions. In the region 1 (above both clines), the motion is southwest, in the region 2 it is southeast, and in the region 3 northeast. On the cline I, the motion is due south, and on the cline II due east. Regardless of the initial point (α, β), all trajectories tend to the semi-trivial solution $(a, 0)$, as $t \to \infty$. (The trajectories starting in the region 3, will enter the region 2, and then tend to $(a, 0)$. The trajectories starting in the region 2, will stay in that region and tend to $(a, 0)$. The trajectories starting in the region 1, will tend to $(a, 0)$ by either staying in this region, or through the region 2.)

This case corresponds to the extinction of deer, and rabbits reaching their maximum sustainable number a.

In case the null cline II lies above of I, when $a/b < d$ and $a < d/c$, a similar analysis shows that all solutions tend to the semi-trivial solution $(0, d)$, as $t \to \infty$. So that the species, which has its null-cline on top, wins the competition, and drives the other one to extinction.

Case 2: The null-clines intersect (in the first quadrant). Their point of intersection $(\frac{a-bd}{1-bc}, \frac{d-ac}{1-bc})$ is a solution of the system (2.5), a *rest point*. Its stability depends on which of the following sub-cases hold. Observe that the null clines intersect the x and y axes at four points, two of which correspond to the semi-trivial solutions.



Intersecting null-clines, with the semitrivial solutions on the inside (circled)

Sub-case (a): The semi-trivial solutions lie on the inside (relative to the other two points of intersection), so that $a < d/c$, and $d < a/b$. The null clines divide the first quadrant into four regions. In all four regions, the motion is eventually toward the rest point. Solutions starting in the region 2, will always stay in this region. Indeed, on the boundary between the regions 1 and 2, the motion is due south, and on the border between the regions 2 and 3, the trajectories travel due east. So that solutions, starting in the region 2, stay in this region, and they tend to the rest point $(\frac{a-bd}{1-bc}, \frac{d-ac}{1-bc})$. Similarly, solutions starting in the region 4, never leave this region, and they tend to the same rest point, as $t \to \infty$. If a solution starts in the region 1, it may stay in this region, and tend to the rest point, or it may enter either one of the regions 2 or 4, and then tend to the rest point. Similarly, solutions which begin in the region 3, will tend to the rest point $(\frac{a-bd}{1-bc}, \frac{d-ac}{1-bc})$, either by staying in this region, or through the regions 2 or 4 (depending on the initial conditions).

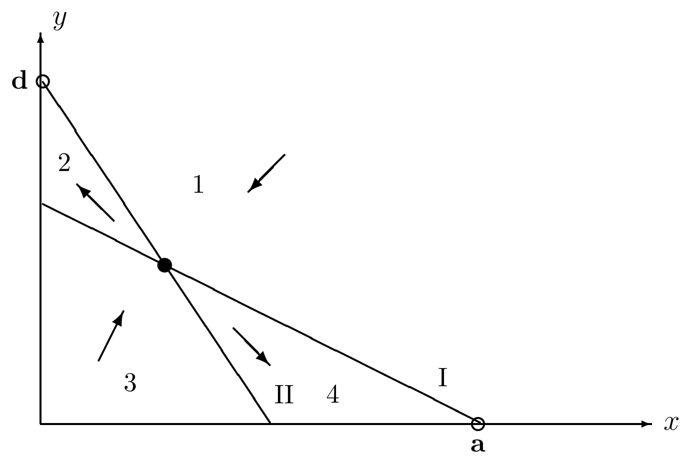
This is the case of *co-existence of the species*. For any initial point (α, β), with $\alpha > 0$ and $\beta > 0$, we have $\lim_{t\to\infty} x(t) = \frac{a-bd}{1-bc}$, and $\lim_{t\to\infty} y(t) = \frac{d-ac}{1-bc}$.

Sub-case (b): The semi-trivial solutions are on the outside, so that $a > d/c$ and $d > a/b$. In the regions 2 and 4, the motion is now away from the rest point. On the lower boundary of the region 2, the motion is due north, and on the upper border, the motion is due west. So that the trajectories starting in the region 2, stay in this region, and tend to the semitrivial solution $(0, d)$. Similarly, solutions starting in the region 4, will stay in this region, and tend to the semitrivial solution $(a, 0)$. A typical solution starting in the region 1, will either enter the region 2 and tend to $(0, d)$, or it will enter the region 4 and tend to $(a, 0)$ (this will depend on the initial conditions). The same conclusion holds for the region 3. The result is that one of the species dies out, what is known as *competitive exclusion of the species*. (Linearizing the Lotka-Volterra system (2.3) at the rest point, one calculates that the rest point is a saddle, in view of the Hartman-Grobman theorem. Hence, there is a solution curve entering the rest point, while a typical trajectory tends to one of the semi-trivial solutions.)

What is the reason behind the drastic difference in the long time dynamics for the above sub-cases? The second equation in (2.3) tells us that the effective carrying capacity of the second species is $d - cx(t) < d$, so that $y(t) < d$ in the long run. For the first species, the effective carrying capacity is $a - by(t)$, and in the sub-case *(a)* (when $bd < a$), we have

$$a - by(t) > a - bd > 0\,,$$

so that the first species can survive even when the number of the second species is at its maximum sustainable level. Similarly, one checks that the second species can survive, assuming that the number of the first species is at its maximum sustainable level. In this sub-case, the competing species do not affect each other too strongly, and so they can co-exist.



Intersecting null-clines, with the semitrivial solutions on the outside (circled)

6.3 An Application to Epidemiology

Suppose that a group of people comes down with an infectious disease. Will the number of sick people grow and cause an epidemic? What measures should public health officials take? We shall analyze a way to model the spread of an infectious disease.

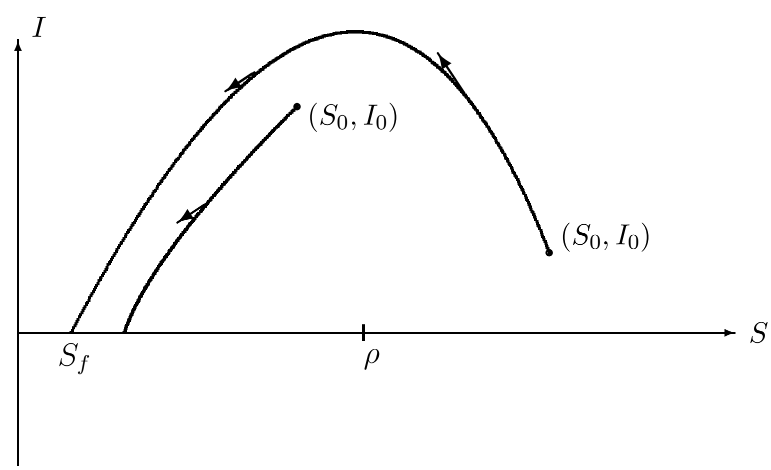
Let $I(t)$ be the number of *infected* people at time t that live in the community (sick people that are hospitalized, or otherwise *removed* from the community, do not count in $I(t)$). Let $S(t)$ be the number of *susceptible* people, the ones at risk of catching the disease. The following model was proposed in 1927 by W.O. Kermack and A.G. McKendrick

$$
\begin{aligned}
\frac{dS}{dt} &= -rSI \\
\frac{dI}{dt} &= rSI - \gamma I\,,
\end{aligned}
\tag{3.1}
$$

with some positive constants r and γ. The first equation reflects the fact that the number of susceptible people decreases, as some people catch the infection (and so they join the group of infected people). The rate of decrease of $S(t)$ is proportional to the number of "encounters" between the infected and susceptible people, which in turn is proportional to the product SI. The number $r > 0$ gives the *infection rate*. The first term in the second equation tells us that the number of infected people would increase at exactly the same rate, if it was not for the second term. The second term, $-\gamma I$, is due to some infected people being *removed* from the population (people who died from the disease, people who have recovered and developed immunity, and sick people who are isolated from the community), which decreases the infection rate dI/dt. The coefficient γ is called the *removal rate*. To the equations (3.1) we add the initial conditions

$$S(0) = S_0, \quad I(0) = I_0\,, \tag{3.2}$$

with given numbers S_0 - the initial number of susceptible people, and I_0 - the initial number of infected people. Solving the equations (3.1) and (3.2), will give us a pair of functions $(S(t), I(t))$, which determines a parametric curve in the (S, I) plane. Alternatively, this curve can be described, if we express I as a function of S.



The solution curves of the system (3.1)

Express

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{rSI - \gamma I}{-rSI} = -1 + \frac{\gamma}{r}\frac{1}{S} = -1 + \rho\,\frac{1}{S}\,,$$

denoting $\gamma/r = \rho$. Taking the antiderivative, and using the initial conditions (3.2), we obtain

$$I = -S + \rho \ln S + I_0 + S_0 - \rho \ln S_0\,. \tag{3.3}$$

Depending on the initial point (S_0, I_0), we get a different integral curve from (3.3). On all of these curves, the maximum value of I is achieved at $S = \rho$ ($\frac{dI}{dS}(\rho) = 0$). The motion on the integral curves is from right to left, because we see from the first equation in (3.1) that $S(t)$ is a decreasing function of time t. If the initial point (S_0, I_0) satisfies $S_0 > \rho$, then the function $I(t)$ grows at first, and then declines, see the picture. In this case we say that an *epidemic occurs*. If, on the other hand, the initial number of susceptible people is below ρ, then the number of infected people $I(t)$ declines and tends to zero, and we say that the initial outbreak has been successfully *contained*. The number ρ is called the *threshold value*.

To avoid an epidemic, public health officials should try to increase the threshold value ρ (to make it more likely that $S_0 < \rho$), by increasing the removal rate γ (recall that $\rho = \gamma/r$), which is achieved by isolating sick people. Notice also the following harsh conclusion: if a disease kills people quickly, then the removal rate γ is high, and such a disease may be easier to contain.

In some cases it is easy to estimate the number of people, that will get sick during an epidemic. Assume that I_0 is so small that we can take $I_0 = 0$, while S_0 is a little larger than the threshold value ρ, so that $S_0 = \rho + \nu$, where $\nu > 0$ is a small value. As time t increases, $I(t)$ tends to zero, while $S(t)$ approaches some final number, call it S_f (look at the integral curves again). We conclude from (3.3) that

$$S_f - \rho \ln S_f = S_0 - \rho \ln S_0\,. \tag{3.4}$$

The function $I(S) = S - \rho \ln S$ takes its global maximum at $S = \rho$. Such a function is almost symmetric with respect to ρ, for S close to ρ, as can be seen from the three term Taylor series expansion of $I(S)$ at $S = \rho$. It follows from (3.4) that the points S_0 and S_f are approximately equidistant from ρ, so that $S_f \approx \rho - \nu$. The total number of people, who will get sick during an epidemic is then $S_0 - S_f \approx 2\nu$. (This fact is known as the *Threshold Theorem of epidemiology*, see for example M. Braun [5].)

6.4 Lyapunov's Stability

We consider a nonlinear system of equations for the unknown functions $x(t)$ and $y(t)$:

$$
\begin{aligned}
x' &= f(x,y), \quad x(0)=\alpha \\
y' &= g(x,y), \quad y(0)=\beta \,,
\end{aligned}
\tag{4.1}
$$

with some given differentiable functions $f(x,y)$ and $g(x,y)$, and the initial values α and β. By the existence and uniqueness Theorem 6.1.1, the problem (4.1) has a unique solution.

Recall that a point (x_0,y_0) is called *rest point* if

$$f(x_0,y_0)=g(x_0,y_0)=0\,.$$

Clearly, the pair of constant functions $x(t)=x_0$ and $y(t)=y_0$ is a solution of (4.1). If we solve the system (4.1) with the initial data $x(0)=x_0$ and $y(0)=y_0$, then $x(t)=x_0$ and $y(t)=y_0$ for all t (by uniqueness of the solution), so that our system is at rest for all time. Now suppose that the initial conditions are perturbed from (x_0,y_0). Will our system come back to rest at (x_0,y_0)?

A differentiable function $L(x,y)$ is called *Lyapunov's function* at (x_0,y_0), if the following two conditions hold:

$$
\begin{gathered}
L(x_0,y_0)=0\,, \\
L(x,y)>0, \quad \text{for } (x,y) \text{ in some neighborhood of } (x_0,y_0)\,.
\end{gathered}
$$

How do the level lines

$$L(x,y)=c \tag{4.2}$$

look, near the point (x_0,y_0)? If $c=0$, then (4.2) is satisfied only at (x_0,y_0). If $c>0$, and small, the level lines are closed curves around (x_0,y_0), and the smaller c is, the closer the level line is to (x_0,y_0).

Along a solution $(x(t),y(t))$ of our system (4.1), Lyapunov's function is a function of t: $L(x(t),y(t))$. Now assume that for all solutions $(x(t),y(t))$ starting near (x_0,y_0), we have

$$\frac{d}{dt}L(x(t),y(t))<0\,, \quad \text{for all } t>0\,, \tag{4.3}$$

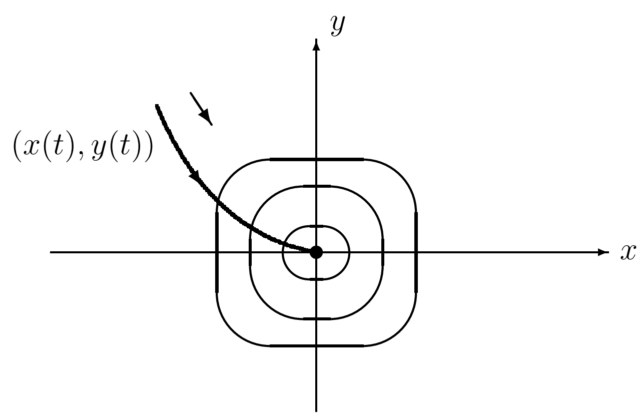
so that the function $L(x,y)$ is decreasing along the solutions. Then one expects that $(x(t),y(t)) \to (x_0,y_0)$, as $t\to\infty$, and we say that the rest point (x_0, y_0) is *asymptotically stable*. (As t increases, the solution point $(x(t), y(t))$ moves to the level lines that are closer and closer to (x_0, y_0).) Using the chain rule and the equations (4.1), we rewrite (4.3) as

$$\frac{d}{dt}L(x(t), y(t)) = L_x x' + L_y y' = L_x f(x, y) + L_y g(x, y) < 0\,. \tag{4.4}$$

The following *Lyapunov's theorem* is now intuitively clear (a proof can be found in the book of M.W. Hirsh and S. Smale [13]).

Theorem 6.4.1 *The rest point (x_0, y_0) is asymptotically stable, provided that there is a Lyapunov function, satisfying*

$$L_x(x, y)f(x, y) + L_y(x, y)g(x, y) < 0\,, \quad \textit{for all } (x, y) \textit{ near } (x_0, y_0)\,.$$



An integral curve cutting inside the level lines of $L(x, y)$

One typically assumes that $(x_0, y_0) = (0, 0)$, which can always be accomplished by declaring the point (x_0, y_0) to be the origin. Then $L(x, y) = ax^2 + cy^2$, with suitable positive constants a and c, is often a good choice of a Lyapunov's function.

Example 1 The system

$$x' = -2x + xy^2$$
$$y' = -y - 3x^2 y$$

has a unique rest point $(0, 0)$. With $L(x, y) = ax^2 + cy^2$, calculate

$$\frac{d}{dt}L(x(t), y(t)) = 2axx' + 2cyy' = 2ax(-2x + xy^2) + 2cy(-y - 3x^2 y)\,.$$

If we choose $a = 3$ and $c = 1$, this simplifies to

$$\frac{d}{dt}L(x(t), y(t)) = -12x^2 - 2y^2 < 0\,, \quad \text{for all } (x, y) \neq (0, 0)\,.$$

The rest point $(0, 0)$ is asymptotically stable.

Observe that the integral curves of this system cut inside of any ellipse $3x^2 + y^2 = A$, and as we vary the positive constant A, these ellipses fill out the entire xy-plane. It follows that *all solutions* of this system tend to $(0, 0)$, as $t \to \infty$. One says that the *domain of attraction* of the rest point $(0, 0)$ is the entire xy-plane.

Example 2 The system

$$\begin{aligned} x' &= -2x + y^4 \\ y' &= -y + x^5 \end{aligned}$$

has a rest point $(0, 0)$. (There is one more rest point at $\left(2^{1/19}, 2^{5/19}\right)$.) For small $|x|$ and $|y|$, the nonlinear terms (y^4 and x^5) are negligible. If we drop these nonlinear terms, then both of the resulting equations, $x' = -2x$ and $y' = -y$, have all solutions tending to zero as $t \to \infty$. Therefore, we expect the asymptotic stability of $(0, 0)$. Choosing $L(x, y) = x^2 + y^2$, compute

$$\begin{aligned} \tfrac{d}{dt}L(x(t), y(t)) &= 2xx' + 2yy' = 2x(-2x + y^4) + 2y(-y + x^5) \\ &= -4x^2 - 2y^2 + 2xy^4 + 2x^5y < -2(x^2 + y^2) + 2xy^4 + 2x^5y \\ &= -2r^2 + 2r^5\cos\theta\sin^4\theta + 2r^6\cos^5\theta\sin\theta < 0\,, \end{aligned}$$

provided that the point (x, y) belongs to a disc $B_\delta : x^2 + y^2 < \delta^2$, with a sufficiently small δ, as is clear by using the polar coordinates (r is small, $r < \delta$). Hence, any solution, with the initial point $(x(0), y(0))$ in B_δ, and δ sufficiently small, tends to zero as $t \to \infty$. The rest point $(0, 0)$ is indeed asymptotically stable. Its domain of attraction includes B_δ, with δ sufficiently small. (If the initial point $(x(0), y(0))$ is not close to the origin, solutions of this system will typically go to infinity in finite time.)

Example 3 If we drop the nonlinear terms in the system,

$$\begin{aligned} x' &= y - x^3 \\ y' &= -x - y^5\,, \end{aligned} \tag{4.5}$$

with a unique rest point at $(0, 0)$, the resulting linear system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \tag{4.6}$$

is equivalent to the harmonic oscillator $x''+x=0$, for which all solutions are periodic, and do not tend to zero. (The point $(0,0)$ is a center for (4.6).) It is the nonlinear terms that make the rest point $(0,0)$ asymptotically stable for the original system (4.5). Indeed, taking $L(x,y)=x^2+y^2$, we have

$$\frac{d}{dt}L(x(t),y(t)) = 2xx'+2yy' = -2x^4-2y^6<0\,, \quad \text{for all } (x,y)\neq(0,0)\,.$$

The domain of attraction of the rest point $(0,0)$ is the entire xy-plane.

Example 4 The system

$$\begin{aligned} x' &= -y + y\left(x^2+y^2\right) \\ y' &= x - x\left(x^2+y^2\right) \end{aligned} \tag{4.7}$$

has a rest point $(0,0)$. Multiply the first equation by x, the second one by y, and add the results. Obtain:

$$xx'+yy'=0\,,$$

$$\frac{d}{dt}\left(x^2+y^2\right)=0\,,$$

$$x^2+y^2=c^2\,.$$

The solution curves $(x(t),y(t))$ are circles around the origin. (If $x(0)=a$ and $y(0)=b$, then $x^2(t)+y^2(t)=a^2+b^2$.) If a solution starts near $(0,0)$, it stays near $(0,0)$, but it does not tend to $(0,0)$. In such a case, we say that the rest point $(0,0)$ is *stable*, although it is not asymptotically stable.

Here is a more formal definition: a rest point (x_0,y_0) of (4.1) is called *stable in the sense of Lyapunov* if given any $\epsilon>0$ one can find $\delta>0$, so that the solution curve $(x(t),y(t))$ lies within the distance ϵ of (x_0,y_0) for all $t>0$, provided that $(x(0),y(0))$ is within the distance δ of (x_0,y_0). Otherwise, the rest point (x_0,y_0) is called *unstable*. In addition to the rest point $(0,0)$, the system (4.7) has a whole circle $x^2+y^2=1$ of rest points. All of them are unstable, because the solutions on nearby circles move away from any point on $x^2+y^2=1$.

Example 5 The system

$$\begin{aligned} x' &= -y + x\left(x^2+y^2\right) \\ y' &= x + y\left(x^2+y^2\right) \end{aligned} \tag{4.8}$$

has a unique rest point $(0,0)$. Again, we multiply the first equation by x, the second one by y, and add the results. Obtain:

$$xx' + yy' = \left(x^2 + y^2\right)^2 .$$

Denoting $\rho = x^2 + y^2$, we rewrite this as

$$\frac{1}{2}\frac{d\rho}{dt} = \rho^2 .$$

Then $\dfrac{d\rho}{dt} > 0$, so that the function $\rho = \rho(t)$ is increasing. Integration gives

$$\rho(t) = \frac{\rho_0}{1 - 2\rho_0 t} , \quad \text{where } \rho_0 = x^2(0) + y^2(0) > 0 .$$

Solutions do not remain near $(0,0)$, no matter how close to this point we start. In fact, solutions move infinitely far away from $(0,0)$, as $t \to \dfrac{1}{2\rho_0}$. It follows that the rest point $(0,0)$ is *unstable*. It turns out that solutions of the system (4.8) spiral out of the rest point $(0,0)$.

To show that solutions move on spirals, we compute the derivative of the polar angle, $\theta = \tan^{-1}\dfrac{y}{x}$, along the solution curves. Using the chain rule

$$\frac{d\theta}{dt} = \theta_x x' + \theta_y y' = \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)x' + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{1}{x}\right)y'$$

$$= \frac{xy' - yx'}{x^2 + y^2} = \frac{x\left(x + y(x^2+y^2)\right) - y\left(-y + x(x^2+y^2)\right)}{x^2 + y^2} = 1 ,$$

so that $\theta = t + c$. It follows that $\theta \to -\infty$ as $t \to -\infty$, and the point $(x(t), y(t))$ moves on a spiral. We see that solutions of the system (4.8) spiral out (counterclockwise) of the rest point $(0,0)$ (corresponding to $t \to -\infty$), and tend to infinity, as $t \to \dfrac{1}{2\rho_0}$.

Example 6 For the system with three variables

$$\begin{aligned} x' &= -3x + y\left(x^2 + z^2 + 1\right) \\ y' &= -y - 2x\left(x^2 + z^2 + 1\right) \\ z' &= -z - x^2 z^3 , \end{aligned}$$

one checks that $(0,0,0)$ is the only rest point. We search for a Lyapunov function in the form $L(x,y,z) = \frac{1}{2}\left(ax^2+by^2+cz^2\right)$. Compute

$$\frac{dL}{dt} = axx'+byy'+czz' = -3ax^2+xy\left(x^2+z^2+1\right)(a-2b)-by^2-cx^2z^4-cz^2\,.$$

We have $\frac{dL}{dt} < 0$ for $(x,y,z) \neq (0,0,0)$, if $a = 2b$. In particular, we may select $b = 1$, $a = 2$, and $c = 1$, to prove that $(0,0,0)$ is asymptotically stable. (Lyapunov's theorem holds for systems with three or more variables too. Solutions cut inside the level surfaces of $L(x,y,z)$.)

6.4.1 Stable Systems

An $n \times n$ matrix A is called a *stable matrix*, if all of its eigenvalues are either negative, or they are complex numbers with negative real parts (which can also be stated as $\operatorname{Re}\lambda_i < 0$, for any eigenvalue of A). For a stable matrix A, all entries of e^{At} tend exponentially to zero as $t \to \infty$:

$$(4.9) \qquad |\left(e^{At}\right)_{ij}| \le ae^{-bt}\,,$$

for some positive constants a and b, and for all i and j. (Here $\left(e^{At}\right)_{ij}$ denotes the ij-element of e^{At}.) Indeed, $x(t) = e^{At}x(0)$ gives solution of the system

$$(4.10) \qquad x' = Ax\,,$$

so that e^{At} is the normalized fundamental solution matrix, and each column of e^{At} is a solution of (4.10). On the other hand, each solution of (4.10) contains factors of the type $e^{\operatorname{Re}\lambda_i t}$, as was developed in Chapter 5, justifying (4.9). For a stable matrix A, all solutions of the system $x' = Ax$ tend to zero, as $t \to \infty$, exponentially fast. If a matrix A is stable, so is its transpose A^T, because the eigenvalues of A^T are the same as those of A, and so the estimate (4.9) holds for A^T too.

We now solve the following *matrix equation*: given a stable matrix A, find a positive definite matrix B so that

$$(4.11) \qquad A^TB + BA = -I\,,$$

where I is the identity matrix. We shall show that a solution of (4.11) is given by

$$(4.12) \qquad B = \int_0^\infty e^{A^Tt}e^{At}\,dt\,.$$

By definition, to integrate a matrix, we integrate all of its entries. In view of the estimate (4.9), all of these integrals in (4.12) are convergent as $t \to \infty$. We have $B^T = B$ (using that $\left(e^{At}\right)^T = e^{A^T t}$), and

$$x^T B x = \int_0^\infty x^T e^{A^T t} e^{At} x \, dt = \int_0^\infty \left(e^{At} x\right)^T e^{At} x \, dt = \int_0^\infty ||e^{At} x||^2 \, dt > 0 \,,$$

for any $x \neq 0$, proving that the matrix B is positive definite. (Recall that $||y||$ denotes the length of a vector y.) Express

$$A^T B + BA = \int_0^\infty \left[A^T e^{A^T t} e^{At} + e^{A^T t} e^{At} A\right] dt$$

$$= \int_0^\infty \frac{d}{dt} \left[e^{A^T t} e^{At}\right] dt = e^{A^T t} e^{At}\Big|_0^\infty = -I,$$

as claimed (the upper limit vanishes by (4.9)).

We now consider a nonlinear system

$$x' = Ax + h(x) \,, \tag{4.13}$$

with a constant $n \times n$ matrix A, and a column vector function $h(x) = [h_1(x)\ h_2(x)\ \ldots\ h_n(x)]^T$, where $x = (x_1, x_2, \ldots, x_n)$. (So that $h(x)$ is a given vector function of the variables $x_1, x_2, \ldots, x_n$.) We assume that $h(0) = 0$, so that the system (4.13) has a trivial solution $x = 0$ ($x = 0$ is a rest point). We shall denote by $||x||$ and $||h(x)||$, the length of the vectors x and $h(x)$ respectively.

Theorem 6.4.2 *Assume that the matrix A is stable, $h(0) = 0$, and*

$$\lim_{x \to 0} \frac{||h(x)||}{||x||} = 0 \,. \tag{4.14}$$

Then the trivial solution of (4.13), $x = 0$, is asymptotically stable, so that any solution $x(t)$, with $||x(0)||$ small enough, satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof: Let the positive definite matrix B be given by (4.12), so that (4.11) holds. The quadratic form $L(x) = x^T B x$ is a Lyapunov function, because $L(0) = 0$ and $L(x) > 0$ for all $x \neq 0$. We shall show that $\frac{d}{dt} L(x(t)) < 0$, so that Lyapunov's theorem applies. Taking the transpose of the equation (4.13), obtain

$$\left(x^T\right)' = x^T A^T + h^T \,.$$

Using (4.11), and then the condition (4.14), gives

$$\frac{d}{dt}L(x(t)) = \left(x^T\right)' Bx + x^T Bx' = \left(x^T A^T + h^T\right) Bx + x^T B(Ax+h)$$

$$= x^T\left(A^T B + BA\right)x + h^T Bx + x^T Bh = -||x||^2 + h^T Bx + x^T Bh < 0\,,$$

provided that $||x||$ is small enough. (By (4.14), for any $\epsilon > 0$, we have $||h(x)|| < \epsilon||x||$, for $||x||$ small enough. Then $||h^T Bx|| \le ||h||\,||B||\,||x|| < \epsilon||B||\,||x||^2$, and the term $x^T Bh$ is estimated similarly. The norm of a matrix is defined as $||B|| = \sqrt{\sum_{i,j=1}^n b_{ij}^2}$, see the Appendix.) ♢

6.5 Limit Cycles

We consider the system (with $x = x(t)$, $y = y(t)$)

$$\begin{aligned} x' &= f(x,y)\,, \quad x(0) = x_0 \\ y' &= g(x,y)\,, \quad y(0) = y_0\,. \end{aligned} \tag{5.1}$$

Here $f(x,y)$ and $g(x,y)$ are given differentiable functions. Observe that these functions do not change with t (unlike $f(t,x,y)$ and $g(t,x,y)$). Systems like (5.1) are called *autonomous*. The initial point (x_0, y_0) is also given. By the existence and uniqueness Theorem 6.1.1, this problem has a solution $(x(t), y(t))$, which defines a curve (a *trajectory*) in the (x,y) plane, parameterized by t. If this curve is closed, we call the solution a *limit cycle*. (The functions $x(t)$ and $y(t)$ are then periodic.) If $(x(t), y(t))$ is a solution of (5.1), the same is true for $(x(t-\alpha), y(t-\alpha))$, where α is any number. Indeed, $(x(t), y(t))$ satisfies the system (5.1) at any t, and in particular at $t - \alpha$.

Example 1 One verifies directly that the unit circle $x = \cos t$, $y = \sin t$ is a limit cycle for the system

$$\begin{aligned} x' &= -y + x\left(1 - x^2 - y^2\right) \\ y' &= x + y\left(1 - x^2 - y^2\right)\,. \end{aligned} \tag{5.2}$$

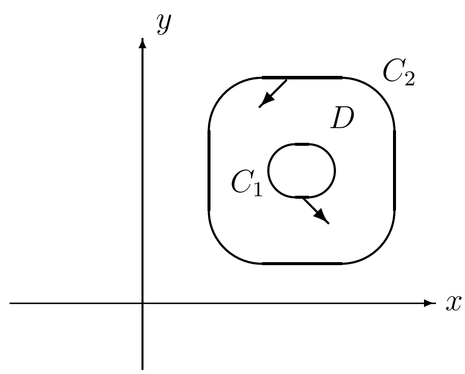
To see the dynamical significance of this limit cycle, we multiply the first equation by x, the second one by y, add the results, and call $\rho = x^2 + y^2 > 0$ (ρ is the square of the distance from the point (x,y) to the origin). Obtain

$$\frac{1}{2}\rho' = \rho(1-\rho)\,. \tag{5.3}$$

The origin $(0,0)$ is a rest point of (5.2), corresponding to the rest point $\rho = 0$ of (5.3). The equation (5.3) also has a rest point $\rho = 1$, corresponding to the limit cycle $x = \cos t$, $y = \sin t$. When $0 < \rho < 1$, it follows from the equation (5.3) that $\rho'(t) > 0$ and $\rho(t)$ is increasing. When $\rho > 1$, $\rho'(t) < 0$ and $\rho(t)$ is decreasing. It follows that $\rho(t) \to 1$ as $t \to \infty$ for all solutions of (5.3) (with $\rho(0) > 0$). We conclude that all solutions of (5.2) tend to the limit cycle, which is called *asymptotically orbitally stable.* Notice that asymptotic orbital stability does not imply asymptotic stability (which means that all solutions, starting sufficiently close to some solution, tend to it as $t \to \infty$). Indeed, a solution tending to the unit circle may tend to $x = \cos(t - \alpha)$, $y = \sin(t - \alpha)$ for some $0 < \alpha < 2\pi$, instead of $x = \cos t$, $y = \sin t$.

The vector field $F(x, y) = (f(x, y), g(x, y))$ is tangent to the solution curves of (5.1), and so $F(x, y)$ gives the direction in which the solution curve travels at the point (x, y) (observe that $F(x, y) = (x', y')$, the velocity).

The following classical theorem gives conditions for the existence of a stable limit cycle.



The trapping region D

Poincare-Bendixson Theorem *Suppose that D is a region of the $xy-$ plane lying between two simple closed curves C_1 and C_2. Assume that the system (5.1) has no rest points in D, and that at all points of C_1 and C_2 the vector field $F(x, y)$ points toward the interior of D. Then (5.1) has a limit cycle inside of D. Moreover, each trajectory of (5.1), originating in D, is either a limit cycle, or it tends to a limit cycle, which is contained in D.*

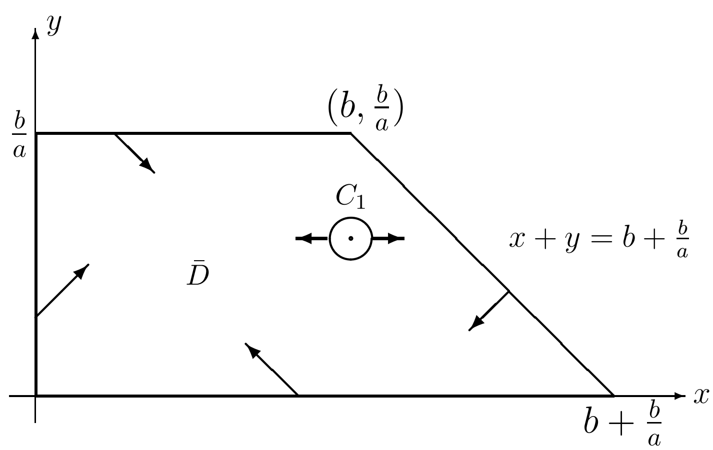
A proof can be found in I.G. Petrovskii [22]. The region D is often called the *trapping region.*

For practice, let us apply the Poincare-Bendixson theorem to the system (5.2) (we already know that $x = \cos t$, $y = \sin t$ is a limit cycle). One checks that $(0,0)$ is the only rest point. (Setting the right hand sides of (5.2) to zero, one gets $y/x = -x/y$, so that $x = y = 0$.) Let C_1 be the circle of radius $1/2$ around the origin, C_2 the circle of radius 2 around the origin, and D the region between them. On C_1, $F(x,y) = \left(-y + \frac{3}{4}x, x + \frac{3}{4}y\right)$. The scalar product of this vector with (x,y) is $\frac{3}{4}x^2 + \frac{3}{4}y^2 > 0$. Hence, $F(x,y)$ points outside of C_1, and into D. On C_2, $F(x,y) = (-y - 3x, x - 3y)$. The scalar product of this vector with (x,y) is $-3x^2 - 3y^2 < 0$, and $F(x,y)$ points into D. We conclude that there is a limit cycle in D, confirming what we already know.

Example 2 To model oscillations connected to *glycolysis* (the process of cells breaking down sugar to obtain energy) the following model was proposed by E.E. Sel'kov [28]

$$
\begin{aligned}
x' &= -x + ay + x^2 y \\
y' &= b - ay - x^2 y\,.
\end{aligned}
\tag{5.4}
$$

Here a and b are positive parameters. The unknown functions $x(t)$ and $y(t)$ represent some biological quantities, and are also assumed to be positive.



The trapping region $\bar{D} \setminus D_1$ (D_1 is the disc inside C_1)

To prove the existence of a limit cycle, we shall construct a trapping region. Consider the four-sided polygon $\bar{D}$ in the xy-plane bounded by the part of x-axis, $0 < x < b + \frac{b}{a}$, by the part of y-axis, $0 < y < \frac{b}{a}$, by the horizontal line $y = \frac{b}{a}$, with $0 < x < b$, and finally by the line $x + y = b + \frac{b}{a}$, see the figure. We claim that the vector field of this system, $F(x,y) = (-x + ax + x^2y, b - ay - x^2y)$, points inside of $\bar{D}$, on the boundary of $\bar{D}$. Indeed, $y' > 0$ when $y = 0$ (on the lower side) and the trajectories go up, $y' < 0$ when $y = \frac{b}{a}$ (on the upper side) and the trajectories go down. On the left side, $x = 0$, we have $x' > 0$, and the trajectories travel to the right. Turning to the right hand side, observe that by adding the equations in (5.4), we get

$$(x+y)' = b - x < 0\,, \quad \text{for } x > b\,.$$

Hence, the trajectories travel from the line $x + y = b + \frac{b}{a}$ (the right side) toward the lines $x + y = c$, with $c < b + \frac{b}{a}$, which corresponds to the interior of $\bar{D}$.

We now look for the rest points. Setting the right hand side of the first equation to zero, we get $y = \frac{x}{x^2+a}$. Similarly, from the second equation, $y = \frac{b}{x^2+a}$. We conclude that $x = b$, and $y = \frac{b}{b^2+a} < \frac{b}{a}$. The only rest point $(b, \frac{b}{b^2+a})$ lies inside $\bar{D}$. To determine the stability of this rest point, we consider its linearized system, with the Jacobian matrix $A = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$ evaluated at the rest point $(b, \frac{b}{b^2+a})$:

$$A = \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{bmatrix} \Bigg|_{\left(b,\, \frac{b}{b^2+a}\right)} = \begin{bmatrix} -1 + 2\frac{b^2}{b^2+a} & a + b^2 \\ -2\frac{b^2}{b^2+a} & -a - b^2 \end{bmatrix}.$$

The eigenvalues of A satisfy $\lambda_1 \lambda_2 = \det A = a + b^2 > 0$, $\lambda_1 + \lambda_2 = \operatorname{tr} A$. If

$$\text{(5.5)} \qquad \operatorname{tr} A = -\frac{b^4 + (2a-1)b^2 + a + a^2}{b^2 + a} > 0\,,$$

then λ_1 and λ_2 are either both positive, or both are complex numbers with positive real parts. In the first case, the rest point $(b, \frac{b}{b^2+a})$ is an unstable node, and in the second case it is an unstable spiral, for both the linearized system and for (5.3), in view of the Hartman-Grobman theorem. Hence, on a small circle C_1 around the rest point, bounding the disc we call D_1, trajectories point out of D_1 (outside of C_1). Let now D denote the region $\bar{D}$, with the disc D_1 removed, $D = \bar{D} \setminus D_1$. Then, under the condition (5.5), D is a trapping region, and by the Poincare-Bendixson theorem there exists a stable limit cycle of (5.2), lying in D.

The condition (5.5) is equivalent to

$$b^2 - b^4 - 2ab^2 - a - a^2 > 0\,,$$

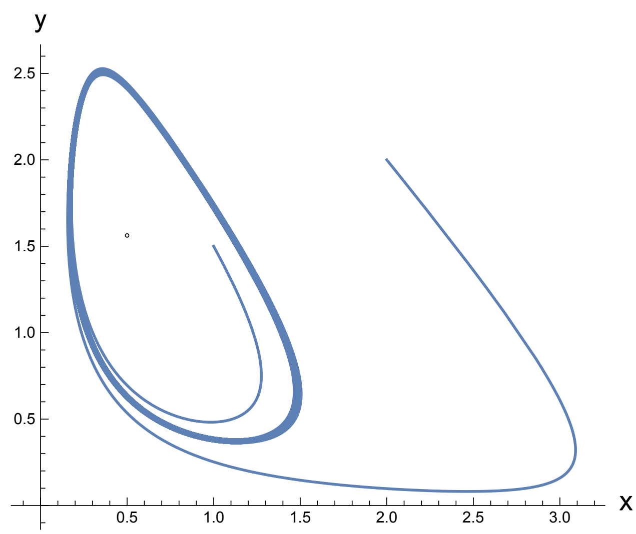


Figure 6.2: The limit cycle of Sel'kov system (5.4) for $a = 0.07$, $b = 0.5$

which holds if b is small, and a is even smaller. Still this result is biologically significant. Computations show that the limit cycle is unique, and it attracts all trajectories originating in the first quadrant of the xy-plane. In Figure 6.2 we present *Mathematica*'s computation of the limit cycle (thick) for Sel'kov system (5.4), with $a = 0.07$, $b = 0.5$. One sees that the trajectories both on the inside and on outside of the limit cycle converge quickly to it. The rest point at $(0.5, 1.5625)$ is marked.

Sometimes one wishes to prove that limit cycles do not exist in some region. Recall that a region is called *simply-connected* if it has no holes.

Dulac-Bendixson Criterion *Assume that $f(x,y)$, $f_x(x,y)$ and $g_y(x,y)$ are continuous in some simply-connected region R of the xy-plane, and*

$$f_x(x,y) + g_y(x,y)$$

does not change sign on R (it is either positive for all points in R, or negative everywhere on R). Then the system (5.1) has no closed trajectories inside R.

Proof: Any solution $(x(t), y(t))$ of (5.1), for $a \le t \le b$, determines a curve

C (or a trajectory) in the xy-plane. We evaluate the following line integral

$$\int_C g(x,y)\,dx - f(x,y)\,dy = \int_a^b \left[g(x(t),y(t))\,x'(t) - f(x(t),y(t))\,y'(t)\right]\,dt = 0\,, \tag{5.6}$$

using the equations (5.1). If $(x(t), y(t))$ is a limit cycle inside R, then (5.6) holds, moreover the curve C is closed, and it encloses some region P inside R. By Green's formula, the line integral

$$\int_C g(x,y)\,dx - f(x,y)\,dy = -\iint_P \left[f_x(x,y) + g_y(x,y)\right]\,dA$$

is either positive or negative, contradicting (5.6). ♢

Exactly the same proof produces a more general result.

Theorem 6.5.1 *(Dulac-Bendixson Criterion) Assume that $f(x,y)$, $f_x(x,y)$ and $g_y(x,y)$ are continuous in some region R of the xy-plane, and there is a differentiable function $h(x,y)$, which is positive on R, so that*

$$\frac{\partial}{\partial x}\left[h(x,y)f(x,y)\right] + \frac{\partial}{\partial y}\left[h(x,y)g(x,y)\right] \tag{5.7}$$

does not change sign on R. Then the system (5.1) has no closed trajectories inside R.

Example 3 The Lotka-Volterra model of two competing species

$$\begin{aligned} x' &= x\,(a - bx - cy) \\ y' &= y\,(d - ex - ky)\ , \end{aligned} \tag{5.8}$$

with positive constants a,b,c,d,e and k, has no limit cycles in the first quadrant of the xy-plane. Indeed, select $h(x,y) = \frac{1}{xy} > 0$ for $x, y > 0$. Then the expression (5.7) becomes

$$\frac{\partial}{\partial x}\left(\frac{a - bx - cy}{y}\right) + \frac{\partial}{\partial y}\left(\frac{d - ex - ky}{x}\right) = -\frac{b}{y} - \frac{k}{x} < 0\,,$$

for $x, y > 0$. It follows that the system (5.8) has no limit cycles in the first quadrant of the xy-plane.

6.6 Periodic Population Models

Population models become much harder to analyze in case the coefficients vary with time t. However, if all coefficients are periodic functions of the same period, it is still possible to obtain detailed description of the solutions.

We begin by considering the logistic equation

$$x' = x\left(a(t) - b(t)x\right), \tag{6.1}$$

modeling the number of rabbits $x(t)$ at time t. We are only interested in positive solutions, $x(t) > 0$ for all t. The given continuous functions $a(t)$ and $b(t)$ are assumed to be periodic, with the period p, so that $a(t+p) = a(t)$, and $b(t+p) = b(t)$ for all t. The periodicity of $a(t)$ and $b(t)$ can be attributed to seasonal variations. For example, the carrying capacity $a(t)$ is likely to be higher in summer, and lower, or even negative, in winter. We assume that the average value of $a(t)$ is positive, so that

$$\int_0^p a(s)\, ds > 0\,, \tag{6.2}$$

and that the self-limitation coefficient $b(t)$ satisfies

$$b(t) > 0\,, \quad \text{for all } t\,. \tag{6.3}$$

This equation is of Bernoulli's type. We divide it by x^2

$$\frac{x'}{x^2} = \frac{a(t)}{x} - b(t)\,,$$

and set $y = \dfrac{1}{x}$. Then $y' = -\dfrac{x'}{x^2}$, and we obtain a linear equation

$$y' + a(t)y = b(t) \tag{6.4}$$

that is easy to analyze.

Lemma 6.6.1 *The problem (6.4) has a positive solution of period p. This solution is unique, and it attracts all other solutions of (6.4), as $t \to \infty$.*

Proof: With the integrating factor $\mu = e^{\int_0^t a(s)\, ds}$, the solution of (6.4), satisfying an initial condition $y(0) = y_0$, is found as follows

$$\frac{d}{dt}\left[\mu(t)y(t)\right] = \mu(t)b(t)\,,$$

$$y(t) = \frac{1}{\mu(t)} y_0 + \frac{1}{\mu(t)} \int_0^t \mu(s) b(s) \, ds \, .$$

This solution is periodic, provided that $y(p) = y(0) = y_0$ (as justified in Problems), implying that

$$y_0 = \frac{1}{\mu(p)} y_0 + \frac{1}{\mu(p)} \int_0^p \mu(s) b(s) \, ds \, ,$$

which we write as

$$(\mu(p) - 1) y_0 = \int_0^p \mu(s) b(s) \, ds \, .$$

Since $\mu(p) > 1$ (by the assumption (6.2)), we can solve this equation for y_0

$$y_0 = \frac{1}{\mu(p) - 1} \int_0^p \mu(s) b(s) \, ds > 0 \, ,$$

obtaining the initial value y_0, which leads to a positive solution $y(t)$ of period p.

If $z(t)$ is another solution of (6.4), the difference $w(t) = z(t) - y(t)$ satisfies

$$w' + a(t) w = 0 \, .$$

Integrating, $w(t) = e^{-\int_0^t a(s) \, ds} w(0) \to 0$, as $t \to \infty$ (by the assumption (6.2)), proving that all solutions tend to $y(t)$. In particular, this fact implies that the periodic solution $y(t)$ is unique. $\diamondsuit$

This lemma makes possible the following complete description of the dynamics for the logistic equation.

Theorem 6.6.1 *Assume that the continuous p-periodic functions $a(t)$, $b(t)$ satisfy the conditions (6.2) and (6.3). Then the equation (6.1) has a positive solution of period p. This solution is unique, and it attracts all other positive solutions of (6.1), as $t \to \infty$.*

Proof: By Lemma 6.6.1, there is a positive p-periodic solution $y(t)$ of (6.4). Then $x(t) = \dfrac{1}{y(t)}$ gives a positive p-periodic solution of (6.1). If $z(t)$ is another positive solution of (6.1), then the same lemma tells us that $\dfrac{1}{x(t)} - \dfrac{1}{z(t)} \to 0$ as $t \to \infty$, which implies that $x(t) - z(t) \to 0$ as $t \to \infty$. (Observe that $z(t)$ is bounded. Indeed, it follows from the equation (6.1) that $z(t)$ is decreasing for large enough z, so that $a(t) - b(t) z < 0$, which holds if $z > \frac{\max a(t)}{\min b(t)}$. Then $z \leq \frac{\max a(t)}{\min b(t)}$.) It follows that the p-periodic solution $x(t)$ attracts all other positive solutions of (6.1), and there is only one p-periodic solution. ♢

The following corollary says that an increase in carrying capacity will increase the p-periodic solution of (6.1). This is natural, because the p-periodic solution attracts all other solutions ("a rising tide lifts all boats").

Corollary 6.6.1 *Let $x_1(t)$ be the positive p-periodic solution of*

$$x_1' = x_1(a_1(t) - b(t)x_1)\,,$$

where the p-periodic function $a_1(t)$ satisfies $a_1(t) > a(t)$ for all t. Then $x_1(t) > x(t)$ for all t ($x(t)$ is the positive p-periodic solution of (6.1)). Moreover, if $a_1(t)$ is close to $a(t)$, then $x_1(t)$ is close to $x(t)$.

Proof: Set $y_1 = \dfrac{1}{x_1}$, and $y = \dfrac{1}{x}$. As before, $y' + a(t)y = b(t)$, and

$$y_1' + a_1(t)y_1 = b(t)\,.$$

Let $z(t) = y(t) - y_1(t)$. Then $z(t)$ is a p-periodic solution of

$$z' + a(t)z = (a_1(t) - a(t))y_1(t) > 0\,. \tag{6.5}$$

By Lemma 6.6.1, the periodic solution of this equation is positive, so that $z(t) > 0$ for all t, and then $y(t) > y_1(t)$, which implies that $x(t) < x_1(t)$ for all t.

Turning to the second statement, we are now given that the right hand side of (6.5) is small. Going over the construction of the p-periodic solution in Lemma 6.6.1, we see that $z(t)$ is small as well. ♢

We consider next another model for a population $x(t) > 0$ of rabbits

$$x' = x\,(x - a(t))\,(b(t) - x)\,. \tag{6.6}$$

The given continuous functions $a(t)$ and $b(t)$ are assumed to be positive, periodic with period p, and satisfying $0 < a(t) < b(t)$ for all t. If $0 < x(t) < a(t)$, it follows from the equation that $x'(t) < 0$, and $x(t)$ decreases. When $a(t) < x(t) < b(t)$, the population grows. So that $a(t)$ gives a *threshold for population growth*. (If the number of rabbits falls too low, they have a problem meeting the "significant others".)

Let us assume additionally that the maximum value of $a(t)$ lies below the minimum value of $b(t)$:

$$\max_{-\infty<t<\infty} a(t) < \min_{-\infty<t<\infty} b(t)\,. \tag{6.7}$$

Theorem 6.6.2 *If $a(t)$ and $b(t)$ are continuous p-periodic functions satisfying (6.7), then the equation (6.6) has exactly two positive solutions of period p.*

Proof: We denote by $x(t, x_0)$ the solution of (6.6), satisfying the initial condition $x(0) = x_0$, so that $x(0, x_0) = x_0$. To prove the existence of two solutions, we define the *Poincaré map* $x_0 \to T(x_0)$, by setting $T(x_0) = x(p, x_0)$. The function $T(x_0)$ is continuous (by the continuous dependence of solutions, with respect to the initial condition). Define the numbers $A = \min_{-\infty<t<\infty} b(t) - \epsilon$, $B = \max_{-\infty<t<\infty} b(t) + \epsilon$, and the interval $I = (A, B)$. If $\epsilon > 0$ is chosen so small that $\max_{-\infty<t<\infty} a(t) < \min_{-\infty<t<\infty} b(t) - \epsilon$, then we claim that the map T takes the interval I into itself. Indeed, if $x_0 = A$, then from the equation $x'(0) > 0$, and $x(t, x_0)$ is increasing, for small t. At future times, the solution curve cannot cross below x_0, because again we have $x'(t) > 0$, if $x(t) = x_0$. It follows that $x(p, x_0) > x_0$, or $T(A) > A$. Similarly, we show that $T(B) < B$. The continuous function $T(x) - x$ is positive at $x = A$, and negative at $x = B$. By the intermediate value theorem $T(x) - x$ has a root $\bar{x}$ on the interval $I = (A, B)$, so that there is a *fixed point* $\bar{x}$ such that $T(\bar{x}) = \bar{x}$. Then $x(p, \bar{x}) = \bar{x}$, which implies that $x(t, \bar{x})$ is a p-periodic solution.

The second periodic solution is obtained by considering the map T_1, defined by setting $T_1(x_0) = x(-p, x_0)$, corresponding to solving the equation (6.6) backward in time. As in the case of T, we see that T_1 is a continuous map, taking the interval $T_1 = (\min\limits_{-\infty<t<\infty} a(t) - \epsilon, \max\limits_{-\infty<t<\infty} a(t) + \epsilon)$ into itself (for small $\epsilon > 0$). T_1 has a fixed point on I_1, giving us the second p-periodic solution. So that the equation (6.6) has at least two positive p-periodic solutions.

To prove that there are at most two positive p-periodic solutions of (6.6), we need the following two lemmas, which are also of independent interest.

Lemma 6.6.2 *Consider the equation (for $w(t)$)*

$$w' = c(t)w \,, \tag{6.8}$$

with a given continuous p-periodic function $c(t)$. This equation has a non-zero p-periodic solution, if and only if

$$\int_0^p c(s)\, ds = 0 \,.$$

Proof: Integrating the equation (6.8), gives $w(t) = w(0)e^{\int_0^t c(s)\,ds}$. Using the periodicity of $c(t)$, we see that

$$
\begin{aligned}
w(t+p) &= w(0)e^{\int_0^{t+p} c(s)\,ds} = w(0)e^{\int_0^t c(s)\,ds + \int_t^{t+p} c(s)\,ds} \\
&= w(0)e^{\int_0^t c(s)\,ds} e^{\int_0^p c(s)\,ds} = w(0)e^{\int_0^t c(s)\,ds} = w(t)\,,
\end{aligned}
$$

exactly when $\int_0^p c(s)\,ds = 0$. ♢

Lemma 6.6.3 *Consider the nonlinear equation (with a continuous function $f(t,x)$, which is twice differentiable in x)*

$$x' = f(t,x)\,. \tag{6.9}$$

Assume that the function $f(t,x)$ is p-periodic in t, and convex in x:

$$
\begin{aligned}
&f(t+p,x) = f(t,x), \quad \text{for all } t, \text{ and } x > 0\,, \\
&f_{xx}(t,x) > 0, \quad \text{for all } t, \text{ and } x > 0\,.
\end{aligned}
$$

Then the equation (6.9) has at most two positive p-periodic solutions.

Proof: Assume, on the contrary, that we have three positive p-periodic solutions: $x_1(t) < x_2(t) < x_3(t)$ (solutions of (6.9) do not intersect, by the existence and uniqueness theorem). Set $w_1 = x_2 - x_1$, and $w_2 = x_3 - x_2$. These functions are p-periodic, and they satisfy

$$
\begin{aligned}
w_1' &= f(t,x_2) - f(t,x_1) = \int_0^1 \frac{d}{d\theta} f\left(t, \theta x_2 + (1-\theta)x_1\right)\,d\theta \\
&= \int_0^1 f_x\left(t, \theta x_2 + (1-\theta)x_1\right)\,d\theta\; w_1 \equiv c_1(t)w_1\,, \\
w_2' &= f(t,x_3) - f(t,x_2) = \int_0^1 \frac{d}{d\theta} f\left(t, \theta x_3 + (1-\theta)x_2\right)\,d\theta \\
&= \int_0^1 f_x\left(t, \theta x_3 + (1-\theta)x_2\right)\,d\theta\; w_2 \equiv c_2(t)w_2\,.
\end{aligned}
$$

(We denoted by $c_1(t)$ and $c_2(t)$ the corresponding integrals.) The function f_x is increasing in x (because $f_{xx} > 0$). It follows that $c_2(t) > c_1(t)$ for all t, and so $c_1(t)$ and $c_2(t)$ cannot both satisfy the condition of Lemma 6.6.2. We have a contradiction with Lemma 6.6.2. (Observe that $w_1(t)$ and $w_2(t)$ are p-periodic solutions of the equations $w_1' = c_1(t)w_1$, and $w_2' = c_2(t)w_2$.) ♢

Returning to the proof of the theorem, we rewrite (6.6) as

$$x' = -x^3 + (a(t) + b(t))x^2 - a(t)b(t)x\,.$$

Divide by x^3:

$$\frac{x'}{x^3} = -1 + \frac{(a(t) + b(t))}{x} - \frac{a(t)b(t)}{x^2}\,.$$

We set here $u = \dfrac{1}{x^2}$, so that $u' = -2x^{-3}x'$, and obtain

$$u' = 2 - 2(a(t) + b(t))\sqrt{u} + 2a(t)b(t)u\,. \tag{6.10}$$

Clearly, the positive p-periodic solutions of the equations (6.6) and (6.10) are in one-to-one correspondence. But the right hand side in (6.10) is convex in u. By Lemma 6.6.3, both of the equations (6.10) and (6.6) have at most two positive p-periodic solutions. It follows that the equation (6.6) has exactly two positive p-periodic solutions. ♢

Let $x_1(t) < x_2(t)$ denote the two positive p-periodic solutions of (6.6), provided by the above theorem, and let $x(t, x_0)$ denote again the solution of this equation, with the initial condition $x(0) = x_0$. It is not hard to show that $x(t, x_0) \to x_2(t)$ as $t \to \infty$, if x_0 belongs to either one of the intervals $(x_2(0), \infty)$, or $(x_1(0), x_2(0))$. On the other hand, $x(t, x_0) \to 0$ as $t \to \infty$, if $x_0 \in (0, x_1(0))$. So that the larger p-periodic solution $x_2(t)$ is *asymptotically stable*, while the smaller one $x_1(t)$ is *unstable*.

We consider next the case of two competing species, with the populations $x(t)$ and $y(t)$, satisfying the Lotka-Volterra system

$$\begin{aligned} x' &= x(a(t) - bx - cy) \\ y' &= y(d(t) - ex - fy)\,. \end{aligned} \tag{6.11}$$

The given functions $a(t)$ and $d(t)$ are assumed to be periodic, with the period p, so that $a(t+p) = a(t)$, and $d(t+p) = d(t)$ for all t. We do not assume $a(t)$ and $d(t)$ to be positive, but assume that they have positive averages over $(0, p)$. The positive numbers b, c, e and f are given. As before, the periodicity of $a(t)$ and $d(t)$ may be attributed to seasonal variations. The numbers c and e are called the *interaction coefficients*. They quantify how detrimental are the species for each other. It seems reasonable to assume that the *self-limitation coefficients* b and f, as well as the interaction coefficients c and e change less with the seasons than the carrying capacities, and so we assumed them to be constants in this model.

We may regard the first equation as a logistic equation for $x(t)$, with the carrying capacity equal to $a(t) - cy(t)$. We see that each species in effect diminishes the carrying capacity of the other one, as they compete for food, hiding places, etc. The largest population of the first species occurs, if the second one goes extinct, so that $y(t) = 0$ for all t. Then the first equation becomes

$$x' = x(a(t) - bx)\,. \tag{6.12}$$

By the Theorem 6.6.1, this equation has a unique positive p-periodic solution, which we denote by $X(t)$. The pair $(X(t), 0)$ is a p-periodic solution of our system (6.11), called a *semi-trivial solution*. If $Y(t)$ denotes the unique positive p-periodic solution of

$$y' = y(d(t) - fy)\,, \tag{6.13}$$

then $(0, Y(t))$ is the other semi-trivial solution.

The following theorem describes a case, when the dynamics of the Lotka-Volterra system is similar to that of a single logistic equation.

Theorem 6.6.3 *Denoting $A = \int_0^p a(t)\,dt$, $D = \int_0^p d(t)\,dt$, assume that $A > 0$, $D > 0$, and*

$$fA - cD > 0, \quad \text{and} \quad bD - eA > 0\,. \tag{6.14}$$

Then the system (6.11) has a unique positive (in both components) p-periodic solution, to which all other positive solutions of (6.11) tend, as $t \to \infty$.

Proof: The largest possible periodic solutions $X(t)$ and $Y(t)$ (defined by (6.12) and (6.13)) occur when the competitor species is extinct. Denote by $\xi(t)$ the positive p-periodic solution of the logistic equation

$$x' = x(a(t) - bx - cY(t))\,.$$

Here the first species is forced to compete with the maximal periodic solution for the second species. Similarly, we denote by $\eta(t)$ the positive p-periodic solution of the logistic equation

$$y' = y(d(t) - eX - fy)\,.$$

(To prove the existence of $\xi(t)$, we need to show that $\int_0^p (a(t) - cY(t))\,dt > 0$. Dividing the equation $Y' = Y(d(t) - fY)$ by Y and integrating, we have $\int_0^p Y(t)\,dt = \frac{D}{f}$. Then $\int_0^p (a(t) - cY(t))\,dt = A - c\frac{D}{f} > 0$, by the first condition in (6.14). The existence of $\eta(t)$ is proved similarly.)

We shall construct a p-periodic solution of our system (6.11) as the limit of the p-periodic approximations (x_n, y_n), defined as follows: $x_1 = X(t)$, $y_1 = \eta(t)$, while $(x_2(t), y_2(t))$ are respectively the p-periodic solutions of the following two logistic equations

$$\begin{aligned} x_2' &= x_2(a(t) - bx_2 - cy_1)\,, \\ y_2' &= y_2(d(t) - ex_2 - fy_2)\,. \end{aligned}$$

So that we compute $x_2(t)$, and immediately use it to compute $y_2(t)$. Using the Corollary 6.6.1, we see that

$$x_2(t) < x_1(t), \text{ and } y_1(t) < y_2(t) \quad \text{for all } t\,.$$

(Or think "ecologically": x_2 is computed when the first species faces some competition, while $x_1 = X(t)$ was computed without competition, so that $x_2(t) < x_1(t)$; y_2 is computed when the second species faces weaker competition, than when y_1 was computed, so that $y_1(t) < y_2(t)$.) In general, once $(x_n(t), y_n(t))$ is computed, we obtain $(x_{n+1}(t), y_{n+1}(t))$ by finding the p-periodic solutions of the logistic equations

$$\begin{aligned} x_{n+1}' &= x_{n+1}(a(t) - bx_{n+1} - cy_n)\,, \\ y_{n+1}' &= y_{n+1}(d(t) - ex_{n+1} - fy_{n+1})\,. \end{aligned} \tag{6.15}$$

By the same reasoning as above, we show that for all t

$$\xi(t) < x_n(t) < \cdots < x_2(t) < x_1(t), \text{ and } y_1(t) < y_2(t) < \cdots < y_n(t) < Y(t)\,.$$

At each t, $x_n(t)$ is a monotone and bounded sequence of numbers, which has a limit. We denote $x_p(t) = \lim_{t\to\infty} x_n(t)$, and similarly $y_p(t) = \lim_{t\to\infty} y_n(t)$. Passing to the limit in the equations (6.15) (or rather in their integral versions), we see that $(x_p(t), y_p(t))$ is a positive p-periodic solution of (6.11).

Next, we prove that there is only one positive p-periodic solution of (6.11). Let $(x(t), y(t))$ be any positive p-periodic solution of (6.11). We divide the first equation in (6.11) by $x(t)$, and integrate over $(0, p)$. By periodicity, $\displaystyle\int_0^p \frac{x'(t)}{x(t)}\,dt = \ln x(t)\big|_0^p = 0$. Then

$$b\int_0^p x(t)\,dt + c\int_0^p y(t)\,dt = A\,.$$

Similarly, from the second equation in (6.11)

$$e\int_0^p x(t)\,dt + f\int_0^p y(t)\,dt = D\,.$$

Solving these two equations for the integrals, we get

$$(6.16)\qquad \int_0^p x(t)\,dt = \frac{fA-cD}{bf-ce} > 0, \quad \int_0^p y(t)\,dt = \frac{bD-eA}{bf-ec} > 0\,.$$

(Observe that our conditions (6.14) imply that $bf - ec > 0$.) Let now $(\bar{x}(t), \bar{y}(t))$ be another positive p-periodic solution of (6.11). Clearly, $\bar{x}(t) < x_1(t) = X(t)$, $\bar{y}(t) > y_1(t) = \eta(t)$. We prove inductively that $\bar{x}(t) < x_n(t)$, and $\bar{y}(t) > y_n(t)$. Letting $n \to \infty$, we have $\bar{x}(t) \le x_p(t)$, and $\bar{y}(t) \ge y_p(t)$. Since by (6.16),

$$\int_0^p x_p(t)\,dt = \int_0^p \bar{x}(t)\,dt\,, \text{ and } \int_0^p y_p(t)\,dt = \int_0^p \bar{y}(t)\,dt\,,$$

we conclude that $\bar{x}(t) = x_p(t)$, and $\bar{y}(t) = y_p(t)$.

Turning to the stability of $(x_p(t), y_p(t))$, we now define another sequence of iterates $(\hat{x}_n, \hat{y}_n)$. Beginning with $\hat{x}_1 = \xi(t)$, $\hat{y}_1 = Y(t)$, once the iterate $(\hat{x}_n(t), \hat{y}_n(t))$ is computed, we obtain $(\hat{x}_{n+1}(t), \hat{y}_{n+1}(t))$ by calculating the p-periodic solutions of the following two logistic equations

$$\begin{aligned} \hat{y}'_{n+1} &= \hat{y}_{n+1}(d(t) - e\hat{x}_n - f\hat{y}_{n+1})\,, \\ \hat{x}'_{n+1} &= \hat{x}_{n+1}(a(t) - b\hat{x}_{n+1} - c\hat{y}_{n+1})\,. \end{aligned}$$

(So that we compute $\hat{y}_{n+1}(t)$, and immediately use it to compute $\hat{x}_{n+1}(t)$.) By the same reasoning as above, we show that for all n

$$\hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_n < x_n(t) < \cdots < x_2(t) < x_1(t)\,,$$

and

$$y_1(t) < y_2(t) < \cdots < y_n(t) < \hat{y}_n(t) < \cdots < \hat{y}_2(t) < \hat{y}_1(t)\,.$$

As before, $(\hat{x}_n, \hat{y}_n)$ tends to a positive p-periodic solution of (6.11), which by the uniqueness must be $(x_p(t), y_p(t))$. We conclude that the periodic solution is approximated from both below and above by the *monotone sequences*, $x_p(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \hat{x}_n(t)$, and $y_p(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \hat{y}_n(t)$.

Next, we sketch a proof that any positive solution, $(x(t), y(t))$ of (6.11), tends to the unique positive p-periodic solution $(x_p(t), y_p(t))$, as $t \to \infty$. The idea is to show inductively that for any integer n, and any $\epsilon > 0$

$$(6.17)\quad \hat{x}_n(t) - \epsilon < x(t) < x_n(t) + \epsilon, \text{ and } y_n(t) - \epsilon < y(t) < \hat{y}_n(t) + \epsilon\,,$$

for t large enough. Indeed, the inequality $y_n(t) - \epsilon < y(t)$ implies that

$$x' < x(a(t) - bx - cy_n + c\epsilon)\,.$$

Then $x(t)$ lies below the solution of

$$x' = x(a(t) - bx - cy_n + c\epsilon)\,, \tag{6.18}$$

with the same initial condition. Any positive solution of (6.18) tends to the p-periodic solution of that equation, which, by the Corollary 6.6.1, is close to the p-periodic solution of

$$x' = x(a(t) - bx - cy_n)\,,$$

or close to x_{n+1}. So that the estimate of $y(t)$ from below in (6.17) leads to the estimate of $x(t)$ from above in (6.17), at $n+1$. This way we establish the inequalities (6.17) at the next value of n. ♢

This theorem appeared first in the author's paper [17]. The idea to use monotone iterations, to prove that all positive solutions tend to the periodic one, is due to E.N. Dancer [7], who used it in another context.

6.6.1 Problems

I.

1. (i) Find and classify the rest points of

$$x'(t) = x(x+1)(x-2)\,.$$

Hint: The sign of $x'(t)$ changes at $x = -1$, $x = 0$, and $x = 2$.

Answer. The rest points are $x = -1$ (unstable), $x = 0$ (stable), and $x = 2$ (unstable).

(ii) Let $y(t)$ be the solution of

$$y'(t) = y(y+1)(y-2)\,, \quad y(0) = 3\,.$$

Find $\lim_{t\to\infty} y(t)$, and $\lim_{t\to-\infty} y(t)$.

Answer. ∞, and 2.

(iii) What is the domain of attraction of the rest point $x = 0$?

Answer. $(-1, 2)$.

2. Consider a *population model with a threshold for growth*

$$x' = x(x-1)(5-x)\,.$$

(i) Find and classify the rest points.

Answer. The rest points are $x = 0$ (stable), $x = 1$ (unstable), and $x = 5$ (stable).

(ii) Calculate $\lim_{t\to\infty} x(t)$ for the following cases:

(a) $x(0) \in (0,1)$, (b) $x(0) > 1$.

Answer. (a) $\lim_{t\to\infty} x(t) = 0$, (b) $\lim_{t\to\infty} x(t) = 5$.

3. (i) Find and classify the rest points of

$$x' = x^2(2-x)\,.$$

Answer. The rest points are $x = 2$ (stable) and $x = 0$ (neither stable or unstable).

(ii) Calculate $\lim_{t\to\infty} x(t)$ for the following cases:

(a) $x(0) < 0$, (b) $x(0) > 0$.

Answer. (a) $\lim_{t\to\infty} x(t) = 0$, (b) $\lim_{t\to\infty} x(t) = 2$.

4. (i) Find and classify the rest point(s) of

$$x' = -x^2\,.$$

(ii) Solve this equation to show that $\lim_{t\to\infty} x(t) = 0$, for any value of $x(0)$.

5. (i) Show that the rest point $(0,0)$ is asymptotically stable for the system

$$\begin{aligned} x_1' &= -2x_1 + x_2 + x_1x_2 \\ x_2' &= x_1 - 2x_2 + x_1^3\,. \end{aligned}$$

(ii) Find the general solution of the corresponding linearized system

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 - 2x_2\,, \end{aligned}$$

and discuss its behavior as $t \to \infty$.

6. Show that the rest point $(0,0)$ is asymptotically stable for the system

$$x' = -5y - x(x^2+y^2)$$
$$y' = x - y(x^2+y^2)\,,$$

and its domain of attraction is the entire xy-plane.

Hint: Use $L(x,y) = \frac{1}{5}x^2 + y^2$.

7. (i) The equation (for $y = y(t)$)

$$y' = y^2(1-y)$$

has the rest points $y = 0$ and $y = 1$. Discuss their Lyapunov's stability.

Answer: $y = 1$ is asymptotically stable, $y = 0$ is unstable.

(ii) Find $\lim\limits_{t\to\infty} y(t)$ in the following cases:

(a) $y(0) < 0$, (b) $0 < y(0) < 1$, (c) $y(0) > 1$.

Answer. (a) $\lim_{t\to\infty} y(t) = 0$, (b) $\lim_{t\to\infty} y(t) = 1$, (c) $\lim_{t\to\infty} y(t) = 1$.

8. Show that the rest point $(0,0)$ is stable, but not asymptotically stable, for the system

$$x' = -y$$
$$y' = x^3\,.$$

Hint: Use $L(x,y) = \frac{1}{2}x^4 + y^2$.

9. (i) Convert the nonlinear equation

$$y'' + f(y)y' + y = 0$$

into a system, by letting $y = x_1$, and $y' = x_2$. Answer:

$$x_1' = x_2$$
$$x_2' = -x_1 - f(x_1)x_2\,.$$

(ii) Show that the rest point $(0,0)$ of this system is asymptotically stable, provided that $f(x_1) > 0$ for all $x_1 \neq 0$. What does this imply for the original equation?

Hint: Use $L = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$.

10. Show that the rest point $(0,0)$ is asymptotically stable for the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -g(x_1) - f(x_1)x_2 \,, \end{aligned}$$

provided that $f(x_1) > 0$, and $x_1 g(x_1) > 0$, for all $x_1 \neq 0$.

Hint: Use $L = \int_0^{x_1} g(s)\,ds + \frac{1}{2}x_2^2$. Observe that $L(x_1,x_2) > 0$, for all $(x_1,x_2) \neq (0,0)$.

What conclusion can one draw for the equation

$$y'' + f(y)y' + g(y) = 0\,?$$

11. Consider the system

$$\begin{aligned} x' &= -x^3 + 4y\left(z^2+1\right) \\ y' &= -y^5 - x\left(z^2+1\right) \\ z' &= -z - x^4 z^3 \,. \end{aligned}$$

(i) Show that $(0,0,0)$ is the only rest point, and it is asymptotically stable.

(ii) If we drop the nonlinear terms, we get a linear system

$$\begin{aligned} x' &= 4y \\ y' &= -x \\ z' &= -z \,. \end{aligned}$$

Show that any solution of this system moves on an elliptic cylinder, and it tends to the xy-plane, as $t \to \infty$. Conclude that the rest point $(0,0,0)$ is not asymptotically stable. Is it stable?

12. (i) Show that the rest point $(0,0,0)$ is Lyapunov stable, but not asymptotically stable for the system

$$\begin{aligned} x_1' &= x_2 + x_3 + x_2 x_3 \\ x_2' &= -x_1 + x_3 - 2x_1 x_3 \\ x_3' &= -x_1 - x_2 + x_1 x_2 \,. \end{aligned} \tag{6.19}$$

Hint: Solutions of (6.19) lie on spheres $x_1^2 + x_2^2 + x_3^2 = c$ (where $c = x_1^2(0) + x_2^2(0) + x_3^2(0)$).

(ii) Find all of the rest points of (6.19).

Consider the 3×3 algebraic system obtained by setting the right hand sides of (6.19) to zero. If one of the variables is zero, so are the other two. In case $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$, obtain a linear system with infinitely many solutions for $y_1 = \frac{1}{x_1}$, $y_2 = \frac{1}{x_2}$, $y_3 = \frac{1}{x_3}$ dividing the first equation by $x_2 x_3$, the second one by $x_1 x_3$, and the third one by $x_1 x_2$.

Answer. $(x_1, x_2, x_3) = \left(t, \frac{t}{t-1}, \frac{t}{1-2t}\right)$, where t is arbitrary, but $t \neq 1$, $t \neq \frac{1}{2}$.

13. Show that the rest point $(1, 1)$ is asymptotically stable for the system

$$\begin{aligned} x' &= -3x - y + xy + 3 \\ y' &= -2x - y + x^2 + 2 \,. \end{aligned}$$

Hint: Set $x = X + 1$, $y = X + 1$, to get a system for (X, Y).

II.

1. Show that for any positive solution of the system

$$\begin{aligned} x' &= x\,(5 - x - 2y) \\ y' &= y\,(2 - 3x - y) \end{aligned}$$

we have $\lim\limits_{t \to \infty} (x(t), y(t)) = (5, 0)$.

2. Show that for any positive solution of the system

$$\begin{aligned} x' &= x\left(2 - x - \tfrac{1}{2}y\right) \\ y' &= y\,(3 - x - y) \end{aligned}$$

we have $\lim\limits_{t \to \infty} (x(t), y(t)) = (1, 2)$.

3. Find $\lim\limits_{t \to \infty} (x(t), y(t))$ for the initial value problem

$$\begin{aligned} x' &= x\,(3 - x - y)\,, \quad x(0) = \tfrac{5}{2} \\ y' &= y\,(4 - 2x - y)\,, \quad y(0) = \tfrac{1}{4}\,. \end{aligned}$$

What if the initial conditions are $x(0) = 0.1$ and $y(0) = 3$?

Answer. $(3, 0)$. For the other initial conditions: $(0, 4)$.

4. Discuss the long term behavior (as $t \to \infty$) of positive solutions of the system

$$\begin{aligned} x' &= x\,(3 - x - y) \\ y' &= y\,(4 - 2x - y)\;. \end{aligned}$$

5. Show that any solution with $x(0) > 0$ and $y(0) > 0$ satisfies $x(t) > 0$ and $y(t) > 0$ for all $t > 0$, and then discuss the long term behavior of positive solutions of the system

$$\begin{aligned} x' &= x\,(6 - 3x - 2y) \\ y' &= y\left(2 - \tfrac{1}{8}x^2 - y\right). \end{aligned}$$

Hint: The second null-cline is a parabola, but the same analysis applies.

Answer. Any positive solution satisfies $\lim_{t\to\infty} x(t) = 6-2\sqrt{7}$, and $\lim_{t\to\infty} y(t) = 3\sqrt{7} - 6$.

III.

1. Show that the ellipse $x = 2\cos t$, $y = \sin t$ is a stable limit cycle for the system

$$\begin{aligned} x' &= -2y + x\left(1 - \tfrac{1}{4}x^2 - y^2\right) \\ y' &= \tfrac{1}{2}x + y\left(1 - \tfrac{1}{4}x^2 - y^2\right). \end{aligned}$$

Hint: Show that $\rho = \frac{1}{4}x^2 + y^2$ satisfies $\rho' = 2\rho(1-\rho)$.

2. Consider the system

$$\begin{aligned} x' &= x - y - x^3 \\ y' &= x + y - y^3 . \end{aligned}$$

(i) Show that the origin $(0, 0)$ is the only rest point.

Hint: Show that the curves $x - y - x^3 = 0$ and $x + y - y^3 = 0$ intersect only at the origin. This can be done either by hand, or (better) by computer.

(ii) Show that the origin $(0, 0)$ is an unstable spiral.

(iii) Show that the system has a limit cycle.

Hint: Compute the scalar product of $F = (x - y - x^3, x + y - y^3)$, and the vector (x, y), then switch to the polar coordinates. Obtain

$$\begin{aligned} x^2 + y^2 - x^4 - y^4 &= x^2 + y^2 - \left(x^2 + y^2\right)^2 + 2x^2y^2 = r^2 - r^4 + 2r^4\cos^2\theta\sin^2\theta \\ &< r^2 - \frac{1}{2}r^4 < 0\,, \quad \text{for } r \text{ large}. \end{aligned}$$

Conclude that the annulus $\rho^2 \le x^2 + y^2 \le R^2$ is a trapping region, provided that $\rho > 0$ is small, and $R > 0$ is large. Apply the Hartman-Grobman theorem.

3. Show that the system

$$x' = x\left(2 - x - y^3\right)$$
$$y' = y\left(4x - 3y - x^2\right)$$

has no limit cycles in the positive quadrant $x, y > 0$.

Hint: Use the Theorem 6.5.1, with $h(x,y) = \dfrac{1}{xy}$.

4. Show that the equation (for $x = x(t)$)

$$x'' + f(x)x' + g(x) = 0$$

has no periodic solutions, provided that either $f(x) > 0$, or $f(x) < 0$, for all real x.

Hint: Periodic solutions would imply limit cycles for the corresponding system (for $x_1 = x$ and $x_2 = x'$). Use the Dulac-Bendixson criterion.

5. Show that the condition (5.5) holds, provided that $a < \frac{1}{8}$, and b^2 lies between the roots of the quadratic $x^2 + (2a-1)x + a + a^2$.

6. Consider a *gradient system*

$$\begin{aligned} x_1' &= -V_{x_1}(x_1, x_2) \\ x_2' &= -V_{x_2}(x_1, x_2)\,, \end{aligned} \tag{6.20}$$

where $V(x_1, x_2)$ is a given twice differentiable function. (Denoting $x = (x_1, x_2)$ and $V = V(x)$, one may write this system in the gradient form $x' = -\nabla V(x)$.)

(i) Show that a point $P = (x_1^0, x_2^0)$ is a rest point of (6.20) if and only if P is a critical point of $V(x_1, x_2)$.

(ii) Show that $V(x_1(t), x_2(t))$ is a strictly decreasing function of t for any solution $(x_1(t), x_2(t))$, except if $(x_1(t), x_2(t))$ is a rest point.

(iii) Show that no limit cycles are possible for gradient system (6.20).

(iv) Let (a, b) be a point of strict local minimum of $V(x_1, x_2)$. Show that (a, b) is asymptotically stable rest point of (6.20).

Hint: Use $L(x_1, x_2) = V(x_1, x_2) - V(a, b)$ as Lyapunov's function.

(v) Show that the existence and uniqueness Theorem 6.1.1 applies to (6.20).

7. Consider a *Hamiltonian system*

$$x_1' = V_{x_1}(x_1, x_2)$$
$$x_2' = -V_{x_2}(x_1, x_2)\,, \tag{6.21}$$

where $V(x_1, x_2)$ is a given twice differentiable function.

(i) Show that

$$V(x_1(t), x_2(t)) = \text{constant}\,,$$

for any solution $(x_1(t), x_2(t))$.

(ii) Show that a point $P = (x_1^0, x_2^0)$ is a rest point of (6.21) if and only if P is a critical point of $V(x_1, x_2)$.

(iii) Show that no asymptotically stable rest points are possible for Hamiltonian system (6.21).

(iv) Let (a, b) be a point of strict local minimum or maximum of $V(x_1, x_2)$. Show that (a, b) is a center for (6.21).

(v) Show that the trajectories of (6.20) are *orthogonal* to the trajectories of (6.21), at all points (x_1, x_2).

8. In the Lotka-Volterra predator-prey system (1.1) let $p = \ln x$, and $q = \ln y$. Show that for the new unknowns $p(t)$ and $q(t)$ one obtains a Hamiltonian system, with $V(p, q) = c\,p - d\,e^p + a\,q - b\,e^q$.

9. Consider the system ($x(t)$ is a vector in R^n)

$$x' = [A + B(t)]\,x\,, \quad t > t_0\,, \tag{6.22}$$

where A is an $n \times n$ matrix with constant entries, and the $n \times n$ matrix $B(t)$ satisfies $\int_{t_0}^{\infty} ||B(t)||\,dt < \infty$. Assume that the eigenvalues of A are either negative or have negative real parts (recall that such matrices are called stable). Show that $\lim_{t\to\infty} x(t) = 0$, for any solution of (6.22).

Hint: Treating the $B(t)x(t)$ term as known, one can regard (6.22) as a non-homogeneous system, and write its solution as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}B(s)x(s)\,ds\,.$$

By (4.9), the norm $||e^{At}|| \le ae^{-bt}$, for some constants $a > 0$ and $b > 0$, with $t > 0$. Then

$$||x(t)|| \le Ke^{-bt} + a\int_{t_0}^{t} e^{-b(t-s)}||B(s)||\,||x(s)||\,ds\,,$$

where $K = ae^{bt_0}||x(t_0)||$. Letting $u(t) = e^{bt}||x(t)||$, obtain

$$||u(t)|| \leq K + a \int_{t_0}^{t} ||B(s)||\, ||u(s)||\, ds\,.$$

Apply the Bellman-Gronwall lemma.

IV. 1. Consider the linear equation

$$x' = a(t)x + b(t)\,, \tag{6.23}$$

with given p-periodic functions $a(t)$ and $b(t)$.

(i) Let $x(t)$ be some solution of (6.23). Show that $x(t+p)$ is also a solution of (6.23).

Hint: $x(t)$ satisfies (6.23) at all t, in particular at $t+p$.

(ii) Let $x(t)$ be a solution of (6.23), such that $x(p) = x(0)$. Show that $x(t)$ is a p-periodic function.

Hint: $x(t)$ and $x(t+p)$ are two solutions of (6.23) that coincide at $t = 0$.

2. Consider a *generalized logistic model*

$$x' = x^2 \left(a(t) - b(t)x\right),$$

with positive p-periodic functions $a(t)$ and $b(t)$. Show that there is a unique positive p-periodic solution, which attracts all other positive solutions, as $t \to \infty$.

Hint: Show that the Poincaré map $x_0 \to x(p, x_0)$ takes the interval (ϵ, M) into itself, provided that $\epsilon > 0$ is small and M is large. For the uniqueness part, show that $y(t) = \frac{1}{x(t)}$ satisfies

$$y' = -a(t) + b(t)\frac{1}{y}\,.$$

The difference of any two positive p-periodic solutions of the last equation satisfies $w' = c(t)w$, with a negative $c(t)$.

3. (Another example of *Fredholm alternative.*) Consider the equation

$$x' + a(t)x = 1\,, \tag{6.24}$$

with a p-periodic function $a(t)$.

(i) Assume that $\int_0^p a(t)\,dt = 0$. Show that the problem (6.24) has no p-periodic solution.

Hint: Let $z(t) > 0$ be a p-periodic solution of

$$z' - a(t)z = 0\,.$$

Combining this equation with (6.24), conclude that $\int_0^p z(t)\,dt = 0$, which is a contradiction.

(ii) Assume that $\int_0^p a(t)\,dt \neq 0$. Show that the problem (6.24) has a p-periodic solution, and moreover this solution satisfies $\int_0^p x(t)\,dt \neq 0$.

Hint: Solve (6.24), with initial condition $x(0) = \alpha$, and select α so that $x(p) = x(0) = \alpha$.

4. Consider the *logistic model*

$$x' = a(t)x - b(t)x^2\,,$$

with p-periodic functions $a(t)$ and $b(t)$. Assume that $\int_0^p a(t)\,dt = 0$, and $b(t) > 0$ for all t. Show that this equation has no non-trivial p-periodic solutions.

Hint: Any non-trivial solution satisfies either $x(t) > 0$ or $x(t) < 0$, for all t. Divide the equation by $x(t)$, and integrate over $(0, p)$.

Chapter 7

The Fourier Series and Boundary Value Problems

The central theme of this chapter involves various types of Fourier series, and the method of separation of variable, which is prominent in egineering and science. The three main equations of mathematical physics, the wave, the heat, and the Laplace equations, are derived and studied in detail. The Fourier transform method is developed, and applied to problems on infinite domains. Non-standard applications include studying temperatures inside the Earth, and the isoperimetric inequality.

7.1 The Fourier Series for Functions of an Arbitrary Period

Recall that in Chapter 2 we studied the Fourier series for functions of period 2π. Namely, if a function $g(t)$ has period 2π, it can be represented by the Fourier series

$$g(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt \right) \,,$$

with the coefficients given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt \,,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)\, \cos nt\, dt \,,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t)\, \sin nt\, dt \,.$$

Observe that the knowledge of a 2π-periodic function $g(t)$ over the interval $(-\pi, \pi]$ is sufficient to describe this function for all $t \in (-\infty, \infty)$.

Suppose now that $f(x)$ has a period $2L$, where $L > 0$ is any number. Consider an auxiliary function $g(t) = f(\frac{L}{\pi}t)$. Then $g(t)$ has period 2π, and we can represent it by the Fourier series

$$f(\frac{L}{\pi}t) = g(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) \,,$$

with the coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}t)\, dt \,,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\, \cos nt\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}t)\, \cos nt\, dt \,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\, \sin nt\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}t)\, \sin nt\, dt \,.$$

Set here

$$x = \frac{L}{\pi}t, \quad \text{or} \;\; t = \frac{\pi}{L}x \,.$$

Then the Fourier series takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\right) \,, \tag{1.1}$$

and making a change of variables $t \to x$, by setting $t = \frac{\pi}{L}x$ with $dt = \frac{\pi}{L}dx$, we express the coefficients as

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \,,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\, \cos \frac{n\pi}{L}x\, dx \,, \tag{1.2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\, \sin \frac{n\pi}{L}x\, dx \,.$$

The formula (1.1) gives the desired *Fourier series for functions of period* $2L$. Its coefficients are computed using the formulas (1.2). Observe that one needs the values of $f(x)$ only on the interval $(-L, L)$, when computing the coefficients.

Suppose that a function $g(x)$ is defined on some interval $(-L, L)$. The function $G(x)$ is called the *periodic extension* of $g(x)$, provided that

$$\begin{aligned} &(i)\ \ G(x) = g(x)\,, \ \text{ for } -L < x < L \\ &(ii)\ \ G(x) \text{ is periodic with period } 2L\,. \end{aligned}$$

Observe that $G(x)$ is defined for all x, except for $x = n\pi$ with integer n.

Example Let $f(x)$ be the function of period 6, which on the interval $(-3, 3)$ is equal to x.

Here $L = 3$, and $f(x) = x$ on the interval $(-3, 3)$. The Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{3} x + b_n \sin \frac{n\pi}{3} x \right)\,.$$

The functions x, and $x \cos \frac{n\pi}{3} x$ are odd, and so

$$a_0 = \frac{1}{6} \int_{-3}^{3} x\, dx = 0\,,$$

$$a_n = \frac{1}{3} \int_{-3}^{3} x\, \cos \frac{n\pi}{3} x\, dx = 0\,.$$

The function $x \sin \frac{n\pi}{3} x$ is even, giving

$$b_n = \frac{1}{3} \int_{-3}^{3} x\, \sin \frac{n\pi}{3} x\, dx = \frac{2}{3} \int_{0}^{3} x\, \sin \frac{n\pi}{3} x\, dx$$

$$= \frac{2}{3} \left[-\frac{3}{n\pi} x \cos \frac{n\pi}{3}\, x + \frac{9}{n^2\pi^2} \sin \frac{n\pi}{3}\, x \right] \Big|_0^3 = -\frac{6}{n\pi} \cos n\pi = \frac{6}{n\pi} (-1)^{n+1}\,,$$

because $\cos n\pi = (-1)^n$. We conclude that

$$f(x) = \sum_{n=1}^{\infty} \frac{6}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{3} x\,.$$

Restricting to the interval $(-3, 3)$, we have

$$x = \sum_{n=1}^{\infty} \frac{6}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{3} x, \quad \text{for } -3 < x < 3\,.$$

Outside of the interval $(-3, 3)$ this Fourier series converges not to x, but to the *periodic extension* of x, which is the function $f(x)$ that we started with.

We see that it is sufficient to know $f(x)$ on the interval $(-L, L)$, in order to compute its Fourier coefficients. If $f(x)$ is defined only on $(-L, L)$, it can still be represented the Fourier series (1.1). Outside of $(-L, L)$, this Fourier series converges to the $2L$-periodic extension of $f(x)$.

7.1.1 Even and Odd Functions

Our computations in the preceding example were aided by the nice properties of even and odd functions, which we review next.

A function $f(x)$ is called *even* if

$$f(-x) = f(x) \quad \text{for all } x \, .$$

Examples include $\cos x$, x^2, x^4, and in general x^{2n}, for any even power $2n$. The graph of an even function is symmetric with respect to the y axis. It follows that

$$\int_{-L}^{L} f(x)\, dx = 2 \int_0^L f(x)\, dx \, ,$$

for any even function $f(x)$, and any constant L.

A function $f(x)$ is called *odd* if

$$f(-x) = -f(x) \quad \text{for all } x \neq 0 \, .$$

(This definition allows $f(x)$ to be discontinuous at $x = 0$; but if $f(x)$ is continuous at $x = 0$, then it implies that $f(0) = 0$.) Examples include $\sin x$, $\tan x$, x, x^3, and in general x^{2n+1}, for any odd power $2n + 1$. (The even functions "eat" minus, while the odd ones "pass it through.") The graph of an odd function is symmetric with respect to the origin. It follows that

$$\int_{-L}^{L} f(x)\, dx = 0 \, ,$$

for any odd function $f(x)$, and any constant L. Products of even and odd functions are either even or odd:

$$\textit{even} \cdot \textit{even} = \textit{even} \, , \qquad \textit{even} \cdot \textit{odd} = \textit{odd} \, , \qquad \textit{odd} \cdot \textit{odd} = \textit{even} \, .$$

If $f(x)$ is even, then $b_n = 0$ for all n (as integrals of odd functions), and the Fourier series (1.1) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \, ,$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx\,,$$

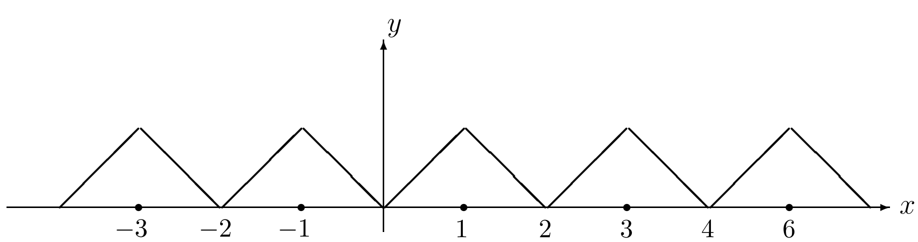
$$a_n = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi}{L}x\,dx\,.$$

If $f(x)$ is odd, then $a_0 = 0$ and $a_n = 0$ for all n, and the Fourier series (1.1) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\,,$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi}{L}x\,dx\,.$$



The periodic extension of $|x|$ as a function of period 2

Example Let $f(x)$ be a function of period 2, which on the interval $(-1,1)$ is equal to $|x|$.

Here $L = 1$, and $f(x) = |x|$ on the interval $(-1,1)$. The function $f(x)$ is even, so that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x\,.$$

Observing that $|x| = x$ on the interval $(0,1)$, we compute the coefficients

$$a_0 = \int_0^1 x\,dx = \frac{1}{2}\,,$$

$$a_n = 2\int_0^1 x\,\cos n\pi x\,dx = 2\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]\Big|_0^1 = \frac{2(-1)^n - 2}{n^2\pi^2}\,.$$

Restricting to the interval $(-1, 1)$

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 2}{n^2\pi^2} \cos n\pi x, \quad \text{for } -1 < x < 1\,.$$

Outside of the interval $(-1, 1)$, this Fourier series converges to the periodic extension of $|x|$, or to the function $f(x)$.

Observing that $a_n = 0$ for even n, one can also write the answer as

$$|x| = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos(2k+1)\pi x\,, \quad \text{for } -1 < x < 1\,.$$

(All odd n can be obtained in the form $n = 2k + 1$, with $k = 0, 1, 2, \ldots$.)

7.1.2 Further Examples and the Convergence Theorem

Even and odd functions are very special. A "general" function is neither even nor odd.

Example 1 On the interval $(-2, 2)$, represent the function

$$f(x) = \begin{cases} 1 & \text{for } -2 < x \le 0 \\ x & \text{for } 0 < x < 2 \end{cases}$$

by its Fourier series.

This function is neither even nor odd (and also it is not continuous, with a jump at $x = 0$). Here $L = 2$, and the Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{2} x + b_n \sin \frac{n\pi}{2} x \right)\,.$$

Compute

$$a_0 = \frac{1}{4} \int_{-2}^{0} 1\, dx + \frac{1}{4} \int_0^2 x\, dx = 1\,,$$

where we broke the interval of integration into two pieces, according to the definition of $f(x)$. Similarly

$$a_n = \frac{1}{2} \int_{-2}^{0} \cos \frac{n\pi}{2} x + \frac{1}{2} \int_0^2 x \cos \frac{n\pi}{2} x\, dx = \frac{2\left(-1 + (-1)^n\right)}{n^2\pi^2}\,,$$

$$b_n = \frac{1}{2} \int_{-2}^{0} \sin \frac{n\pi}{2} x + \frac{1}{2} \int_0^2 x \sin \frac{n\pi}{2} x\, dx = \frac{\left(-1 - (-1)^n\right)}{n\pi}\,.$$

On the interval $(-2, 2)$ we have

$$f(x) = 1 + \sum_{n=1}^{\infty} \left(\frac{2\left(-1 + (-1)^n\right)}{n^2\pi^2} \cos \frac{n\pi}{2} x + \frac{\left(-1 - (-1)^n\right)}{n\pi} \sin \frac{n\pi}{2} x \right) .$$

The quantity $-1 + (-1)^n$ is equal to zero if n is even, and to -2 if n is odd. The quantity $-1 - (-1)^n$ is equal to zero if n is odd, and to -2 if n is even. All even n can be obtained in the form $n = 2k$, with $k = 1, 2, 3, \ldots$, and all odd n can be obtained in the form $n = 2k - 1$, with $k = 1, 2, 3, \ldots$. We can then rewrite the Fourier series as

$$f(x) = 1 - \sum_{k=1}^{\infty} \left(\frac{4}{(2k-1)^2\pi^2} \cos \frac{(2k-1)\pi}{2} x + \frac{1}{k\pi} \sin k\pi x \right), \text{ for } -2 < x < 2 .$$

Outside of the interval $(-2, 2)$, this series converges to the extension of $f(x)$, as a function of period 4.

Example 2 Find the Fourier series of $f(x) = 2 \sin x + \sin^2 x$, on the interval $(-\pi, \pi)$.

Here $L = \pi$, and the Fourier series takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) .$$

Let us spell out several terms of this series:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots .$$

Using the trigonometric formula $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$, write

$$f(x) = \frac{1}{2} - \frac{1}{2} \cos 2x + 2 \sin x .$$

This is the desired Fourier series! Here $a_0 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$, $b_1 = 2$, and all other coefficients are zero. In effect, this function is its own Fourier series.

Example 3 Find the Fourier series of $f(x) = 2 \sin x + \sin^2 x$, on the interval $(-2\pi, 2\pi)$.

This time $L = 2\pi$, and the Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n}{2} x + b_n \sin \frac{n}{2} x \right) .$$

As before, we rewrite $f(x)$

$$f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x + 2\sin x\,.$$

And again this is the desired Fourier series! This time $a_0 = \frac{1}{2}$, $a_4 = -\frac{1}{2}$, $b_2 = 2$, and all other coefficients are zero.

To discuss the convergence properties of Fourier series, we need the concept of *piecewise smooth functions*. These are functions that are continuous and differentiable, except for discontinuities at some isolated points. In case a discontinuity happens at some point x_0, we assume that the limit from the left $f(x_0-)$ exists, as well as the limit from the right $f(x_0+)$. (So that at a point of discontinuity either $f(x_0)$ is not defined, or $f(x_0-) \neq f(x_0+)$, or $f(x_0) \neq \lim_{x\to x_0} f(x)$.)

Theorem 7.1.1 *Let f(x) be a piecewise smooth function of period $2L$. Then its Fourier series*

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

converges to $f(x)$ at any point x where $f(x)$ is continuous. If $f(x)$ has a discontinuity at x, the Fourier series converges to

$$\frac{f(x-) + f(x+)}{2}\,.$$

The proof can be found in the book of H.F. Weinberger [36]. (At jump points, the Fourier series tries to be fair, and it converges to the average of the limits from the left and from the right.)

Let now $f(x)$ be defined on $[-L, L]$. Let us extend it as a function of period $2L$. Unless it so happens that $f(-L) = f(L)$, the extended function will have jumps at $x = -L$ and $x = L$. Then this theorem implies the next one.

Theorem 7.1.2 *Let f(x) be a piecewise smooth function defined on $[-L, L]$. Let x be a point inside $(-L, L)$. Then its Fourier series*

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

converges to $f(x)$ at any point x where $f(x)$ is continuous. If $f(x)$ has a discontinuity at x, the Fourier series converges to

$$\frac{f(x-)+f(x+)}{2}\,.$$

At both end points, $x=-L$ and $x=L$, the Fourier series converges to

$$\frac{f(-L+)+f(L-)}{2}\,.$$

(The average of the limits from the right and from the left, at $-L$ and L respectively.)

7.2 The Fourier Cosine and the Fourier Sine Series

Suppose a function $f(x)$ is defined on the interval $(0,L)$. How do we represent $f(x)$ by a Fourier series? We can compute Fourier series for functions defined on $(-L,L)$, but $f(x)$ "lives" only on $(0,L)$.

One possibility is to extend $f(x)$ as an arbitrary function on $(-L,0)$ (by drawing randomly any graph on $(-L,0)$). This gives us a function defined on $(-L,L)$, which we may represent by its Fourier series, and then use this series only on the interval $(0,L)$, where $f(x)$ lives. So that there are infinitely many ways to represent $f(x)$ by a Fourier series on the interval $(0,L)$. However, two of these Fourier series stand out, the ones when the extension produces either an even or an odd function.

Let $f(x)$ be defined on the interval $(0,L)$. We define its *even extension* to the interval $(-L,L)$, as follows

$$f_e(x)=\begin{cases} f(x) & \text{for } 0<x<L \\ f(-x) & \text{for } -L<x<0 \end{cases}\,.$$

(Observe that $f_e(0)$ is left undefined.) The graph of $f_e(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the y-axis. The function $f_e(x)$ is even on $(-L,L)$, and so its Fourier series has the form

$$f_e(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{L}x\,, \tag{2.1}$$

with the coefficients

$$a_0=\frac{1}{L}\int_0^L f_e(x)\,dx\;=\frac{1}{L}\int_0^L f(x)\,dx\,,$$

$$a_n = \frac{2}{L}\int_0^L f_e(x)\,\cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi}{L}x\,dx\,,$$

because $f_e(x) = f(x)$ on the interval of integration $(0, L)$. We now restrict the series (2.1) to the interval $(0, L)$, obtaining

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x\,, \quad \text{for } 0 < x < L\,, \tag{2.2}$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx\,, \tag{2.3}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi}{L}x\,dx\,. \tag{2.4}$$

The series (2.2), with the coefficients computed using the formulas (2.3) and (2.4), is called the *Fourier cosine series* of $f(x)$.

Where is $f_e(x)$ now? It disappeared. We used it as an artifact of construction, like scaffolding.

Example 1 Find the Fourier cosine series of $f(x) = x + 2$, on the interval $(0, 3)$.

The series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{3}x \quad \text{for } 0 < x < 3\,.$$

Compute

$$a_0 = \frac{1}{3}\int_0^3 (x+2)\,dx = \frac{7}{2}\,,$$

$$a_n = \frac{2}{3}\int_0^3 (x+2)\,\cos\frac{n\pi}{3}x\,dx = \frac{6\left(-1+(-1)^n\right)}{n^2\pi^2}\,.$$

Answer:

$$x+2 = \frac{7}{2} + \sum_{n=1}^{\infty} \frac{6\left(-1+(-1)^n\right)}{n^2\pi^2}\cos\frac{n\pi}{3}x = \frac{7}{2} - 12\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi}{3}x\,.$$

Assume again that $f(x)$ is defined only on the interval $(0, L)$. We now define its *odd extension* to the interval $(-L, L)$, as follows

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 < x < L \\ -f(-x) & \text{for } -L < x < 0 \end{cases}\,.$$

The graph of $f_o(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the origin. Observe that $f_o(0)$ is not defined. (If $f(0)$ is defined, but $f(0) \neq 0$, this extension is still discontinuous at $x = 0$.) The function $f_o(x)$ is odd on $(-L, L)$, and so its Fourier series has only the sine terms:

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \,, \tag{2.5}$$

with the coefficients

$$b_n = \frac{2}{L} \int_0^L f_o(x) \, \sin \frac{n\pi}{L} x \, dx = \frac{2}{L} \int_0^L f(x) \, \sin \frac{n\pi}{L} x \, dx \,,$$

because on the interval of integration $(0, L)$, $f_o(x) = f(x)$. We restrict the series (2.5) to the interval $(0, L)$, obtaining

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \,, \quad \text{for } 0 < x < L \,, \tag{2.6}$$

with

$$b_n = \frac{2}{L} \int_0^L f(x) \, \sin \frac{n\pi}{L} x \, dx \,. \tag{2.7}$$

The series (2.6), with the coefficients computed using the formula (2.7), is called the *Fourier sine series* of $f(x)$.

Example 2 Find the Fourier sine series of $f(x) = x + 2$, on the interval $(0, 3)$.

Compute

$$b_n = \frac{2}{3} \int_0^3 (x + 2) \, \sin \frac{n\pi}{3} x \, dx = \frac{4 - 10(-1)^n}{n\pi} \,.$$

We conclude that

$$x + 2 = \sum_{n=1}^{\infty} \frac{4 - 10(-1)^n}{n\pi} \sin \frac{n\pi}{3} x, \quad \text{for } 0 < x < 3 \,.$$

Clearly this series does not converge to $f(x)$ at the end points $x = 0$ and $x = 3$ of our interval $(0, 3)$. But inside $(0, 3)$, we do have convergence.

We now discuss the convergence issue. The Fourier sine and cosine series were developed by using the Fourier series on $(-L, L)$. It follows from the convergence Theorem 7.1.1 that inside of $(0, L)$, both of these series converge to $f(x)$ at points of continuity, and to $\dfrac{f(x-) + f(x+)}{2}$ if $f(x)$ is discontinuous at x. At both end points $x = 0$ and $x = L$, the Fourier sine series converges to 0 (as can be seen directly from the series), while the Fourier cosine series converges to $f(0+)$ and $f(L-)$ respectively. (The extension of $f_e(x)$ as a function of period $2L$ has no jumps at $x = 0$ and $x = L$.)

7.3 Two Point Boundary Value Problems

We shall need to find non-trivial solutions $y = y(x)$ of the problem

$$\begin{aligned} y'' + \lambda y = 0, \quad 0 < x < L \\ y(0) = y(L) = 0\,, \end{aligned} \tag{3.1}$$

on an interval $(0, L)$. Here λ is a real number. Unlike initial value problems, where the values of the solution and its derivative are prescribed at some point, here we prescribe that the solution vanishes at $x = 0$ and at $x = L$, which are the end-points (the *boundary points*) of the interval $(0, L)$. The problem (3.1) is an example of a *boundary value problem*. Of course, $y(x) = 0$ is a solution of our problem (3.1), which is called the *trivial solution*. We wish to find non-trivial solutions. What are the values of the *parameter* λ, for which non-trivial solutions are possible?

The form of the general solution depends on whether λ is positive, negative or zero, so that there are three cases to consider.

Case 1. $\lambda < 0$. We may write $\lambda = -\omega^2$, with some $\omega > 0$ ($\omega = \sqrt{-\lambda}$), and our equation takes the form

$$y'' - \omega^2 y = 0\,.$$

Its general solution is $y = c_1 e^{-\omega x} + c_2 e^{\omega x}$. The boundary conditions

$$\begin{aligned} y(0) &= c_1 + c_2 = 0\,, \\ y(L) &= e^{-\omega L} c_1 + e^{\omega L} c_2 = 0\,, \end{aligned}$$

give us two equations to determine c_1 and c_2. From the first equation $c_2 = -c_1$, and then from the second equation $c_1 = 0$. So that $c_1 = c_2 = 0$, and the only solution is $y = 0$, the trivial solution.

Case 2. $\lambda = 0$. The equation takes the form

$$y'' = 0\,.$$

Its general solution is $y = c_1 + c_2 x$. The boundary conditions

$$\begin{aligned} y(0) &= c_1 = 0\,, \\ y(L) &= c_1 L + c_2 = 0\,, \end{aligned}$$

give us $c_1 = c_2 = 0$, so that $y = 0$. We struck out again in the search for a non-trivial solution.

Case 3. $\lambda > 0$. We may write $\lambda = \omega^2$, with some $\omega > 0$ ($\omega = \sqrt{\lambda}$), and our equation takes the form

$$y'' + \omega^2 y = 0 .$$

Its general solution is $y = c_1 \cos \omega x + c_2 \sin \omega x$. The first boundary condition,

$$y(0) = c_1 = 0 ,$$

tells us that $c_1 = 0$. We update the general solution: $y = c_2 \sin \omega x$. The second boundary condition gives

$$y(L) = c_2 \sin \omega L = 0 .$$

One possibility for this product to be zero, is $c_2 = 0$. That would lead again to the trivial solution. What saves us is that $\sin \omega L = 0$, for some "lucky" ω's, namely when $\omega L = n\pi$, or $\omega = \omega_n = \frac{n\pi}{L}$, and then $\lambda_n = \omega_n^2 = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$. The corresponding solutions are $c_2 \sin \frac{n\pi}{L} x$, or we can simply write them as $\sin \frac{n\pi}{L} x$, because a constant multiple of a solution is also a solution of (3.1).

To recapitulate, non-trivial solutions of the boundary value problem (3.1) occur at the infinite sequence of λ's, $\lambda_n = \frac{n^2\pi^2}{L^2}$, called the *eigenvalues*, and the corresponding solutions $y_n = \sin \frac{n\pi}{L} x$ are called the *eigenfunctions*.

Next, we search for non-trivial solutions of the problem

$$\begin{gathered} y'' + \lambda y = 0, \quad 0 < x < L \\ y'(0) = y'(L) = 0 , \end{gathered}$$

in which the boundary conditions are different. As before, we see that in case $\lambda < 0$, there are no non-trivial solutions. The case $\lambda = 0$ turns out to be different: any non-zero constant is a non-trivial solution. So that $\lambda_0 = 0$ is an eigenvalue, and $y_0 = 1$ is the corresponding eigenfunction. In case $\lambda > 0$, we get infinitely many eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, with the corresponding eigenfunctions $y_n = \cos \frac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$.

7.3.1 Problems

I. 1. Is the integral $\displaystyle\int_{-1}^{3/2} \tan^{15} x\, dx$ positive or negative?

Hint: Consider first $\int_{-1}^{1} \tan^{15} x\, dx$. Answer. Positive.

2. Show that any function can be written as a sum of an even function and an odd function.

Hint: $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$.

3. Let $g(x) = x$ on the interval $[0, 3)$.

(i) Find the even extension of $g(x)$. Answer. $g_e(x) = |x|$, defined on $(-3, 3)$.

(ii) Find the odd extension of $g(x)$. Answer. $g_o(x) = x$, defined on $(-3, 3)$.

4. Let $h(x) = -x^3$ on the interval $[0, 5)$. Find its even and odd extensions, and state the interval on which they are defined.

Answer. $h_e(x) = -|x|^3$ and $h_0(x) = -x^3$, both defined on $(-5, 5)$.

5. Let $f(x) = x^2$ on the interval $[0, 1)$. Find its even and odd extensions, and state the interval on which they are defined.

Answer. $f_o(x) = x|x|$, defined on $(-1, 1)$.

6. Differentiate the functions $f(x) = x|x|^{p-1}$ and $g(x) = |x|^p$, with $p > 1$.

Hint: The function $f(x)$ is odd, so that $f'(x)$ is even. Begin by computing $f'(x)$ for $x > 0$.

Answer. $f'(x) = p|x|^{p-1}$, $g'(x) = px|x|^{p-2}$.

7. Assume that $f(x)$ has period 2π. Show that the function $\int_0^x f(t)\, dt$ is also 2π-periodic, if and only if $\int_0^{2\pi} f(t)\, dt = 0$.

8. Assume that $f(x)$ has period T. Show that for any constant a

$$\int_a^{T+a} f'(x) e^{f(x)}\, dx = 0\,.$$

II. Find the Fourier series of a given function over the indicated interval.

1. $f(x) = \sin x \cos x + \cos^2 2x$ on $(-\pi, \pi)$.

Answer. $f(x) = \frac{1}{2} + \frac{1}{2}\cos 4x + \frac{1}{2}\sin 2x$.

2. $f(x) = \sin x \cos x + \cos^2 2x$ on $(-2\pi, 2\pi)$.

Answer. $f(x) = \frac{1}{2} + \frac{1}{2}\cos 4x + \frac{1}{2}\sin 2x$.

3. $f(x) = \sin x \cos x + \cos^2 2x$ on $(-\pi/2, \pi/2)$.

Answer. $f(x) = \frac{1}{2} + \frac{1}{2}\cos 4x + \frac{1}{2}\sin 2x$.

4. $f(x) = x + x^2$ on $(-\pi, \pi)$.

Answer. $f(x) = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\dfrac{4(-1)^n}{n^2}\cos nx + \dfrac{2(-1)^{n+1}}{n}\sin nx\right)$.

5. (i) $f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases}$ on $(-\pi, \pi)$.

Answer. $f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$.

(ii) Set $x = \frac{\pi}{2}$ in the last series, to conclude that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

6. $f(x) = 1 - |x|$ on $(-2, 2)$.

Answer. $f(x) = \sum_{n=1}^{\infty} \dfrac{4}{n^2\pi^2}\left(1 - (-1)^n\right)\cos \dfrac{n\pi}{2}x$.

7. $f(x) = x|x|$ on $(-1, 1)$.

Answer. $f(x) = \sum_{n=1}^{\infty} \dfrac{-2\left(n^2\pi^2 - 2\right)(-1)^n - 4}{n^3\pi^3}\sin n\pi x$.

8. Let $f(x) = \begin{cases} 1 & \text{for } -1 < x < 0 \\ 0 & \text{for } 0 < x < 1 \end{cases}$ on $(-1, 1)$. Sketch the graphs of $f(x)$ and of its Fourier series. Then calculate the Fourier series of $f(x)$ on $(-1, 1)$.

Answer. $f(x) = \dfrac{1}{2} - \sum_{k=1}^{\infty} \dfrac{2}{(2k-1)\pi}\sin(2k-1)\pi x$.

9. Let $f(x) = \begin{cases} x & \text{for } -2 < x < 0 \\ -1 & \text{for } 0 < x < 2 \end{cases}$ on $(-2, 2)$. Sketch the graphs of $f(x)$ and of its Fourier series. Then calculate the Fourier series of $f(x)$ on $(-2, 2)$.

III. Find the Fourier cosine series of a given function over the indicated interval.

1. $f(x) = \cos 3x - \sin^2 3x$ on $(0, \pi)$.

Answer. $f(x) = -\dfrac{1}{2} + \cos 3x + \dfrac{1}{2} \cos 6x$.

2. $f(x) = \cos 3x - \sin^2 3x$ on $(0, \frac{\pi}{3})$.

Answer. $f(x) = -\dfrac{1}{2} + \cos 3x + \dfrac{1}{2} \cos 6x$.

3. $f(x) = x$ on $(0, 2)$.

Answer. $f(x) = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{-4 + 4(-1)^n}{n^2 \pi^2} \cos \frac{n\pi}{2} x$.

4. $f(x) = \sin x$ on $(0, 2)$.

Hint: $\sin ax \cos bx = \frac{1}{2} \sin(a - b)x + \frac{1}{2} \sin(a + b)x$.

Answer. $f(x) = \dfrac{1}{2}(1 - \cos 2) + \displaystyle\sum_{n=1}^{\infty} \frac{4\left((-1)^n \cos 2 - 1\right)}{n^2 \pi^2 - 4} \cos \frac{n\pi}{2} x$.

5. $f(x) = \sin^4 x$ on $(0, \frac{\pi}{2})$.

Answer. $f(x) = \dfrac{3}{8} - \dfrac{1}{2} \cos 2x + \dfrac{1}{8} \cos 4x$.

IV. Find the Fourier sine series of a given function over the indicated interval.

1. $f(x) = 5 \sin x \cos x$ on $(0, \pi)$.

Answer. $f(x) = \dfrac{5}{2} \sin 2x$.

2. $f(x) = 1$ on $(0, 3)$.

Answer. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - (-1)^n\right) \sin \frac{n\pi}{3} x$.

3. $f(x) = x$ on $(0, 2)$.

Answer. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{4}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{2} x$.

4. $f(x) = \sin x$ on $(0, 2)$.

Hint: $\sin ax \sin bx = \frac{1}{2} \cos(a - b)x - \frac{1}{2} \cos(a + b)x$.

Answer. $f(x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n\pi \sin 2}{n^2 \pi^2 - 4} \sin \frac{n\pi}{2} x$.

5. $f(x) = \sin^3 x$ on $(0, \pi)$.

Hint: $e^{i3x} = \left(e^{ix}\right)^3$. Use Euler's equation on both sides, then compare the imaginary parts.

Answer. $\frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

6. $f(x) = \begin{cases} x & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$ on $(0, \pi)$.

Answer. $f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \, \sin nx$.

7. $f(x) = x - 1$ on $(0, 3)$.

Answer. $f(x) = -\sum_{n=1}^{\infty} \frac{2 + 4(-1)^n}{n\pi} \sin \frac{n\pi}{3} x$.

V.

1. Find the eigenvalues and the eigenfunctions of

$$y'' + \lambda y = 0, \quad 0 < x < L, \quad y'(0) = y(L) = 0\,.$$

Answer. $\lambda_n = \frac{\pi^2 (n + \frac{1}{2})^2}{L^2}$, $y_n = \cos \frac{\pi (n + \frac{1}{2})}{L} x$.

2. Find the eigenvalues and the eigenfunctions of

$$y'' + \lambda y = 0, \quad 0 < x < L, \quad y(0) = y'(L) = 0\,.$$

Answer. $\lambda_n = \frac{\pi^2 (n + \frac{1}{2})^2}{L^2}$, $y_n = \sin \frac{\pi (n + \frac{1}{2})}{L} x$.

3. Find the eigenvalues and the eigenfunctions of

$$y'' + \lambda y = 0, \quad y(x) \text{ is a } 2\pi \text{ periodic function}\,.$$

Answer. $\lambda_0 = 0$ with $y_0 = 1$, and $\lambda_n = n^2$ with $y_n = a_n \cos nx + b_n \sin nx$.

4. Find all non-trivial solutions of an *integro-differential equation*

$$y'' + \left(\int_0^1 y^2(x)\, dx\right) y = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0\,.$$

Hint: $\int_0^1 y^2(x)\, dx$ is a constant.

Answer. $y = \pm\sqrt{2}n \, \sin n\pi x$, with integer $n \geq 1$.

5*. Show that the fourth order problem (with $a > 0$)

$$y'''' - a^4 y = 0, \ \ 0 < x < 1, \ \ y(0) = y'(0) = y(1) = y'(1) = 0$$

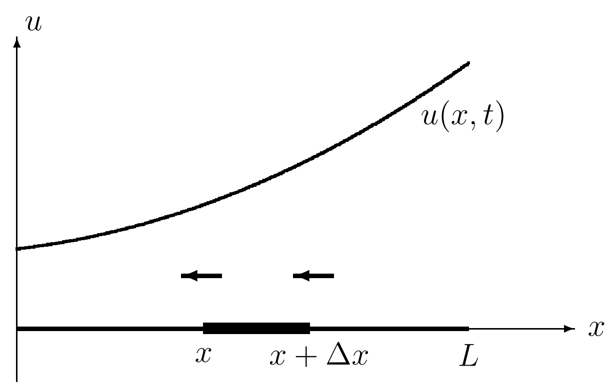
has non-trivial solutions (eigenfunctions) if and only if a satisfies

$$\cos a = \frac{1}{\cosh a} \, .$$

Show graphically that there are infinitely many such a's, and calculate the corresponding eigenfunctions.

Hint: The general solution is $y(x) = c_1 \cos ax + c_2 \sin ax + c_3 \cosh ax + c_4 \sinh ax$. From the boundary conditions obtain two equation for c_3 and c_4.

7.4 The Heat Equation and the Method of Separation of Variables



Heat flows in at $x + \Delta x$, and escapes at x

Suppose we have a rod of length L, so thin that we may assume it to be one dimensional, extending along the x-axis, for $0 \leq x \leq L$. Assume that the surface of the rod is insulated, so that heat can travel only to the left or to the right along the rod. We wish to determine the temperature $u = u(x, t)$ at any point x of the rod, and at any time $t > 0$. Consider an element $(x, x + \Delta x)$ of the rod, of length Δx. The amount of heat (in calories) that this element holds we approximate by

$$cu(x, t)\Delta x \, .$$

Indeed, the amount of heat ought to be proportional to the temperature $u = u(x,t)$, and to the length Δx. A physical constant $c > 0$ reflects the rod's capacity to store heat (c also makes the physical units right, so that the product is in calories). The rate of change of the amount of heat is

$$cu_t(x,t)\Delta x\,,$$

where $u_t(x,t)$ denotes the partial derivative in t. The change in the amount of heat occurs because of the heat flow through the end-points of the interval $(x, x+\Delta x)$. The function $u(x,t)$ is likely to be monotone over the small interval $(x, x+\Delta x)$, so let us assume that $u(x,t)$ in increasing in x over $(x, x+\Delta x)$ (think of t as fixed). At the right end-point $x+\Delta x$, heat flows into our element, because to the right of this point the temperatures are higher. The heat flow per unit time (called the *heat flux*) is assumed to be

$$c_1 u_x(x+\Delta x, t)\,,$$

or proportional to the rate of temperature increase ($c_1 > 0$ is another physical constant). Similarly, at the left end-point x

$$c_1 u_x(x,t)$$

calories of heat are lost per unit time. The equation of heat balance is then

$$cu_t(x,t)\Delta x = c_1 u_x(x+\Delta x, t) - c_1 u_x(x,t)\,.$$

Divide by $c\Delta x$, and call $\dfrac{c_1}{c} = k$

$$u_t(x,t) = k\,\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\,.$$

And finally, we let $\Delta x \to 0$, obtaining the *heat equation*

$$u_t = k u_{xx}\,.$$

It gives an example of a *partial differential equation*, or a PDE for short. (So far we studied *ordinary differential equations*, or ODE's, with unknown functions depending on only one variable.)

If $u_1(x,t)$ and $u_2(x,t)$ are two solutions of the heat equation, then so is $c_1u_1(x,t) + c_2u_2(x,t)$, for any constants c_1 and c_2, as can be seen by a direct substitution. The situation is similar in case of three or more solutions. This *superposition property of solutions* could be taken as a definition of *linear*

PDE. More formally, an equation is called linear if it depends linearly on the unknown function and its derivatives. We shall consider only linear PDE's in this chapter.

Suppose now that initially, or at the time $t = 0$, the temperatures inside the rod could be obtained from a given function $f(x)$, while the temperatures at the end-points, $x = 0$ and $x = L$, are kept at 0 degree Celsius at all time t (think that the end-points are kept on ice). To determine the temperature $u(x,t)$ at all points $x \in (0, L)$, and all time $t > 0$, we need to solve

$$
\begin{aligned}
u_t &= k u_{xx} \qquad \text{for } 0 < x < L, \text{ and } t > 0 \\
u(x,0) &= f(x) \qquad \text{for } 0 < x < L \\
u(0,t) &= u(L,t) = 0 \qquad \text{for } t > 0\,.
\end{aligned} \tag{4.1}
$$

Here the second line represents the *initial condition*, and the third line gives the *boundary conditions*.

Separation of Variables

We search for a solution of (4.1) in the form $u(x,t) = F(x)G(t)$, with the functions $F(x)$ and $G(t)$ to be determined. From the equation (4.1)

$$F(x)G'(t) = kF''(x)G(t)\,.$$

Divide by $kF(x)G(t)$:

$$\frac{G'(t)}{kG(t)} = \frac{F''(x)}{F(x)}\,.$$

On the left we have a function of t only, while on the right we have a function of x only. In order for them to be the same, they must be both equal to the same constant, which we denote by $-\lambda$

$$\frac{G'(t)}{kG(t)} = \frac{F''(x)}{F(x)} = -\lambda\,.$$

This gives us two differential equations, for $F(x)$ and $G(t)$,

$$\frac{G'(t)}{kG(t)} = -\lambda\,, \tag{4.2}$$

and

$$F''(x) + \lambda F(x) = 0\,.$$

From the boundary condition at $x = 0$,

$$u(0,t) = F(0)G(t) = 0\,.$$

This implies that $F(0) = 0$ (setting $G(t) = 0$, would give $u = 0$, which does not satisfy the initial condition in (4.1)). Similarly, we have $F(L) = 0$, using the other boundary condition. So that $F(x)$ satisfies

$$F''(x) + \lambda F(x) = 0, \quad F(0) = F(L) = 0\,. \tag{4.3}$$

We studied this problem in the preceding section. Non-trivial solutions occur only at $\lambda = \lambda_n = \dfrac{n^2\pi^2}{L^2}$. Corresponding solutions are

$$F_n(x) = \sin \frac{n\pi}{L}\, x \quad \text{(and their multiples)}\,.$$

With $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$ the equation (4.2) becomes

$$\frac{G'(t)}{G(t)} = -k\frac{n^2\pi^2}{L^2}\,. \tag{4.4}$$

Solving these equations for all n

$$G_n(t) = b_n e^{-k\frac{n^2\pi^2}{L^2}\,t}\,,$$

where b_n's are arbitrary constants. We have constructed infinitely many functions

$$u_n(x,t) = G_n(t)F_n(x) = b_n e^{-k\frac{n^2\pi^2}{L^2}\,t} \sin \frac{n\pi}{L}\, x\,,$$

which satisfy the PDE in (4.1), and the boundary conditions. By linearity, their sum

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}\,t} \sin \frac{n\pi}{L}\, x \tag{4.5}$$

also satisfies the PDE in (4.1), and the boundary conditions. We now turn to the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}\, x = f(x)\,. \tag{4.6}$$

We need to represent $f(x)$ by its Fourier sine series, which requires

$$b_n = \frac{2}{L}\int_0^L f(x)\, \sin \frac{n\pi}{L} x\, dx\,. \tag{4.7}$$

Conclusion: the series

$$u(x,t)=\sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}\,t}\sin\frac{n\pi}{L}x\,, \tag{4.8}$$

with b_n's calculated using (4.7), gives the solution of our problem (4.1). Observe that going from the Fourier sine series of $f(x)$ to the solution of our problem (4.1) involves just putting in the additional factors $e^{-k\frac{n^2\pi^2}{L^2}\,t}$. In practice, one should memorize the formula (4.8).

Example 1 Find the temperature $u(x,t)$ satisfying

$$\begin{aligned} u_t &= 5u_{xx} \qquad \text{for } 0<x<2\pi, \text{ and } t>0\\ u(x,0) &= 2\sin x-3\sin x\cos x \qquad \text{for } 0<x<2\pi\\ u(0,t) &= u(2\pi,t)=0 \qquad \text{for } t>0\,. \end{aligned}$$

Here $k=5$, and $L=2\pi$. The Fourier sine series on $(0,2\pi)$ has the form $\sum_{n=1}^{\infty} b_n \sin\frac{n}{2}x$. Writing the initial temperatures as

$$2\sin x-3\sin x\cos x=2\sin x-\frac{3}{2}\sin 2x\,,$$

we see that this function is its own Fourier sine series, with $b_2=2$, $b_4=-\frac{3}{2}$, and all other coefficients equal to zero. According to (4.8), the solution is

$$u(x,t)=2e^{-5\frac{2^2\pi^2}{(2\pi)^2}\,t}\sin x-\frac{3}{2}e^{-5\frac{4^2\pi^2}{(2\pi)^2}\,t}\sin 2x=2e^{-5t}\sin x-\frac{3}{2}e^{-20t}\sin 2x\,.$$

By the time $t=1$, the first term of the solution totally dominates the second one, so that $u(x,t)\approx 2e^{-5t}\sin x$ for $t>1$.

Example 2 Solve

$$\begin{aligned} u_t &= 2u_{xx} \qquad \text{for } 0<x<3, \text{ and } t>0\\ u(x,0) &= x-1 \qquad \text{for } 0<x<3\\ u(0,t) &= u(3,t)=0 \qquad \text{for } t>0\,. \end{aligned}$$

Here $k=2$, and $L=3$. We begin by calculating the Fourier sine series

$$x-1=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{3}x\,,$$

with

$$b_n=\frac{2}{3}\int_0^3 (x-1)\,\sin\frac{n\pi}{3}x\,dx=-\frac{2+4(-1)^n}{n\pi}\,.$$

Then, we put in the appropriate exponential factors, according to (4.8). Solution:

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{2+4(-1)^n}{n\pi} e^{-2\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x \,.$$

What is the value of this solution? The initial temperatures, $u(x,0) = x-1$, are negative for $0 < x < 1$, and positive for $1 < x < 3$. Again, very quickly (by the time $t = 1$), the first term ($n = 1$) dominates all others, and then

$$u(x,t) \approx \frac{2}{\pi} e^{-\frac{2\pi^2}{9}t} \sin \frac{\pi}{3} x \,,$$

so that the temperatures become positive at all points, because the *first harmonic* $\sin \frac{\pi}{3} x > 0$, on the interval $(0,3)$. For large t, the temperatures tend exponentially to zero, while retaining the shape of the first harmonic.

Assume now that the rod is insulated at the end-points $x = 0$ and $x = L$. Recall that the flux at $x = 0$ (the amount of heat flowing per unit time) is proportional to $u_x(0,t)$. Since there is no heat flow at $x = 0$ for all t, we have $u_x(0,t) = 0$, and similarly $u_x(L,t) = 0$. If the initial temperatures are prescribed by $f(x)$, one needs to solve

$$\begin{aligned} u_t &= k u_{xx} \qquad \text{for } 0 < x < L, \text{ and } t > 0 \\ u(x,0) &= f(x) \qquad \text{for } 0 < x < L \\ u_x(0,t) &= u_x(L,t) = 0 \qquad \text{for } t > 0 \,. \end{aligned} \tag{4.9}$$

It is natural to expect that in the long run the temperatures inside the rod will average out, and be equal to the average of the initial temperatures, $\frac{1}{L}\int_0^L f(x)\,dx$.

Again, we search for a solution in the form $u(x,t) = F(x)G(t)$. Separation of variables shows that $G(t)$ still satisfies (4.4), while $F(x)$ needs to solve

$$F''(x) + \lambda F(x) = 0, \quad F'(0) = F'(L) = 0 \,.$$

Recall that nontrivial solutions of this problem occur only at $\lambda = \lambda_0 = 0$, and at $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$. The corresponding solutions are

$$F_0(x) = 1, \;\; F_n(x) = \cos \frac{n\pi}{L} x \quad \text{(and their multiples)}\,.$$

Solving (4.4) for $n = 0$, and for all $n = 1, 2, 3, \ldots$, gives

$$G_0 = a_0, \;\; G_n(t) = a_n e^{-k\frac{n^2\pi^2}{L^2}t} \,,$$

where a_0 and a_n's are arbitrary constants. We constructed infinitely many functions

$$u_0(x,t) = G_0(t)F_0(x) = a_0, \quad u_n(x,t) = G_n(t)F_n(x) = a_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos \frac{n\pi}{L}x \,,$$

satisfying the PDE in (4.9), and the boundary conditions. By linearity, their sum

$$(4.10) \quad u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos \frac{n\pi}{L}x$$

also satisfies the PDE in (4.9), and the boundary conditions. To satisfy the initial condition

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}x = f(x)$$

one needs to represent $f(x)$ by its Fourier cosine series, for which we calculate

$$(4.11) \qquad a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\,.$$

Conclusion: the series

$$(4.12) \qquad u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-k\frac{n^2\pi^2}{L^2}t} \cos \frac{n\pi}{L}x \,,$$

with a_n's computed using (4.11), gives the solution of our problem (4.9). Observe that going from the Fourier cosine series of $f(x)$ to the solution of the problem (4.9) involves just putting in the additional factors $e^{-k\frac{n^2\pi^2}{L^2}t}$. As $t \to \infty$, $u(x,t) \to a_0$, which is equal to the average of the initial temperatures. The first term of the series dominates all others, and so $u(x,t) \approx a_0 + a_1 e^{-k\frac{\pi^2}{L^2}t}\cos\frac{\pi}{L}x$, for t not too small, say for $t > 1$.

Example 3 Solve

$$\begin{gathered}
u_t = 3u_{xx} \qquad \text{for } 0 < x < \pi/2, \text{ and } t > 0 \\
u(x,0) = 2\cos^2 x - 3\cos^2 2x \qquad \text{for } 0 < x < \pi/2 \\
u_x(0,t) = u_x(\pi/2,t) = 0 \qquad \text{for } t > 0\,.
\end{gathered}$$

Here $k = 3$, and $L = \pi/2$. The Fourier cosine series has the form $a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\,x$. Writing

$$2\cos^2 x - 3\cos^2 2x = -\frac{1}{2} + \cos 2x - \frac{3}{2}\cos 4x\,,$$

we see that this function is its own Fourier cosine series, with $a_0 = -\frac{1}{2}$, $a_1 = 1$, $a_2 = -\frac{3}{2}$, and all other coefficients equal to zero. Putting in the exponential factors, according to (4.12), we arrive at the solution:

$$u(x,t) = -\frac{1}{2} + e^{-12t}\cos 2x - \frac{3}{2}e^{-48t}\cos 4x\,.$$

Example 4 Solve

$$\begin{aligned} u_t &= 3u_{xx} - au \qquad \text{for } 0 < x < \pi, \text{ and } t > 0 \\ u(x,0) &= 2\cos x + x^2 \qquad \text{for } 0 < x < \pi \\ u_x(0,t) &= u_x(\pi,t) = 0 \qquad \text{for } t > 0\,, \end{aligned}$$

where a is a positive constant. The extra term $-au$ is an example of a *lower order term*. Its physical significance is that the rod is no longer insulated, and heat freely *radiates* through its side, with the ambient temperature outside of the rod being zero. Indeed, the heat balance relation leading to this equation is

$$u_t\,\Delta x = 3\left[u_x(x+\Delta x,t) - u_x(x,t)\right] - au(x,t)\Delta x\,.$$

If $u(x,t) > 0$, heat radiates out, and if $u(x,t) < 0$, heat enters through the side of the rod.

Let $v(x,t)$ be the new unknown function, defined by

$$u(x,t) = e^{-at}v(x,t)\,.$$

Calculate $u_t = -ae^{-at}v + e^{-at}v_t$, and $u_{xx} = e^{-at}v_{xx}$. Then $v(x,t)$ solves the problem

$$\begin{aligned} v_t &= 3v_{xx} \qquad \text{for } 0 < x < \pi, \text{ and } t > 0 \\ v(x,0) &= 2\cos x + x^2 \qquad \text{for } 0 < x < \pi \\ v_x(0,t) &= v_x(\pi,t) = 0 \qquad \text{for } t > 0\,, \end{aligned} \tag{4.13}$$

which we know how to handle. Because $2\cos x$ is its own Fourier cosine series on the interval $(0,\pi)$, we expand x^2 in the Fourier cosine series (and then add $2\cos x$). Obtain

$$x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,x\,,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}\,,$$

$$a_n = \frac{2}{\pi} \int_0^\pi x^2 \cos nx \, dx = \frac{4(-1)^n}{n^2} .$$

Then

$$2\cos x + x^2 = 2\cos x + \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\,x = \frac{\pi^2}{3} - 2\cos x + \sum_{n=2}^{\infty} \frac{4(-1)^n}{n^2} \cos n\,x .$$

Using (4.12), the solution of (4.13) is

$$v(x,t) = \frac{\pi^2}{3} - 2e^{-3t} \cos x + \sum_{n=2}^{\infty} \frac{4(-1)^n}{n^2} e^{-3n^2 t} \cos n\,x .$$

Answer: $u(x,t) = \dfrac{\pi^2}{3} e^{-at} - 2e^{-(3+a)t} \cos x + \displaystyle\sum_{n=2}^{\infty} \frac{4(-1)^n}{n^2} e^{-(3n^2+a)t} \cos n\,x.$

7.5 Laplace's Equation

We now study heat conduction in a thin two-dimensional rectangular plate: $0 \le x \le L$, $0 \le y \le M$. Assume that both sides of the plate are insulated, so that heat travels only in the xy-plane. Let $u(x,y,t)$ denote the temperature at a point (x,y), and time $t > 0$. It is natural to expect that the heat equation in two dimensions takes the form

$$u_t = k\left(u_{xx} + u_{yy}\right) . \tag{5.1}$$

Indeed, one can derive (5.1) similarly to the way we have derived the one-dimensional heat equation, see e.g., the book of L. Evans [9].

The boundary of our plate consists of four line segments. Let us assume that the side lying on the x-axis is kept at 1 degree Celsius, so that $u(x,0) = 1$ for $0 \le x \le L$, while the other three sides are kept at 0 degree Celsius (they are kept on ice), so that $u(x,M) = 0$ for $0 \le x \le L$, and $u(0,y) = u(L,y) = 0$ for $0 \le y \le M$. The heat will flow from the warmer side toward the three sides on ice. While the heat will continue its flow indefinitely, eventually the temperatures will stabilize (we can expect temperatures to be close to 1 near the warm side, and close to 0 near the icy sides). Stable temperatures do not change with time, so that $u = u(x,y)$. Then $u_t = 0$, and the equation (5.1) becomes *Laplace's equation*:

$$u_{xx} + u_{yy} = 0 . \tag{5.2}$$

one of the three main equations of mathematical physics (along with the heat and the wave equations). Mathematicians use the notation: $\Delta u = u_{xx}+u_{yy}$, while engineers seem to prefer $\nabla^2 u = u_{xx} + u_{yy}$. The latter notation has to do with the fact that computing the divergence of the gradient of $u(x, y)$ gives $\nabla \cdot \nabla u = u_{xx}+u_{yy}$. Solutions of Laplace's equation are called *harmonic functions*.

To find the *steady state temperatures* $u = u(x, y)$ for our example, we need to solve the problem

$$\begin{gathered} u_{xx} + u_{yy} = 0 \qquad \text{for } 0 < x < L, \text{ and } 0 < y < M \\ u(x,0) = 1 \qquad \text{for } 0 < x < L \\ u(x,M) = 0 \qquad \text{for } 0 < x < L \\ u(0,y) = u(L,y) = 0 \qquad \text{for } 0 < y < M\,. \end{gathered}$$

We apply the separation of variables technique, looking for a solution in the form

$$u(x, y) = F(x)G(y)\,,$$

with the functions $F(x)$ and $G(y)$ to be determined. Substitution into the Laplace equation gives

$$F''(x)G(y) = -F(x)G''(y)\,.$$

Divide both sides by $F(x)G(y)$:

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}\,.$$

The function on the left depends on x only, while the one the right depends only on y. In order for them to be the same, they must be both equal to the same constant, which we denote by $-\lambda$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda\,.$$

Using the boundary conditions, we obtain

$$F'' + \lambda F = 0, \quad F(0) = F(L) = 0\,, \tag{5.3}$$

$$G'' - \lambda G = 0, \quad G(M) = 0\,. \tag{5.4}$$

Nontrivial solutions of (5.3) occur at $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$, and they are $F_n(x) = B_n \sin \frac{n\pi}{L} x$, with arbitrary constants B_n. We solve the problem (5.4) with

$\lambda = \frac{n^2\pi^2}{L^2}$, obtaining $G_n(y) = \sinh \frac{n\pi}{L}(y-M)$. (Recall that the general solution of the equation in (5.4) may be taken in the form $G = c_1 \sinh \frac{n\pi}{L}(y+c_2)$.) We conclude that the functions

$$u_n(x,y) = F_n(x)G_n(y) = B_n \sin \frac{n\pi}{L}x \sinh \frac{n\pi}{L}(y-M)$$

satisfy Laplace's equation, and the three zero boundary conditions. The same is true for their sum

$$u(x,y) = \sum_{n=1}^{\infty} F_n(x)G_n(y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{L}(y-M) \sin \frac{n\pi}{L}x\,.$$

It remains to satisfy the boundary condition at the warm side:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{L}(-M) \sin \frac{n\pi}{L}x = 1\,.$$

We need to choose B_n's, so that $B_n \sinh \frac{n\pi}{L}(-M)$ are the Fourier sine series coefficient of $f(x) = 1$, i.e.,

$$B_n \sinh \frac{n\pi}{L}(-M) = \frac{2}{L}\int_0^L \sin \frac{n\pi}{L}x\,dx = \frac{2\left(1-(-1)^n\right)}{n\pi}\,,$$

which gives

$$B_n = -\frac{2\left(1-(-1)^n\right)}{n\pi \sinh \frac{n\pi M}{L}}\,.$$

Answer:

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{2\left(1-(-1)^n\right)}{n\pi \sinh \frac{n\pi M}{L}} \sinh \frac{n\pi}{L}(y-M) \sin \frac{n\pi}{L}x\,.$$

Recall that the general solution of (ω is a constant, $y = y(x)$)

$$y'' - \omega^2 y = 0$$

can be written in three ways: $y = c_1 e^{-\omega x} + c_2 e^{\omega x}$, $y = c_1 \cosh \omega x + c_2 \sinh \omega x$, and $y = c_1 \sinh \omega(x+c_2)$. We used the third form when solving for $G(y)$, while the second one is convenient if the initial conditions are prescribed at $x = 0$.

Example Find the steady state temperatures

$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad & \text{for } 0 < x < 1, \text{ and } 0 < y < 2 \\ u(x,0) = 0 \quad & \text{for } 0 < x < 1 \\ u(x,2) = 0 \quad & \text{for } 0 < x < 1 \\ u(0,y) = 0 \quad & \text{for } 0 < y < 2 \\ u(1,y) = y \quad & \text{for } 0 < y < 2\,. \end{aligned}$$

(The warm side is now $x = 1$.) Look for a solution in the form $u(x,y) = F(x)G(y)$. After separating the variables, it is convenient to use λ (instead of $-\lambda$) to denote the common value of two functions

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \lambda\,.$$

Using the boundary conditions, obtain

$$G'' + \lambda G = 0, \quad G(0) = G(2) = 0\,,$$

$$F'' - \lambda F = 0, \quad F(0) = 0\,.$$

The first problem has non-trivial solutions at $\lambda = \lambda_n = \frac{n^2\pi^2}{4}$, and they are $G_n(y) = B_n \sin \frac{n\pi}{2} y$. We then solve the second equation with $\lambda = \frac{n^2\pi^2}{4}$, obtaining $F_n(x) = \sinh \frac{n\pi}{2} x$. It follows that the functions

$$u_n(x,y) = F_n(x)G_n(y) = B_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y$$

satisfy the Laplace equation, and the three zero boundary conditions. The same is true for their sum

$$u(x,y) = \sum_{n=1}^{\infty} F_n(x)G_n(y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y\,.$$

It remains to satisfy the boundary condition at the fourth side:

$$u(1,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2} \sin \frac{n\pi}{2} y = y\,.$$

We need to choose B_n's, so that $B_n \sinh \frac{n\pi}{2}$ are the Fourier sine series coefficient of y on the interval $(0,2)$:

$$B_n \sinh \frac{n\pi}{2} = \int_0^2 y \sin \frac{n\pi}{2} y\, dy = \frac{4(-1)^{n+1}}{n\pi}\,,$$

which gives

$$B_n = \frac{4(-1)^{n+1}}{n\pi \sinh \frac{n\pi}{2}} .$$

Answer: $u(x,y) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi \sinh \frac{n\pi}{2}} \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y.$

Our computations in the above examples were aided by the fact that three out of the four boundary conditions were zero (homogeneous). A general *boundary value problem* has the form

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } 0 < x < L, \text{ and } 0 < y < M \\ u(x,0) &= f_1(x) \qquad \text{for } 0 < x < L \\ u(x,M) &= f_2(x) \qquad \text{for } 0 < x < L \\ u(0,y) &= g_1(y) \qquad \text{for } 0 < y < M \\ u(L,y) &= g_2(y) \qquad \text{for } 0 < y < M, \end{aligned} \tag{5.5}$$

with given functions $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(y)$. Because this problem is linear, we can break it into four sub-problems, each time keeping one of the boundary conditions, and setting the other three to zero. Namely, we look for solution in the form

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y) ,$$

where u_1 is found by solving

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } 0 < x < L, \text{ and } 0 < y < M \\ u(x,0) &= f_1(x) \qquad \text{for } 0 < x < L \\ u(x,M) &= 0 \qquad \text{for } 0 < x < L \\ u(0,y) &= 0 \qquad \text{for } 0 < y < M \\ u(L,y) &= 0 \qquad \text{for } 0 < y < M, \end{aligned}$$

u_2 is computed from

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } 0 < x < L, \text{ and } 0 < y < M \\ u(x,0) &= 0 \qquad \text{for } 0 < x < L \\ u(x,M) &= f_2(x) \qquad \text{for } 0 < x < L \\ u(0,y) &= 0 \qquad \text{for } 0 < y < M \\ u(L,y) &= 0 \qquad \text{for } 0 < y < M, \end{aligned}$$

u_3 is computed from

$$\begin{gathered}
u_{xx}+u_{yy}=0 \qquad \text{for } 0<x<L, \text{ and } 0<y<M \\
u(x,0)=0 \qquad \text{for } 0<x<L \\
u(x,M)=0 \qquad \text{for } 0<x<L \\
u(0,y)=g_1(y) \qquad \text{for } 0<y<M \\
u(L,y)=0 \qquad \text{for } 0<y<M,
\end{gathered}$$

and u_4 is obtained by solving

$$\begin{gathered}
u_{xx}+u_{yy}=0 \qquad \text{for } 0<x<L, \text{ and } 0<y<M \\
u(x,0)=0 \qquad \text{for } 0<x<L \\
u(x,M)=0 \qquad \text{for } 0<x<L \\
u(0,y)=0 \qquad \text{for } 0<y<M \\
u(L,y)=g_2(y) \qquad \text{for } 0<y<M.
\end{gathered}$$

We solve each of these four problems, using separation of variables, as in the examples considered previously.

7.6 The Wave Equation

We consider vibrations of a guitar string (or a similar elastic string). We assume that the motion of the string is *transverse*, so that it goes only up and down (and not sideways). Let $u(x,t)$ denote the displacement of the string at a point x and time t, and our goal is to calculate $u(x,t)$. The motion of an element of the string $(x, x+\Delta x)$ is governed by Newton's second law of motion

$$ma = f\,. \tag{6.1}$$

The acceleration $a = u_{tt}(x,t)$. If ρ denotes the density of the string, then the mass of the element is $m = \rho\Delta x$. (The string is assumed to be *uniform*, so that $\rho > 0$ is a constant.) We also assume that the *internal tension* is the only force acting on this element, and that the magnitude of the tension T is constant throughout the string. Our final assumption is that the slope of the string $u_x(x,t)$ is small, for all x and t. Observe that $u_x(x,t) = \tan\theta$, the slope of the tangent line to the function $u = u(x)$, with t fixed. The vertical (acting) component of the force at the right end-point of our element is

$$T\sin\theta \approx T\tan\theta = Tu_x(x+\Delta x, t)\,,$$

because for small angles θ, $\sin\theta \approx \tan\theta \approx \theta$. At the left end-point, the vertical component of the force is $Tu_x(x,t)$, and the equation (6.1) becomes

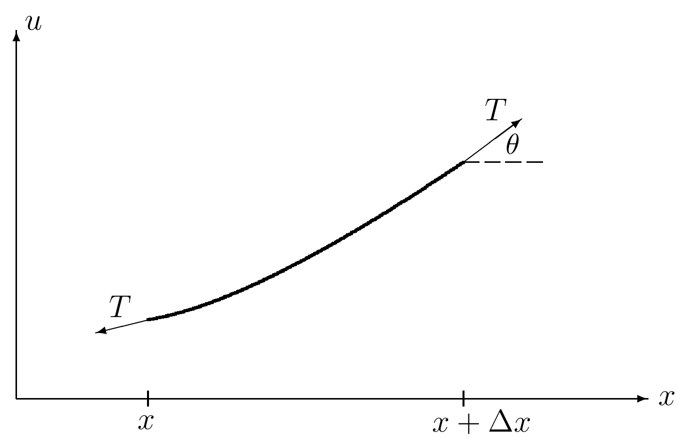
$$\rho\Delta x\, u_{tt}(x,t) = Tu_x(x+\Delta x,t) - Tu_x(x,t)\,.$$

Divide both sides by $\rho\Delta x$, and denote $T/\rho = c^2$

$$u_{tt}(x,t) = c^2 \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\,.$$

Letting $\Delta x \to 0$, we obtain the *wave equation*

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)\,.$$



Forces acting on an element of a string

We consider now the vibrations of a string of length L, which is fixed at the end-points $x = 0$ and $x = L$, with given initial displacement $f(x)$, and given initial velocity $g(x)$:

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \qquad \text{for } 0 < x < L, \text{ and } t > 0\\
u(0,t) &= u(L,t) = 0 \qquad \text{for } t > 0\\
u(x,0) &= f(x) \qquad \text{for } 0 < x < L\\
u_t(x,0) &= g(x) \qquad \text{for } 0 < x < L\,.
\end{aligned}$$

Perform separation of variables, by setting $u(x,t) = F(x)G(t)$, and obtaining

$$F(x)G''(t) = c^2 F''(x)G(t)\,,$$

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = -\lambda \,.$$

Using the boundary conditions, gives

$$F'' + \lambda F = 0, \qquad F(0) = F(L) = 0\,,$$
$$G'' + \lambda c^2 G = 0\,.$$

Nontrivial solutions of the first problem occur at $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$, and they are $F_n(x) = \sin\frac{n\pi}{L}x$. The second equation then takes the form

$$G'' + \frac{n^2\pi^2}{L^2}c^2 G = 0\,.$$

Its general solution is

$$G_n(t) = b_n \cos\frac{n\pi c}{L}\,t + B_n \sin\frac{n\pi c}{L}\,t\,,$$

where b_n and B_n are arbitrary constants. The function

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x)G_n(t) = \sum_{n=1}^{\infty}\left(b_n\cos\frac{n\pi c}{L}\,t + B_n\sin\frac{n\pi c}{L}\,t\right)\sin\frac{n\pi}{L}x$$

satisfies the wave equation, and the boundary conditions. It remains to satisfy the initial conditions. Compute

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x = f(x)\,,$$

which requires that

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\,, \tag{6.2}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi}{L}x = g(x)\,,$$

which implies that

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx\,. \tag{6.3}$$

Answer:

$$u(x,t)=\sum_{n=1}^{\infty}\left(b_n\cos\frac{n\pi c}{L}t+B_n\sin\frac{n\pi c}{L}t\right)\sin\frac{n\pi}{L}x\,, \tag{6.4}$$

with b_n's computed by (6.2), and B_n's by (6.3).

The last formula shows that the motion of the string is periodic in time, similarly to the harmonic motion of a spring (the period is $\frac{2\pi L}{\pi c}$). This is understandable, because we did not account for the dissipation of energy in our model of vibrating string.

Example Find the displacements $u=u(x,t)$

$$\begin{aligned}
&u_{tt}=9u_{xx} \qquad \text{for } 0<x<\pi, \text{ and } t>0\\
&u(0,t)=u(\pi,t)=0 \qquad \text{for } t>0\\
&u(x,0)=2\sin x \qquad \text{for } 0<x<\pi\\
&u_t(x,0)=0 \qquad \text{for } 0<x<\pi\,.
\end{aligned}$$

Here $c=3$ and $L=\pi$. Because $g(x)=0$, all $B_n=0$, while b_n's are the Fourier sine series coefficients of $2\sin x$ on the interval $(0,\pi)$, so that $b_1=2$, and all other $b_n=0$. Using (6.4), gives the answer: $u(x,t)=2\cos 3t\sin x$.

To interpret this answer, we use a trigonometric identity to write

$$u(x,t)=\sin(x-3t)+\sin(x+3t)\,.$$

The graph of $y=\sin(x-3t)$ in the xy-plane is obtained by translating the graph of $y=\sin x$ by $3t$ units to the right. Drawing these graphs on the same screen, for different times t, we see a *wave traveling to the right with speed* 3. Similarly, the graph of $\sin(x+3t)$ is a *wave traveling to the left with speed* 3. Our solution is the sum, or the *superposition*, of these two waves. For the general wave equation, the *wave speed* is given by c.

7.6.1 Non-Homogeneous Problems

Let us solve the following problem

$$\begin{aligned}
&u_{tt}-4u_{xx}=x \qquad \text{for } 0<x<3, \text{ and } t>0\\
&u(0,t)=1 \qquad \text{for } t>0\\
&u(3,t)=2 \qquad \text{for } t>0\\
&u(x,0)=0 \qquad \text{for } 0<x<3\\
&u_t(x,0)=1 \qquad \text{for } 0<x<3\,.
\end{aligned}$$

This problem does not fit the pattern studied so far. Indeed, the x term on the right makes the equation non-homogeneous, and the boundary conditions are non-homogeneous (non-zero) too. We look for solution in the form

$$u(x,t) = U(x) + v(x,t)\,.$$

We ask of the function $U(x)$ to take care of all of our problems (to remove the non-homogeneous terms), and satisfy

$$\begin{gathered}-4U'' = x \\ U(0) = 1, \;\; U(3) = 2\,.\end{gathered}$$

Integrating twice, gives

$$U(x) = -\frac{1}{24}x^3 + \frac{17}{24}x + 1\,.$$

Then the function $v(x,t) = u(x,t) - U(x)$ satisfies

$$\begin{gathered}v_{tt} - 4v_{xx} = 0 \qquad \text{for } 0 < x < 3, \text{ and } t > 0 \\ v(0,t) = 0 \qquad \text{for } t > 0 \\ v(3,t) = 0 \qquad \text{for } t > 0 \\ v(x,0) = -U(x) = \tfrac{1}{24}x^3 - \tfrac{17}{24}x - 1 \qquad \text{for } 0 < x < 3 \\ v_t(x,0) = 1 \qquad \text{for } 0 < x < 3\,.\end{gathered}$$

This is a homogeneous problem, of the type considered in the preceding section! Here $c = 2$ and $L = 3$. Separation of variables (or the formula (6.4)) gives

$$v(x,t) = \sum_{n=1}^{\infty}\left(b_n \cos\frac{2n\pi}{3}\,t + B_n \sin\frac{2n\pi}{3}\,t\right)\sin\frac{n\pi}{3}x\,,$$

with

$$b_n = \frac{2}{3}\int_0^3 \left(\frac{1}{24}x^3 - \frac{17}{24}x - 1\right)\sin\frac{n\pi}{3}x\,dx = -\frac{2}{n\pi} + \frac{27 + 8n^2\pi^2}{2n^3\pi^3}(-1)^n\,,$$

$$B_n = \frac{1}{n\pi}\int_0^3 \sin\frac{n\pi}{3}x\,dx = \frac{3 - 3(-1)^n}{n^2\pi^2}\,.$$

Answer:

$$u(x,t) = -\frac{1}{24}x^3 + \frac{17}{24}x + 1$$

$$+\sum_{n=1}^{\infty}\left[\left(-\frac{2}{n\pi}+\frac{27+8n^2\pi^2}{2n^3\pi^3}(-1)^n\right)\cos\frac{2n\pi}{3}t+\frac{3-3(-1)^n}{n^2\pi^2}\sin\frac{2n\pi}{3}t\right]\sin\frac{n\pi}{3}x\,.$$

In the non-homogeneous wave equation

$$u_{tt}-c^2u_{xx}=F(x,t)\,, \tag{6.5}$$

the term $F(x,t)$ represents the *acceleration of an external force* applied to the string. Indeed, the $ma=f$ equation for an element of a string, in the presence of an external force, takes the form

$$\rho\Delta x\,u_{tt}=T\left[u_x(x+\Delta x,t)-u_x(x,t)\right]+\rho\Delta x\,F(x,t)\,.$$

Dividing by $\rho\Delta x$, and letting $\Delta x\to 0$ (as before), we obtain (6.5). It follows that the $\rho\Delta x\,F(x,t)$ term is an extra force, and $F(x,t)$ is its acceleration.

Non-homogeneous problems for the heat equation are solved similarly.

Example Let us solve the problem

$$\begin{aligned} u_t-2u_{xx}=1 &\quad \text{for } 0<x<1, \text{ and } t>0\\ u(x,0)=x &\quad \text{for } 0<x<1\\ u(0,t)=0 &\quad \text{for } t>0\\ u(1,t)=3 &\quad \text{for } t>0\,. \end{aligned}$$

Again, look for solution in the form

$$u(x,t)=U(x)+v(x,t)\,,$$

with $U(x)$ satisfying

$$\begin{aligned} -2U''&=1\\ U(0)=0,\;\; U(1)&=3\,. \end{aligned}$$

Integrating, we calculate

$$U(x)=-\frac{1}{4}x^2+\frac{13}{4}x\,.$$

The function $v(x,t)=u(x,t)-U(x)$ satisfies

$$\begin{aligned} v_t-2v_{xx}=0 &\quad \text{for } 0<x<1, \text{ and } t>0\\ v(x,0)=x-U(x)=\tfrac{1}{4}x^2-\tfrac{9}{4}x &\quad \text{for } 0<x<1\\ v(0,t)=v(1,t)=0 &\quad \text{for } t>0\,. \end{aligned}$$

To solve the last problem, we begin by expanding the initial temperature $v(x,0)$ in its Fourier sine series

$$\frac{1}{4}x^2 - \frac{9}{4}x = \sum_{n=1}^{\infty} b_n \sin n\pi x \,,$$

with

$$b_n = 2\int_0^1 \left(\frac{1}{4}x^2 - \frac{9}{4}x\right) \sin n\pi x \, dx = \frac{-1 + \left(1 + 4n^2\pi^2\right)(-1)^n}{n^3\pi^3} \,.$$

Then, using (4.8),

$$v(x,t) = \sum_{n=1}^{\infty} \frac{-1 + \left(1 + 4n^2\pi^2\right)(-1)^n}{n^3\pi^3} e^{-2n^2\pi^2 t} \sin n\pi x \,.$$

Answer:

$$u(x,t) = -\frac{1}{4}x^2 + \frac{13}{4}x + \sum_{n=1}^{\infty} \frac{-1 + \left(1 + 4n^2\pi^2\right)(-1)^n}{n^3\pi^3} e^{-2n^2\pi^2 t} \sin n\pi x \,.$$

7.6.2 Problems

I. Solve the following problems, and explain their physical significance.

1.
$$\begin{aligned} u_t &= 2u_{xx} \qquad \text{for } 0 < x < \pi, \text{ and } t > 0 \\ u(x,0) &= \sin x - 3\sin x \cos x \qquad \text{for } 0 < x < \pi \\ u(0,t) &= u(\pi,t) = 0 \qquad \text{for } t > 0 \,. \end{aligned}$$

Answer. $u(x,t) = e^{-2t}\sin x - \frac{3}{2}e^{-8t}\sin 2x$.

2.
$$\begin{aligned} u_t &= 2u_{xx} \qquad \text{for } 0 < x < 2\pi, \text{ and } t > 0 \\ u(x,0) &= \sin x - 3\sin x \cos x \qquad \text{for } 0 < x < 2\pi \\ u(0,t) &= u(2\pi,t) = 0 \qquad \text{for } t > 0 \,. \end{aligned}$$

Answer. $u(x,t) = e^{-2t}\sin x - \frac{3}{2}e^{-8t}\sin 2x$.

3.
$$\begin{aligned} u_t &= 5u_{xx} \qquad \text{for } 0 < x < 2, \text{ and } t > 0 \\ u(x,0) &= x \qquad \text{for } 0 < x < 2 \\ u(0,t) &= u(2,t) = 0 \qquad \text{for } t > 0 \,. \end{aligned}$$

Answer. $u(x,t)=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}e^{-\frac{5n^2\pi^2}{4}t}\sin\frac{n\pi}{2}x.$

4.
$$\begin{aligned}
u_t &= 3u_{xx} \qquad \text{for } 0<x<\pi, \text{ and } t>0\\
u(x,0) &= \begin{cases} x & \text{for } 0<x<\frac{\pi}{2} \\ \pi-x & \text{for } \frac{\pi}{2}<x<\pi \end{cases} \qquad \text{for } 0<x<\pi\\
u(0,t) &= u(\pi,t)=0 \qquad \text{for } t>0\,.
\end{aligned}$$

Answer. $u(x,t)=\sum_{n=1}^{\infty}\frac{4}{\pi n^2}e^{-3n^2t}\sin\frac{n\pi}{2}\,\sin nx.$

5.
$$\begin{aligned}
u_t &= u_{xx} \qquad \text{for } 0<x<3, \text{ and } t>0\\
u(x,0) &= x+2 \qquad \text{for } 0<x<3\\
u(0,t) &= u(3,t)=0 \qquad \text{for } t>0\,.
\end{aligned}$$

Answer. $u(x,t)=\sum_{n=1}^{\infty}\frac{4-10(-1)^n}{n\pi}\,e^{-\frac{n^2\pi^2}{9}t}\sin\frac{n\pi}{3}x.$

6.
$$\begin{aligned}
u_t &= u_{xx} \qquad \text{for } 0<x<3, \text{ and } t>0\\
u(x,0) &= x+2 \qquad \text{for } 0<x<3\\
u_x(0,t) &= u_x(3,t)=0 \qquad \text{for } t>0\,.
\end{aligned}$$

Answer. $u(x,t)=\frac{7}{2}+\sum_{n=1}^{\infty}\frac{6(-1+(-1)^n)}{n^2\pi^2}\,e^{-\frac{n^2\pi^2}{9}t}\cos\frac{n\pi}{3}x.$

7.
$$\begin{aligned}
u_t &= 2u_{xx} \qquad \text{for } 0<x<\pi, \text{ and } t>0\\
u(x,0) &= \cos^4 x \qquad \text{for } 0<x<\pi\\
u_x(0,t) &= u_x(\pi,t)=0 \qquad \text{for } t>0\,.
\end{aligned}$$

Answer. $u(x,t)=\frac{3}{8}+\frac{1}{2}e^{-8t}\cos 2x+\frac{1}{8}e^{-32t}\cos 4x.$

8.
$$\begin{aligned}
u_t &= 3u_{xx}+u \qquad \text{for } 0<x<2, \text{ and } t>0\\
u(x,0) &= 1-x \qquad \text{for } 0<x<2\\
u_x(0,t) &= u_x(2,t)=0 \qquad \text{for } t>0\,.
\end{aligned}$$

Answer. $u(x,t)=\sum_{k=1}^{\infty}\frac{8}{\pi^2(2k-1)^2}e^{\left(-\frac{3(2k-1)^2\pi^2}{4}+1\right)t}\cos\frac{(2k-1)\pi}{2}\,x.$

9. Show that the following functions are harmonic: $u(x,y) = a$, $v(x,y) = \sin mx \sinh my$, $w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ (a and m are constants).

10. If $u(x,y)$ is a harmonic function, show that $v(x,y) = u\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ is also harmonic. (The map $(x,y) \to \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ is called the *Kelvin transform* with respect to the unit circle.)

11.
$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{for } 0 < x < 2, \text{ and } 0 < y < 3 \\ u(x,0) = u(x,3) = 5 \qquad & \text{for } 0 < x < 2 \\ u(0,y) = u(2,y) = 5 \qquad & \text{for } 0 < y < 3\,. \end{aligned}$$

Hint: Look for a simple solution.

12.
$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{for } 0 < x < 2, \text{ and } 0 < y < 3 \\ u(x,0) = 5 \qquad & \text{for } 0 < x < 2 \\ u(x,3) = 0 \qquad & \text{for } 0 < x < 2 \\ u(0,y) = u(2,y) = 0 \qquad & \text{for } 0 < y < 3\,. \end{aligned}$$

13.
$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{for } 0 < x < 2, \text{ and } 0 < y < 1 \\ u(x,0) = u(x,1) = 0 \qquad & \text{for } 0 < x < 2 \\ u(0,y) = y \qquad & \text{for } 0 < y < 1 \\ u(2,y) = 0 \qquad & \text{for } 0 < y < 1\,. \end{aligned}$$

14.
$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{for } 0 < x < \pi, \text{ and } 0 < y < 1 \\ u(x,0) = 0 \qquad & \text{for } 0 < x < \pi \\ u(x,1) = 3\sin 2x \qquad & \text{for } 0 < x < \pi \\ u(0,y) = u(\pi,y) = 0 \qquad & \text{for } 0 < y < 1\,. \end{aligned}$$

Answer. $u(x,y) = \dfrac{3}{\sinh 2} \sin 2x \sinh 2y$.

15.
$$\begin{aligned} u_{xx} + u_{yy} = 0 \qquad & \text{for } 0 < x < 2\pi, \text{ and } 0 < y < 2 \\ u(x,0) = \sin x \qquad & \text{for } 0 < x < 2\pi \\ u(x,2) = 0 \qquad & \text{for } 0 < x < 2\pi \\ u(0,y) = 0 \qquad & \text{for } 0 < y < 2 \\ u(2\pi,y) = y \qquad & \text{for } 0 < y < 2\,. \end{aligned}$$

Answer. $u(x,y) = -\dfrac{1}{\sinh 2}\sinh(y-2)\sin x + \displaystyle\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi \sinh n\pi^2} \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y$.

16.

$$
\begin{aligned}
u_{tt} - 4u_{xx} = 0 \qquad & \text{for } 0 < x < \pi, \text{ and } t > 0 \\
u(x,0) = \sin 2x \qquad & \text{for } 0 < x < \pi \\
u_t(x,0) = -4\sin 2x \qquad & \text{for } 0 < x < \pi \\
u(0,t) = u(\pi,t) = 0 \qquad & \text{for } t > 0\,.
\end{aligned}
$$

Answer. $u(x,t) = \cos 4t \sin 2x - \sin 4t \sin 2x$.

17.

$$
\begin{aligned}
u_{tt} - 4u_{xx} = 0 \qquad & \text{for } 0 < x < 1, \text{ and } t > 0 \\
u(x,0) = 0 \qquad & \text{for } 0 < x < 1 \\
u_t(x,0) = x \qquad & \text{for } 0 < x < 1 \\
u(0,t) = u(1,t) = 0 \qquad & \text{for } t > 0\,.
\end{aligned}
$$

Answer. $u(x,t) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2} \sin 2n\pi t \sin n\pi x$.

18.

$$
\begin{aligned}
u_{tt} - 4u_{xx} = 0 \qquad & \text{for } 0 < x < 1, \text{ and } t > 0 \\
u(x,0) = -3 \qquad & \text{for } 0 < x < 1 \\
u_t(x,0) = x \qquad & \text{for } 0 < x < 1 \\
u(0,t) = u(1,t) = 0 \qquad & \text{for } t > 0\,.
\end{aligned}
$$

Answer. $u(x,t) = \sum_{n=1}^{\infty} \left[\frac{6}{n\pi} \left((-1)^n - 1\right) \cos 2n\pi t + \frac{(-1)^{n+1}}{n^2\pi^2} \sin 2n\pi t \right] \sin n\pi x$.

19. Use separation of variables to obtain the solution of

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} = 0 \qquad & \text{for } 0 < x < L, \text{ and } t > 0 \\
u(x,0) = f(x) \qquad & \text{for } 0 < x < L \\
u_t(x,0) = g(x) \qquad & \text{for } 0 < x < L \\
u_x(0,t) = u_x(L,t) = 0 \qquad & \text{for } t > 0
\end{aligned}
$$

in the form $u(x,t) = a_0 + A_0 t + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c}{L} t + A_n \sin \frac{n\pi c}{L} t \right) \cos \frac{n\pi}{L} x$, and express the coefficients as integrals.

20. Solve

$$
\begin{aligned}
u_{tt} - 9u_{xx} = 0 \qquad & \text{for } 0 < x < \pi, \text{ and } t > 0 \\
u(x,0) = 4 \qquad & \text{for } 0 < x < \pi \\
u_t(x,0) = \cos^2 x \qquad & \text{for } 0 < x < \pi \\
u_x(0,t) = u_x(\pi,t) = 0 \qquad & \text{for } t > 0\,.
\end{aligned}
$$

Answer. $u(x,t) = 4 + \frac{1}{2}t + \frac{1}{12}\sin 6t \cos 2x$.

II. Solve the following non-homogeneous problems. You may leave the complicated integrals unevaluated.

1.
$$\begin{aligned} u_t = 5u_{xx}, \quad & \text{for } 0 < x < 1, \text{ and } t > 0 \\ u(0,t) = 0 \quad & \text{for } t > 0 \\ u(1,t) = 1 \quad & \text{for } t > 0 \\ u(x,0) = 0 \quad & \text{for } 0 < x < 1\,. \end{aligned}$$

Answer. $u(x,t) = x + \displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-5n^2\pi^2 t} \sin n\pi x$.

2.
$$\begin{aligned} u_t = 2u_{xx} \quad & \text{for } 0 < x < \pi, \text{ and } t > 0 \\ u(x,0) = \tfrac{x}{\pi} \quad & \text{for } 0 < x < \pi \\ u(0,t) = 0 \quad & \text{for } t > 0 \\ u(\pi,t) = 1 \quad & \text{for } t > 0\,. \end{aligned}$$

Hint: $U(x) = \dfrac{x}{\pi}$.

3.
$$\begin{aligned} u_t = 2u_{xx} + 4x \quad & \text{for } 0 < x < 1, \text{ and } t > 0 \\ u(x,0) = 0 \quad & \text{for } 0 < x < 1 \\ u(0,t) = 0 \quad & \text{for } t > 0 \\ u(1,t) = 0 \quad & \text{for } t > 0\,. \end{aligned}$$

Hint: $U(x) = \frac{1}{3}(x - x^3)$.

4.
$$\begin{aligned} u_{tt} = 4u_{xx} + x \quad & \text{for } 0 < x < 4, \text{ and } t > 0 \\ u(x,0) = x \quad & \text{for } 0 < x < 4 \\ u_t(x,0) = 0 \quad & \text{for } 0 < x < 4 \\ u(0,t) = 1 \quad & \text{for } t > 0 \\ u(4,t) = 3 \quad & \text{for } t > 0\,. \end{aligned}$$

Hint: $U(x) = 1 + \frac{7}{6}x - \frac{1}{24}x^3$.

5.
$$\begin{aligned} u_t = ku_{xx} + f(x,t) \quad & \text{for } 0 < x < \pi, \text{ and } t > 0 \\ u(x,0) = 0 \quad & \text{for } 0 < x < \pi \\ u(0,t) = 0 \quad & \text{for } t > 0 \\ u(\pi,t) = 0 \quad & \text{for } t > 0\,. \end{aligned}$$

Here $f(x,t)$ is a given function, $k > 0$ is a given number.

Hint: Expand $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin nx$, with $f_n(t) = \frac{2}{\pi} \int_0^{\pi} f(x,t) \sin nx \, dx$. Writing $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$, one has

$$u_n' + kn^2 u_n = f_n(t), \quad u_n(0) = 0\,.$$

6. Solve

$$\begin{aligned} u_t &= u_{xx} + t \sin 3x \qquad \text{for } 0 < x < \pi, \text{ and } t > 0 \\ u(x,0) &= 0 \qquad \text{for } 0 < x < \pi \\ u(0,t) &= 0 \qquad \text{for } t > 0 \\ u(\pi,t) &= 0 \qquad \text{for } t > 0\,. \end{aligned}$$

Answer. $u(x,t) = \left(\frac{t}{9} - \frac{1}{81} + \frac{1}{81} e^{-9t} \right) \sin 3x$.

7.7 Calculating Earth's Temperature and Queen Dido's Problem

7.7.1 The Complex Form of the Fourier Series

Recall that a real valued function $f(x)$, defined on $(-L, L)$, can be represented by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right), \tag{7.1}$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx\,,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx\,.$$

Using Euler's formulas: $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, write (7.1) as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}}{2} + b_n \frac{e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x}}{2i} \right).$$

Combining the similar terms, we rewrite this as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}\,, \tag{7.2}$$

where $c_0 = a_0$, and

$$c_n = \begin{cases} \frac{a_n}{2} - \frac{b_n i}{2} & \text{for } n > 0 \\ \frac{a_{-n}}{2} + \frac{b_{-n} i}{2} & \text{for } n < 0 \end{cases} .$$

We see that $\bar{c}_m = c_{-m}$, for any integer m, where the bar denotes the complex conjugate. (The same fact also follows by taking the complex conjugate of (7.2).)

Using the formulas for a_n and b_n, and Euler's formula, we have for $n > 0$

$$c_n = \frac{1}{L} \int_{-L}^{L} f(x) \frac{\cos \frac{n\pi}{L} x - i \sin \frac{n\pi}{L} x}{2} \, dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi}{L} x} \, dx .$$

In case $n \leq 0$, the formula for c_n is exactly the same.

The series (7.2), with the coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi}{L} x} \, dx, \quad n = 0, \pm 1, \pm 2, \ldots \tag{7.3}$$

is called the *complex form of the Fourier series.*

Recall that for any real a

$$\int e^{iax} \, dx = \frac{e^{iax}}{a} + c \,, \tag{7.4}$$

as follows by Euler's formula.

Example Find the complex form of the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for } -2 < x < 0 \\ 1 & \text{for } 0 < x < 2 \end{cases} , \quad \text{defined on } (-2, 2).$$

Here $L = 2$. Calculate $c_0 = \dfrac{1}{4} \displaystyle\int_{-2}^{2} f(x) \, dx = 0$. Using (7.3), (7.4) and Euler's formula, calculate for $n \neq 0$

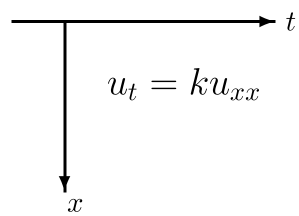
$$c_n = \frac{1}{4} \left(\int_{-2}^{0} (-1) e^{-i \frac{n\pi}{2} x} \, dx + \int_{0}^{2} 1 e^{-i \frac{n\pi}{2} x} \, dx \right) = \frac{i}{n\pi} \left[-1 + (-1)^n \right] ,$$

so that (with the sum taken over $n \neq 0$)

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{n\pi} \left[-1 + (-1)^n \right] e^{i \frac{n\pi}{2} x} .$$

7.7.2 The Temperatures Inside the Earth and Wine Cellars

Suppose that the average daily temperature in some area, for the t-th day of the year, is given by the function $f(t)$, $0 \le t \le 365$. (So that $t = 34$ corresponds to February 3.) We assume $f(t)$ to be periodic, with the period $T = 365$. What is the temperature x cm inside the Earth, for any t?



The sideways heat equation

Assume that x is not too large, so that we may ignore the geothermal effects. Direct the x-axis downward, inside the Earth, with $x = 0$ corresponding to Earth's surface. Direct the t-axis horizontally, and solve the heat equation for $u = u(x,t)$

$$u_t = ku_{xx}, \quad u(0,t) = f(t) \; \text{ for } x > 0, \text{ and } -\infty < t < \infty \,. \tag{7.5}$$

Geologists tell us that $k = 2 \cdot 10^{-3} \frac{cm^2}{sec}$ (see [30]).

Observe that the "initial condition" is now prescribed along the t-axis, and the "evolution" happens along the x-axis. This is sometimes referred to as the *sideways heat equation*. We represent $f(t)$ by its complex Fourier series ($L = \frac{T}{2}$, for T-periodic functions, corresponding to the interval $(-\frac{T}{2}, \frac{T}{2})$)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2n\pi}{T}t} \,.$$

Similarly, we expand the solution $u = u(x,t)$

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x) e^{i\frac{2n\pi}{T}t} \,.$$

The coefficients $u_n(x)$ are complex valued functions of x. Using this series in (7.5), gives for $n \neq 0$

$$u_n'' = p_n^2 u_n, \quad u_n(0) = c_n, \quad \text{with } p_n^2 = \frac{2in\pi}{kT} \,. \tag{7.6}$$

Depending on whether n is positive or negative, we can write

$$2in = (1 \pm i)^2 |n| \,,$$

and then

$$p_n = (1 \pm i) q_n, \;\; \text{with } q_n = \sqrt{\frac{|n|\pi}{kT}} \,.$$

(It is plus in case $n > 0$, and minus for $n < 0$.) Solving the equation in (7.6), gives

$$u_n(x) = a_n e^{(1\pm i)q_n x} + b_n e^{-(1\pm i)q_n x} \,.$$

We must set here $a_n = 0$, to avoid solutions with complex modulus becoming infinite as $x \to \infty$. Then, using the initial condition in (7.6), for $n \neq 0$

$$u_n(x) = c_n e^{-(1\pm i)q_n x} \,.$$

In case $n = 0$, the bounded solution of (7.6) is

$$u_0(x) = c_0 \,.$$

Obtain

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{-q_n x} e^{i\left[\frac{2n\pi}{T}t - (\pm) q_n x\right]} \,. \tag{7.7}$$

Write c_n in its polar form, for $n > 0$,

$$c_n = |c_n| e^{i\gamma_n} \,,$$

with some real numbers γ_n, and transform (7.7) as follows:

$$\begin{aligned} u(x,t) &= c_0 + \textstyle\sum_{n=1}^{\infty} e^{-q_n x} \left[c_n e^{i\left(\frac{2n\pi}{T}t - q_n x\right)} + c_{-n} e^{-i\left(\frac{2n\pi}{T}t - q_n x\right)} \right] \\ &= c_0 + \textstyle\sum_{n=1}^{\infty} e^{-q_n x} \left[c_n e^{i\left(\frac{2n\pi}{T}t - q_n x\right)} + \overline{c_n e^{i\left(\frac{2n\pi}{T}t - q_n x\right)}} \right] \\ &= c_0 + \textstyle\sum_{n=1}^{\infty} 2|c_n| e^{-q_n x} \cos\left(\frac{2n\pi}{T}t + \gamma_n - q_n x\right) \,, \end{aligned}$$

using that $\bar{c}_n = c_{-n}$, and that $z + \bar{z} = 2Re(z)$.

We see that the amplitude $2|c_n|e^{-q_n x}$ of the n-th term is damped exponentially in x, and this damping is increasing with n, so that the first term is dominant, giving

$$u(x,t) \approx c_0 + 2|c_1| e^{-q_1 x} \cos\left(\frac{2\pi}{T}t + \gamma_1 - q_1 x\right) \,.$$

When x changes, the cosine term is a shift of the function $\cos \frac{2\pi}{T} t$, giving us a *wave along the x-axis.* If we select x so that $\gamma_1 - q_1 x = 0$, we have a *complete phase shift*, so that the warmest temperatures at this depth occur in winter (when $t \approx 0$), and the coolest temperatures in summer (when $t \approx T/2 = 182.5$). This value of x is a good depth for a wine cellar. Not only the seasonal variations are very small, but they will also counteract any influence of air flow into the cellar.

The material of this section is based on the book of A. Sommerfeld [30], see p. 68. I became aware of this application through the wonderful lectures of Henry P. McKean at Courant Institute, NYU, in the late seventies.

7.7.3 The Isoperimetric Inequality

Complex Fourier series can be used to prove the following *Wirtinger's inequality*: any continuously differentiable function $x(s)$, which is periodic of period 2π, and has average zero, so that $\int_0^{2\pi} x(s)\, ds = 0$, satisfies

$$\int_0^{2\pi} x^2(s)\, ds \le \int_0^{2\pi} x'^2(s)\, ds\,.$$

Indeed, we represent $x(s)$ by its complex Fourier series $x(s) = \sum_{n=-\infty}^{\infty} x_n e^{ins}$ with the coefficients satisfying $x_{-n} = \bar{x}_n$ for $n \neq 0$, and

$$x_0 = \frac{1}{2\pi} \int_0^{2\pi} x(s)\, ds = 0\,.$$

Calculate:

$$\int_0^{2\pi} x^2(s)\, ds = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} x_n e^{ins} \sum_{m=-\infty}^{\infty} x_m e^{ims}\, ds$$

$$= \sum_{n,m=-\infty}^{\infty} x_n x_m \int_0^{2\pi} e^{i(n+m)s}\, ds = 2\pi \sum_{n=-\infty}^{\infty} x_n x_{-n} = 2\pi \sum_{n=-\infty}^{\infty} |x_n|^2\,,$$

because the integral $\int_0^{2\pi} e^{iks}\, ds$ is zero for any integer $k \neq 0$, and is equal to 2π for $k = 0$. (So that $\int_0^{2\pi} e^{i(n+m)s}\, ds$ is equal to 2π if $m = -n$, and to zero for all other m.)

Since $x'(s) = \sum_{n=-\infty}^{\infty} inx_n e^{ins}$, a similar computation gives

$$\int_0^{2\pi} {x'}^2(s)\, ds = 2\pi \sum_{n=-\infty}^{\infty} n^2 |x_n|^2 \,,$$

and Wirtinger's inequality follows. (The term corresponding to $n = 0$ is zero in both series.)

According to Virgil, the queen Dido of Carthage (circa 800 B.C.) had a long rope to enclose land, which would become hers. Dido used the rope to form a circle, which became the city of Carthage. We shall show that she made the optimal choice: *among all closed curves of length L, circle encloses the largest area.* If a circle has length L, then its radius is $r = \frac{L}{2\pi}$, and its area is $\pi r^2 = \frac{L^2}{4\pi}$. We wish to show that the area A of any region enclosed by a rope of length L satisfies

$$A \le \frac{L^2}{4\pi} \,.$$

This fact is known as the *isoperimetric inequality.*

We may assume that $L = 2\pi$, corresponding to $r = 1$, by declaring $r = \frac{L}{2\pi}$ to be the new unit of length. Then we need to show that the area A of any region enclosed by a closed curve of length $L = 2\pi$ satisfies

$$A \le \pi \,.$$

If we use the arc-length parameterization for any such curve $(x(s), y(s))$, then $x(s)$ and $y(s)$ are periodic functions of the period 2π (after traveling the distance 2π, we come back to the original point on the curve). Recall that for the arc-length parameterization, the tangent vector $(x'(s), y'(s))$ is of unit length, so that ${x'}^2(s) + {y'}^2(s) = 1$ for all s. Then

$$(7.8) \quad 2\pi = L = \int_0^{2\pi} \sqrt{{x'}^2(s) + {y'}^2(s)}\, ds = \int_0^{2\pi} \left[{x'}^2(s) + {y'}^2(s) \right] \, ds \,.$$

We may assume that $\gamma = \int_0^{2\pi} x(s)\, ds = 0$. (If not, consider the shifted curve $(x(s) - \frac{\gamma}{2\pi}, y(s))$, for which this condition holds.)

According to Green's formula, the area A enclosed by a closed curve $(x(s), y(s))$ is given by the line integral $\int x\,dy$ over this curve, which evaluates to $\int_0^{2\pi} x(s)y'(s)\,ds$. Using the numerical inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, Wirtinger's inequality and (7.8), we conclude

$$A = \int_0^{2\pi} x(s)y'(s)\,ds \leq \frac{1}{2}\int_0^{2\pi} \left[x^2(s) + y'^2(s)\right]\,ds$$
$$\leq \frac{1}{2}\int_0^{2\pi} \left[x'^2(s) + y'^2(s)\right]\,ds = \pi\,,$$

justifying the isoperimetric inequality.

7.8 Laplace's Equation on Circular Domains

Polar coordinates (r, θ) will be appropriate for circular domains, and it turns out that the Laplacian in the polar coordinates is

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,. \tag{8.1}$$

To justify (8.1), we begin by writing

$$u(x,y) = u(r(x,y), \theta(x,y))\,,$$

with

$$r = r(x,y) = \sqrt{x^2+y^2}, \quad \theta = \theta(x,y) = \arctan\frac{y}{x}\,. \tag{8.2}$$

By the chain rule

$$u_x = u_r r_x + u_\theta \theta_x\,,$$
$$u_{xx} = u_{rr}r_x^2 + 2u_{r\theta}r_x\theta_x + u_{\theta\theta}\theta_x^2 + u_r r_{xx} + u_\theta\theta_{xx}\,.$$

Similarly

$$u_{yy} = u_{rr}r_y^2 + 2u_{r\theta}r_y\theta_y + u_{\theta\theta}\theta_y^2 + u_r r_{yy} + u_\theta\theta_{yy}\,,$$

and so

$$u_{xx} + u_{yy} = u_{rr}\left(r_x^2 + r_y^2\right) + 2u_{r\theta}\left(r_x\theta_x + r_y\theta_y\right) + u_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right)$$
$$+ u_r\left(r_{xx} + r_{yy}\right) + u_\theta\left(\theta_{xx} + \theta_{yy}\right)\,.$$

Straightforward differentiation, using (8.2), shows that

$$r_x^2 + r_y^2 = 1\,,$$

$$r_x\theta_x + r_y\theta_y = 0\,,$$

$$\theta_x^2 + \theta_y^2 = \frac{1}{r^2}\,,$$

$$r_{xx} + r_{yy} = \frac{1}{r}\,,$$

$$\theta_{xx} + \theta_{yy} = 0\,,$$

and the formula (8.1) follows.

We now consider a circular plate: $x^2 + y^2 < R^2$, or $r < R$ in polar coordinates ($R > 0$ is its radius). The boundary of the plate consists of the points (R, θ), with $0 \le \theta < 2\pi$. Assume that the temperatures $u(R, \theta)$ at the boundary points are prescribed by a given function $f(\theta)$, of period 2π. What are the steady state temperatures $u(r, \theta)$ inside the plate?

We are searching for the function $u(r, \theta)$ of period 2π in θ that solves what is known as the *Dirichlet boundary value problem* (when the value of the unknown function is prescribed on the boundary):

$$\begin{aligned} &\Delta u = u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, \quad \text{for } r < R \\ &u(R, \theta) = f(\theta)\,. \end{aligned} \tag{8.3}$$

Perform separation of variables, looking for solution in the form $u(r, \theta) = F(r)G(\theta)$. Substituting this $u(r, \theta)$ into the equation (8.3), gives

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) = -\frac{1}{r^2}F(r)G''(\theta)\,.$$

Multiply both sides by r^2, and divide by $F(r)G(\theta)$:

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = \lambda\,.$$

This gives

$$\begin{aligned} &G'' + \lambda G = 0, \quad G(\theta) \text{ is } 2\pi \text{ periodic,} \\ &r^2F''(r) + rF'(r) - \lambda F(r) = 0\,. \end{aligned} \tag{8.4}$$

The first problem, of eigenvalue type, was considered in a problem set previously. It has non-trivial solutions when $\lambda = \lambda_n = n^2$ (n is a positive integer), and when $\lambda = \lambda_0 = 0$, and these solutions are

$$G_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad G_0 = A_0\,,$$

with arbitrary constants A_0, A_n and B_n. The second equation in (8.4), when $\lambda = \lambda_n = n^2$, becomes

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0\,.$$

This is Euler's equation! Its general solution is

$$F(r) = c_1 r^n + c_2 r^{-n}\,. \tag{8.5}$$

We need to select $c_2 = 0$, to avoid infinite temperature at $r = 0$, so that $F_n(r) = r^n$. When $\lambda = \lambda_0 = 0$, the second equation in (8.4) is

$$r^2 F''(r) + rF'(r) = 0\,,$$

for which the general solution is $F(r) = c_1 \ln r + c_2$. Again, we need to select $c_1 = 0$, and then $F_0(r) = 1$. The function

$$u(r,\theta) = F_0(r)G_0(\theta) + \sum_{n=1}^{\infty} F_n(r)G_n(\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

satisfies the Laplace equation for $r < R$. The boundary condition requires

$$u(R,\theta) = A_0 + \sum_{n=1}^{\infty} R^n \left(A_n \cos n\theta + B_n \sin n\theta\right) = f(\theta)\,.$$

Expand $f(\theta)$ in its Fourier series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right)\,, \tag{8.6}$$

with the coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta\,, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta\,, \tag{8.7}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \quad (n \geq 1)\,.$$

(For 2π-periodic functions, integrals over $(-\pi, \pi)$ are equal to the corresponding integrals over $(0, 2\pi)$.) Then we need to select $A_0 = a_0$, $A_n R^n = a_n$ and $B_n R^n = b_n$, so that $A_n = \dfrac{1}{R^n} a_n$ and $B_n = \dfrac{1}{R^n} b_n$. Conclusion: the solution of (8.3) is given by

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \,, \tag{8.8}$$

with the coefficients calculated by (8.7).

We see that the solution of (8.3) can be obtained from the Fourier series of $f(\theta)$, by just putting in the additional factors, $\left(\frac{r}{R}\right)^n$. Observe also that $f(\theta)$ needs to be defined only on $(0, 2\pi)$, according to (8.7), so that the requirement that $f(\theta)$ is 2π-periodic can now be removed.

Example 1 Solve

$$\begin{aligned} \Delta u = 0 \quad & \text{for } x^2 + y^2 < 4 \\ u = x^2 - 3y \quad & \text{on } x^2 + y^2 = 4 \,. \end{aligned}$$

Using that $x = 2\cos\theta$, and $y = 2\sin\theta$ on the boundary, write

$$f(\theta) = x^2 - 3y = 4\cos^2\theta - 6\sin\theta = 2 + 2\cos 2\theta - 6\sin\theta \,.$$

The last function is its own Fourier series, with $a_0 = 2$, $a_2 = 2$, $b_1 = -6$, and all other coefficients are zero. The formula (8.8) gives the solution (here $R = 2$):

$$u(r, \theta) = 2 + 2\left(\frac{r}{2}\right)^2 \cos 2\theta - 6\frac{r}{2}\sin\theta = 2 + \frac{1}{2}r^2\cos 2\theta - 3r\sin\theta \,.$$

In Cartesian coordinates this solution is (using that $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

$$u(x, y) = 2 + \frac{1}{2}(x^2 - y^2) - 3y \,.$$

Consider next the *exterior problem*

$$\begin{aligned} \Delta u = u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0 \,, & \quad \text{for } r > R \\ u(R, \theta) = f(\theta) \,. & \end{aligned} \tag{8.9}$$

Physically, we have a plate with the disc $r < R$ removed. Outside of this disc, the plate is so large, that we can think that it extends to infinity.

Temperatures are prescribed by $f(\theta)$ on the boundary of the plate, and the solution of (8.9) will give the steady state temperatures.

Perform separation of variables, following the same steps as above, and in (8.5) this time select $c_1 = 0$, to avoid infinite temperatures as $r \to \infty$. Conclusion: the solution of the exterior problem (8.9) is given by

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \ ,$$

with the coefficients taken from the Fourier series of $f(\theta)$, as given in (8.7). Again, going from the Fourier series of $f(\theta)$ to the solution of (8.9), involves just putting in the additional factors $\left(\frac{R}{r}\right)^n$.

Example 2 Solve the exterior problem ($R = 3$)

$$\begin{aligned} \Delta u = 0, &\quad \text{for } x^2 + y^2 > 9 \\ u = x^2 &\quad \text{on } x^2 + y^2 = 9\,. \end{aligned}$$

Writing $f(\theta) = x^2 = (3\cos\theta)^2 = 9\cos^2\theta = \frac{9}{2} + \frac{9}{2}\cos 2\theta$, obtain

$$u = \frac{9}{2} + \frac{9}{2}\left(\frac{3}{r}\right)^2 \cos 2\theta = \frac{9}{2} + \frac{81}{2}\frac{r^2\left(\cos^2\theta - \sin^2\theta\right)}{r^4} = \frac{9}{2} + \frac{81}{2}\frac{x^2 - y^2}{\left(x^2+y^2\right)^2}\,.$$

Consider next the *Neumann boundary value problem*

$$\text{(8.10)} \qquad \begin{aligned} \Delta u = u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0, &\quad \text{for } r < R \\ u_r(R,\theta) = f(\theta)\,. & \end{aligned}$$

It describes the steady state temperatures inside the disc $r < R$, with the heat flux $f(\theta)$ prescribed on the boundary of the disc. By separation of variables, obtain as above

$$\text{(8.11)} \qquad u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)\ .$$

The boundary condition requires

$$\text{(8.12)} \qquad u_r(R,\theta) = \sum_{n=1}^{\infty} nR^{n-1} (A_n \cos n\theta + B_n \sin n\theta) = f(\theta)\,.$$

This is impossible to arrange, unless the constant term is zero in the Fourier series of $f(\theta)$, which implies that

$$\int_0^{2\pi} f(\theta)\, d\theta = 0\,. \tag{8.13}$$

(Observe that the constant term is zero in the series on the left in (8.12). The same must be true for $f(\theta)$ on the right.) The condition (8.13) is a *necessary condition* for the Neumann problem to have solutions. If the condition (8.13) holds, we choose A_n and B_n to satisfy $nR^{n-1}A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$, and $nR^{n-1}B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$, while A_0 is *arbitrary*.

To recapitulate, the Neumann problem (8.10) is solvable only if the condition (8.13) holds, otherwise it has no solutions. If the condition (8.13) holds, the formula (8.11) gives infinitely many solutions ("feast-or-famine").

7.9 Sturm-Liouville Problems

Let us recall the eigenvalue problem

$$\begin{gathered} y'' + \lambda y = 0, \quad 0 < x < L \\ y(0) = y(L) = 0\,, \end{gathered}$$

on some interval $(0, L)$. Its eigenfunctions, $\sin \frac{n\pi}{L}x$, $n = 1, 2, \ldots$, are the building blocks of the Fourier sine series on $(0, L)$. These eigenfunctions are orthogonal on $(0, L)$, which means that $\int_0^L \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x\, dx = 0$ for any $m \neq n$. Similarly, the eigenfunctions of

$$\begin{gathered} y'' + \lambda y = 0, \quad 0 < x < L \\ y'(0) = y'(L) = 0\,, \end{gathered}$$

which are: 1, $\cos\frac{n\pi}{L}x$, $n = 1, 2, \ldots$, give rise to the Fourier cosine series on $(0, L)$. It turns out that under some conditions, solutions of other eigenvalue problems lead to their own types of Fourier series on $(0, L)$.

Recall that given a general linear second order equation

$$P(x)y'' + Q(x)y' + R(x)y = 0\,,$$

with $P(x) > 0$, one can divide this equation by $P(x)$, and then multiply by the integrating factor $p(x) = e^{\int \frac{Q(x)}{P(x)}\, dx}$, to put it into the *self-adjoint form*

$$\left(p(x)y'\right)' + r(x)y = 0\,,$$

with $r(x)=p(x)\frac{R(x)}{P(x)}$.

On an arbitrary interval (a,b), we consider an eigenvalue problem for equations in the self-adjoint form

$$(p(x)y')' + \lambda r(x)y = 0\,, \quad \text{for } a<x<b\,, \tag{9.1}$$

together with the homogeneous boundary conditions

$$\alpha y(a)+\beta y'(a)=0,\;\; \gamma y(b)+\delta y'(b)=0\,. \tag{9.2}$$

The differentiable function $p(x)$ and the continuous function $r(x)$ are given, and both are assumed to be positive on $[a,b]$. The boundary conditions in (9.2) are called *separated boundary conditions*, with the first one involving the values of the solution and its derivative only at $x=a$, and the other one uses the values only at the right end-point $x=b$. The constants α, β, γ and δ are given, however to prevent the possibility that both constants in the same boundary condition are zero, we assume that $\alpha^2+\beta^2\neq 0$, and $\gamma^2+\delta^2\neq 0$. By the *eigenfunctions* we mean non-trivial (non-zero) solutions of (9.1), satisfying the boundary conditions in (9.2), and the corresponding values of λ are called the *eigenvalues*.

Theorem 7.9.1 *Assume that $y(x)$ is an eigenfunction of the problem (9.1), (9.2), corresponding to an eigenvalue λ, while $z(x)$ is an eigenfunction corresponding to an eigenvalue μ, and $\lambda\neq\mu$. Then $y(x)$ and $z(x)$ are orthogonal on (a,b) with weight $r(x)$, which means that*

$$\int_a^b y(x)z(x)\,r(x)\,dx = 0\,.$$

Proof: The eigenfunction $z(x)$ satisfies

$$\begin{aligned} &(p(x)z')' + \mu r(x)z = 0 \\ &\alpha z(a)+\beta z'(a)=0,\;\; \gamma z(b)+\delta z'(b)=0\,. \end{aligned} \tag{9.3}$$

Multiply the equation (9.1) by $z(x)$, and subtract from that the equation (9.3) multiplied by $y(x)$. Obtain

$$(p(x)y')'\,z(x) - (p(x)z')'\,y(x) + (\lambda-\mu)r(x)y(x)z(x) = 0\,.$$

Rewrite this as

$$\left[p(y'z-yz')\right]' + (\lambda-\mu)r(x)y(x)z(x) = 0\,.$$

Integrating over $[a, b]$, gives

$$[p(y'z - yz')] \big|_a^b + (\lambda - \mu) \int_a^b y(x)z(x)\, r(x)\, dx = 0\,. \tag{9.4}$$

We shall show that

$$p(b)\left(y'(b)z(b) - y(b)z'(b)\right) = 0\,, \tag{9.5}$$

and

$$p(a)\left(y'(a)z(a) - y(a)z'(a)\right) = 0\,. \tag{9.6}$$

Then the first term in (9.4) is zero. It follows that the second term in (9.4) is also zero, and therefore $\int_a^b y(x)z(x)\, r(x)\, dx = 0$, because $\lambda - \mu \neq 0$.

We shall justify (9.5), while the proof of (9.6) is similar. Consider first the case when $\delta = 0$. Then the corresponding boundary conditions simplify to read $y(b) = 0$, $z(b) = 0$, and (9.5) follows. In the other case, when $\delta \neq 0$, we express $y'(b) = -\frac{\gamma}{\delta} y(b)$, $z'(b) = -\frac{\gamma}{\delta} z(b)$, and then

$$y'(b)z(b) - y(b)z'(b) = -\frac{\gamma}{\delta} y(b)z(b) + \frac{\gamma}{\delta} y(b)z(b) = 0\,,$$

and (9.5) follows, which concludes the proof. ♢

Theorem 7.9.2 *The eigenvalues of the problem (9.1), (9.2) are real numbers.*

Proof: Assume, on the contrary, that an eigenvalue λ is not real, so that $\bar{\lambda} \neq \lambda$, and the corresponding eigenfunction $y(x)$ is complex valued. Taking the complex conjugates of (9.1) and (9.2), gives

$$\left(p(x)\bar{y}'\right)' + \bar{\lambda} r(x) \bar{y} = 0$$

$$\alpha \bar{y}(a) + \beta \bar{y}'(a) = 0, \;\; \gamma \bar{y}(b) + \delta \bar{y}'(b) = 0\,.$$

It follows that $\bar{\lambda}$ is also an eigenvalue, and $\bar{y}$ is the corresponding eigenfunction. By the preceding Theorem 7.9.1

$$\int_a^b y(x)\bar{y}(x)\, r(x)\, dx = \int_a^b |y(x)|^2\, r(x)\, dx = 0\,.$$

The second integral involves a non-negative function, and it can be zero only if $y(x) = 0$, for all x. But an eigenfunction cannot be zero function. We have a contradiction, which was caused by the assumption that λ is not real. It follows that only real eigenvalues are possible. $\diamondsuit$

Example On the interval $(0, \pi)$, we consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad 0 < x < \pi \tag{9.7}$$
$$y(0) = 0, \quad y'(\pi) - y(\pi) = 0\,,$$

and calculate its eigenvalues and the corresponding eigenvectors.

The general solution depends on the sign of λ.

Case 1. $\lambda < 0$. We may write $\lambda = -k^2$, with $k > 0$. The general solution is then $y = c_1 e^{-kx} + c_2 e^{kx}$. Using the boundary conditions, compute $c_1 = c_2 = 0$, so that $y = 0$, and there are no negative eigenvalues.

Case 2. $\lambda = 0$. The general solution is $y = c_1 x + c_2$. Again, calculate $c_1 = c_2 = 0$, and $\lambda = 0$ is not an eigenvalue.

Case 3. $\lambda > 0$. We may write $\lambda = k^2$, with $k > 0$. The general solution is then $y = c_1 \cos kx + c_2 \sin kx$, and $c_1 = 0$ by the first boundary condition. The second boundary condition implies that

$$c_2\,(k \cos k\pi - \sin k\pi) = 0\,.$$

We need $c_2 \neq 0$, to get a non-trivial solution, therefore the quantity in the bracket must be zero, which implies that

$$\tan k\pi = k\,. \tag{9.8}$$

This equation has infinitely many solutions, $0 < k_1 < k_2 < k_3 < \ldots$, as can be seen by drawing the graphs of $y = k$ and $y = \tan k\pi$ in the ky-plane. We obtain infinitely many eigenvalues $\lambda_i = k_i^2$, and the corresponding eigenfunctions $y_i = \sin k_i x$, $i = 1, 2, 3, \ldots$. (Observe that $-k_i$'s are also solutions of (9.8), but they lead to the same eigenvalues and eigenfunctions.) Using that $\tan k_i\pi = k_i$, or $\sin k_i\pi = k_i \cos k_i\pi$, and recalling two trigonometric identities, we verify that for all $i \neq j$

$$\int_0^\pi \sin k_i x \sin k_j x \, dx = \frac{1}{2} \int_0^\pi \left[\cos(k_i - k_j)x - \cos(k_i + k_j)x\right] \, dx$$

$$= \frac{\sin(k_i - k_j)\pi}{2(k_i - k_j)} - \frac{\sin(k_i + k_j)\pi}{2(k_i + k_j)}$$

$$= \frac{1}{k_i^2 - k_j^2}\,(k_j \cos k_j\pi \sin k_i\pi - k_i \cos k_i\pi \sin k_j\pi) = 0\,,$$

proving directly that the eigenfunctions are orthogonal, which serves to illustrate the Theorem 7.9.1 above.

It is known that the problem (9.1), (9.2) has infinitely many eigenvalues, and the corresponding eigenfunctions $y_j(x)$ allow us to do Fourier series. This means that we can represent on the interval (a,b) any $f(x)$, for which $\int_a^b f^2(x)r(x)\,dx$ is finite, as

$$f(x) = \sum_{j=1}^{\infty} c_j y_j(x)\,, \quad \text{for } a < x < b\,. \tag{9.9}$$

One says that the *eigenfunctions $y_j(x)$ form a complete set.* To find the coefficients, multiply both sides by $y_i(x)r(x)$, and integrate over (a,b)

$$\int_a^b f(x)y_i(x)r(x)\,dx = \sum_{j=1}^{\infty} c_j \int_a^b y_j(x)y_i(x)r(x)\,dx\,.$$

By the Theorem 7.9.1, for all $j \neq i$, the integrals on the right are zero. So that the sum on the right is equal to $c_i \int_a^b y_i^2(x)r(x)\,dx$. Therefore

$$c_i = \frac{\int_a^b f(x)y_i(x)r(x)\,dx}{\int_a^b y_i^2(x)r(x)\,dx}\,. \tag{9.10}$$

For the example (9.7) considered above, the corresponding Fourier series takes the form

$$f(x) = \sum_{j=1}^{\infty} c_i \sin k_i x\,, \quad \text{for } 0 < x < \pi\,,$$

for any $f(x)$ satisfying $\int_0^\pi f^2(x)\,dx < \infty$, with the coefficients

$$c_i = \frac{\int_0^\pi f(x)\sin k_i x\,dx}{\int_0^\pi \sin^2 k_i x\,dx}\,.$$

Using a symbolic software, like *Mathematica*, it is easy to compute approximately k_i's, and the integrals for c_i's. The book of H. Weinberger [36] has more information on the validity of the expansion (9.9).

7.9.1 The Fourier-Bessel Series

Consider the following eigenvalue problem: on the interval $(0,R)$ determine non-trivial solutions $F = F(r)$ of

$$\begin{gathered} F'' + \tfrac{1}{r}F' + \lambda F = 0 \\ F'(0) = 0, \quad F(R) = 0\,. \end{gathered} \tag{9.11}$$

Rewrite this equation in the self-adjoint form (9.1) as

$$\left(rF'\right)' + \lambda r F = 0\,.$$

Then, by the Theorem 7.9.1, any two eigenfunctions of (9.11), corresponding to different eigenvalues, are orthogonal on $(0,R)$ with weight r.

We shall reduce the equation in (9.11) to Bessel's equation of order zero. To this end, make a change of variables $r \to x$, by letting $r = \frac{1}{\sqrt{\lambda}}x$. By the chain rule

$$F_r = F_x\sqrt{\lambda}, \;\; F_{rr} = F_{xx}\lambda\,.$$

Then the problem (9.11) becomes

$$\lambda F_{xx} + \frac{1}{\frac{1}{\sqrt{\lambda}}x}\sqrt{\lambda}F_x + \lambda F = 0$$
$$F_x(0) = 0, \;\; F(\sqrt{\lambda}R) = 0\,.$$

Divide by λ, and use primes again to denote the derivatives in x

$$F'' + \frac{1}{x}F_x + F = 0$$
$$F'(0) = 0, \;\; F(\sqrt{\lambda}R) = 0\,.$$

This equation is Bessel's equation of order zero. The Bessel function $J_0(x)$, which was considered in Chapter 3, satisfies this equation, as well as the condition $F'(0) = 0$. Recall that the function $J_0(x)$ has infinitely many positive roots $r_1 < r_2 < r_3 < \cdots$. In order to satisfy the second boundary condition, we need

$$\sqrt{\lambda}R = r_i\,, \quad i = 1, 2, 3, \ldots\,,$$

so that $\lambda = \lambda_i = \frac{r_i^2}{R^2}$. Returning to the original variable r (observe that $F(x) = F(\sqrt{\lambda}r)$), gives us the eigenvalues and the corresponding eigenfunctions of the problem (9.11):

$$\lambda_i = \frac{r_i^2}{R^2}, \;\; F_i(r) = J_0\left(\frac{r_i}{R}r\right), \quad i = 1, 2, 3, \ldots\,.$$

The Fourier-Bessel series is then the following expansion, using the eigenfunctions $J_0\left(\frac{r_i}{R}r\right)$,

$$f(r) = \sum_{j=1}^{\infty} c_i J_0\left(\frac{r_i}{R}r\right)\,, \quad \text{for } 0 < r < R\,, \tag{9.12}$$

and the coefficients c_i are given by (in view of (9.10))

$$c_i = \frac{\int_0^R f(r) J_0(\frac{r_i}{R} r) r\, dr}{\int_0^R J_0^2(\frac{r_i}{R} r) r\, dr} . \tag{9.13}$$

This expansion is valid for any $f(r)$, with $\int_0^R f^2(r) r\, dr$ finite. Using *Mathematica*, it is easy to compute c_i's numerically, and to work with the expansion (9.12).

7.9.2 Cooling of a Cylindrical Tank

It is known that the heat equation in three spacial dimensions is (see e.g., the book of H. Weinberger [36])

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right), \quad k > 0 \text{ is a constant}.$$

Suppose that we have a cylindrical tank $x^2 + y^2 \leq R^2$, $0 \leq z \leq H$, and the temperatures inside it are independent of z, so that $u = u(x, y, t)$. The heat equation then becomes

$$u_t = k\left(u_{xx} + u_{yy}\right) .$$

Assume also that the boundary of the cylinder is kept at zero temperature, while the initial temperatures, $u(x, y, 0)$, are prescribed to be $f(r)$, where r is the polar radius. Because the initial temperatures do not depend on the polar angle θ, it is natural to expect that $u = u(x, y, t)$ is independent of θ too, so that $u = u(r, t)$. Then the Laplacian becomes

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = u_{rr} + \frac{1}{r} u_r .$$

Under these assumptions, we need to solve the following problem

$$\begin{aligned} & u_t = k\left(u_{rr} + \tfrac{1}{r} u_r\right) \quad \text{for } 0 < r < R \text{, and } t > 0 \\ & u_r(0, t) = 0, \;\; u(R, t) = 0 \quad \text{for } t > 0 \\ & u(r, 0) = f(r) , \end{aligned} \tag{9.14}$$

with a given function $f(r)$. The condition $u_r(0, t) = 0$ was added, because we expect the temperatures to have a critical point in the middle of the tank, for all time t.

Use separation variables, writing $u(r, t) = F(r) G(t)$. Substitute this product into our equation, then divide both sides by $kF(r)G(t)$:

$$F(r) G'(t) = k\left(F''(r)) + \frac{1}{r} F'(r)\right) G(t) ,$$

$$\frac{G'(t)}{kG(t)} = \frac{F''(r)) + \frac{1}{r}F'(r)}{F(r)} = -\lambda\,,$$

which gives

$$F'' + \frac{1}{r}F' + \lambda F = 0 \qquad (9.15)$$
$$F'(0) = 0, \quad F(R) = 0\,,$$

and

$$\frac{G'(t)}{kG(t)} = -\lambda\,. \qquad (9.16)$$

The eigenvalue problem (9.15) for $F(r)$ was solved in the preceding section, giving the eigenvalues $\lambda_i = \frac{r_i^2}{R^2}$ and the corresponding eigenfunctions $F_i(r) = J_0(\frac{r_i}{R}r)$. Using $\lambda = \lambda_i = \frac{r_i^2}{R^2}$ in (9.16), compute

$$G_i(t) = c_i e^{-k\frac{r_i^2}{R^2}t}\,.$$

The function

$$u(r,t) = \sum_{i=1}^{\infty} c_i e^{-k\frac{r_i^2}{R^2}t} J_0(\frac{r_i}{R}r) \qquad (9.17)$$

satisfies our equation, and the boundary conditions. The initial condition

$$u(r,0) = \sum_{i=1}^{\infty} c_i J_0(\frac{r_i}{R}r) = f(r)$$

will hold, if we choose c_i's to be the coefficients of the Fourier-Bessel series, given by (9.13). Conclusion: the series in (9.17), with c_i's computed by (9.13), gives the solution to our problem.

7.9.3 Cooling of a Rectangular Bar

Consider a function of two variables $f(x,y)$ defined on a rectangle $0 < x < L$, $0 < y < M$. Regarding x as a primary variable, we may represent $f(x,y)$ by the Fourier sine series on $(0,L)$

$$f(x,y) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi}{L}x\,, \qquad (9.18)$$

where
$$f_n = \frac{2}{L} \int_0^L f(x, y) \sin \frac{n\pi}{L} x \, dx \, .$$
Clearly each f_n is a function of y, $f_n = f_n(y)$. We now represent $f_n(y)$ by Fourier sine series on $(0, M)$

$$f_n(y) = \sum_{m=1}^{\infty} b_{nm} \sin \frac{m\pi}{M} y \, , \tag{9.19}$$

with the constants

$$b_{nm} = \frac{2}{M} \int_0^M f_n(y) \sin \frac{m\pi}{M} y \, dy \, .$$

Using (9.19) in (9.18) we obtain the *double Fourier sine series*

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{M} y \, ,$$

where
$$b_{nm} = \frac{4}{LM} \int_0^M \int_0^L f(x, y) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{M} y \, dx \, dy \, . \tag{9.20}$$

Similarly one could develop the double cosine series, or mixed "sine-cosine" series.

Next, we solve the problem (for $u = u(x, y, t)$)

$$\begin{aligned} u_t &= k \left(u_{xx} + u_{yy} \right), \quad 0 < x < L \, , \; 0 < y < M \\ u(x, 0, t) &= u(x, M, t) = 0, \quad 0 < x < L \\ u(0, y, t) &= u(L, y, t) = 0, \quad 0 < y < M \\ u(x, y, 0) &= f(x, y) \, , \end{aligned}$$

describing cooling of a rectangular plate, with all four sides kept on ice (temperature zero), and with the initial temperatures prescribed by a given function $f(x, y)$.

Use separation of variables, setting $u(x, y, t) = F(x)G(y)H(t)$ in our equation, and then divide by $kF(x)G(y)H(t)$:

$$\frac{H'(t)}{kH(t)} = \frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = -\lambda \, . \tag{9.21}$$

The first of these relations gives

$$\frac{H'(t)}{kH(t)} = -\lambda \,. \tag{9.22}$$

In the second one, $\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = -\lambda$, we separate the variables further

$$\frac{F''(x)}{F(x)} = -\lambda - \frac{G''(y)}{G(y)} = -\mu \,,$$

which gives the familiar eigenvalue problems

$$\begin{gathered} F'' + \mu F = 0 \,, \quad 0 < x < L, \quad F(0) = F(L) = 0 \,, \\ G'' + (\lambda - \mu) G = 0 \,, \quad 0 < y < M, \quad G(0) = G(M) = 0 \,. \end{gathered}$$

It follows that $\mu_n = \frac{n^2\pi^2}{L^2}$, $F_n(x) = \sin \frac{n\pi}{L} x$, and then $\lambda_{nm} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2}$, $G_{nm}(y) = \sin \frac{m\pi}{M} y$. From (9.22), $H_{nm}(t) = e^{-k\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2}\right)t}$. The sum

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} e^{-k\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{M^2}\right)t} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{M} y \,,$$

with the coefficients b_{nm} computed by (9.20), gives the solution of our problem.

7.10 Green's Function

We wish to solve the non-homogeneous boundary value problem

$$\begin{gathered} (p(x)y')' + r(x)y = f(x), \quad a < x < b \\ y(a) = 0, \quad y(b) = 0 \,, \end{gathered} \tag{10.1}$$

where the equation is written in self-adjoint form. Here $p(x)$, $r(x)$ and $f(x)$ are given differentiable functions, and we assume that $p(x) > 0$, and $r(x) > 0$ on $[a, b]$. We shall also consider the corresponding homogeneous equation

$$(p(x)y')' + r(x)y = 0 \,, \tag{10.2}$$

and the corresponding homogeneous boundary value problem

$$\begin{gathered} (p(x)y')' + r(x)y = 0 \\ y(a) = 0, \quad y(b) = 0 \,. \end{gathered} \tag{10.3}$$

Recall the concept of the *Wronskian determinant* of two functions $y_1(x)$ and $y_2(x)$, or the *Wronskian*, for short:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)\,.$$

Lemma 7.10.1 *Let $y_1(x)$ and $y_2(x)$ be any two solutions of the homogeneous equation (10.2). Then $p(x)W(x)$ is a constant.*

Proof: We need to show that $(p(x)W(x))' = 0$. Compute

$$\begin{aligned}(p(x)W(x))' &= y_1'p(x)y_2' + y_1\left(p(x)y_2'\right)' - p(x)y_1'y_2' - \left(p(x)y_1'\right)' y_2 \\ &= y_1\left(p(x)y_2'\right)' - \left(p(x)y_1'\right)' y_2 = -r(x)y_1y_2 + r(x)y_1y_2 = 0\,.\end{aligned}$$

On the last step we expressed $(p(x)y_1')' = -r(x)y_1$, and $(p(x)y_2')' = -r(x)y_2$, by using the equation (10.2), which both y_1 and y_2 satisfy. ♢

We make the following fundamental assumption: *the homogeneous boundary value problem (10.3) has only the trivial solution* $y = 0$. Define $y_1(x)$ to be a non-trivial solution of the homogeneous equation (10.2), together with the condition $y(a) = 0$ (which can be computed e.g., by adding a second initial condition $y'(a) = 1$). By our fundamental assumption, $y_1(b) \neq 0$. Similarly, we define $y_2(x)$ to be a non-trivial solution of the homogeneous equation (10.2) together with the condition $y(b) = 0$. By the fundamental assumption, $y_2(a) \neq 0$. The functions $y_1(x)$ and $y_2(x)$ form a fundamental set of the homogeneous equation (10.2) (they are not constant multiples of one another). To find a solution of the non-homogeneous equation (10.1), we use the variation of parameters method, and look for solution in the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)\,, \tag{10.4}$$

with the functions $u_1(x)$ and $u_2(x)$ satisfying the formulas (10.6) and (10.7) below. We shall additionally require that these functions satisfy

$$u_1(b) = 0, \quad u_2(a) = 0\,. \tag{10.5}$$

Then $y(x)$ in (10.4) satisfies our boundary conditions $y(a) = y(b) = 0$, and gives the desired solution of the problem (10.1).

We put the equation (10.1) into the form considered in Chapter 2

$$y'' + \frac{p'(x)}{p(x)}y' + \frac{r(x)}{p(x)}y = \frac{f(x)}{p(x)}\,.$$

Then by the formulas (8.5) in Chapter 2, we have

$$u_1'(x) = -\frac{y_2(x)f(x)}{p(x)W(x)} = -\frac{y_2(x)f(x)}{K}\,, \tag{10.6}$$

$$u_2'(x) = \frac{y_1(x)f(x)}{p(x)W(x)} = \frac{y_1(x)f(x)}{K}\,, \tag{10.7}$$

where W is the Wronskian of $y_1(x)$ and $y_2(x)$, and by K we denote the constant that $p(x)W(x)$ is equal to, by Lemma 7.10.1.

Integrating (10.7), and using the condition $u_2(a) = 0$, we get

$$u_2(x) = \int_a^x \frac{y_1(\xi)f(\xi)}{K}\,d\xi\,.$$

Similarly, integrating (10.6), and using the condition $u_1(b) = 0$, gives

$$u_1(x) = \int_x^b \frac{y_2(\xi)f(\xi)}{K}\,d\xi\,.$$

Using these functions in (10.4), we get the solution of our problem (10.1)

$$y(x) = y_1(x)\int_x^b \frac{y_2(\xi)f(\xi)}{K}\,d\xi + y_2(x)\int_a^x \frac{y_1(\xi)f(\xi)}{K}\,d\xi\,. \tag{10.8}$$

It is customary to define *Green's function*

$$G(x,\xi) = \begin{cases} \frac{y_1(x)y_2(\xi)}{K} & \text{for } a \le x \le \xi \\ \frac{y_2(x)y_1(\xi)}{K} & \text{for } \xi \le x \le b\,, \end{cases} \tag{10.9}$$

so that the solution (10.8) can be written as

$$y(x) = \int_a^b G(x,\xi)f(\xi)d\xi\,. \tag{10.10}$$

(Break this integral as $\int_a^b = \int_a^x + \int_x^b$. In the first integral, $\xi \le x$, so that $G(x,\xi)$ is given by the second formula in (10.9).)

Example 1 Find Green's function, and the solution of the problem

$$y'' + y = f(x)\,, \quad \text{for } 0 < x < 1$$
$$y(0) = 0, \;\; y(1) = 0\,.$$

The function $\sin(x-a)$ solves the corresponding homogeneous equation $y''+y=0$, for any constant a. Therefore, we may take $y_1(x)=\sin x$, and $y_2(x)=\sin(x-1)$ (giving $y_1(0)=y_2(1)=0$). Compute

$$W=y_1(x)y_2'(x)-y_1'(x)y_2(x)=\sin x\cos(x-1)-\cos x\sin(x-1)=\sin 1\,,$$

which follows by using that $\cos(x-1)=\cos x\cos 1+\sin x\sin 1$, and $\sin(x-1)=\sin x\cos 1-\cos x\sin 1$. Then

$$G(x,\xi)=\begin{cases}\frac{\sin x\sin(\xi-1)}{\sin 1} & \text{for } 0\le x\le\xi\\ \frac{\sin(x-1)\sin\xi}{\sin 1} & \text{for } \xi\le x\le 1\,,\end{cases}$$

and the solution is

$$y(x)=\int_0^1 G(x,\xi)f(\xi)d\xi\,.$$

Example 2 Find Green's function, and the solution of the problem

$$x^2y''-2xy'+2y=f(x)\,,\quad \text{for } 1<x<2$$
$$y(1)=0,\;\; y(2)=0\,.$$

The corresponding homogeneous equation

$$x^2y''-2xy'+2y=0$$

is Euler's equation, with the general solution $y(x)=c_1x+c_2x^2$. We then find $y_1(x)=x-x^2$, and $y_2(x)=2x-x^2$ ($y_1(1)=y_2(2)=0$). Compute

$$W=y_1(x)y_2'(x)-y_1'(x)y_2(x)=(x-x^2)(2-2x)-(1-2x)(2x-x^2)=x^2\,.$$

Turning to the construction of $G(x,\xi)$, we observe that our equation is not in the self-adjoint form. To put it into the right form, divide the equation by x^2

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=\frac{f(x)}{x^2}\,,$$

and then multiply the new equation by the integrating factor, $\mu=e^{\int(-\frac{2}{x})\,dx}=e^{-2\ln x}=\dfrac{1}{x^2}$, obtaining

$$\left(\frac{1}{x^2}y'\right)'+\frac{2}{x^4}y=\frac{f(x)}{x^4}\,.$$

Here $p(x) = \dfrac{1}{x^2}$, and $K = p(x)W(x) = 1$. Then

$$G(x,\xi) = \begin{cases} (x - x^2)(2\xi - \xi^2) & \text{for } 1 \le x \le \xi \\ (2x - x^2)(\xi - \xi^2) & \text{for } \xi \le x \le 2\,, \end{cases}$$

and the solution is

$$y(x) = \int_1^2 G(x,\xi)\frac{f(\xi)}{\xi^4}d\xi\,.$$

Finally, we observe that the same construction works for general separated boundary conditions (9.2). If $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding homogeneous equation, satisfying the boundary conditions at $x = a$ and at $x = b$ respectively, then the formula (10.9) gives Green's function.

7.10.1 Problems

I. Find the complex form of the Fourier series for the following functions on the given interval.

1. $f(x) = x$ on $(-2, 2)$.

Answer. $x = \sum_{n=-\infty}^{\infty} \frac{2i(-1)^n}{n\pi}\, e^{i\frac{n\pi}{2}x}$.

2. $f(x) = e^x$ on $(-1, 1)$.

Answer. $e^x = \displaystyle\sum_{n=-\infty}^{\infty} (-1)^n \frac{(1 + in\pi)(e - \frac{1}{e})}{2(1 + n^2\pi^2)}\, e^{in\pi x}$.

3. $f(x) = \sin^2 x$ on $(-\pi, \pi)$.

Answer. $-\frac{1}{4}e^{-i2x} + \frac{1}{2} - \frac{1}{4}e^{i2x}$.

4. $f(x) = \sin 2x \cos 2x$ on $(-\pi/2, \pi/2)$.

Answer. $\frac{i}{4}e^{-i4x} - \frac{i}{4}e^{i4x}$.

5. Suppose a real valued function $f(x)$ is represented by its complex Fourier series on $(-L, L)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{n\pi}{L}x}\,.$$

(i) By taking the complex conjugates of both sides, show that $\bar{c}_n = c_{-n}$ for all n.

(ii) Multiply both sides by $e^{-i\frac{m\pi}{L}x}$, and integrate over $(-L, L)$, to conclude that

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx, \quad n = 0, \pm 1, \pm 2, \dots .$$

6. Let $u(t)$ be a differentiable function of period T, satisfying $\int_0^T u(t)\,dt = 0$. Prove the following *Poincare's inequality*

$$|u(t)| \le \frac{\sqrt{T}}{2\sqrt{3}}\left(\int_0^T u'^2(t)\,dt\right)^{\frac{1}{2}} \quad \text{for all } t\,.$$

Hint: Represent $u(t)$ by its complex Fourier series: $u(t) = \sum_{n\neq 0} c_n e^{i\frac{2\pi}{T}nt}$ (with $c_0 = 0$ by our condition). Then $u'(t) = \sum_{n\neq 0} i\frac{2\pi}{T} n c_n\, e^{i\frac{2\pi}{T}nt}$, and $\int_0^T u'^2(t)\,dt = \frac{4\pi^2}{T}\sum_{n\neq 0} n^2|c_n|^2$. We have (using that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$)

$$|u(t)| \le \sum_{n\neq 0}|c_n| \le \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n\neq 0} n^2|c_n|^2\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{3}}\frac{\sqrt{T}}{2\pi}\left(\int_0^T u'^2(t)\,dt\right)^{\frac{1}{2}}.$$

II. Solve the following problems on circular domains, and describe their physical significance.

1.
$$\Delta u = 0, \quad r < 3$$
$$u(3,\theta) = 4\cos^2\theta\,.$$

Answer. $u = 2 + \frac{2}{9}r^2\cos 2\theta = 2 + \frac{2}{9}(x^2 - y^2)$.

2.
$$\Delta u = 0, \quad r > 3$$
$$u(3,\theta) = 4\cos^2\theta\,.$$

Answer. $u = 2 + \frac{18}{r^2}\cos 2\theta = 2 + 18\frac{x^2 - y^2}{(x^2+y^2)^2}$.

3.
$$\Delta u = 0, \quad r < 2$$
$$u(2,\theta) = y^2\,.$$

Answer. $u = 2 - \frac{1}{2}(x^2 - y^2)$.

4.
$$\Delta u = 0, \quad r > 2$$
$$u(2,\theta) = y^2\,.$$

Answer. $u = 2 - 8\dfrac{x^2-y^2}{(x^2+y^2)^2}$.

5.
$$\Delta u = 0, \quad r < 1$$
$$u(1,\theta) = \cos^4\theta\,.$$

Hint: $\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2$.

6.
$$\Delta u = 0, \quad r < 1$$
$$u(1,\theta) = \theta\,.$$

Hint: Extend $f(\theta) = \theta$ as a 2π periodic function, equal to θ on $[0, 2\pi]$. Then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\int_0^{2\pi} \theta\cos n\theta d\theta\,,$$

and compute similarly a_0, and b_n's.

Answer. $u = \pi - \sum_{n=1}^{\infty} \dfrac{2}{n} r^n \sin n\theta$.

7. Solve the exterior problem

$$\Delta u = 0, \quad r > 3$$
$$u(3,\theta) = \theta + 2\,.$$

Answer. $u = \pi + 2 - 2\sum_{n=1}^{\infty} \dfrac{3^n}{n} r^{-n} \sin n\theta$.

8. Solve the problem, and write the answer in the Cartesian coordinates

$$u_{xx} + u_{yy} = 0 \quad \text{inside } r < 2,$$
$$u = x^2 - y \quad \text{on } r = 2\,.$$

Answer. $u(x,y) = 1 + \frac{1}{4}\left(x^2 - y^2\right) + y$.

9. Find the steady state temperatures inside the disc $x^2 + y^2 < 9$, if the temperatures on its boundary are prescribed by the function $y^2 - x$.

Answer. $u(x,y) = \dfrac{9}{2} - x - \dfrac{1}{2}(x^2 - y^2)$.

10. Determine if the following Neumann problem is solvable, and if it is, find its solutions

$$\Delta u = 0, \quad r < 3$$
$$u_r(3,\theta) = \sin\theta\cos\theta - 2\sin 3\theta\,.$$

Answer. $u = \dfrac{1}{12}r^2\sin 2\theta - \dfrac{2}{27}r^3\sin 3\theta + c$.

11. Determine if the following Neumann problem is solvable, and if it is, find its solutions

$$\Delta u = 0, \quad r < 1$$
$$u_r(1,\theta) = \sin^2\theta - 2\sin 3\theta\,.$$

III. 1. Find the eigenvalues and the eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \;\; y(\pi) + y'(\pi) = 0\,.$$

Answer. $\lambda_n = n^2$, $y_n = \sin nx - n\cos nx$, and also $\lambda = -1$ with $y = e^{-x}$.

2. Identify graphically the eigenvalues, and find the eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \;\; y(\pi) = 0\,.$$

3. (i) Find the eigenvalues and the eigenfunctions of (a is a constant)

$$y'' + ay' + \lambda y = 0, \quad y(0) = y(L) = 0\,.$$

Answer. $\lambda_n = \frac{a^2}{4} + \frac{n^2\pi^2}{L^2}$, $y_n(x) = e^{-\frac{a}{2}x}\sin\frac{n\pi}{L}x$.

(ii) Use separation of variables to solve

$$u_t = u_{xx} + au_x, \;\; 0 < x < L$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)\,.$$

Answer. $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{a^2}{4}+\frac{n^2\pi^2}{L^2}\right)t} e^{-\frac{a}{2}x}\sin\frac{n\pi}{L}x$, $b_n = \dfrac{\int_0^L f(x)y_n(x)\,dx}{\int_0^L y_n^2(x)\,dx}$.

4. (i) Find the eigenvalues and the eigenfunctions of

$$x^2y'' + 3xy' + \lambda y = 0, \quad y(1) = y(e) = 0\,.$$

Answer. $\lambda_n = 1 + n^2\pi^2$, $y_n(x) = x^{-1}\sin(n\pi \ln x)$.

(ii) Put this equation into the self-adjoint form $(p(x)y')' + \lambda r(x)y = 0$, and verify that the eigenfunctions are orthogonal with weight $r(x)$.

Hint: Divide the equation by x^2, and verify that x^3 is the integrating factor, so that $p(x) = x^3$ and $r(x) = x$.

(iii) Use separation of variables to solve

$$\begin{aligned} u_t &= x^2 u_{xx} + 3x u_x\,, \quad 1 < x < e \\ u(1,t) &= u(e,t) = 0 \\ u(x,0) &= f(x)\,. \end{aligned}$$

Answer. $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} y_n(x)$, $b_n = \frac{\int_1^e f(x) y_n(x)\, x\, dx}{\int_1^e y_n^2(x)\, x\, dx}$.

5. Show that the eigenfunctions of

$$y'''' + \lambda y = 0, \quad y(0) = y'(0) = y(L) = y'(L) = 0$$

corresponding to different eigenvalues, are orthogonal on $(0, L)$.

Hint:

$$y''''z - yz'''' = \frac{d}{dx}\left(y'''z - y''z' + y'z'' - yz'''\right)\,.$$

6. Consider an eigenvalue problem

$$(p(x)y')' + \lambda r(x)y = 0, \qquad \alpha y(0) + \beta y'(0) = 0, \quad \gamma y(\pi) + \delta y'(\pi) = 0\,.$$

Assume that the given functions $p(x)$ and $r(x)$ are positive, while α and β are non-zero constants of different sign, and γ and δ are non-zero constants of the same sign. Show that all eigenvalues are positive.

Hint: Multiply the equation by $y(x)$ and integrate over $(0, \pi)$. Perform an integration by parts.

7. Find the eigenvalues and the eigenfunctions of $(u = u(r))$

$$u''(r) + \frac{2}{r}u'(r) + \lambda u(r) = 0\,, \quad 0 < r < \pi\,, \quad u'(0) = u(\pi) = 0\,.$$

Hint: Write the equation in the form $(ru)'' + \lambda\,(ru) = 0$.

Answer. $\lambda_m = m^2$, $y_m = \frac{\sin mr}{r}$, $m = 1, 2, 3, \ldots$.

8. Find the eigenvalues and the eigenfunctions of

$$u''(r) + \frac{n-1}{r}u'(r) + \lambda u(r) = 0\,, \quad 0 < r < 1\,, \quad u'(0) = u(1) = 0\,,$$

where n is a positive integer.

Hint: The change of variables $u(r) = r^{\frac{2-n}{2}}v(r)$ transforms this equation into Bessel's equation of order $\frac{n-2}{2}$, with solution $v(r) = J_{\frac{n-2}{2}}\left(\sqrt{\lambda}\,r\right)$.

Answer. $u_m(r) = r^{\frac{2-n}{2}}J_{\frac{n-2}{2}}\left(\lambda_{\frac{n-2}{2},m}\,r\right)$, $\lambda_m = \lambda^2_{\frac{n-2}{2},m}$, $m = 1, 2, 3, \ldots$, where $\lambda_{\frac{n-2}{2},m}$ denotes the m-th root of $J_{\frac{n-2}{2}}(r)$.

9. Find the eigenvalues and the eigenfunctions of ($F = F(t)$, α is a constant)

$$F'' + \frac{1}{t}F' + \lambda t^{2\alpha}F = 0\,, \quad 0 < t < 1\,, \quad F'(0) = F(1) = 0\,.$$

Hint: Show that the change of variables $r = \frac{t^{\alpha+1}}{\alpha+1}$ transforms this problem into (9.11), with $R = \frac{1}{\alpha+1}$.

Answer. $\lambda_i = (\alpha+1)^2 r_i$, $F_i = J_0\left(r_i t^{\alpha+1}\right)$, where r_i are the roots of J_0.

10. Solve

$$\begin{aligned}
u_t &= 3\left(u_{xx} + u_{yy}\right), \quad 0 < x < \pi\,,\ 0 < y < \pi \\
u(x,0,t) &= u(x,\pi,t) = 0, \quad 0 < x < \pi \\
u(0,y,t) &= u(\pi,y,t) = 0, \quad 0 < y < \pi \\
u(x,y,0) &= \sin x \cos x \sin y\,.
\end{aligned}$$

Answer. $u(x,y,t) = \frac{1}{2}e^{-15t}\sin 2x \sin y$.

11. (i) Solve

$$\begin{aligned}
u_t &= u_{xx} + u_{yy}, \quad 0 < x < 3\,,\ 0 < y < 2 \\
u(x,0,t) &= u(x,2,t) = 0, \quad 0 < x < 3 \\
u(0,y,t) &= u(3,y,t) = 0, \quad 0 < y < 2 \\
u(x,y,0) &= xy - y\,.
\end{aligned}$$

Answer.

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \frac{8(-1)^m + 16(-1)^{n+m}}{nm\pi^2}\, e^{-\left(\frac{n^2\pi^2}{9}+\frac{m^2\pi^2}{4}\right)t} \sin\frac{n\pi}{3}x \sin\frac{m\pi}{2}y\,.$$

(ii) Find the eigenvalues and the corresponding eigenfunctions of the Laplacian on the rectangle $(0,3)\times(0,2)$

$$u_{xx}+u_{yy}+\lambda u=0,\;\; 0<x<3\,,\; 0<y<2$$
$$u(x,0,t)=u(x,2,t)=0,\;\; 0<x<3$$
$$u(0,y,t)=u(3,y,t)=0,\;\; 0<y<2\,.$$

Answer. $\lambda_{mn}=\frac{n^2\pi^2}{9}+\frac{m^2\pi^2}{4}$, $u_{nm}(x,y)=\sin\frac{n\pi}{3}x\sin\frac{m\pi}{2}y$ $(m,n=1,2,\ldots)$.

IV. Find Green's function and the solution of the following problems.

1.
$$y''+y=f(x) \qquad a<x<b$$
$$y(a)=0,\;\; y(b)=0\,.$$

Answer. $G(x,\xi)=\begin{cases}\frac{\sin(x-a)\sin(\xi-b)}{\sin(b-a)} & \text{for } a\le x\le\xi\\ \frac{\sin(x-b)\sin(\xi-a)}{\sin(b-a)} & \text{for } \xi\le x\le b\,.\end{cases}$

2.
$$y''+y=f(x) \qquad 0<x<2$$
$$y(0)=0,\;\; y'(2)+y(2)=0\,.$$

Hint: $y_1(x)=\sin x$, $y_2(x)=-\sin(x-2)+\cos(x-2)$.

3.
$$x^2y''+4xy+2y=f(x) \qquad 1<x<2$$
$$y(1)=0,\;\; y(2)=0\,.$$

Answer. $G(x,\xi)=\begin{cases}(x^{-1}-x^{-2})(\xi^{-1}-2\xi^{-2}) & \text{for } 1\le x\le\xi\\ (\xi^{-1}-\xi^{-2})(x^{-1}-2x^{-2}) & \text{for } \xi\le x\le 2\,,\end{cases}$

$y(x)=\int_1^2 G(x,\xi)\xi^2 f(\xi)\,d\xi$.

7.11 The Fourier Transform

This section develops the concept of the Fourier Transform, a very important tool for both theoretical and applied PDE. Applications are made to physically significant problems on infinite domains.

Recall the complex form of the Fourier series. A function $f(x)$, defined on $(-L,L)$, can be represented by the series

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{i\frac{n\pi}{L}x}\,, \tag{11.1}$$

with the coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{-i\frac{n\pi}{L}\xi}\,d\xi, \quad n = 0, \pm 1, \pm 2, \dots . \tag{11.2}$$

We substitute (11.2) into (11.1):

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{i\frac{n\pi}{L}(x-\xi)}\,d\xi\,. \tag{11.3}$$

Now assume that the interval $(-\infty, \infty)$ along some axis, which we call the s-axis, is subdivided into pieces, using the subdivision points $s_n = \frac{n\pi}{L}$. The length of each interval is $\Delta s = \frac{\pi}{L}$. We rewrite (11.3) as

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \int_{-L}^{L} f(\xi)e^{is_n(x-\xi)}\,d\xi\Delta s\,. \tag{11.4}$$

so that we can regard $f(x)$ as a Riemann sum of a certain function of s, over the interval $(-\infty, \infty)$. Let now $L \to \infty$. Then $\Delta s \to 0$, and the Riemann sum in (11.4) converges to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{is(x-\xi)}\,d\xi ds\,.$$

This formula is known as the *Fourier integral*. Our derivation of it is made rigorous in more advanced books. Rewrite this integral as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{-is\xi}\,d\xi\right) ds\,. \tag{11.5}$$

We define the *Fourier transform* by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{-is\xi}\,d\xi\,.$$

The *inverse Fourier transform* is then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}F(s)\,ds\,.$$

As with the Laplace transform, we use capital letters to denote the Fourier transforms. We shall also use the *operator notation* $\mathcal{F}(f(x)) = F(s)$.

Example Let $f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a\,. \end{cases}$

Using Euler's formula, we compute the Fourier transform:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-is\xi}\, d\xi = \frac{2}{\sqrt{2\pi}} \frac{e^{ias} - e^{-ias}}{2is} = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} .$$

Assume that $f(x) \to 0$, as $x \to \pm\infty$. Integrating by parts

$$\mathcal{F}\left(f'(x)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(\xi) e^{-is\xi}\, d\xi = \frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-is\xi}\, d\xi = isF(s) .$$

(The boundary term $\frac{1}{\sqrt{2\pi}} f(\xi) e^{-is\xi}\, |_{-\infty}^{\infty}$ is zero, because $|e^{-is\xi}| = 1$.) It follows that

$$\mathcal{F}\left(f''(x)\right) = is\mathcal{F}\left(f'(x)\right) = -s^2 F(s) . \tag{11.6}$$

These formulas for $\mathcal{F}\left(f'(x)\right)$ and $\mathcal{F}\left(f''(x)\right)$ are similar to the corresponding formulas for the Laplace transform.

7.12 Problems on Infinite Domains

7.12.1 Evaluation of Some Integrals

The following integral occurs often

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} . \tag{12.1}$$

We justify this formula as follows:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, drd\theta = \pi ,$$

and (12.1) follows. We used polar coordinates to evaluate the double integral.

We shall show that for any x

$$y(x) \equiv \int_0^{\infty} e^{-z^2} \cos xz\, dz = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{4}} . \tag{12.2}$$

Using integration by parts, compute (d denotes the differential)

$$y'(x) = \int_0^{\infty} e^{-z^2} \left(-z \sin xz\right)\, dz = \frac{1}{2} \int_0^{\infty} \sin xz\, d\left(e^{-z^2}\right)$$

$$= -\frac{x}{2}\int_0^\infty e^{-z^2}\cos xz\,dz\,,$$

which implies that

$$y'(x) = -\frac{x}{2}y(x)\,. \tag{12.3}$$

By (12.1)

$$y(0) = \frac{\sqrt{\pi}}{2}\,. \tag{12.4}$$

Solving the differential equation (12.3), together with the initial condition (12.4), justifies the formula (12.2).

The last integral we need, is just a Laplace transform (a is a constant)

$$\int_0^\infty e^{-sy}\cos as\,ds = \frac{y}{y^2+a^2}\,. \tag{12.5}$$

7.12.2 The Heat Equation for $-\infty < x < \infty$

We shall solve the initial value problem

$$\begin{aligned} u_t &= ku_{xx} \qquad -\infty < x < \infty,\ \ t > 0 \\ u(x,0) &= f(x) \qquad -\infty < x < \infty\,. \end{aligned} \tag{12.6}$$

Here $u(x,t)$ gives the temperature at a point x, and time t, for an infinite bar. (The bar is very long, so that we assume it to be infinite.) The initial temperatures are prescribed by the given function $f(x)$, and $k > 0$ is a given constant.

The Fourier transform of the solution

$$\mathcal{F}(u(x,t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty u(x,t)e^{-isx}\,dx = U(s,t) \tag{12.7}$$

depends on s and also on t, which we may regard as a parameter, at the moment. Observe that $\mathcal{F}(u_t(x,t)) = U_t(s,t)$, as follows by differentiating (12.7) in t. Applying the Fourier transform to the problem (12.6), and using (11.6), obtain

$$\begin{gathered} U_t = -ks^2U \\ U(s,0) = \mathcal{F}(u(x,0)) = \mathcal{F}(f(x)) = F(s)\,, \end{gathered}$$

where $F(s)$ is the Fourier transform of $f(x)$. Integrating this initial value problem for U as a function of t (we now regard s as a parameter)

$$U(s,t) = F(s)e^{-ks^2 t}\,.$$

To obtain the solution of (12.6), we apply the inverse Fourier transform, and get

$$\begin{aligned} u(x,t) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} F(s) e^{-ks^2 t}\, ds = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{is(x-\xi)-ks^2 t} f(\xi)\, d\xi ds \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{is(x-\xi)-ks^2 t}\, ds \right) f(\xi)\, d\xi\,, \end{aligned}$$

after switching the order of integration. We denote by K the integral in the brackets, $K = \int_{-\infty}^{\infty} e^{is(x-\xi)-ks^2 t}\, ds$. To evaluate K, we use Euler's formula

$$K = \int_{-\infty}^{\infty} \left[\cos s(x-\xi) + i \sin s(x-\xi)\right] e^{-ks^2 t}\, ds = 2 \int_0^{\infty} \cos s(x-\xi) e^{-ks^2 t}\, ds\,,$$

because $\cos s(x-\xi)$ is an even function of s, and $\sin s(x-\xi)$ is an odd function of s. To evaluate the last integral, we make a change of variables $s \to z$, by setting

$$\sqrt{kt}\, s = z\,,$$

and then use the integral in (12.2):

$$K = \frac{2}{\sqrt{kt}} \int_0^{\infty} e^{-z^2} \cos\left(\frac{x-\xi}{\sqrt{kt}}\, z \right) dz = \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{(x-\xi)^2}{4kt}}\,.$$

With K evaluated, we get the solution of (12.6):

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)\, d\xi\,. \tag{12.8}$$

This formula is important for both PDE's and Probability Theory. The function $K(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$ is known as the *heat kernel*. (Recall from Chapter 4 that one may write $u(x,t) = K(x,t) * f(x)$, with $\lim_{t\to 0} K(x,t) = \delta(x)$, and $\lim_{t \to 0} u(x,t) = f(x)$, where $*$ denotes the convolution.)

Assume now that the function $f(x)$, giving the initial temperatures, is positive on some small interval $(-\epsilon, \epsilon)$, and is zero outside of this interval. Then $u(x,t) > 0$ for all $x \in (-\infty, \infty)$ and $t > 0$. Not only the temperatures become positive far from the heat source, this happens practically instantaneously! This is known as the *infinite propagation speed*, which points to an imperfection of our model. Observe, however, that for this $f(x)$, the temperatures given by (12.8) are negligible for large $|x|$.

7.12.3 Steady State Temperatures for the Upper Half Plane

We shall solve the boundary value problem

$$(12.9)\qquad \begin{aligned} u_{xx} + u_{yy} &= 0 \qquad -\infty < x < \infty, \;\; y > 0 \\ u(x,0) &= f(x) \qquad -\infty < x < \infty \,. \end{aligned}$$

Here $u(x,y)$ will provide the steady state temperature, at a point (x,y) of an infinite plate, occupying the upper half of the xy-plane. The given function $f(x)$ prescribes the temperatures at the boundary $y = 0$ of the plate. We looking for the solution that is bounded, as $y \to \infty$. (Without this assumption the solution is not unique: if $u(x,y)$ is a solution of (12.9), then so is $u(x,y) + cy$, for any constant c.)

Applying the Fourier transform in x, $U(s,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\xi,y) e^{-is\xi}\, d\xi$, gives (observe that $\mathcal{F}(u_{yy}(x,t)) = U_{yy}(s,t)$)

$$(12.10)\qquad \begin{aligned} U_{yy} - s^2 U &= 0 \\ U(s,0) &= F(s)\,. \end{aligned}$$

The general solution of the equation in (12.10) is

$$U(s,y) = c_1 e^{-sy} + c_2 e^{sy}\,.$$

When $s > 0$, we select $c_2 = 0$, so that U (and therefore u) is bounded as $y \to \infty$. Then $c_1 = F(s)$, from the initial condition in (12.10), giving us

$$U(s,y) = F(s) e^{-sy}, \;\; \text{when } s > 0\,.$$

When $s < 0$, we select $c_1 = 0$, to get a bounded solution. Then $c_2 = F(s)$, giving us

$$U(s,y) = F(s) e^{sy}, \;\; \text{when } s < 0\,.$$

Combining both cases, we conclude that the bounded solution of (12.10) is

$$U(s,y) = F(s) e^{-|s|y}\,.$$

It remains to compute the inverse Fourier transform

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx-|s|y} F(s)\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \left(\int_{-\infty}^{\infty} e^{is(x-\xi)-|s|y}\, ds \right) d\xi\,,$$

after switching the order of integration. We evaluate the integral in the brackets by using Euler's formula, the fact that $\cos s(x-\xi)$ is even in s, and $\sin s(x-\xi)$ is odd in s, and (on the last step) the formula (12.5):

$$\int_{-\infty}^{\infty} e^{is(x-\xi)-|s|y}\, ds = 2\int_0^{\infty} e^{-sy} \cos s(x-\xi)\, ds = \frac{2y}{(x-\xi)^2 + y^2}\,.$$

The solution of (12.9), known as *Poisson's formula*, is then

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{(x-\xi)^2+y^2}\,.$$

This integral converges, provided that $f(\xi)$ does not grow too fast as $\xi \to \pm\infty$.

7.12.4 Using the Laplace Transform for a Semi-Infinite String

Imagine a string extending for $0 < x < \infty$, which is initially at rest. Its left end-point, $x = 0$, undergoes periodic vibrations, with the displacements given by $A \sin \omega t$, where A and ω are constants. We wish to find the displacements $u(x,t)$ at any point $x > 0$ and time $t > 0$, assuming that the displacements are bounded.

We need to solve the following *initial-boundary value problem* for the wave equation, with a given $c > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}\,, \qquad x > 0 \\ u(x,0) &= u_t(x,0) = 0\,, \qquad x > 0 \\ u(0,t) &= A \sin \omega t\,, \qquad t > 0\,. \end{aligned}$$

Take the Laplace transform of the equation in the variable t, denoting $U(x,s) = \mathcal{L}\left(u(x,t)\right)$. Using the initial and boundary conditions, we get

$$\begin{aligned} s^2 U &= c^2 U_{xx} \\ U(0,s) &= \frac{A\omega}{s^2+\omega^2}\,. \end{aligned} \tag{12.11}$$

The general solution of the equation in (12.11) is

$$U(x,s) = c_1 e^{\frac{s}{c}x} + c_2 e^{-\frac{s}{c}x}\,.$$

To get a solution bounded as $x \to +\infty$, we select $c_1 = 0$. Then $c_2 = \frac{A\omega}{s^2+\omega^2}$ by the initial condition in (12.11), giving

$$U(x,s) = e^{-\frac{x}{c}s}\frac{A\omega}{s^2+\omega^2}\,.$$

Taking the inverse Laplace transform, and using the formula (2.5) from Chapter 4, gives the solution

$$u(x,t) = A u_{x/c}(t) \sin \omega(t - x/c)\,,$$

where $u_{x/c}(t)$ is the Heaviside step function. This formula shows that at any point $x > 0$, the solution is zero for $0 < t < x/c$ (the time it takes for the signal to travel from 0 to x). For $t > x/c$, the motion of the string at x is identical with the motion at $x = 0$, but is *delayed in time*, by the value of x/c.

7.12.5 Problems

1. Find the Fourier transform of the function $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$.

Answer. $F(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2}(1 - \cos s)$.

2. Find the Fourier transform of $f(x) = e^{-|x|}$.

Answer. $F(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2 + 1}$.

3. Find the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$.

Answer. $F(s) = e^{-\frac{s^2}{2}}$. (Hint: Use the formula (12.2).)

4. Find the Fourier transform of $f(x) = e^{-ax^2}$, where $a > 0$ is a constant.

Answer. $F(s) = \frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}$.

5. Solve the heat conduction problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad -\infty < x < \infty, \ \ t > 0 \\ u(x, 0) &= e^{-x^2} \quad -\infty < x < \infty\,. \end{aligned}$$

Answer. $u(x,t) = \frac{1}{\sqrt{1 + 4t}} e^{-\frac{x^2}{1+4t}}$.

6. Show that for any constant a

(i) $\mathcal{F}(f(x)e^{iax}) = F(s - a)$.

(ii) $\mathcal{F}(f(ax)) = \frac{1}{a} F(\frac{s}{a}) \quad (a \neq 0)$.

7. Find a non-trivial solution of the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad -\infty < x < \infty, \ \ y > 0 \\ u(x, 0) &= 0 \quad -\infty < x < \infty\,. \end{aligned}$$

Hint: Assume that u depends only on y.

This example shows that our physical intuition may fail for unbounded domains.

8. Use Poisson's formula to solve

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty, \ \ y > 0$$
$$u(x,0) = f(x) \quad -\infty < x < \infty \,,$$

where $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$.

Answer. $u(x,y) = \dfrac{1}{\pi}\left(\tan^{-1}\dfrac{x+1}{y} - \tan^{-1}\dfrac{x-1}{y}\right)$.

9. The famous *Black-Scholes equation* for the price of a stock option is (here $V = V(S,t)$)

$$V_t + aS^2 V_{SS} + bSV_S - rV = 0 \,,$$

where a, b and r are positive constants. By a change of variables, reduce this equation to the heat equation.

Hint: If the V_t term was not present, we would have Euler's equation. This suggests to set $x = \ln s$. Then let $\tau = -at$. Obtain:

$$V_\tau = V_{xx} + 2\alpha V_x - \frac{r}{a} V \,,$$

where we denoted $2\alpha = b/a - 1$. Multiply the last equation by $e^{\alpha x}$, and denote $w = e^{\alpha x} V$. Obtain:

$$w_\tau = w_{xx} - \left(\alpha^2 + \frac{r}{a}\right) w \,.$$

Finally, multiply this equation by the integrating factor $e^{\left(\alpha^2 + \frac{r}{a}\right)\tau}$, and denote $z = e^{\left(\alpha^2 + \frac{r}{a}\right)\tau} w$. Conclude:

$$z_\tau = z_{xx} \,.$$

Chapter 8

Elementary Theory of PDE

This chapter continues the study of the three main equations of mathematical physics: wave, heat and Laplace's equations. We now deal with the theoretical aspects: propagation and reflection of waves, maximum principles, harmonic functions, Poisson's integral formulas, variational approach. Classification theory is presented, and it shows that the three main equations are representative of all linear second order equations. First order PDE's are solved by reducing them to ODE's along the characteristic lines.

8.1 Wave Equation: Vibrations of an Infinite String

Waves

The graph of $y = (x-1)^2$ is a translation of the parabola $y = x^2$, by one unit to the right. The graph of $y = (x-t)^2$ is a translation of the same parabola by t units. If we think of t as time, and draw these translations on the same screen, we get a *wave* of speed one, traveling to the right. Similarly, $y = (x-ct)^2$ is a wave of speed c. The same reasoning applies for other functions. So that $y = f(x-ct)$ is *a wave of speed c traveling to the right*, while $y = f(x+ct)$ describes *a wave of speed c traveling to the left.*

Transverse Vibrations of a Guitar String: d'Alembert's Formula

Assume that an elastic string extends along the x-axis, for $-\infty < x < \infty$, and we wish to find its transverse displacements $u = u(x,t)$, as a function of the position x and the time t. As in Chapter 7, we need to solve the wave equation, together with the initial conditions:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } -\infty < x < \infty, \text{ and } t > 0 \tag{1.1}$$

$$u(x,0) = f(x) \qquad \text{for } -\infty < x < \infty$$
$$u_t(x,0) = g(x) \qquad \text{for } -\infty < x < \infty\,.$$

Here $f(x)$ is given initial displacement, $g(x)$ is given initial velocity, and $c > 0$ is a given constant.

We look for *classical solutions*, which means that $u(x,t)$ has two continuous derivatives in x and t (u_x, u_t, u_{xx}, u_{xt}, and u_{tt} are continuous). We perform a change of variables $(x,t) \to (\xi,\eta)$, with the new variables (ξ,η) given by

$$\xi = x - ct$$
$$\eta = x + ct\,.$$

Compute the partial derivatives: $\xi_x = 1$, $\eta_x = 1$, $\xi_t = -c$, and $\eta_t = c$. We may think of solution as $u(x,t) = u(\xi(x,t), \eta(x,t))$. Using the chain rule, we express

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta\,,$$

$$u_{xx} = (u_x)_x = (u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

(using that $u_{\eta\xi} = u_{\xi\eta}$). Similarly

$$u_t = u_\xi \xi_t + u_\eta \eta_t = -cu_\xi + cu_\eta\,,$$

$$u_{tt} = -c\left(-cu_{\xi\xi} + cu_{\eta\xi}\right) + c\left(-cu_{\xi\eta} + cu_{\eta\eta}\right) = c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right)\,.$$

Substituting these expressions of u_{tt} and u_{xx} into the wave equation, and simplifying, we get

$$u_{\xi\eta} = 0\,.$$

Since $(u_\xi)_\eta = 0$, integration in η gives

$$u_\xi = F(\xi)\,,$$

where $F(\xi)$ is an arbitrary function. Integrating once more

$$u(\xi,\eta) = \int F(\xi)d\xi + G(\eta) = F(\xi) + G(\eta)\,.$$

Here $G(\eta)$ is an arbitrary function of η. The antiderivative of $F(\xi)$ is an arbitrary function, which we again denote by $F(\xi)$. Returning to the original variables, we have the general solution

$$u(x,t) = F(x - ct) + G(x + ct)\,. \tag{1.2}$$

Observe that the wave equation has an *overwhelmingly large number of solutions*, because its general solution depends on *two arbitrary functions.*

Turning to the initial conditions, compute

$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)\,.$$

We have

$$\begin{aligned} u(x,0) &= F(x) + G(x) = f(x) \\ u_t(x,0) &= -cF'(x) + cG'(x) = g(x)\,. \end{aligned} \tag{1.3}$$

Integrating the second equation in (1.3) gives

$$-cF(x) + cG(x) = \int_\gamma^x g(\tau)d\tau\,,$$

where γ is any constant. Adding to this formula the first equation in (1.3), multiplied by c, produces

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_\gamma^x g(\tau)d\tau\,.$$

From the first equation in (1.3)

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_\gamma^x g(\tau)d\tau\,.$$

Using these expressions in (1.2), we get

$$u(x,t) = \frac{1}{2}f(x-ct) - \frac{1}{2}\int_\gamma^{x-ct} g(\tau)d\tau + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_\gamma^{x+ct} g(\tau)d\tau\,.$$

Writing $-\int_\gamma^{x-ct} g(\tau)d\tau = \int_{x-ct}^\gamma g(\tau)d\tau$, we combine both integrals into one:

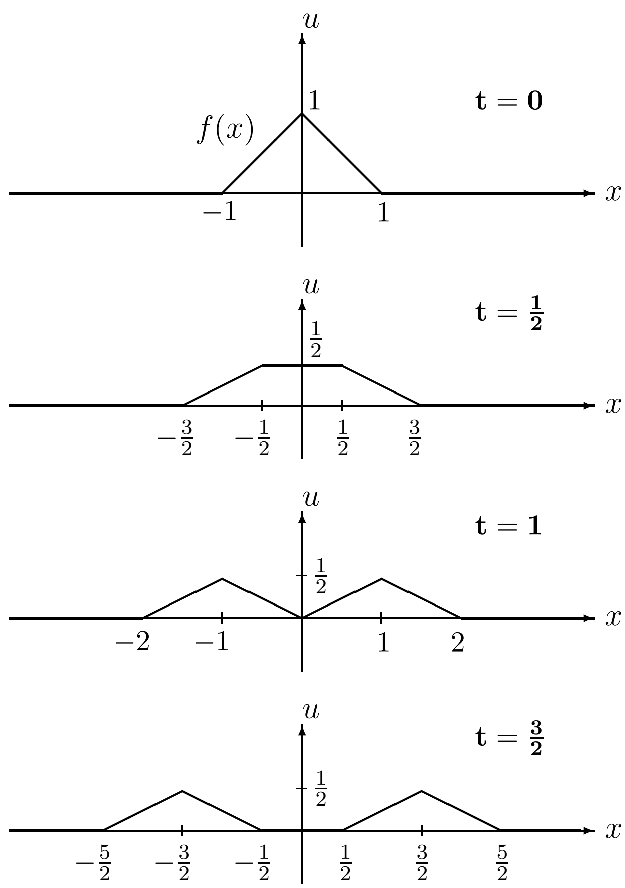
$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau\,.$$

We derived the famous *d'Alembert formula* (published in 1747, see the Wikipedia article).

In case $g(x) = 0$, this formula gives

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2}\,,$$

which is a superposition (sum) of a wave traveling to the right and a wave traveling to the left, both of speed c. (The same conclusion is true for general $g(x)$, just look at the formula (1.2).)



Snapshots of $u(x,t)$ in case of pinched string

Example: *Pinched String.* We solve

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0 \qquad \text{for } -\infty < x < \infty, \text{ and } t > 0 \\
&u(x,0) = f(x) \qquad \text{for } -\infty < x < \infty \\
&u_t(x,0) = 0 \qquad \text{for } -\infty < x < \infty\,,
\end{aligned}
\tag{1.4}
$$

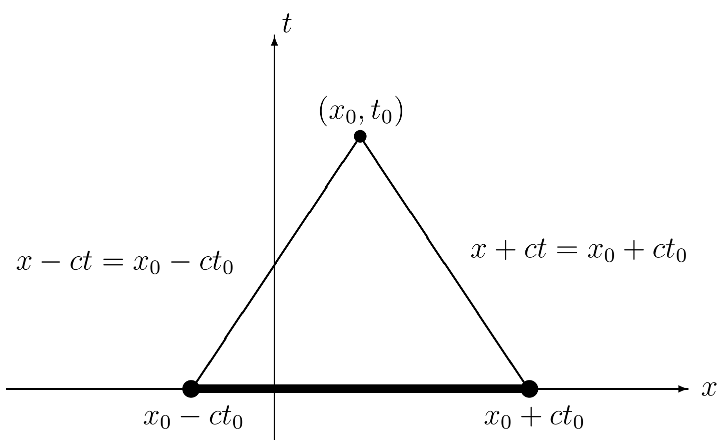
where

$$f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| > 1\,. \end{cases}$$

The initial displacement $f(x)$ resembles a "pinch" (see the snapshot at $t = 0$), while the initial velocity is zero. By d'Alembert's formula

$$u(x,t) = \frac{1}{2} f(x-t) + \frac{1}{2} f(x+t)\,.$$

This expression implies that the initial "pinch" breaks into two pinches of similar shape, but half of the original magnitude, with one of them traveling to the right, and the other one to the left, both with speed 1. We present the snapshots of $u(x,t)$ at $t = 0$ (the initial pinch), $t = \frac{1}{2}$, $t = 1$, and $t = \frac{3}{2}$.



The domain of dependence (thick) of the point (x_0, t_0)

We now define the important concept of *characteristic lines*. A family of parallel straight lines in the (x,t) plane (with $t > 0$, and α a constant)

$$x - ct = \alpha$$

are called the *left characteristic lines*, or the *left characteristics* for short. They all have the slope $\frac{1}{c} > 0$, and varying the constant α produces a specific line, parallel to all others. Given any point (x_0, t_0) (with $t_0 > 0$), we can select a left characteristic, passing through it, namely

$$x - ct = x_0 - ct_0\,.$$

Let us follow this line for decreasing t until it intersects the x-axis. This happens at $x = x_0 - ct_0$. Similarly, a family of parallel straight lines in the (x, t) plane, given by $x + ct = \alpha$, are called the *right characteristics.* They all have the slope $-\frac{1}{c} < 0$. The right characteristic passing through (x_0, t_0) is

$$x + ct = x_0 + ct_0 \,.$$

It intersects the x-axis at $x = x_0 + ct_0$. The string's displacement at any point (x_0, t_0) is (according to d'Alembert's formula)

$$u(x_0, t_0) = \frac{f(x_0 - ct_0) + f(x_0 + ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} g(\tau) d\tau \,.$$

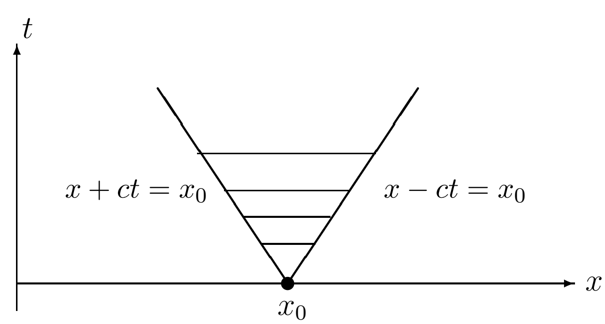
Geometrically, $u(x_0, t_0)$ is equal to the average of the values of $f(x)$ at the points where the characteristics, passing through (x_0, y_0), intersect the x-axis, plus $\frac{1}{2c}$ times the integral of $g(x)$ between these points. One calls the interval $[x_0 - ct_0, x_0 + ct_0]$ *the domain of dependence of the point* (x_0, t_0).

Given a point $(x_0, 0)$ on the x-axis, the characteristics passing through it are $x + ct = x_0$ and $x - ct = x_0$. The region between these characteristics is called *the region of influence of the point* $(x_0, 0)$. If a point (x, t) lies outside of this region, the value of the solution $u(x, t)$ is not influenced by the values of $f(x)$ and $g(x)$ at (or near) x_0.

We say that a function $f(x)$ has *compact support*, if $f(x)$ is identically zero outside of some bounded interval $[a, b]$. In such a case, it is customary to say that $f(x)$ *lives on* $[a, b]$.

Lemma 8.1.1 *Assume that the initial data $f(x)$ and $g(x)$ are of compact support. Then the solution $u(x, t)$ of the problem (1.1) is of compact support, for any fixed t.*

Proof: If $f(x)$ and $g(x)$ live on $[a, b]$, then $u(x, t)$ lives on $[a - ct, b + ct]$, for any fixed t, as follows by d'Alembert's formula (just draw the regions of influence of $(a, 0)$, and of $(b, 0)$). ♢



The region of influence of the point $(x_0, 0)$

We define the *energy of a string* to be

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[u_t^2(x,t) + c^2 u_x^2(x,t) \right] dx \, .$$

Theorem 8.1.1 *Assume that the initial data $f(x)$ and $g(x)$ are of compact support. Then the energy of a string is constant, for any solution $u(x,t)$ of the wave equation in (1.1).*

Proof: We shall show that $E'(t) = 0$ for all t. Indeed,

$$E'(t) = \int_{-\infty}^{\infty} \left[u_t u_{tt} + c^2 u_x u_{xt} \right] dx = \int_{-\infty}^{\infty} \left[u_t u_{tt} - c^2 u_{xx} u_t \right] dx$$

$$= \int_{-\infty}^{\infty} u_t (u_{tt} - c^2 u_{xx}) \, dx = 0 \, .$$

On the second step we performed integration by parts, with the boundary terms vanishing by the Lemma 8.1.1. On the last step we used that $u(x,t)$ satisfies the wave equation. ♢

This theorem implies that for all t

$$E(t) = E(0) = \frac{1}{2} \int_{-\infty}^{\infty} \left[u_t^2(x,0) + c^2 u_x^2(x,0) \right] \tag{1.5}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[g^2(x) + c^2 f'^2(x) \right] dx \, .$$

Theorem 8.1.2 *The problem (1.1) has a unique solution.*

Proof: Assume that $v(x,t)$ is another solution of (1.1), in addition to the solution $u(x,t)$ given by d'Alembert's formula. Call $w(x,t) = u(x,t) - v(x,t)$. Then $w(x,t)$ satisfies the wave equation (1.1) with zero initial data ($w(x,0) = w_t(x,0) = 0$). By (1.5)

$$E(t) = \int_{-\infty}^{\infty} \left[w_t^2(x,t) + c^2 w_x^2(x,t) \right] dx = E(0) = 0 \, ,$$

for all t. It follows that $w_t(x,t) = 0$, and $w_x(x,t) = 0$ for all x and t, so that $w(x,t) = \textit{constant}$. Setting $t = 0$, we see that this constant is zero. We conclude that $w(x,t) = 0$ for all x and t, which means that $v(x,t)$ is identical to $u(x,t)$. ♢

8.2 Semi-Infinite String: Reflection of Waves

Suppose that a string extends along the positive x-axis, for $0 < x < \infty$, and we assume that its left end-point is kept in a fixed position, so that its displacement is zero at $x = 0$, for all t, $u(0,t) = 0$ (the *Dirichlet boundary condition*). To find the displacements $u(x,t)$, we need to solve the wave equation, together with initial and boundary conditions:

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } 0 < x < \infty, \text{ and } t > 0 \\
u(x,0) &= f(x) \quad \text{for } 0 < x < \infty \\
u_t(x,0) &= g(x) \quad \text{for } 0 < x < \infty \\
u(0,t) &= 0 \quad \text{for } t > 0\,,
\end{aligned}
\tag{2.1}
$$

with given initial displacement $f(x)$, and initial velocity $g(x)$.

Recall the concept of the *odd extension*. If $f(x)$ is defined on $(0,\infty)$, then its odd extension

$$
f_o(x) = \begin{cases} f(x) & \text{for } x > 0 \\ -f(-x) & \text{for } x < 0 \end{cases}
$$

is defined for all $x \neq 0$. Geometrically, this amounts to reflecting the graph of $f(x)$ with respect to the origin. ($f_o(x)$ is left undefined at $x = 0$.) The resulting function $f_o(x)$ is *odd*, satisfying $f_o(-x) = -f_o(x)$ for all $x \neq 0$.

If $f_o(x)$ and $g_o(x)$ are the odd extensions of the functions $f(x)$ and $g(x)$ respectively, then we claim that the solution of the problem (2.1) is

$$
u(x,t) = \frac{f_o(x-ct) + f_o(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_o(\tau)d\tau\,. \tag{2.2}
$$

Indeed, we already know that this formula gives a solution of the wave equation, and that $u(x,0) = f_o(x) = f(x)$, and $u_t(x,0) = g_o(x) = g(x)$, for $x > 0$. As for the boundary condition, we have

$$
u(0,t) = \frac{f_o(-ct) + f_o(ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} g_o(\tau)d\tau = 0\,.
$$

Example 1 Solve

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0 \quad \text{for } 0 < x < \infty, \text{ and } t > 0 \\
u(x,0) &= x \quad \text{for } 0 < x < \infty \\
u_t(x,0) &= x^2 \quad \text{for } 0 < x < \infty \\
u(0,t) &= 0 \quad \text{for } t > 0\,.
\end{aligned}
$$

We have (here $c = 1$)

$$u(x,t) = \frac{f_o(x-t) + f_o(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g_o(\tau)d\tau\,, \tag{2.3}$$

with $f_o(x) = x$, and $g_o(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ -x^2 & \text{for } x < 0 \end{cases}$.

Case 1. $t \leq x$. Then $x - t \geq 0$, and

$$u(x,t) = \frac{x-t+x+t}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tau^2 d\tau \;= x + x^2 t + \frac{1}{3}t^3\,.$$

Case 2. $t > x$. Then $x - t < 0$, and the integral term in (2.3) needs to be split into two pieces:

$$u(x,t) = \frac{x-t+x+t}{2} - \frac{1}{2}\int_{x-t}^{0} \tau^2 d\tau + \frac{1}{2}\int_{0}^{x+t} \tau^2 d\tau \;= x + xt^2 + \frac{1}{3}x^3\,.$$

Answer. $u(x,t) = \begin{cases} x + x^2 t + \frac{1}{3}t^3 & \text{for } t \leq x \\ \\ x + xt^2 + \frac{1}{3}x^3 & \text{for } t > x \end{cases}$.

We now return to the formula (2.2). If $x - ct \geq 0$, then we can replace $f_o(x)$ and $g_o(x)$ by $f(x)$ and $g(x)$ respectively. In case $x - ct < 0$, we claim that the formula (2.2) gives

$$u(x,t) = \frac{-f(-x+ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{-x+ct}^{x+ct} g(\tau)d\tau\,. \tag{2.4}$$

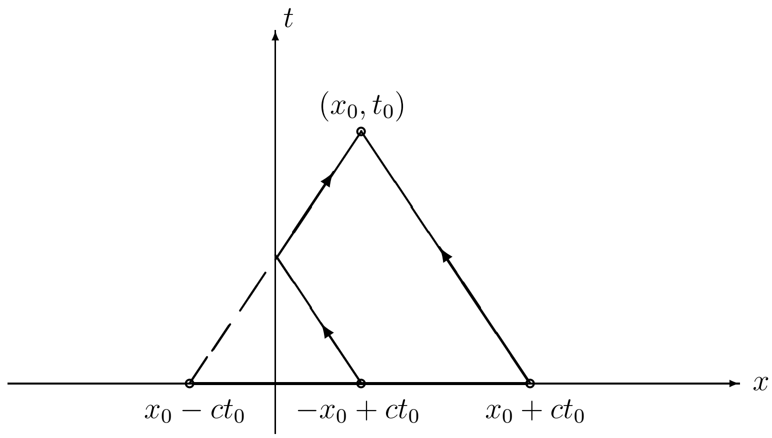
Indeed, observe that $-x + ct = -(x - ct) > 0$, and $\int_{x-ct}^{-x+ct} g_o(\tau)d\tau = 0$. Then

$$\int_{x-ct}^{x+ct} g_o(\tau)d\tau = \int_{x-ct}^{-x+ct} g_o(\tau)d\tau + \int_{-x+ct}^{x+ct} g_o(\tau)d\tau = \int_{-x+ct}^{x+ct} g(\tau)d\tau\,.$$

In case $g(x) = 0$, it follows from (2.4) that

$$u(x_0,t_0) = \frac{-f(-x_0+ct_0) + f(x_0+ct_0)}{2}\,, \tag{2.5}$$

so that instead of computing $f(x)$ at $x_0 - ct_0$, we compute $f(x)$ at the point symmetric with respect to $x = 0$. We say that the left characteristic got *reflected* when it hit the t-axis, and the opposite sign is the way to account (or the "price to pay") for a reflected wave.



Reflection at the boundary point $x = 0$

Example 2 *Pinched string.* We solve

$$\begin{aligned} u_{tt} - u_{xx} = 0 \qquad & \text{for } 0 < x < \infty, \text{ and } t > 0 \\ u(x,0) = f(x) \qquad & \text{for } 0 < x < \infty \\ u_t(x,0) = 0 \qquad & \text{for } 0 < x < \infty \\ u(0,t) = 0 \qquad & \text{for } t > 0\,, \end{aligned}$$

where

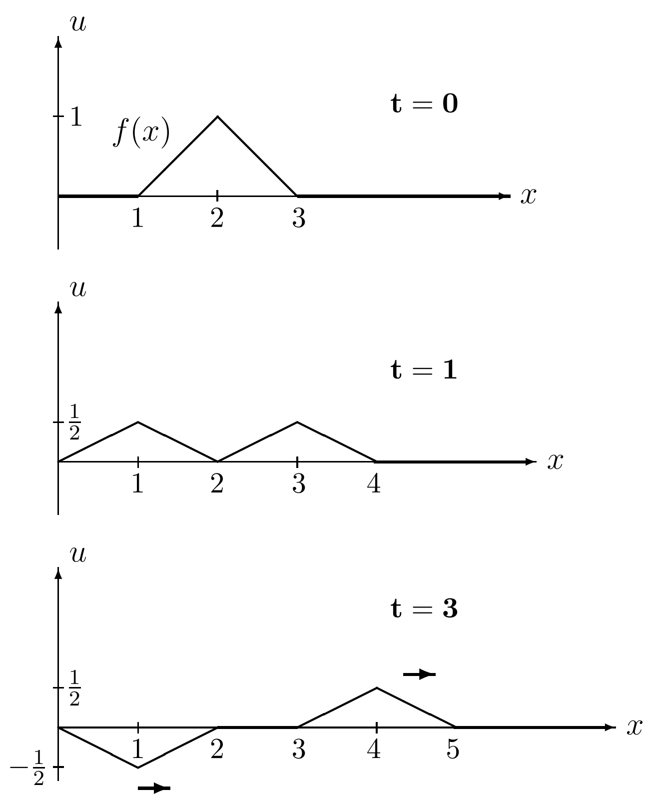
$$f(x) = \begin{cases} x - 1 & \text{if } 1 \le x \le 2 \\ -x + 3 & \text{if } 2 \le x \le 3 \\ 0 & \text{for all other } x \end{cases} . \tag{2.6}$$

(This is the pinch considered earlier, shifted two units to the right, and centered at $x = 2$.) Using the odd extension $f_o(x)$, we write the solution of this problem on $(-\infty, \infty)$:

$$u(x,t) = \frac{f_o(x-t) + f_o(x+t)}{2} . \tag{2.7}$$

On the interval $(-\infty, \infty)$, the graph of $f_o(x)$ includes the original positive pinch on the interval $(1, 3)$, and a negative pinch of the same shape over $(-3, -1)$ ($f_o(x)$ is zero for other x). By (2.7), each pinch breaks into two half-pinches, and the four half-pinches set in motion, as above. We then translate our results to the original (physical) interval $(0, \infty)$.

Conclusion: the original "pinch" $f(x)$ breaks into two pinches of similar shape, but half of the magnitude, with one of them traveling to the right, and the other one moving to the left, both with speed 1. At the time $t = 1$, the left half-pinch reaches the $x = 0$ end-point. By the time $t = 3$, it completely reflects and becomes negative, of the same triangle shape. Then both half-pinches (one of them is positive, and the other one negative) travel to the right, for all $t > 3$.



Snapshots of a semi-infinite pinched string

8.3 Bounded String: Multiple Reflections

Assume that the string is finite, extending over some interval $0 < x < L$, and at both end-points the displacement is zero for all time. We need to solve

$$
\begin{aligned}
(3.1) \qquad & u_{tt} - c^2 u_{xx} = 0 \qquad \text{for } 0 < x < L, \text{ and } t > 0 \\
& u(x,0) = f(x) \qquad \text{for } 0 < x < L \\
& u_t(x,0) = g(x) \qquad \text{for } 0 < x < L \\
& u(0,t) = u(L,t) = 0 \qquad \text{for } t > 0\,,
\end{aligned}
$$

with given initial displacement $f(x)$, and the initial velocity $g(x)$.

Let $f_o(x)$ be the odd extension of $f(x)$ from $(0,L)$ to $(-L,L)$, and then we extend $f_o(x)$ to $(-\infty,\infty)$ as a function of period $2L$. We call this new extended function $\bar{f}(x)$. Similarly, we define the extension $\bar{g}(x)$ of $g(x)$. On the original interval $(0,L)$ these extensions agree with $f(x)$ and $g(x)$ respectively. Clearly, $\bar{f}(x)$ and $\bar{g}(x)$ are odd on $(-\infty,\infty)$. In addition, both of these functions are *odd with respect to* L, which means that

$$\bar{f}(L+x) = -\bar{f}(L-x), \text{ and } \bar{g}(L+x) = -\bar{g}(L-x), \text{ for all } x\,.$$

It turns out that the solution of (3.1) is

$$(3.2) \qquad u(x,t) = \frac{\bar{f}(x-ct)+\bar{f}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(\tau)\,d\tau\,.$$

Indeed, comparing with d'Alemberts's formula, we see that this function satisfies the wave equation, and the initial conditions. Turning to the boundary conditions, we have

$$u(0,t) = \frac{\bar{f}(-ct)+\bar{f}(ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} \bar{g}(\tau)\,d\tau = 0\,,$$

because $\bar{f}(x)$ and $\bar{g}(x)$ are odd. At the other end point

$$u(L,t) = \frac{\bar{f}(L-ct)+\bar{f}(L+ct)}{2} + \frac{1}{2c}\int_{L-ct}^{L+ct} \bar{g}(\tau)\,d\tau = 0\,,$$

because $\bar{f}(x)$ and $\bar{g}(x)$ are odd with respect to L.

Consider now the case $g(x) = 0$. Then

$$(3.3) \qquad u(x,t) = \frac{\bar{f}(x-ct)+\bar{f}(x+ct)}{2}\,.$$

Similarly to the above, we reflect the characteristics when they reach either the t-axis, or the line $x = L$. This time, when we continue the characteristics backward in time, multiple reflections are possible, from both the t-axis (the line $x = 0$), and from the line $x = L$, before the x-axis is reached. By examining the graph of $\bar{f}(x)$, one can see that the formula (3.3) implies that the result (or the "price") of each reflection is change of sign. So, if after 3 reflections the left characteristic arrives at a point A inside $(0, L)$, then its contribution is $\bar{f}(x - ct) = -f(A)$. If it took 10 reflections for the right characteristic to arrive at a point $B \in (0, L)$, then we have $\bar{f}(x+ct) = f(B)$.

Example 1 Find $u(1/4, 1)$, $u(1/4, 2)$ and $u(1/4, 3)$ for the problem

$$
\begin{gathered}
u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < 2, \text{ and } t > 0 \\
u(x, 0) = f(x) = x^2 \qquad \text{for } 0 < x < 2 \\
u_t(x, 0) = 0 \qquad \text{for } 0 < x < 2 \\
u(0, t) = u(2, t) = 0 \qquad \text{for } t > 0\,.
\end{gathered}
$$

Here $c = 1$, so that the left characteristics have slope 1, and the right ones have slope -1. When finding the solution at $(1/4, 1)$, the left characteristic is reflected once, coming down at $x = 3/4$, while the right one is not reflected, coming down at $x = 5/4$, giving

$$u(1/4, 1) = -\frac{1}{2} f(3/4) + \frac{1}{2} f(5/4) = \frac{1}{2}\,.$$

To find the solution at $(1/4, 2)$, both characteristics are reflected once, and both are coming down at the same point $x = 7/4$, giving

$$u(1/4, 2) = -\frac{1}{2} f(7/4) - \frac{1}{2} f(7/4) = -\frac{49}{16}\,.$$

When computing $u(1/4, 3)$, the left characteristic is reflected twice, coming down at $x = 5/4$. The right characteristic is reflected once, coming down at $x = 3/4$, giving

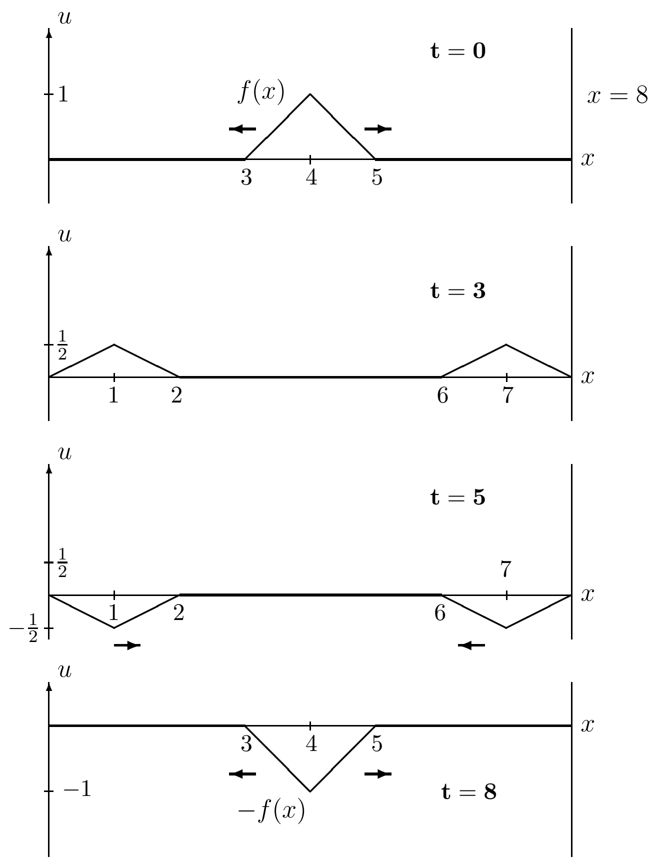
$$u(1/4, 3) = \frac{1}{2} f(5/4) - \frac{1}{2} f(3/4) = \frac{1}{2}\,.$$

Example 2 *Pinched string.* We solve

$$
\begin{gathered}
u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < 8, \text{ and } t > 0 \\
u(x, 0) = f(x) \qquad \text{for } 0 < x < 8 \\
u_t(x, 0) = 0 \qquad \text{for } 0 < x < 8 \\
u(0, t) = u(8, t) = 0 \qquad \text{for } t > 0\,,
\end{gathered}
$$

with the initial displacement

$$f(x) = \begin{cases} x-3 & \text{if } 3 \le x \le 4 \\ -x+5 & \text{if } 4 \le x \le 5 \\ 0 & \text{for all other } x \in [0,8] \end{cases} .$$



Snapshots of a bounded pinched string

The same pinch we considered above is now centered at $x = 4$ (see the snapshot at $t = 0$). Reasoning as in the case of semi-infinite string, the formula (3.3) implies that the initial "pinch" breaks into two pinches of similar shape, but half of the magnitude, with one of them traveling to the right, and the other one to the left, both with speed 1. When the left half-pinch reaches the $x = 0$ end-point, at the time $t = 3$, it gradually reflects and at $t = 5$ becomes negative, of the same shape. When the right half-pinch reaches the $x = 8$ end-point, at the same time $t = 3$, it also reflects and becomes negative, of the same shape. Then both half-pinches travel toward each other, turning at $t = 8$ into the exact negative of the original pinch. Then the negative pinch splits up into two halves, traveling to the left and to the right, and becoming positive after the next round of reflections. Then both half-pinches travel toward each other, turning at $t = 16$ into exactly the original pinch. Then everything is repeated. The result is periodic in time motion (of the period 16), consistent with the formulas obtained previously by separation of variables.

8.4 Neumann Boundary Conditions

We consider again a semi-infinite string, $0 < x < \infty$. Assume that at the $x = 0$ end-point, the string is allowed to slide freely up and down, but it is attached to a clamp, which makes its slope zero. So that the condition $u_x(0,t) = 0$ is prescribed at the boundary point $x = 0$, which is referred to as the *Neumann boundary condition*. We are led to solve the problem

$$
\begin{aligned}
(4.1) \qquad u_{tt} - c^2 u_{xx} &= 0 \qquad \text{for } 0 < x < \infty, \text{ and } t > 0\\
u(x,0) &= f(x) \qquad \text{for } 0 < x < \infty\\
u_t(x,0) &= g(x) \qquad \text{for } 0 < x < \infty\\
u_x(0,t) &= 0 \qquad \text{for } t > 0\,,
\end{aligned}
$$

with given initial displacement $f(x)$, and the initial velocity $g(x)$.

Define $f_e(x)$, the *even extension of* $f(x)$, by

$$
f_e(x) = \begin{cases} f(x) & \text{for } x > 0 \\ f(-x) & \text{for } x < 0 \end{cases}.
$$

The function $f_e(x)$ is even, defined for all $x \neq 0$. The graph of $f_e(x)$ can be obtained by reflecting the graph of $f(x)$ with respect to the y-axis. ($f_e(x)$ is left undefined at $x = 0$.) Similarly, define $g_e(x)$ to be the even extension of $g(x)$. We claim that the solution of (4.1) is given by

$$
u(x,t) = \frac{f_e(x-ct) + f_e(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_e(\tau)\,d\tau\,.
$$

Indeed, we know (comparing with d'Alembert's solution) that this formula gives a solution of the wave equation, and that $u(x,0) = f_e(x) = f(x)$, and

$u_t(x,0) = g_e(x) = g(x)$, for $x > 0$. Turning to the boundary condition, compute

$$u_x(x,t) = \frac{f_e'(x-ct) + f_e'(x+ct)}{2} + \frac{1}{2c}\left[g_e(x+ct) - g_e(x-ct)\right] \,,$$

and therefore

$$u_x(0,t) = \frac{f_e'(-ct) + f_e'(ct)}{2} + \frac{1}{2c}\left[g_e(ct) - g_e(-ct)\right] = 0 \,,$$

using that the derivative of an even function is an odd function.

Example 1 Solve

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \qquad \text{for } 0 < x < \infty, \text{ and } t > 0 \\ u(x,0) &= x^2 \qquad \text{for } 0 < x < \infty \\ u_t(x,0) &= x \qquad \text{for } 0 < x < \infty \\ u_x(0,t) &= 0 \qquad \text{for } t > 0 \,. \end{aligned}$$

We have $(x^2)_e = x^2$ and $(x)_e = |x|$. The solution is

$$u(x,t) = \frac{(x-2t)^2 + (x+2t)^2}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} |\tau| \, d\tau \,.$$

Considering two cases, depending on the sign of $x - 2t$, we calculate

$$u(x,t) = \begin{cases} x^2 + 4t^2 + xt & \text{for } x - 2t \geq 0 \\ \frac{5}{4}x^2 + 5t^2 & \text{for } x - 2t < 0 \,. \end{cases}$$

In case $g(x) = 0$, the solution of (4.1) is

$$u(x,t) = \begin{cases} \frac{f(x-ct)+f(x+ct)}{2} & \text{for } x \geq ct \\ \\ \frac{f(-x+ct)+f(x+ct)}{2} & \text{for } x < ct \,. \end{cases}$$

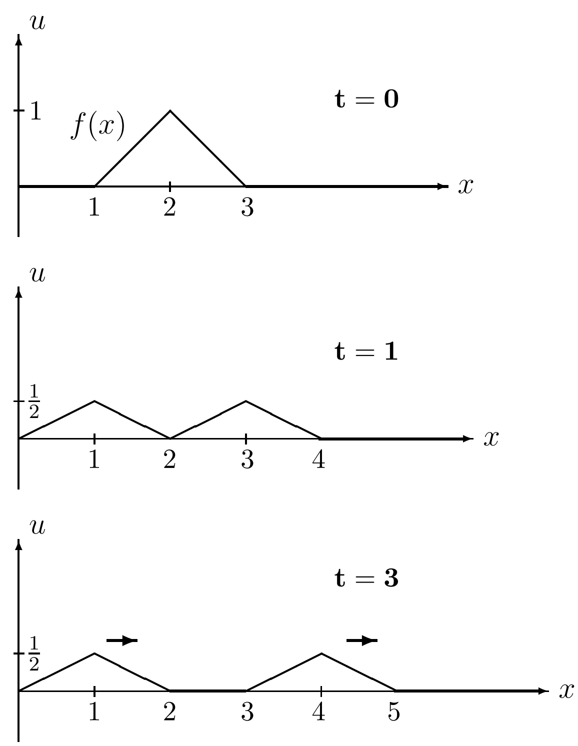
If a wave is reflected, we evaluate $f(x)$ at the point where the reflected wave comes down on the x-axis (and do not change the sign).

Only a small adjustment is required for bounded strings:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad \text{for } 0 < x < L, \text{ and } t > 0 \\ u(x,0) &= f(x) \qquad \text{for } 0 < x < L \\ u_t(x,0) &= g(x) \qquad \text{for } 0 < x < L \\ u_x(0,t) = u_x(L,t) &= 0 \qquad \text{for } t > 0 \,. \end{aligned} \tag{4.2}$$

Let $f_e(x)$ be the even extension of $f(x)$ from $(0, L)$ to $(-L, L)$, and then we extend $f_e(x)$ to $(-\infty, \infty)$ as a function of period $2L$. We call this new extended function $\hat{f}(x)$. Similarly, we define the extension $\hat{g}(x)$ of $g(x)$. On the original interval $(0, L)$ these extensions agree with $f(x)$ and $g(x)$ respectively. Clearly, $\hat{f}(x)$ and $\hat{g}(x)$ are even functions on $(-\infty, \infty)$. In addition, both of these functions are *even with respect to* L, which means that

$$\hat{f}(L+x) = \hat{f}(L-x), \quad \text{and} \quad \hat{g}(L+x) = \hat{g}(L-x), \quad \text{for all } x\,.$$



Semi-infinite pinched string with the Neumann condition

It is straightforward to verify that the solution of (4.2) is given by

$$u(x,t) = \frac{\hat{f}(x-ct) + \hat{f}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{g}(\tau)d\tau\,. \tag{4.3}$$

It is now a simple exercise to draw pictures for a pinched string, for both semi-infinite and bounded strings (at end-points, the reflected half-pinch keeps the same sign).

Example 2 *Pinched string.* We solve

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{for } 0 < x < \infty, \text{ and } t > 0 \\ u(x,0) &= f(x) \qquad \text{for } 0 < x < \infty \\ u_t(x,0) &= 0 \qquad \text{for } 0 < x < \infty \\ u_x(0,t) &= 0 \qquad \text{for } t > 0\,, \end{aligned}$$

where $f(x)$ is the familiar pinch, centered at $x = 2$:

$$f(x) = \begin{cases} x - 1 & \text{if } 1 \le x \le 2 \\ -x + 3 & \text{if } 2 \le x \le 3 \\ 0 & \text{for all other } x \end{cases} .$$

Using the formula (4.3) (here $\hat{g}(x) = 0$) we conclude that the original "pinch" $f(x)$ breaks into two pinches of similar shape, but half of the magnitude, with one of them traveling to the right, and the other one moving to the left, both with speed 1. At the time $t = 1$, the left half-pinch reaches the $x = 0$ end-point. By the time $t = 3$, it completely bounces off the left end-point and stays positive, of the same triangle shape. Then both positive half-pinches travel to the right, for all $t > 3$.

8.5 Non-Homogeneous Wave Equation

Let us recall Green's formula from calculus. If a closed curve C encloses a region D in the xt-plane, then for continuously differentiable functions $P(x,t)$ and $Q(x,t)$ we have

$$\int_C P(x,t)\,dx + Q(x,t)\,dt = \iint_D \left[Q_x(x,t) - P_t(x,t)\right] dxdt\,.$$

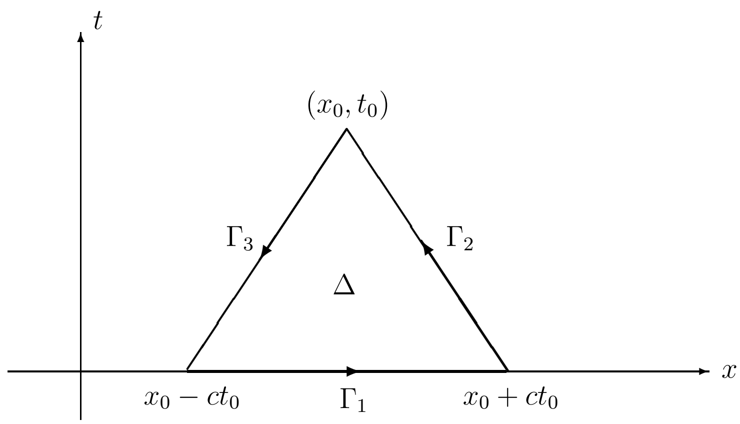
We now consider non-homogeneous equations

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t) \qquad \text{for } -\infty < x < \infty, \text{ and } t > 0 \\ u(x,0) &= f(x) \qquad \text{for } -\infty < x < \infty \\ u_t(x,0) &= g(x) \qquad \text{for } -\infty < x < \infty\,. \end{aligned} \tag{5.1}$$

Here $F(x,t)$ is given acceleration of the external force acting on the string, as was explained in Chapter 7. The initial displacement $f(x)$, and the initial velocity $g(x)$ are also given.

For any point (x_0, t_0), we denote by Δ the *characteristic triangle*, formed by the characteristic lines passing through (x_0, t_0), and the x-axis, with the vertices at $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$ and (x_0, t_0). By Γ we denote the boundary curve of Δ. Using Green's formula, and our equation

$$(5.2) \quad \int_\Gamma u_t\,dx + c^2 u_x\,dt = -\iint_D \left(u_{tt} - c^2 u_{xx}\right)\,dxdt = -\iint_\Delta F(x,t)\,dxdt\,.$$



The characteristic triangle Δ

We now calculate the line integral on the left, by breaking the boundary Γ into three pieces Γ_1, Γ_2 and Γ_3, which are the line segments joining the vertices of Δ. The integral over Γ is the sum of the integrals over Γ_1, Γ_2 and Γ_3. Along Γ_1, we have $t = 0$ and $dt = 0$. Then

$$\int_{\Gamma_1} u_t\,dx + c^2 u_x\,dt = \int_{\Gamma_1} u_t(x,0)\,dx = \int_{x-ct_0}^{x_0+ct_0} g(\tau)d\tau\,.$$

The equation of Γ_2 is $x + ct = x_0 + ct_0$, and so $dx + cdt = 0$. We replace $dx = -cdt$, and $dt = -\frac{1}{c}dx$, obtaining

$$\begin{aligned} &\textstyle\int_{\Gamma_2} u_t\,dx + c^2 u_x\,dt = -c\int_{\Gamma_2} u_x\,dx + u_t\,dt = -c\int_{\Gamma_2} du \\ &= -c\left[u(x_0,t_0) - u(x_0+ct_0,0)\right] = -c\left[u(x_0,t_0) - f(x_0+ct_0)\right]\,. \end{aligned}$$

The equation of Γ_3 is $x - ct = x_0 - ct_0$, and so $dx - cdt = 0$. We replace $dx = cdt$, and $dt = \frac{1}{c}dx$, obtaining

$$\int_{\Gamma_3} u_t\,dx + c^2 u_x\,dt = c\int_{\Gamma_3} u_x\,dx + u_t\,dt = c\int_{\Gamma_3} du = c\left[f(x_0 - ct_0) - u(x_0,t_0)\right]\,.$$

Using these three integrals in (5.2), we express

$$u(x_0,t_0)=\frac{f(x_0-ct_0)+f(x_0+ct_0)}{2}+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau+\frac{1}{2c}\iint_\Delta F(x,t)\,dxdt\,.$$

Finally, we replace $(x_0,t_0)\to(x,t)$, and in the double integral rename the dummy variables $(x,t)\to(\xi,\eta)$, obtaining the solution

$$\text{(5.3)}\qquad u(x,t)=\frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

$$+\frac{1}{2c}\iint_\Delta F(\xi,\eta)\,d\xi d\eta\,.$$

This formula reduces to d'Alembert's formula in case $F(x,t)=0$. Observe also that the characteristic triangle Δ depends on x and t.

Example Solve

$$\begin{aligned} u_{tt}-9u_{xx}=x &\qquad \text{for } -\infty<x<\infty,\ \text{and } t>0\\ u(x,0)=0 &\qquad \text{for } -\infty<x<\infty\\ u_t(x,0)=x^2 &\qquad \text{for } -\infty<x<\infty\,. \end{aligned}$$

By (5.3):

$$u(x_0,t_0)=\frac{1}{6}\int_{x_0-3t_0}^{x_0+3t_0}\tau^2 d\tau+\frac{1}{6}\iint_\Delta x\,dxdt\,.$$

The first integral is equal to $\frac{1}{18}\left[(x_0+3t_0)^3-(x_0-3t_0)^3\right]=x_0^2t_0+3t_0^3$. The double integral is evaluated as follows:

$$\iint_\Delta x\,dxdt=\int_0^{t_0}\left(\int_{3t+x_0-3t_0}^{-3t+x_0+3t_0} x\,dx\right)dt$$

$$=\frac{1}{2}\int_0^{t_0}\left[(-3t+x_0+3t_0)^2-(3t+x_0-3t_0)^2\right]dt$$

$$=-\frac{1}{18}(-3t+x_0+3t_0)^3\Big|_0^{t_0}-\frac{1}{18}(3t+x_0-3t_0)^3\Big|_0^{t_0}=3x_0t_0^2\,.$$

So that $u(x_0,t_0)=x_0^2t_0+3t_0^3+\frac{1}{2}x_0t_0^2$. Replacing $(x_0,t_0)\to(x,t)$, we conclude

$$u(x,t)=x^2t+3t^3+\frac{1}{2}xt^2\,.$$

Duhamel's Principle

Consider the problem

$$(5.4) \qquad \begin{aligned} u_{tt} - c^2 u_{xx} &= F(x,t) \quad \text{for } -\infty < x < \infty, \text{ and } t > 0 \\ u(x,0) &= 0 \quad \text{for } -\infty < x < \infty \\ u_t(x,0) &= 0 \quad \text{for } -\infty < x < \infty \,, \end{aligned}$$

with homogeneous (zero) initial conditions. Its solution at a point (x_0, t_0), written as a repeated integral, is

$$u(x_0,t_0) = \frac{1}{2c} \iint_\Delta F(x,t)\, dxdt = \frac{1}{2c} \int_0^{t_0} \left(\int_{ct+x_0-ct_0}^{-ct+x_0+ct_0} F(x,t)\, dx \right) dt$$
$$= \int_0^{t_0} \left(\frac{1}{2c} \int_{c\eta+x_0-ct_0}^{-c\eta+x_0+ct_0} F(x,\eta)\, dx \right) d\eta \,.$$

On the last step we changed the "dummy" variable t to η. We now consider a family of problems, depending on a parameter η: find $U(x,t)$ solving

$$(5.5) \qquad \begin{aligned} U_{tt} - c^2 U_{xx} &= 0 \quad \text{for } -\infty < x < \infty, \text{ and } t > \eta \\ U(x,\eta) &= 0 \quad \text{for } -\infty < x < \infty \\ U_t(x,\eta) &= F(x,\eta) \quad \text{for } -\infty < x < \infty \,. \end{aligned}$$

Here the initial conditions are prescribed at the time moment $t = \eta$, and the force term $F(x,t)$ now acts as the initial velocity. The solution U depends also on the parameter η, so that $U = U(x,t,\eta)$. By adjusting d'Alembert's formula (the initial time is now $t = \eta$), the solution of (5.5) at a point (x_0, t_0) is

$$U(x_0,t_0,\eta) = \frac{1}{2c} \int_{c\eta+x_0-ct_0}^{-c\eta+x_0+ct_0} F(x,\eta)\, dx \,.$$

(The left and the right characteristics are continued backward from the point (x_0, t_0) until they intersect the line $t = \eta$.) The solution of the original problem (5.4) is then

$$u(x_0,t_0) = \int_0^{t_0} U(x_0,t_0,\eta)\, d\eta \,.$$

We see that the force term $F(x,t)$ is being distributed as the initial velocities at times η $(0 < \eta < t_0)$, and at the time t_0 we integrate (sum up) the effects of these initial velocities.

Similar approach works for many other *evolution equations*, which are equations involving the "time" variable.

8.5.1 Problems

1. Solve the initial value problem, and describe its physical significance. Here $u = u(x,t)$. Simplify your answer.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \quad -\infty < x < \infty, \ \ t \geq 0 \\ u(x,0) &= x \quad -\infty < x < \infty, \\ u_t(x,0) &= \cos x \quad -\infty < x < \infty. \end{aligned}$$

Answer. $u(x,t) = x + \frac{1}{2}\cos x \sin 2t$.

2. Find the values of the solution $u(3,1)$ and $u(1,3)$ for the following problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) &= x^2 \quad 0 < x < \infty, \\ u_t(x,0) &= x \quad 0 < x < \infty, \\ u(0,t) &= 0 \quad t \geq 0\,. \end{aligned}$$

Answer. $u(3,1) = 13$, $u(1,3) = 9$.

3. Solve the initial-boundary value problem, and describe its physical significance

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) &= x \quad 0 < x < \infty, \\ u_t(x,0) &= \cos x \quad 0 < x < \infty, \\ u_x(0,t) &= 0 \quad t \geq 0\,. \end{aligned}$$

Answer. $u(x,t) = \frac{|x-2t|+|x+2t|}{2} + \frac{1}{2}\cos x \sin 2t$.

4. Solve the non-homogeneous boundary value problem, and describe its physical significance. Simplify your answer.

$$\begin{aligned} u_{tt} - 4u_{xx} &= x \quad -\infty < x < \infty, \ \ t \geq 0 \\ u(x,0) &= 0 \quad -\infty < x < \infty, \\ u_t(x,0) &= 0 \quad -\infty < x < \infty. \end{aligned}$$

Answer. $u(x,t) = \frac{1}{2}xt^2$.

5. Solve the non-homogeneous boundary value problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= x + 3t \quad -\infty < x < \infty, \ \ t \geq 0 \\ u(x,0) &= 0 \quad -\infty < x < \infty, \\ u_t(x,0) &= \cos x \quad -\infty < x < \infty. \end{aligned}$$

Answer. $u(x,t) = \frac{1}{2}\left(xt^2 + t^3 + \cos x \sin 2t\right)$.

6. Solve the initial-boundary value problem, and describe its physical significance

$$\begin{gathered} u_{tt} - 4u_{xx} = 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) = x \quad 0 < x < \infty, \\ u_t(x,0) = \sin x \quad 0 < x < \infty, \\ u(0,t) = 0 \quad t \geq 0\,. \end{gathered}$$

Answer. $u(x,t) = x + \frac{1}{2}\sin x \sin 2t$.

7. Solve the following initial-boundary value problem, and describe its physical significance

$$\begin{gathered} u_{tt} - 4u_{xx} = 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) = x^2 \quad 0 < x < \infty, \\ u_t(x,0) = \cos x \quad 0 < x < \infty, \\ u(0,t) = 0 \quad t \geq 0\,. \end{gathered}$$

8. Find $u(3,1)$ and $u(1,3)$ for the solution of the following problem

$$\begin{gathered} u_{tt} - 4u_{xx} = 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) = x + 1 \quad 0 < x < \infty, \\ u_t(x,0) = 0 \quad 0 < x < \infty \\ u(0,t) = 0\,. \end{gathered}$$

Answer. $u(3,1) = 4$, and $u(1,3) = 1$.

9. Solve

$$\begin{gathered} u_{tt} = u_{xx}, \quad 0 < x < \infty, \ \ t > 0 \\ u(x,0) = x^2, \\ u_t(x,0) = x, \\ u(0,t) = 0\,. \end{gathered}$$

Answer. $u(x,t) = \begin{cases} x^2 + xt + t^2 & \text{for } t \leq x \\ \\ 3xt & \text{for } t > x \end{cases}$.

10. Find $u(3,1)$ and $u(1,3)$ for the solution of the following problem

$$\begin{gathered} u_{tt} - 4u_{xx} = 0 \quad 0 < x < \infty, \ \ t \geq 0 \\ u(x,0) = x+1 \quad 0 < x < \infty, \\ u_t(x,0) = 0 \quad 0 < x < \infty \\ u_x(0,t) = 0\,. \end{gathered}$$

Answer. $u(3,1) = 4$, and $u(1,3) = 7$.

11. Find $u(1/2,2)$ and $u(1/3,3)$ for the following problem

$$\begin{gathered} u_{tt} - u_{xx} = 0 \quad 0 < x < 2, \ \ t \geq 0 \\ u(x,0) = x \quad 0 < x < 2, \\ u_t(x,0) = 0 \quad 0 < x < 2, \\ u(0,t) = u(2,t) = 0 \quad t \geq 0\,. \end{gathered}$$

Answer. $u(1/2,2) = -\frac{3}{2}$, $u(1/3,3) = \frac{1}{3}$.

12. Find $u(1/2,2)$ and $u(1/3,3)$ for the following problem

$$\begin{gathered} u_{tt} - u_{xx} = 0 \quad 0 < x < 2, \ \ t \geq 0 \\ u(x,0) = x \quad 0 < x < 2, \\ u_t(x,0) = 0 \quad 0 < x < 2, \\ u_x(0,t) = u_x(2,t) = 0 \quad t \geq 0\,. \end{gathered}$$

Answer. $u(1/2,2) = \frac{3}{2}$, $u(1/3,3) = 1$.

13. Consider a wave equation with a lower order term ($a > 0$ is a constant)

$$u_{tt} - 4u_{xx} + au_t = 0 \quad -\infty < x < \infty, \ \ t \geq 0\,.$$

Assume that the solution $u(x,t)$ is of compact support. Show that the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_t^2 + 4u_x^2\right)\,dx$ is a decreasing function.

14. (*Equipartition of energy*). For the initial value problem

$$\begin{gathered} u_{tt} - c^2u_{xx} = 0 \qquad \text{for } -\infty < x < \infty, \text{ and } t > 0 \\ u(x,0) = f(x), \ \ u_t(x,0) = g(x), \quad \text{for } -\infty < x < \infty \end{gathered}$$

assume that $f(x)$ and $g(x)$ are of compact support. Define the *kinetic energy* $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$, and the *potential energy* $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} c^2u_x^2(x,t)\,dx$ (so that $E(t) = k(t)+p(t)$ is the total energy, considered before). Show that

$$k(t) = p(t)\,, \quad \text{for large enough time } t\,.$$

Hint: From d'Alembert's formula

$$u_x = \frac{f'(x-ct)+f'(x+ct)}{2} + \frac{g(x+ct)-g(x-ct)}{2c}\,,$$

$$u_t = \frac{-cf'(x-ct)+cf'(x+ct)}{2} + \frac{g(x+ct)+g(x-ct)}{2}\,.$$

Then

$$\begin{aligned} u_t^2 - c^2u_x^2 &= (u_t - cu_x)\,(u_t + cu_x) \\ &= (g(x-ct) - cf'(x-ct))\,(cf'(x+ct) + g(x+ct)) = 0\,, \end{aligned}$$

for large t, because both $x - ct$ and $x + ct$ will leave the intervals on which $f(x)$ and $g(x)$ live.

14. Let $u(x,t)$ be a solution of the heat equation

$$\begin{aligned} u_t &= 5u_{xx} \quad 0 < x < 1,\, t > 0 \\ u(0,t) &= u(1,t) = 0\,. \end{aligned}$$

Show that $E(t) = \int_0^1 u^2(x,t)\,dx$ is a decreasing function.

15. Show that $u(x,t) = 0$ is the only solution of the nonlinear equation

$$\begin{aligned} u_t &= 5u_{xx} - u^3 + uu_x \quad 0 < x < 1,\, t > 0 \\ u(0,t) &= u(1,t) = 0 \quad t > 0 \\ u(x,0) &= 0 \quad 0 < x < 1\,. \end{aligned}$$

Hint: Show that $E(t) = \int_0^1 u^2(x,t)\,dx$ is a decreasing function.

16. Think of some function. Then write down two solutions of the wave equation

$$u_{tt} - 9u_{xx} = 0\,,$$

connected to that function.

Hint: I thought of $f(z) = \dfrac{z}{\sin 5z}$, and obtained two solutions $u_1(x,t) = \dfrac{x-3t}{\sin 5(x-3t)}$, and $u_2(x,t) = \dfrac{x+3t}{\sin 5(x+3t)}$.

17. Let $v(x,t)$ be a complex-valued solution of a nonlinear *Schroedinger's equation* ($i = \sqrt{-1}$)

$$iv_t + v_{xx} + 2v|v|^2 = 0\,,$$

where $|v|$ denotes the complex modulus of $v(x,t)$. Find the *standing wave solution* in the form $v(x,t) = e^{imt}u(x)$, with a real valued $u(x)$, and a constant $m > 0$.

Hint: Recall that $u(x) = \frac{\sqrt{m}}{\cosh \sqrt{m}(x-c)}$ are homoclinic solutions of

$$u'' - mu + 2u^3 = 0\,.$$

Answer. $v(x,t) = e^{imt}\frac{\sqrt{m}}{\cosh \sqrt{m}(x-c)}$, with arbitrary constant c. (Other solutions of Schroedinger's equation are also possible.)

8.6 First Order Linear Equations

Recall that curves in the xy-plane can be described by parametric equations $x = x(s)$ and $y = y(s)$, where s is a parameter, $a \le s \le b$. Along such a curve, any function $u = u(x,y)$ becomes a function of s, $u = u(x(s), y(s))$, and by the chain rule

$$\frac{d}{ds}u(x(s),y(s)) = u_x(x(s),y(s))\,x'(s) + u_y(x(s),y(s))\,y'(s)\,.$$

We wish to find $u = u(x,y)$, solving the first order equation

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)\,, \tag{6.1}$$

where continuously differentiable functions $a(x,y)$ and $b(x,y)$, and continuous functions $c(x,y)$ and $f(x,y)$ are given.

Consider a system of two ODE's

$$\begin{aligned} \frac{dx}{ds} &= a(x,y) \\ \frac{dy}{ds} &= b(x,y)\,, \end{aligned} \tag{6.2}$$

depending on some parameter s, with the initial conditions

$$x(0) = x_0, \quad y(0) = y_0\,. \tag{6.3}$$

By the existence and uniqueness Theorem 6.1.1, there exists a unique solution (at least locally near the initial point (x_0, y_0)) $x = x(s)$ and $y = y(s)$, which defines a curve, called the *characteristic curve* or the *characteristic*, for short. So that we can find a characteristic, passing through any point (x_0, y_0). Along the characteristic curve, our equation (6.1) becomes

$$\frac{du}{ds} + c(x(s), y(s))u = f(x(s), y(s)) \,.$$

The *original PDE becomes an ODE along the characteristic curve!*

One often chooses either x or y as the parameter on characteristics. Observe that from (6.2)

$$\frac{dy}{dx} = \frac{dy/ds}{dx/ds} = \frac{b(x,y)}{a(x,y)} \,. \tag{6.4}$$

If x is chosen as the parameter on the characteristics, then $y = y(x)$, and

$$\frac{d}{dx}u(x,y) = u_x + u_y \frac{dy}{dx} = u_x + u_y \, \frac{b(x,y)}{a(x,y)} \,.$$

Dividing (6.1) by $a(x,y)$, we rewrite it as (we assume that $a(x,y) \neq 0$)

$$\frac{du}{dx} + \frac{c(x,y)}{a(x,y)}\, u = \frac{f(x,y)}{a(x,y)} \,.$$

Then we solve this ODE along the characteristics, beginning at a point where $u(x,y)$ is prescribed. (Here $y = y(x)$, and we solve for $u = u(x)$.)

If y is chosen as the parameter, then $x = x(y)$, and by (6.4)

$$\frac{d}{dy}u(x,y) = u_x \frac{dx}{dy} + u_y = u_x \, \frac{a(x,y)}{b(x,y)} + u_y \,.$$

Dividing (6.1) by $b(x,y)$, we rewrite it as (assuming that $b(x,y) \neq 0$)

$$\frac{du}{dy} + \frac{c(x,y)}{b(x,y)}\, u = \frac{f(x,y)}{b(x,y)} \,,$$

giving an ODE for $u = u(y)$.

Example 1 Find $u = u(x,y)$, solving

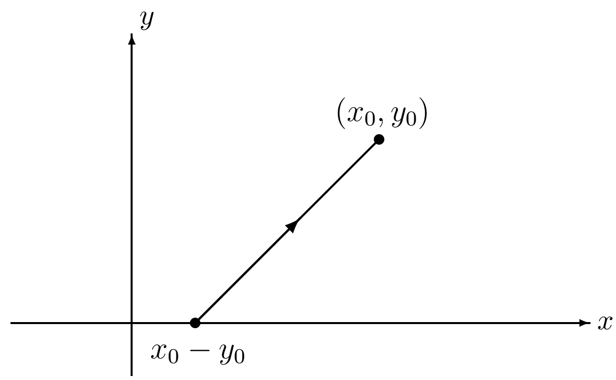
$$\begin{aligned} u_x + u_y &= 1 \\ u(x,0) &= e^x \,. \end{aligned} \tag{6.5}$$

Here the solution (or the "data") is prescribed along the x-axis. By (6.4), the equation to find the characteristics is

$$\frac{dy}{dx} = 1 \,.$$

The characteristics are the straight lines $y = x + c$. The one passing through a point (x_0, y_0) is

$$y = x + y_0 - x_0 \,.$$



Integrating along the characteristic line

It intersects the x axis at $x = x_0 - y_0$. Choosing x as the parameter, the equation in (6.5) becomes

$$\frac{du}{dx} = 1 \,.$$

We integrate this equation along the characteristic line, between the points $(x_0 - y_0, 0)$ and (x_0, y_0), or between the parameter values of $x = x_0 - y_0$ (where $y = 0$) and $x = x_0$ (where $y = y_0$)

$$\int_{x_0 - y_0}^{x_0} \frac{du}{dx}\, dx = \int_{x_0 - y_0}^{x_0} dx \,,$$

$$u(x_0, y_0) - u(x_0 - y_0, 0) = y_0 \,,$$

$$u(x_0, y_0) = u(x_0 - y_0, 0) + y_0 = e^{x_0 - y_0} + y_0 \,.$$

(The data in the second line of (6.5) was used on the last step.) Finally, replace the arbitrary point (x_0, y_0) by (x, y). Answer: $u(x, y) = e^{x-y} + y$.

Example 2 Find $u = u(x, y)$, solving

$$\begin{aligned} u_x + \cos x \, u_y &= \sin x \\ u(0, y) &= \sin y \,. \end{aligned} \tag{6.6}$$

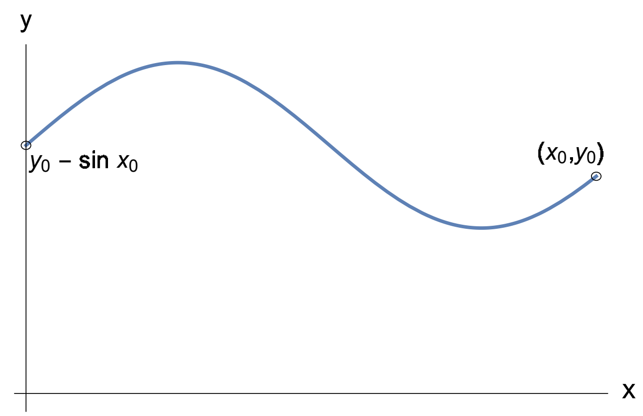


Figure 8.1: A characteristic line for the equation (6.6)

This time the data is given along the y-axis. The characteristics are solutions of

$$\frac{dy}{dx} = \cos x \,,$$

which are $y = \sin x + c$. The one passing through the point (x_0, y_0) is (see Figure 8.1)

$$y = \sin x + y_0 - \sin x_0 \,.$$

It intersects the y-axis at $y = y_0 - \sin x_0$. Choosing x as the parameter, the original equation becomes (along the characteristics)

$$\frac{du}{dx} = \sin x \,.$$

We integrate along the characteristic curve, between the point $(0, y_0 - \sin x_0)$ on the y-axis, where the data is given, and the target point (x_0, y_0)

$$\int_0^{x_0} \frac{du}{dx}\, dx = \int_0^{x_0} \sin x \; dx \,,$$

$$u(x_0, y_0) - u(0, y_0 - \sin x_0) = -\cos x_0 + 1 \,,$$

$$u(x_0, y_0) = u(0, y_0 - \sin x_0) - \cos x_0 + 1 = \sin\left(y_0 - \sin x_0\right) - \cos x_0 + 1 \,.$$

Answer: $u(x, y) = \sin\left(y - \sin x\right) - \cos x + 1$.

Example 3 Find $u = u(x, y)$, solving (here $f(x)$ is a given function)

$$\begin{aligned} &\sin y \, u_x + u_y = e^y \\ &u(x, 0) = f(x) \,. \end{aligned} \tag{6.7}$$

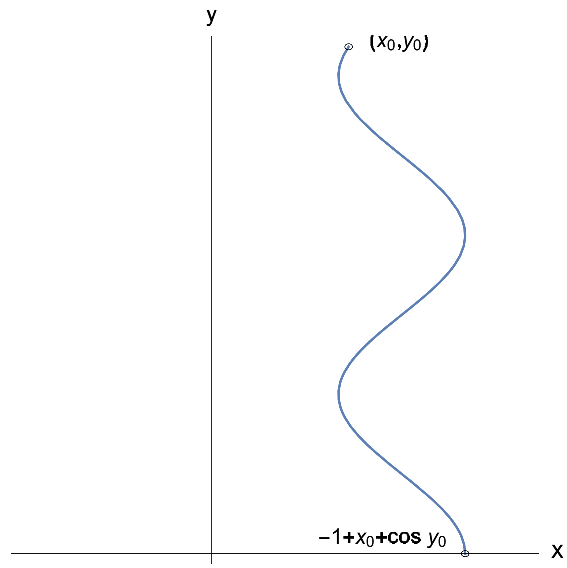


Figure 8.2: A characteristic line for the equation (6.7)

The data $f(x)$ is given along the x-axis. The characteristics are solutions of

$$\frac{dx}{dy} = \sin y\,,$$

which are $x = -\cos y + c$. The one passing through the point (x_0, y_0) is

$$x = -\cos y + x_0 + \cos y_0\,. \tag{6.8}$$

It intersects the x-axis at $x = -1 + x_0 + \cos y_0$. We shall use y as the parameter. (One cannot use x as the parameter, because solving (6.8) for $y = y(x)$ produces multiple answers.) The original equation becomes (along the characteristics)

$$\frac{du}{dy} = e^y\,.$$

We integrate along the characteristic curve, between the points $(-1 + x_0 + \cos y_0, 0)$ and (x_0, y_0), or between the parameter values of $y = 0$ (where $x = -1 + x_0 + \cos y_0$) and $y = y_0$ (where $x = x_0$)

$$\int_0^{y_0} \frac{du}{dy}\,dy = \int_0^{y_0} e^y\;dy\,,$$

$$u(x_0, y_0) - u(-1 + x_0 + \cos y_0, 0) = e^{y_0} - 1\,,$$

$$u(x_0, y_0) = u(-1 + x_0 + \cos y_0, 0) + e^{y_0} - 1 = f(-1 + x_0 + \cos y_0) + e^{y_0} - 1\,.$$

Answer: $u(x, y) = f(-1 + x + \cos y) + e^y - 1$. (This expression may also be seen as the general solution of our equation, considering the function $f(x)$ to be arbitrary.)

Example 4 Find $u = u(x, y)$, solving

$$\begin{aligned} xu_x - yu_y + u &= x \\ u = 1 \;\; &\text{on } y = x\,. \end{aligned} \tag{6.9}$$

The data is given along the line $y = x$. The characteristics are the solutions of

$$\frac{dy}{dx} = -\frac{y}{x}\,,$$

which are the hyperbolas $y = \dfrac{c}{x}$. The one passing through the point (x_0, y_0) is

$$y = \frac{x_0 y_0}{x}\,. \tag{6.10}$$

Let us begin by assuming that the point (x_0, y_0) lies in the first quadrant of the xy-plane, so that $x_0 > 0$ and $y_0 > 0$. The characteristic (6.10) intersects the line $y = x$ at the point $\left(\sqrt{x_0 y_0}, \sqrt{x_0 y_0}\right)$. Taking x as the parameter, our PDE becomes (after dividing by x)

$$\frac{du}{dx} + \frac{1}{x}u = 1\,,$$

or

$$\frac{d}{dx}(xu) = x\,.$$

We integrate along the characteristic curve, between the points $\left(\sqrt{x_0 y_0}, \sqrt{x_0 y_0}\right)$ and (x_0, y_0), or between $x = \sqrt{x_0 y_0}$ (where $y = \sqrt{x_0 y_0}$) and $x = x_0$ (where $y = y_0$), obtaining

$$\int_{\sqrt{x_0 y_0}}^{x_0} \frac{d}{dx}(xu)\, dx = \int_{\sqrt{x_0 y_0}}^{x_0} x\, dx\,,$$

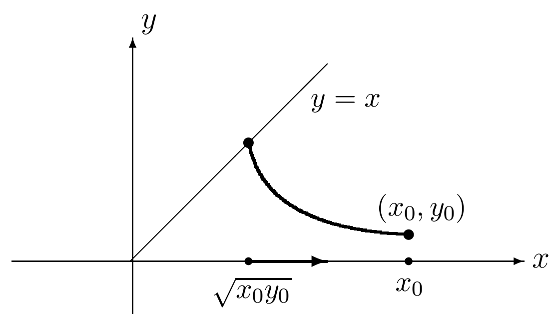
$$x_0 u(x_0, y_0) - \sqrt{x_0 y_0}\, u(\sqrt{x_0 y_0}, \sqrt{x_0 y_0}) = \frac{1}{2}x_0^2 - \frac{1}{2}x_0 y_0\,.$$

(It does not matter which of the limits of integration is larger, $\sqrt{x_0 y_0}$ or x_0.) By the initial condition, $u(\sqrt{x_0 y_0}, \sqrt{x_0 y_0}) = 1$, and then

$$u(x_0, y_0) = \sqrt{\frac{y_0}{x_0}} + \frac{1}{2}x_0 - \frac{1}{2}y_0\,.$$

In case the point (x_0, y_0) lies in the third quadrant, we obtain the same result. In case the point (x_0, y_0) lies in either the second or the fourth quadrants, our method does not apply, because the characteristic hyperbolas do not intersect the line $y = x$.

Answer: $u(x, y) = \sqrt{\frac{y}{x}} + \frac{1}{2}x - \frac{1}{2}y$, in case the point (x, y) lies in either the first or the third quadrants, and no solution exists if the point (x, y) lies in either the second or the fourth quadrants.



Integrating along the characteristic hyperbola

We conclude by observing that the curve, on which the data is prescribed, cannot be a characteristic (or have a part, which is a characteristic). Indeed, if solution is known at some point on a characteristic, it can be computed at all other points along the same characteristic line, and therefore solution cannot be arbitrary prescribed on this characteristic line.

8.6.1 Problems

1. Solve the problem

$$u_x + u_y = 1$$
$$u(x, 0) = e^x\,,$$

by using y as a parameter.

2. Solve the problem

$$xu_x - yu_y + u = x$$
$$u = 1 \text{ on } y = x\,,$$

in case the point (x, y) lies in the third quadrant of the xy-plane.

Answer. $u(x, y) = \sqrt{\frac{y}{x}} + \frac{1}{2}x - \frac{1}{2}y$.

3. Solve for $u = u(x, y)$

$$xu_x + yu_y + u = x,$$
$$u = 1 \text{ on the line } x + y = 1\,.$$

Answer. $u(x, y) = \dfrac{1}{x+y} + \dfrac{x}{2} - \dfrac{x}{2(x+y)^2}$.

4. Find $u = u(x, y)$, solving

$$\sin y\, u_x + u_y = x$$
$$u(x, 0) = x^2\,.$$

Hint: Use y as a parameter. Express x as a function of y, when integrating along the characteristic curve.

Answer. $u(x, y) = (x + \cos y - 1)^2 - \sin y + y\cos y + xy$.

5. Find the general solution of

$$2u_x + u_y = x\,.$$

Hint: Denote by $f(x)$ the values of $u(x, y)$ on the x-axis.

Answer. $u(x, y) = f(x - 2y) + xy - y^2$, where f is an arbitrary function.

6. Show that the following problem has no solution

$$2u_x + u_y = x$$
$$u(x, \tfrac{1}{2}x) = x^2\,.$$

Hint: Compare the data line and the characteristics.

7. (i) Find the general solution of

$$xu_x - yu_y + u = x\,,$$

which is valid in the second quadrant of the xy-plane.

Hint: Let $u = f(x)$ on $y = -x$, where $f(x)$ is an arbitrary function.

Answer. $u(x,y) = \sqrt{-\frac{y}{x}}\, f(-\sqrt{-xy}) + \frac{1}{2}x + \frac{1}{2}y$.

(ii) Show that the problem (6.9) above has no solution in the second quadrant.

8. Solve the problem (here $f(y)$ is an arbitrary function)

$$xu_x + 2yu_y + \tfrac{y}{x}u = 0$$
$$u = f(y) \text{ on the line } x = 1\,.$$

Answer. $u(x,y) = e^{\left(\frac{y}{x^2} - \frac{y}{x}\right)} f\left(\frac{y}{x^2}\right)$.

8.7 Laplace's Equation: Poisson's Integral Formula

A Trigonometric Sum

Let ρ and α be two real numbers, with $0 < \rho < 1$. We claim that

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n\alpha = \frac{1-\rho^2}{2\left(1 - 2\rho\cos\alpha + \rho^2\right)}\,. \tag{7.1}$$

We begin the proof by recalling the geometric series: for any complex number z, with the modulus $|z| < 1$, one has

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\,.$$

Consider a complex number $z = \rho e^{i\alpha}$. Then $|z| = \rho < 1$, and by Euler's formula

$$z^n = \rho^n e^{in\alpha} = \rho^n \left(\cos n\alpha + i \sin n\alpha\right)\,.$$

It follows that

$$\mathrm{Re}\, z^n = \rho^n \cos n\alpha\,,$$

where Re denotes the real part of a complex number. Then

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n\alpha = \mathrm{Re}\left[\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right] = \mathrm{Re}\left[\frac{1}{2} + \frac{1}{1-z} - 1\right] = \mathrm{Re}\left[\frac{1+z}{2(1-z)}\right]$$
$$= \frac{1}{2}\mathrm{Re}\left[\frac{1+\rho e^{i\alpha}}{1-\rho e^{i\alpha}}\right] = \frac{1}{2}\mathrm{Re}\left[\frac{1+\rho\cos\alpha + i\rho\sin\alpha}{1-\rho\cos\alpha - i\rho\sin\alpha}\right] = \frac{1-\rho^2}{2\left(1-2\rho\cos\alpha+\rho^2\right)}\,.$$

On the last step we multiplied both the numerator and the denominator by $1 - \rho\cos\alpha + i\rho\sin\alpha$ (the complex conjugate of the denominator).

Laplace's Equation on a Disc

Recall from Chapter 7 that in order to solve the following boundary value problem in polar coordinates (on a disc $r < R$)

$$\begin{aligned} &\Delta u = u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\theta\theta} = 0, \quad \text{for } r < R,\ 0 \le \theta < 2\pi \\ &u(R,\theta) = f(\theta), \quad \text{for } 0 \le \theta < 2\pi\,, \end{aligned} \tag{7.2}$$

we begin by expanding the given piecewise smooth function $f(\theta)$ into its Fourier series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta\,,$$

with the coefficients given by

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi\,,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos n\phi\, d\phi\,,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin n\phi\, d\phi\,.$$

The solution of the problem (7.2) is then

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta)\ . \tag{7.3}$$

Recall that this solution represents the steady state temperatures inside the disc $x^2 + y^2 < R^2$, provided that the temperatures on the boundary circle $x^2 + y^2 = R^2$ are prescribed by the function $f(\theta)$.

We now substitute the integral formulas for a_n's and b_n's into (7.3), and denote $\rho = \frac{r}{R}$, obtaining

$$u(r,\theta) = \frac{1}{\pi} \int_0^{2\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n (\cos n\phi \cos n\theta + \sin n\phi \sin n\theta) \right] f(\phi)\, d\phi\,.$$

Observing that $\cos n\phi \cos n\theta + \sin n\phi \sin n\theta = \cos n(\theta - \phi)$, and using the formula (7.1), the sum in the square bracket becomes

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos n(\theta - \phi) = \frac{1 - \rho^2}{2\left(1 - 2\rho\cos(\theta - \phi) + \rho^2\right)} = \frac{R^2 - r^2}{2\left(R^2 - 2Rr\cos(\theta - \phi) + r^2\right)}\,.$$

We conclude *Poisson's integral formula*

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(\phi)\, d\phi\,, \tag{7.4}$$

which gives the solution of the boundary value problem (7.2). The function $\dfrac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}$ is called *Poisson's kernel.*

Recall that solutions of the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

are called *harmonic functions.* Poisson's integral formula implies that one can find a harmonic function inside of any disc $x^2+y^2 < R^2$, with arbitrarily prescribed values on the boundary of the disc. Poisson's integral formula is suitable for numerical computations.

Throughout this chapter we consider only the *classical solutions*, which means that $u(x,y)$ has all derivatives in x and y of first and second order, which are continuous functions.

8.8 Some Properties of Harmonic Functions

Setting $r = 0$ in Poisson's formula gives the solution at the origin:

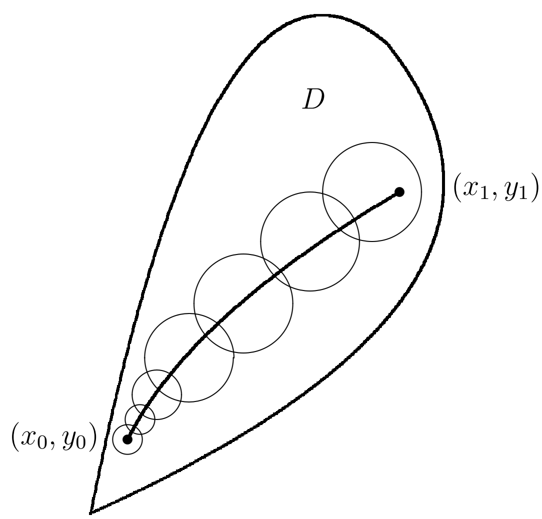
$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi = \frac{1}{2\pi} \int_0^{2\pi} u(R,\phi)\, d\phi\,.$$

So that $u(0,0)$ is equal to the average of the values of u on the circle of any radius R around $(0,0)$. Any point in the xy-plane may be declared to be the origin. We therefore conclude the *mean value property*: the value of a harmonic function at any point (x_0,y_0) is equal to the average of the values of $u(x,y)$ on a circle of any radius R around the point (x_0,y_0).

If a closed curve C encloses a bounded domain D, then we denote $\bar{D} = D \cup C$. $\bar{D}$ is called the *closure* of D. One often writes ∂D to denote the boundary curve C. It is known from calculus that a continuous on $\bar{D}$ function $u(x,y)$ *takes on its maximum and minimum values.* This means that at some point $(x_1,y_1) \in \bar{D}$, $u(x_1,y_1) = \max_{\bar{D}} u(x,y)$, and $u(x_2,y_2) = \min_{\bar{D}} u(x,y)$, at a point $(x_2,y_2) \in \bar{D}$.

Theorem 8.8.1 *(Strong maximum principle) A function $u(x, y)$ which is harmonic in a domain D cannot take on its maximum value at points inside D, unless $u(x, y)$ is a constant.*

Proof: Denote $M = \max_{\bar{D}} u(x, y)$, and assume that $u(x_0, y_0) = M$ at some point $(x_0, y_0) \in D$. We shall show that $u(x, y) = M$ for all points $(x, y) \in D$. Let the number $R > 0$ be so small that the circle of radius R around the point (x_0, y_0) lies inside D. The values of $u(x, y)$ on that circle are $\leq M$, and in fact they have to be equal to M, because otherwise their average would be less than M, but that average is equal to $u(x_0, y_0) = M$. We conclude that $u(x, y) = M$, at all points inside of any circle around (x_0, y_0), which lies inside D. Let now (x_1, y_1) be any other point in D. Join (x_0, y_0) to (x_1, y_1) by any path, and cover that path by small overlapping circles, each lying inside D. Repeating the same argument for all circles, we conclude that $u(x_1, y_1) = M$. $\diamondsuit$



Overlapping circles joining (x_0, y_0) to (x_1, y_1) inside D

The strong maximum principle has the following physical interpretation: for steady state temperatures (which harmonic functions represent), one cannot have a point in D which is hotter than all of its neighbors.

Similarly, one has the *strong minimum principle*: a function $u(x, y)$, which is harmonic in a domain D, cannot take on its minimum value inside

D, unless $u(x,y)$ is a constant. So where do harmonic functions assume their maximum and minimum values? On the boundary ∂D. A function harmonic in the entire plane, like $u(x,y) = x^2 - y^2$, has no points of local maximum and of local minimum in the entire plane, but if you restrict this function to, say, a unit disc $x^2 + y^2 \le 1$, then it takes on its maximum and minimum values on the boundary $x^2 + y^2 = 1$.

We shall need the following estimate of the Poisson kernel:

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \le \frac{R^2 - r^2}{R^2 - 2Rr + r^2} = \frac{(R-r)(R+r)}{(R-r)^2} = \frac{R+r}{R-r}\,,$$

which is obtained by estimating $-\cos(\theta - \phi) \ge -1$, then simplifying. Similarly,

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \ge \frac{R^2 - r^2}{R^2 + 2Rr + r^2} = \frac{R-r}{R+r}\,,$$

which is obtained by estimating $-\cos(\theta - \phi) \le 1$, and simplifying. Combining

$$\frac{R-r}{R+r} \le \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \le \frac{R+r}{R-r}\,.$$

Consider again the boundary value problem (7.2), and assume that $f(\phi) \ge 0$, for all ϕ. Using Poisson's integral formula (7.4), we have

$$\frac{R-r}{R+r}\,\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \le u(r,\theta) \le \frac{R+r}{R-r}\,\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi\,.$$

By the mean value property, we conclude that for any non-negative function $u(r,\theta)$, which is harmonic inside $x^2 + y^2 < R^2$, the following *Harnack's inequalities* hold

$$\frac{R-r}{R+r}\,u(0,0) \le u(r,\theta) \le \frac{R+r}{R-r}\,u(0,0)\,. \tag{8.1}$$

(The assumption that $u(r,\theta) \ge 0$ was needed to assure that $f(\phi) = u(R,\phi) \ge 0$.)

Theorem 8.8.2 *(Liouville's Theorem)* *If a function $u(x,y) \ge 0$ is harmonic in the entire plane, then $u(x,y) = constant$.*

Proof: The estimates (8.1) hold for all R. (Observe that $f(\phi) = u(R,\phi) \ge 0$.) Keeping (r,θ) fixed, we let $R \to \infty$ in (8.1). Then

$$u(r,\theta) = u(0,0) = constant, \quad \text{for any } (r,\theta)\,,$$

as claimed. $\diamondsuit$

Suppose now that a function $u(x,y)$ is harmonic and non-negative, defined on the disc B_R: $x^2+y^2 \le R^2$. We also consider the disc $r \le R/2$ (or $x^2+y^2 \le R^2/4$), which we denote by $B_{R/2}$. Using (8.1), we estimate

$$\max_{B_{R/2}} u(x,y) \le \frac{R+\frac{1}{2}R}{R-\frac{1}{2}R}\, u(0,0) = 3u(0,0)\,.$$

Similarly,

$$\min_{B_{R/2}} u(x,y) \ge \frac{R-\frac{1}{2}R}{R+\frac{1}{2}R}\, u(0,0) = \frac{1}{3}u(0,0)\,.$$

We conclude that

$$\frac{\max_{B_{R/2}} u(x,y)}{\min_{B_{R/2}} u(x,y)} \le 9\,,$$

for *any non-negative harmonic function, defined on the disc* $x^2+y^2 \le R^2$. This fact reflects the strong averaging property of harmonic functions. (More generally, for each bounded domain there is a bound on the ratio of the maximum value over the minimum value for *any non-negative harmonic function* defined on some larger domain.)

8.9 The Maximum Principle

In this section we consider two important classes of functions that include harmonic functions. Not all of these functions satisfy the strong maximum principle. We now describe a substitute property. Recall that D denotes a bounded domain, with the boundary C, and $\bar{D} = D \cup C$.

Theorem 8.9.1 *(Maximum principle) Assume that*

$$\Delta u(x,y) \ge 0 \quad \text{for all } (x,y) \in D\,. \tag{9.1}$$

Then $u(x,y)$ takes on its maximum value on the boundary, so that

$$\max_{\bar{D}} u(x,y) = \max_{C} u(x,y)\,. \tag{9.2}$$

Functions satisfying the inequality (9.1) are called *subharmonic* in D. This theorem asserts that subharmonic functions take on their maximum values at the boundary of the domain. (The possibility that the maximum value is also taken on at points inside D, is not excluded here.)

Proof: Consider, first, an easy case when the inequality in (9.1) is strict

$$\Delta u(x,y) > 0 \qquad \text{for all } (x,y) \in D\,. \tag{9.3}$$

We claim that $u(x,y)$ cannot have points of local maximum inside D. Indeed, if $(x_0,y_0) \in D$ was a point of local maximum, then $u_{xx}(x_0,y_0) \le 0$ and $u_{yy}(x_0,y_0) \le 0$, and therefore

$$\Delta u(x_0,y_0) = u_{xx}(x_0,y_0) + u_{yy}(x_0,y_0) \le 0\,,$$

contradicting (9.3). It follows that the maximum of $u(x,y)$ on $\bar{D}$ is achieved on the boundary curve C, so that (9.2) holds, and moreover

$$u(x,y) < \max_C u(x,y) \qquad \text{for all } (x,y) \in D\,.$$

Turning to the general case, consider the function $v(x,y) = u(x,y) + \epsilon(x^2+y^2)$, with some $\epsilon > 0$. Then

$$\Delta v(x,y) = \Delta u(x,y) + 4\epsilon > 0\,,$$

and by the easy case, considered above,

$$u(x,y) < v(x,y) < \max_C v(x,y) \le \max_C u(x,y) + \epsilon K \quad \text{for all } (x,y) \in D\,,$$

where K is any constant exceeding x^2+y^2 on the bounded closed curve C. Letting $\epsilon \to 0$, we conclude that

$$u(x,y) \le \max_C u(x,y) \qquad \text{for all } (x,y) \in D\,,$$

which implies (9.2). ♢

The following *minimum principle* holds for the *superharmonic functions*, defined as the functions satisfying $\Delta u(x,y) \le 0$ on D.

Theorem 8.9.2 *Assume that*

$$\Delta u(x,y) \le 0 \quad \textit{for all } (x,y) \in D\,.$$

Then $u(x,y)$ takes on its minimum on the boundary, so that

$$\min_{\bar{D}} u(x,y) = \min_C u(x,y)\,.$$

Harmonic functions are both subharmonic and superharmonic, and so they assume their minimum and maximum values on the boundary C. If a harmonic function is zero on C, it has to be zero on D. (This fact also follows from the strong maximum principle.) It follows that the Dirichlet problem for *Poisson's equation*

$$\begin{aligned} \Delta u = f(x,y) \quad & \text{for } (x,y) \in D \\ u = g(x,y) \quad & \text{for } (x,y) \in C \end{aligned}$$

has at most one solution, for any given functions $f(x,y)$ and $g(x,y)$. Indeed, if $u(x,y)$ and $v(x,y)$ are two solutions, then their difference $u(x,y)-v(x,y)$ is harmonic in D function, which is zero on C. It follows that $u(x,y)-v(x,y) = 0$, so that $u(x,y) = v(x,y)$ for all $(x,y) \in D$.

Occasionally one can use the maximum principle to find the maximum value of a function on a bounded domain.

Example Find the maximum value of $u(x,y) = 2x^4 - 3xy^2 + y^6 + x^2 + x - 2y$ on the closed square $[0,1] \times [0,1]$, where $0 \le x, y \le 1$.

Setting the partials u_x and u_y to zero, would lead to an intractable 2×2 nonlinear system. Instead, we calculate

$$\Delta u(x,y) = 24x^2 + 2 - 6x + 30y^4 \ge 24x^2 - 6x + 2 > 0, \text{ for all } (x,y)\,.$$

By the maximum principle, the maximum value of $u(x,y)$ occurs at the boundary of the square. The boundary of the square consists of four line segments, and on each segment $u(x,y)$ is a simple function of one variable. Examining these line segments in turn, one sees that the maximum value of $u(x,y)$ is equal to 4, and it occurs at the point $x=1$, $y=0$.

The reasoning behind the maximum principle may be used to analyze some *nonlinear equations*. Consider, for example, the Dirichlet problem

$$\begin{aligned} \Delta u = u^3 \quad & \text{for } (x,y) \in D \\ u = 0 \quad & \text{for } (x,y) \in C\,. \end{aligned}$$

This problem has the *trivial solution* $u(x,y) = 0$. It turns out that there are no other solutions. Indeed, if a solution $u(x,y)$ was positive at some points in D, it would have a point of global maximum $(x_0,y_0) \in D$, with $u(x_0,y_0) > 0$. At that point, $\Delta u(x_0,y_0) \le 0$, while $u^3(x_0,y_0) > 0$. We have a contradiction with our equation, at the point (x_0,y_0), which implies that $u(x,y)$ cannot take on positive values. Similarly, one shows that $u(x,y)$ cannot take on negative values. It follows that $u(x,y) = 0$ is the only solution.

Maximum and minimum principles also hold in the presence of *lower order terms* (derivatives of the first order).

Theorem 8.9.3 *Assume that for all* $(x, y) \in D$ *we have*

$$\Delta u(x,y) + a(x,y)u_x(x,y) + b(x,y)u_y(x,y) \geq 0\,, \tag{9.4}$$

where $a(x, y)$ *and* $b(x, y)$ *are given continuous functions. Then* $u(x, y)$ *takes on its maximum value on the boundary* C*, so that*

$$\max_{\bar{D}} u(x,y) = \max_{C} u(x,y)\,.$$

Proof: Assume, first, that the inequality in (9.4) is strict. If there was a point of maximum (x_0, y_0) inside D, then $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$, and $u_{xx}(x_0, y_0) \leq 0$, $u_{yy}(x_0, y_0) \leq 0$. Evaluating the strict inequality (9.4) at (x_0, y_0), we would have a contradiction, proving the theorem in this case. The proof of the general case is similar to that for the Theorem 8.9.1. ♢

8.10 The Maximum Principle for the Heat Equation

Recall from Chapter 7 that in case of a bounded interval $(0, L)$, a typical problem involving the heat equation is

$$\begin{aligned} u_t - ku_{xx} &= F(x,t) \quad 0 < x < L,\ 0 < t \leq T \\ u(x,0) &= f(x) \qquad 0 < x < L \\ u(0,t) &= a(t) \qquad 0 < t \leq T \\ u(L,t) &= b(t) \qquad 0 < t \leq T\,, \end{aligned} \tag{10.1}$$

with given continuous functions (called the data) $F(x, t)$, $a(t)$, $b(t)$ and $f(x)$, and a given constant $k > 0$. We assume that the final time $T < \infty$. The data is prescribed on the *parabolic boundary* Γ, which is defined to be consisting of the lines $x = 0$, $x = L$ (for $0 < t \leq T$), and the segment $0 \leq x \leq L$ of the x-axis. The solution must be determined in the *parabolic domain* $D = (0, L) \times (0, T]$, where $0 < x < L$ and $0 < t \leq T$. We shall denote $\bar{D} = D \cup \Gamma$.

Recall from calculus that if a differentiable function $v(t)$, defined on some interval $[0, T]$, has a local maximum at some $t_0 \in (0, T)$ then $v'(t_0) = 0$, while if a local maximum (relative to $[0, T]$) occurs at T, then $v'(T) \geq 0$.

Theorem 8.10.1 *(The maximum principle) Assume that $F(x,t) \le 0$ for all $(x,t) \in D$, or*

$$u_t - ku_{xx} \le 0\,, \quad \textit{for all } (x,t) \in D\,. \tag{10.2}$$

Then $u(x,t)$ takes on its maximum value on the parabolic boundary, so that

$$\max_{\bar{D}} u(x,t) = \max_{\Gamma} u(x,t)\,. \tag{10.3}$$

(In particular, if $u(x,t) \le 0$ on Γ, then $u(x,t) \le 0$ on D.)

Proof: Again, we consider first the case of strict inequality

$$u_t - ku_{xx} < 0\,, \quad \text{for all } (x,t) \in D\,. \tag{10.4}$$

We claim that $u(x,t)$ cannot assume its maximum value at a point (x_0, t_0) in D. Assume, on the contrary, that $(x_0, t_0) \in D$ is a point of maximum of $u(x,t)$. If $t_0 < T$, then $u_t(x_0,t_0) = 0$ and $-u_{xx}(x_0,t_0) \ge 0$, contradicting the inequality (10.4), evaluated at (x_0,t_0). In case $t_0 = T$, we have $u_{xx}(x_0,t_0) \le 0$, and from (10.4) we get $u_t(x_0,t_0) < 0$. But then $u(x_0,t)$ is larger than $u(x_0,t_0)$ at times t a little before t_0, in contradiction with (x_0,t_0) being a point of maximum. So that the point of maximum (of $u(x,t)$ on $\bar{D}$) occurs on the parabolic boundary Γ, and

$$u(x,t) < \max_{\Gamma} u(x,t)\,, \quad \text{for all } (x,t) \in D\,, \tag{10.5}$$

which implies (10.3).

Turning to the general case, we denote $M = \max_{\Gamma} u(x,t)$, and let $v(x,t) = u(x,t) + \epsilon x^2$, with a constant $\epsilon > 0$. Then

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon < 0\,,$$

so that strict inequality holds for $v(x,t)$, and then by (10.5)

$$u(x,t) < v(x,t) < \max_{\Gamma} v(x,t) \le M + \epsilon L^2 \quad \text{for all } (x,t) \in D\,.$$

Letting $\epsilon \to 0$, we conclude that

$$u(x,t) \le M\,, \quad \text{for all } (x,t) \in D\,,$$

which implies (10.3). ♢

Similarly, one establishes the following *minimum principle.*

Theorem 8.10.2 *Assume that $F(x,t) \geq 0$ for all $(x,t) \in D$, or in other words,*

$$u_t - ku_{xx} \geq 0 \quad \textit{for all } (x,t) \in D\,.$$

Then $u(x,t)$ takes on its minimum value on the parabolic boundary, and

$$\min_{\bar{D}} u(x,t) = \min_{\Gamma} u(x,t)\,.$$

(In particular, if $u(x,t) \geq 0$ on Γ, then $u(x,t) \geq 0$ on D.)

For the homogeneous heat equation, where $F(x,t) = 0$, both minimum and maximum principles apply. As a consequence, the problem (10.1) has at most one solution (one shows that the difference of any two solutions is zero).

We have the following *comparison theorem.*

Theorem 8.10.3 *Assume we have two functions $u(x,t)$ and $v(x,t)$, such that*

$$u_t - ku_{xx} \geq v_t - kv_{xx} \;\; \textit{in } D, \;\; \textit{and } u \geq v \;\; \textit{on } \Gamma\,.$$

Then $u(x,t) \geq v(x,t)$ in D.

Proof: The function $w(x,t) = u(x,t) - v(x,t)$ satisfies $w_t - kw_{xx} \geq 0$ in D, and $w \geq 0$ on Γ. By the minimum principle $w \geq 0$ in D. ♢

More information on maximum principles may be found in a nice book of M.H. Protter and H.F. Weinberger [24].

8.10.1 Uniqueness on an Infinite Interval

We begin with discussion of a remarkable function

$$g(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}\,.$$

This function is positive for $t \neq 0$, however $g(0) = 0$, and $g'(0) = g''(0) = g'''(0) = \cdots = 0$, so that all derivatives at $t = 0$ are zero. Indeed,

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{e^{-\frac{1}{t^2}}}{t}\,.$$

Letting $\frac{1}{t} = u$, we evaluate this limit by using L'Hospital's rule:

$$\lim_{t \to 0} \frac{e^{-\frac{1}{t^2}}}{t} = \lim_{u \to \infty} u e^{-u^2} = \lim_{u \to \infty} \frac{u}{e^{u^2}} = \lim_{u \to \infty} \frac{1}{2u e^{u^2}} = 0 .$$

It follows that

$$g'(t) = \begin{cases} \frac{2}{t^3} e^{-\frac{1}{t^2}} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases} .$$

The derivatives $g''(0)$, $g'''(0)$, and so on, are evaluated similarly.

The initial value problem on the entire x-axis

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } -\infty < x < \infty,\ t > 0 \\ u(x,0) &= 0 \quad \text{for } -\infty < x < \infty \end{aligned} \tag{10.6}$$

has the trivial solution $u(x,t) = 0$. Surprisingly, this problem also has non-trivial solutions! Here is one of them:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} , \tag{10.7}$$

where $g(t)$ is the function just defined. (It is not hard to show that this series converges for all x, and all $t > 0$.) Clearly $u(x,0) = 0$, because $g^{(k)}(0) = 0$ for any derivative k. Compute

$$u_{xx} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k-2)!} x^{2k-2} = \sum_{i=0}^{\infty} \frac{g^{(i+1)}(t)}{(2i)!} x^{2i} = u_t ,$$

where we shifted the index of summation, $k \to i$, by letting $k - 1 = i$.

It follows that the problem

$$\begin{aligned} u_t - u_{xx} &= F(x,t) \quad \text{for } -\infty < x < \infty,\ t > 0 \\ u(x,0) &= g(x) \quad \text{for } -\infty < x < \infty \end{aligned} \tag{10.8}$$

has infinitely many solutions, provided that it has one solution. Indeed, to any solution one may add a constant multiple of the function in (10.7), to get other solutions.

The function $g(t)$ appears in other *counterexamples*, or the examples that challenge our intuition. For example, the Maclauren series for $g(t)$ is a sum of zeroes, and it converges to zero, *not to* $g(t)$.

We shall show that the problem (10.8) has at most one *bounded solution*, which means that $|u(x,t)| < M$ for some constant $M > 0$, and all $-\infty < x < \infty$, and $t > 0$. This fact will follow immediately from the following theorem.

Theorem 8.10.4 *Assume that we have a solution $u(x,t)$ of the problem*

$$u_t - ku_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ t > 0 \tag{10.9}$$
$$u(x,0) = 0 \quad \text{for } -\infty < x < \infty\,,$$

which is a bounded (for all x and t) function. Then $u(x,t) = 0$, for all x and t.

Proof: We are given that $|u(x,t)| \le M$ for some $M > 0$, and all x and t (or $-M \le u(x,t) \le M$). In the region $D: -L < x < L$, $0 < t \le T$, with some positive constants L and T, we consider the *comparison function*

$$v(x,t) = \frac{2M}{L^2}\left(\frac{1}{2}x^2 + kt\right) .$$

One calculates

$$v_t - kv_{xx} = 0\,.$$

On the parabolic boundary of the region D we have

$$v(x,0) = \frac{M}{L^2}x^2 \ge 0 = u(x,0)\,,$$

$$v(\pm L,t) = M + \frac{2Mk}{L^2}t \ge M \ge u(\pm L,t)\,.$$

By the comparison Theorem 8.10.3

$$u(x,t) \le v(x,t) \ \text{ in } D\,.$$

The function $-v(x,t)$ satisfies

$$(-v)_t - k(-v)_{xx} = 0\,,$$

$$-v(x,0) = -\frac{M}{L^2}x^2 \le 0 = u(x,0)\,,$$

$$-v(\pm L,t) = -M - \frac{2Mk}{L^2}t \le -M \le u(\pm L,t)\,.$$

Using the comparison Theorem 8.10.3 again, we conclude that

$$-v(x,t) \le u(x,t) \le v(x,t) \ \text{ in } D\,,$$

which gives

$$|u(x,t)| \le v(x,t) = \frac{2M}{L^2}\left(\frac{1}{2}x^2 + kt\right).$$

Letting here $L \to \infty$, we conclude that $u(x,t) = 0$ for any fixed x and t. $\diamondsuit$

As a consequence, we have the following uniqueness theorem.

Theorem 8.10.5 *For any given functions $F(x,t)$ and $f(x)$, the problem*

$$\begin{aligned} u_t - ku_{xx} &= F(x,t) \quad \textit{for } -\infty < x < \infty,\ t > 0 \\ u(x,0) &= f(x) \quad \textit{for } -\infty < x < \infty \end{aligned}$$

has at most one bounded solution.

Proof: The difference of any two bounded solutions would be a bounded solution of the problem (10.9), which is zero by the preceding theorem. $\diamondsuit$

8.11 Dirichlet's Principle

Recall the concept of divergence a vector field $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z)\,.$$

An example of a vector field is given by the gradient of any function $w(x,y,z)$, namely $\nabla w = (w_x(x,y,z), w_y(x,y,z), w_z(x,y,z))$. One calculates

$$\text{div}\,(\nabla w) = w_{xx} + w_{yy} + w_{zz} = \Delta w\,. \tag{11.10}$$

Suppose that a bounded domain D in (x,y,z) space, is bounded by a closed and smooth surface S. The *divergence theorem* reads:

$$\int_D \text{div}\,\mathbf{F}\,dV = \int_S \mathbf{F} \cdot \mathbf{n}\,dS\,.$$

Here $\int_D$ denotes the triple integral over D, $\int_S$ is the surface (double) integral over S, $\mathbf{n}$ is the unit normal vector pointing outside of D ($\mathbf{n}$ is changing from point to point on S), $\mathbf{F} \cdot \mathbf{n}$ denotes the scalar product of two vectors, and

$dV = dx\,dy\,dz$ is the volume element. Applying the divergence theorem to (11.10) gives

$$\int_D \Delta w\, dV = \int_S \nabla w \cdot \mathbf{n}\, dS = \int_S \frac{\partial w}{\partial n}\, dS\,,$$

where $\frac{\partial w}{\partial n}$ denotes the directional derivative in the direction of $\mathbf{n}$.

Given two functions $u(x,y,z)$ and $v(x,y,z)$, one calculates

$$\text{div } (v\nabla u) = \Delta u\, v + \nabla u \cdot \nabla v\,.$$

By the divergence theorem

$$\int_D (\Delta u\, v + \nabla u \cdot \nabla v)\; dV = \int_D \text{div}\,(v \nabla u)\; dV = \int_S v \nabla u \cdot \mathbf{n}\, dS\,,$$

or

$$\int_D \Delta u\, v\, dV = -\int_D \nabla u \cdot \nabla v\, dV + \int_S \frac{\partial u}{\partial n}\, v\, dS\,. \tag{11.11}$$

This formula is called *Green's identity*; it extends the integration by parts formula to higher dimensions.

We now apply Green's identity to give another proof of the uniqueness of solution of the Dirichlet problem for Poisson's equation.

Theorem 8.11.1 *Given any $f(x,y,z)$ and $g(x,y,z)$, there exists at most one solution of the boundary value problem*

$$\begin{aligned} \Delta u &= f(x,y,z) \quad \text{in } D \\ u &= g(x,y,z) \quad \text{on } S\,. \end{aligned} \tag{11.12}$$

Proof: Assume that there are two solutions $u(x,y,z)$ and $v(x,y,z)$. Their difference $w = u - v$ satisfies

$$\Delta w = 0 \;\text{ in } D\,, \;\; w = 0 \;\text{ on } S\,.$$

We multiply the last equation by w, and integrate over D. In view of (11.11)

$$0 = \int_D w \Delta w\, dV = -\int_D \nabla w \cdot \nabla w\, dV + \int_S w\, \frac{\partial w}{\partial n}\, dS = -\int_D |\nabla w|^2\, dV\,.$$

(We used that $\int_S w\, \frac{\partial w}{\partial n}\, dS = 0$, because $w = 0$ on S; $|\nabla w|$ denotes the length of the gradient vector.) It follows that $\nabla w = 0$, so that $w(x,y,z)$ is a constant. This constant is zero, because of the boundary condition. So that $w = u - v \equiv 0$, and then $u \equiv v$ in D. ♢

Dirichlet's principle says that the solution of the boundary value problem (11.12) minimizes the following *energy functional*

$$J(u) = \int_D \left[\frac{1}{2}|\nabla u|^2 + uf\right] dV\,.$$

among all functions satisfying the boundary condition in (11.12). (Here $|\nabla u|$ denotes the length of the gradient vector, $|\nabla u|^2 = \nabla u \cdot \nabla u$.)

Theorem 8.11.2 *Assume that $u(x, y, z)$ is a solution of (11.12). Then*

$$J(u) = \min_w J(w)\,, \quad \text{where } w = g(x, y, z) \text{ on } S\,. \tag{11.13}$$

Conversely, if $u(x, y, z)$ satisfies (11.13), then it is a solution of the boundary value problem (11.12).

Proof: Part 1. If u is a solution of (11.12), then $u - w = 0$ on S. Multiply the equation in (11.12) by $u - w$, integrate over D, and use Green's identity

$$0 = \int_D \left[\Delta u\,(u - w) - f(u - w)\right] dV = \int_D \left[-\nabla u \cdot \nabla(u - w) - f(u - w)\right] dV\,.$$

(Observe that $\int_S \frac{\partial u}{\partial n}\,(u - w)\,dS = 0$.) It follows that

$$\begin{aligned} \int_D \left[|\nabla u|^2 + uf\right] dV &= \int_D \left[\nabla u \cdot \nabla w + wf\right] dV \\ &\le \frac{1}{2}\int_D |\nabla u|^2\,dV + \int_D \left[\frac{1}{2}|\nabla w|^2 + wf\right] dV\,. \end{aligned} \tag{11.14}$$

On the last step we used the Cauchy-Schwarz inequality for vectors $\nabla u \cdot \nabla w \le |\nabla u|\,|\nabla w|$, followed by the numerical inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$. Rearranging the terms in (11.14), we conclude that $J(u) \le J(w)$ for any w, satisfying $w = g$ on S.

Part 2. Conversely, assume that u minimizes $J(u)$ among the functions w, satisfying $w = g$ on S. Fix any $v(x, y, z)$, satisfying $v = 0$ on S. Then for any number ϵ the function $u + \epsilon v$ is equal to g on S, and therefore $J(u) \le J(u + \epsilon v)$. It follows that the function $j(\epsilon) = J(u + \epsilon v)$ has a minimum at $\epsilon = 0$, and therefore $j'(0) = 0$. Calculate

$$\begin{aligned} j(\epsilon) = J(u + \epsilon v) &= \int_D \left[\frac{1}{2}\left(\nabla u + \epsilon \nabla v\right) \cdot \left(\nabla u + \epsilon \nabla v\right) + \left(u + \epsilon v\right) f\right] dV \\ &= \frac{\epsilon^2}{2}\int_D |\nabla v|^2\,dV + \epsilon \int_D \left(\nabla u \cdot \nabla v + vf\right) dV + \int_D \left(\frac{1}{2}|\nabla u|^2 + uf\right) dV\,. \end{aligned}$$

It follows that

$$0 = j'(0) = \int_D (\nabla u \cdot \nabla v + vf)\, dV = \int_D (-\Delta u + f)\, v\, dV\,,$$

using Green's formula on the last step. Since the function v is arbitrary (except for the condition $v = 0$ on S), we conclude that $\Delta u = f$ in D. $\diamondsuit$

One often refers to the functions w satisfying $w = g$ on S as the *competing functions*. Among the competing functions one searches for the minimizer of $J(u)$, which provides the solution of the boundary value problem (11.12). This approach is suitable for numerical computations.

8.12 Classification Theory for Two Variables

The equation

$$\varphi(x, y) = c\,, \tag{12.15}$$

where $\varphi(x, y)$ is a differentiable function of two variables, and c is a constant, defines implicitly a function $y = y(x)$. Differentiating (12.15) using the chain rule gives

$$\varphi_x(x, y) + \varphi_y(x, y)y'(x) = 0\,,$$

and then

$$y'(x) = -\frac{\varphi_x}{\varphi_y}\,, \tag{12.16}$$

assuming that $\varphi_y(x, y) \neq 0$.

We wish a solution $z = z(x, y)$ of the following nonlinear first order PDE

$$a(x, y)z_x^2 + b(x, y)z_x z_y + c(x, y)z_y^2 = 0\,, \tag{12.17}$$

with given continuous functions $a(x, y)$, $b(x, y)$ and $c(x, y)$. Similarly to linear first order equations, one needs to solve an ODE.

Lemma 8.12.1 *Assume that the function $y(x)$, which is implicitly defined by $\varphi(x, y) = c$, solves the equation*

$$a(x, y){y'}^2 - b(x, y)y' + c(x, y) = 0\,, \tag{12.18}$$

and $\varphi_y(x, y) \neq 0$ for all x and y. Then $z = \varphi(x, y)$ is a solution of (12.17).

Proof: Substituting $y'(x) = -\frac{\varphi_x}{\varphi_y}$ into (12.18), and then clearing the denominators gives

$$a(x,y)\varphi_x^2 + b(x,y)\varphi_x\varphi_y + c(x,y)\varphi_y^2 = 0\,,$$

so that $\varphi(x,y)$ is a solution of (12.17). ♢

The equation (12.18) is just a quadratic equation for $y'(x)$. Its solutions are

$$y'(x) = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{12.19}$$

One finds $y(x)$ by integration, in case $b^2 - 4ac \geq 0$.

Example 1 Find two solutions of

$$xz_x^2 + (x+y)z_x z_y + yz_y^2 = 0\,. \tag{12.20}$$

The equation (12.18) takes the form

$$x\left(\frac{dy}{dx}\right)^2 - (x+y)\frac{dy}{dx} + y = 0\,.$$

Solving this quadratic equation gives

$$\frac{dy}{dx} = \frac{x + y \pm \sqrt{(x+y)^2 - 4xy}}{2x} = \frac{x + y \pm (x-y)}{2x}\,.$$

When we take "plus", we obtain $\frac{dy}{dx} = 1$ or $y = x + c$. We put this function into an implicit form $y - x = c$, which gives us the first solution $z_1(x,y) = y - x$. In case of "minus", we get

$$\frac{dy}{dx} = \frac{y}{x}\,.$$

The solution of this equation is $y = cx$, or $\frac{y}{x} = c$ in implicit form. The second solution is $z_2(x,y) = \frac{y}{x}$. There are other solutions of the equation (12.20), for example $z_3(x,y) = c$, or the negatives of $z_1(x,y)$ and $z_2(x,y)$.

Recall that the wave equation was solved by introducing new variables ξ and η, which reduced it to a simpler form. We consider now more general equations

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y = 0\,, \tag{12.21}$$

with given continuous coefficient functions $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$ and $e(x,y)$, and the unknown function $u = u(x,y)$. We make a change of variables $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, and look for the twice differentiable functions $\xi(x, y)$ and $\eta(x, y)$, which will make this equation simpler. We assume that the change of variables is *non-singular*, meaning that one can solve for x and y as functions of ξ and η, so that we can go back to the original variables after solving the equation in the new variables ξ and η. This is known to be the case, provided that the *Jacobian* (or *Jacobian determinant*)

$$J = \xi_x \eta_y - \xi_y \eta_x \neq 0\,, \tag{12.22}$$

a condition we shall assume to hold.

Writing

$$u(x, y) = u(\xi(x, y), \eta(x, y))\,,$$

we use the chain rule to calculate the derivatives:

$$u_x = u_\xi \xi_x + u_\eta \eta_x\,,$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}\,.$$

Similarly

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \eta_x\xi_y\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}\,,$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}\,.$$

We use these expressions in (12.21) to obtain

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + \cdots = 0\,, \tag{12.23}$$

with the new coefficient functions

$$\begin{aligned} A &= a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2\,, \\ B &= 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y\,, \\ C &= a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2\,, \end{aligned} \tag{12.24}$$

and where the terms not shown in (12.23) involve the first derivatives u_ξ and u_η, the lower order terms.

The equation (12.23) will be simpler if one chooses $\xi(x, y)$ and $\eta(x, y)$ to be the solutions of (12.17), called the *characteristic functions* (or *characteristics*, for short). The number of real valued characteristic functions depends on the sign of $b^2(x, y) - 4a(x, y)c(x, y)$ (see (12.19)). The equation (12.21) is called

hyperbolic at a point (x_0, y_0), if $b^2(x_0, y_0) - 4a(x_0, y_0)c(x_0, y_0) > 0$,
parabolic at a point (x_0, y_0), if $b^2(x_0, y_0) - 4a(x_0, y_0)c(x_0, y_0) = 0$,
elliptic at a point (x_0, y_0), if $b^2(x_0, y_0) - 4a(x_0, y_0)c(x_0, y_0) < 0$.

In the the hyperbolic case, the change of variables $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ makes $A = C = 0$, and dividing the equation (12.23) by $B \neq 0$, we obtain the *canonical form*

$$u_{\xi\eta} + \cdots = 0\,,$$

similarly to the wave equation.

How did we know that $B \neq 0$? This fact follows by the formula

$$B^2 - 4AC = (b^2 - 4ac)J^2\,.$$

which is verified by using the above expressions for A, B and C. The same formula also shows that *the type* of an equation is preserved by a non-singular change of variables ($J \neq 0$).

In the the parabolic case, when

$$b^2 - 4ac = 0\,, \tag{12.25}$$

there is only one characteristic function $\xi(x, y)$. Choosing $\xi = \xi(x, y)$, we make $A = 0$, eliminating one term in (12.23). We choose $\eta = \bar{\eta}(x, y)$, where the function $\bar{\eta}(x, y)$ is almost arbitrary, with the only requirement being that the Jacobian J (defined in (12.22)) is non-zero. Comparing (12.19) with (12.16), and using (12.25)

$$\frac{b}{2a} = -\frac{\xi_x}{\xi_y}\,,$$

or

$$2a\xi_x + b\xi_y = 0\,.$$

Then

$$B = \eta_x\left(2a\xi_x + b\xi_y\right) + \eta_y\left(2c\xi_y + b\xi_x\right) = \eta_y\xi_y\left(2c + b\,\frac{\xi_x}{\xi_y}\right)$$
$$= \eta_y\xi_y\left(2c - \frac{b^2}{2a}\right) = \frac{\eta_y\xi_y(4ac - b^2)}{2a} = 0\,.$$

The result is the *canonical form of parabolic equations*

$$u_{\eta\eta} + \cdots = 0\,,$$

which is consistent with the heat equation.

In the elliptic case there are no real characteristic functions, however there are two complex conjugate ones $\xi(x,y)$ and $\bar{\xi}(x,y)$. Making the change of variables $\xi = \xi(x,y)$ and $\eta = \bar{\xi}(x,y)$, we obtain similarly to the hyperbolic case

$$u_{\xi\eta} + \cdots = 0\,. \tag{12.26}$$

(The coefficients of the lower order terms are complex valued, in general.) We now make a further change of variables $(\xi,\eta) \to (\alpha,\beta)$:

$$\alpha = \frac{\xi - \eta}{2i}\,, \quad \beta = \frac{\xi + \eta}{2}\,,$$

which takes (12.26) into the canonical form for elliptic equations

$$u_{\alpha\alpha} + u_{\beta\beta} + \cdots = 0\,. \tag{12.27}$$

The resulting change of variables $(x,y) \to (\alpha,\beta)$ is real valued, and so the lower terms in (12.27) have real valued coefficients.

Example 2 Let us find the canonical form of the equation

$$x^2 u_{xx} - y^2 u_{yy} - 2y u_y = 0\,. \tag{12.28}$$

Here $b^2 - 4ac = 4x^2y^2$, so that the equation is hyperbolic at all points of the xy-plane, except for the coordinate axes. The equation (12.19) takes the form

$$y' = \frac{\pm\sqrt{4x^2y^2}}{2x^2} = \pm\frac{y}{x}\,,$$

assuming that $xy > 0$. The solution of $y' = \frac{y}{x}$ is $y = cx$ or $\frac{y}{x} = c$. So that $\xi = \frac{y}{x}$. The solution of $y' = -\frac{y}{x}$ is $y = \frac{c}{x}$ or $xy = c$. So that $\eta = xy$. The change of variables

$$\xi = \frac{y}{x}\,, \quad \eta = xy$$

produces the canonical form of our equation

$$u_{\xi\eta} + \frac{1}{2\xi}u_\eta = 0\,.$$

One can now solve the original equation (12.28). Setting $v = u_\eta$, we obtain an ODE

$$v_\xi = -\frac{1}{2\xi}v$$

with the solution $v = \xi^{-\frac{1}{2}} F(\eta)$, where $F(\eta)$ is an arbitrary function. So that $u_\eta = \xi^{-\frac{1}{2}} F(\eta)$. Another integration in η gives $u = \xi^{-\frac{1}{2}} F(\eta) + G(\xi)$, where $G(\xi)$ is an arbitrary function. Returning to the original variables, one concludes that $u(x,y) = \sqrt{\frac{x}{y}} F(xy) + G(\frac{y}{x})$ is the general solution of (12.28).

Example 3 The equation

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0\,.$$

is of parabolic type for all x and y. The equation (12.19) becomes

$$y' = -\frac{y}{x}\,,$$

with the general solution $y = \frac{c}{x}$. This leads to $\xi = xy$, and we choose arbitrarily $\eta = y$. (The Jacobian $J = \xi_x \eta_y - \xi_y \eta_x = y \neq 0$, under the assumption that $y \neq 0$.) This change of variables produces the canonical form

$$\eta u_{\eta\eta} + u_\eta = 0\,.$$

Writing the last equation in the form $\left(\eta u_\eta\right)_\eta = 0$, and integrating twice in η, we obtain its general solution

$$u = F(\xi) \log|\eta| + G(\xi)\,,$$

with arbitrary functions $F(\xi)$ and $G(\xi)$. The general solution of the original equation is then

$$u(x,y) = F(xy) \log|y| + G(xy)\,, \quad \text{for } y \neq 0\,.$$

Example 4 *Tricomi's equation*

$$u_{xx} + x u_{yy} = 0$$

changes type: it is elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$. Let us find its canonical form in the elliptic case $x > 0$. The equation (12.19) gives

$$y' = \pm i\sqrt{x}\,,$$

or $y = \pm \frac{2}{3} i x^{\frac{3}{2}} + c$. The complex-valued characteristics are $\xi = y + \frac{2}{3} i x^{\frac{3}{2}}$ and $\bar{\xi} = y - \frac{2}{3} i x^{\frac{3}{2}}$. Then $\alpha = \frac{\xi - \eta}{2i} = \frac{2}{3} x^{\frac{3}{2}}$, and $\beta = \frac{\xi + \eta}{2} = y$. We have $u_x = u_\alpha \alpha_x$, $u_{xx} = u_{\alpha\alpha} \alpha_x^2 + u_\alpha \alpha_{xx} = u_{\alpha\alpha} x + u_\alpha \frac{1}{2} x^{-\frac{1}{2}}$, and $u_{yy} = u_{\beta\beta}$. The equation transforms as

$$x\left(u_{\alpha\alpha} + u_{\beta\beta}\right) + \frac{1}{2} x^{-\frac{1}{2}} u_\alpha = 0\,,$$

which leads to the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\alpha}\, u_\alpha = 0\,.$$

The wave equation is the main example of hyperbolic equations, the heat equation is the best known parabolic equation, and Laplace's equation is an example, as well as the canonical form, for elliptic equations. Our study of these three main equations suggests what to expect of other equations of the same type: sharp signals and finite propagation speed for hyperbolic equations, diffusion and infinite propagation speed for parabolic equations, maximum principles and smooth solutions for elliptic equations. These facts are justified in more advanced PDE books, see e.g., L. Evans [9].

8.12.1 Problems

I. 1. Assume that the function $u(x,y)$ is harmonic in the entire plane, and $u(x,y) > -12$ for all (x,y). Show that $u(x,y)$ is a constant.

Hint: Consider $v(x,y) = u(x,y) + 12$.

2. Assume that the function $u(x,y)$ is harmonic in the entire plane, and $u(x,y) < 0$ for all (x,y). Show that $u(x,y)$ is a constant.

Hint: Consider $v(x,y) = -u(x,y)$.

3. Prove that a harmonic in the entire plane function cannot be bounded from below, or from above, unless it is a constant.

4. Assume that the function $u(x,y)$ is harmonic in D, and $u(x,y) = 5$ on ∂D. Show that $u(x,y) = 5$ in D.

5. Calculate the integral $\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\, d\phi$, where R, r and θ are parameters.

Hint: Identify this integral with the solution of a certain Dirichlet problem, given by Poisson's integral formula.

6. Let D be the square: $-1 < x < 1$, $-1 < y < 1$. Assume that $u(x,y)$ satisfies

$$\Delta u = -1 \text{ in } D\,, \quad u = 0 \text{ on } \partial D\,.$$

Show that $\dfrac{1}{4} \le u(0,0) \le \dfrac{1}{2}$.

Hint: Consider $v(x,y) = u(x,y) + \frac{1}{4}\left(x^2+y^2\right)$, and Δv.

7. Show that any solution of the nonlinear problem

$$\Delta u + u^2(1-u) = 0 \ \text{ in } D, \quad u = 0 \ \text{ on } \partial D$$

satisfies $0 \le u \le 1$.

Hint: Show that $u(x,y)$ cannot take on a maximum value, which is greater than 1, and a negative minimum value.

8. Let $a(x,y) > 0$ be a given positive function. Show that the problem

$$\begin{aligned} \Delta u - a(x)u = 0 \quad & \text{in } D \\ u = 0 \quad & \text{on } \partial D \end{aligned}$$

has only the trivial solution $u(x) = 0$.

9. Find the absolute maximum of $u(x,y) = y^4 + 2x^2y^2 + x^4 - x^2 + y^2$ on the disc $x^2 + y^2 \le 4$.

10. Show that the nonlinear problem

$$\Delta u(x,y) + yu_x(x,y) - 2xu_y(x,y) - u^5(x,y) = 0 \ \text{ in } D, \quad u = 0 \ \text{ on } \partial D$$

has no non-trivial solutions.

11. Show that the solution of

$$\begin{aligned} u_t - 5u_{xx} = x^2 + t^2 - t + 1 \quad & 0 < x < 4, \quad t > 0 \\ u(x,0) = 0 \quad & 0 < x < 4 \\ u(0,t) = 1 \quad & t > 0 \\ u(4,t) = \sin t \quad & t > 0 \end{aligned}$$

is positive for $0 < x < 4$, $0 < t < \pi$.

12. Assume that the function $u(x,y)$ is harmonic, satisfying $u(0,0) = 0$ and $u(1,0) = 3$. Show that $u(x,y)$ cannot be non-negative for all (x,y) satisfying $x^2 + y^2 \le 4$.

Hint: Use the mean value property.

13. Assume that the function $u(x,y)$ is harmonic, satisfying $u(0,0) = 1$ and $u(0,1) = 10$. Show that $u(x,y)$ cannot be non-negative for all (x,y) satisfying $x^2 + y^2 \le 4$.

Hint: Use Harnack's inequality, or (8.1).

14. Assume that $u(x,t)$ satisfies

$$u_t - u_{xx} + c(x,t)u \geq 0 \quad \text{for } 0 < x < L,\, 0 < t < T\,,$$
$$u(x,0) \geq 0, \;\; u(0,t) \geq 0, \;\; u(L,t) \geq 0 \quad \text{for } 0 < x < L,\, 0 < t < T\,,$$

where $c(x,t)$ is *any* function satisfying $|c(x,t)| \leq M$ for all $0 < x < L$, $0 < t < T$, and some constant $M > 0$. Show that

$$u(x,t) \geq 0 \quad \text{for } 0 < x < L,\, 0 < t < T\,.$$

Hint: Assume first that $c(x,t) > 0$, and get a contradiction at any point, where $u(x,t)$ assumes a negative minimum. If $u(x,t) = e^{\alpha t} v(x,t)$, then $v(x,t)$ satisfies

$$v_t - v_{xx} + (\alpha + c(x,t))\, v \geq 0\,,$$

and $\alpha + c(x,t) > 0$, if the constant α is chosen large enough.

15. Show that there is at most one solution of the nonlinear problem

$$u_t - u_{xx} + u^2 = 0 \quad \text{for } (x,t) \in D = (0,L) \times (0,T]\,,$$

if the values of $u(x,t)$ are prescribed on the parabolic boundary Γ.

16. (i) Let $f(v)$ be a convex function for all $v \in R$. Assume that $\varphi(x) > 0$ on (a,b), and $\int_a^b \varphi(x)\, dx = 1$. Prove *Jensen's inequality*

$$f\left(\int_a^b u(x)\varphi(x)\, dx\right) \leq \int_a^b f\left(u(x)\right)\varphi(x)\, dx\,,$$

for any function $u(x)$ defined on (a,b).

Hint: A convex function lies above any of its tangent lines, so that for any p and q

$$f(q) \geq f(p) + f'(p)(q-p)\,.$$

Set here $q = u(x)$ and $p = \int_a^b u(x)\varphi(x)\, dx$

$$f(u(x)) \geq f\left(\int_a^b u(x)\varphi(x)\, dx\right) + f'(p)\left[u(x) - \int_a^b u(x)\varphi(x)\, dx\right].$$

Multiply both sides by $\varphi(x)$, and integrate over (a,b).

(ii) Consider a nonlinear heat equation

$$\begin{gathered} u_t = u_{xx} + f(u)\,, \;\; 0 < x < \pi\,, \;\; t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = u_0(x)\,. \end{gathered}$$

Assume that $f(u)$ is a convex function for all $u \in R$. Assume there is an $\bar{u}$, so that $f(u) - u > 0$ for $u > \bar{u}$, and $\int_{\bar{u}}^{\infty} \frac{du}{f(u)-u} < \infty$. Assume finally that $\frac{1}{2}\int_0^{\pi} u_0(x) \sin x \, dx > \bar{u}$. Show that the solution blows up in finite time.

Hint: Multiply the equation by $\varphi(x) = \frac{1}{2}\sin x$ and integrate over $(0, \pi)$. Denote $v(t) = \frac{1}{2}\int_0^{\pi} u(x,t) \sin x \, dx$. Integrating by parts twice, we express $\frac{1}{2}\int_0^{\pi} u_{xx}(x,t) \sin x \, dx = -\frac{1}{2}\int_0^{\pi} u(x,t) \sin x \, dx = -v(t)$. Applying Jensen's inequality gives

$$\frac{dv}{dt} \geq -v + f(v) \,,$$

or $\frac{dv}{-v+f(v)} \geq dt$. It follows that $v(t)$ becomes infinite by the time $t = \int_{\bar{u}}^{\infty} \frac{dv}{f(v)-v} < \infty$.

(iii) Let $f(u) = u^2$, $u_0(x) = 4 \sin x$. Show that the solution becomes unbounded by the time $t = \ln \frac{\pi}{\pi - 1}$.

Hint: Here $\frac{dv}{dt} \geq -v + v^2$ and $v(0) = \pi$.

II. 1. Find two solutions of

$$z_x^2 - y z_y^2 = 0 \,.$$

Answer. $z = 2\sqrt{y} - x$, and $z = 2\sqrt{y} + x$. (Also, $z = -2\sqrt{y} + x$, and $z = -2\sqrt{y} - x$.)

2. Show that the change of variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ takes the equation

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y = 0$$

into

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + Du_{\xi} + Eu_{\eta} = 0 \,,$$

with A, B and C given by (12.24), and

$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + f\xi_y \,,$$
$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + f\eta_y \,.$$

3. Find the canonical form for the equation

$$u_{xx} - y u_{yy} - \frac{1}{2} u_y = 0$$

in the upper half plane $y > 0$, and then find the general solution.

Answer. $\xi = 2\sqrt{y} - x$, and $\eta = 2\sqrt{y} + x$ leads to $u_{\xi\eta} = 0$, and then $u(x, y) = F(2\sqrt{y} - x) + G(2\sqrt{y} + x)$.

4. Show that *Tricomi's equation*

$$u_{xx} + xu_{yy} = 0$$

is of a different type for $x < 0$, $x = 0$, and $x > 0$. For each type find the corresponding canonical form.

Answer. In case $x < 0$, this equation is of hyperbolic type, and its canonical form is

$$u_{\xi\eta} + \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta) = 0\,.$$

5. Find the canonical form for the equation

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0\,.$$

Hint: The equation is of parabolic type, for all x and y. Calculate $\xi = x\,y$, and choose $\eta = x$ (arbitrarily).

6. (i) Let us re-visit the first order equation

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)\,. \tag{12.29}$$

Assume that the function $y(x)$, which is implicitly defined by $\varphi(x, y) = c$, satisfies the equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}\,.$$

Show that the change of variables $(x, y) \to (\xi, \eta)$, given by $\xi = \varphi(x, y)$ and $\eta = y$ (chosen arbitrarily) transforms the equation (12.29) into

$$bu_\eta + cu = d\,.$$

(ii) Find the general solution of

$$au_x + bu_y = 0\,, \quad a \text{ and } b \text{ are non-zero constants}\,.$$

Answer. $u(x, y) = f(bx - ay)$, where f is an arbitrary function.

(iii) Find the general solution of

$$-xu_x + yu_y + u = x\,.$$

Answer. $u(x,y) = \frac{y}{2} + \frac{f(xy)}{y}$, where f is an arbitrary function.

III. 1. Show that the Neumann problem

$$\Delta u = f(x,y,z) \text{ in } D\,, \quad \frac{\partial u}{\partial n} = 0 \text{ on } S$$

has no solution if $\int_D f(x,y,z)\,dV \neq 0$. (Here and in the problems that follow, we denote $S = \partial D$.)

2. Show that the difference of any two solutions of the Neumann problem

$$\Delta u = f(x,y,z) \text{ in } D\,, \quad \frac{\partial u}{\partial n} = g(x,y,z) \text{ on } S$$

is a constant.

3. Let D be a domain in (x,y,z) space, bounded by a closed and smooth surface S, and let $\mathbf{n} = (n_1, n_2, n_3)$ denote the unit normal vector on S pointing outside of D ($\mathbf{n}$ as well as its components n_1, n_2 and n_3 are functions of (x,y,z)). Consider a vector field $\mathbf{F} = (u(x,y,z), 0, 0)$, with a continuously differentiable function $u(x,y,z)$.

(i) Use the divergence theorem to show that

$$\int_D u_x\,dV = \int_S u\,n_1\,dS\,.$$

Derive similar formulas for $\int_D u_y\,dV$, and for $\int_D u_z\,dV$.

(ii) Show that the nonlinear Dirichlet problem

$$\Delta u + u u_x = 0 \text{ in } D\,, \quad u = 0 \text{ on } S$$

has only the trivial solution $u = 0$.

Hint: Multiply the equation by u, and write $u^2 u_x = \frac{1}{3}\frac{\partial}{\partial x} u^3$. Then integrate over D.

(iii) Let $v(x,y,z)$ be another continuously differentiable function. Derive the *integration by parts* formula

$$\int_D u_x v\,dV = -\int_D u_x v\,dV + \int_S uv\,n_1\,dS\,.$$

Derive similar formulas for $\int_D u_y v\,dV$, and for $\int_D u_z v\,dV$.

(iv) Show that the nonlinear Dirichlet problem

$$\Delta u + x u^2 u_x = 0 \ \text{ in } D\,, \quad u = 0 \ \text{ on } S$$

has only the trivial solution $u = 0$.

Hint: Multiply the equation by u, and write $u^3 u_x = \frac{1}{4}\frac{\partial}{\partial x} u^4$. Then integrate over D.

4. Consider a nonlinear boundary value problem

$$\begin{aligned} \Delta u &= f(u) \ \text{ in } D \\ u &= g(x,y,z) \ \text{ on } S\,, \end{aligned}$$

with an increasing function $f(u)$. Show that there is at most one solution.

Hint: Let v be another solution. Then

$$\Delta(u-v) = f(u) - f(v) \ \text{ in } D\,, \quad u - v = 0 \ \text{ on } S.$$

Multiply by $u - v$, and integrate over D

$$\int_D |\nabla(u-v)|^2 \, dV = -\int_D \left[f(u) - f(v)\right](u-v)\, dV \le 0\,.$$

5. (i) Let D be a three-dimensional domain, bounded by a closed and smooth surface S. Derive the *second Green's identity*

$$\int_D (\Delta u \, v - \Delta v \, u) \, dV = \int_S \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) dS\,.$$

Hint: Interchange u and v in (11.11), then subtract the formulas.

(ii) Consider the nonlinear Dirichlet problem

$$\Delta u = f(u) \ \text{ in } D\,, \quad u = 0 \ \text{ on } S\,.$$

Assume that $\frac{f(u)}{u}$ is increasing for all $u > 0$. Show that it is impossible to have two solutions of this problem satisfying $u(x) > v(x) > 0$ for all $x \in D$.

Hint: Integrate the identity: $\Delta u \, v - \Delta v \, u = uv \left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)$.

6. Assume that the functions $u(x) = u\left(x_1, x_2, \ldots, x_n\right)$ and $w(x) = w\left(x_1, x_2, \ldots, x_n\right)$ are twice continuously differentiable, with $u(x) > 0$. Let $\xi(t)$ be a continuously differentiable function. Derive the following *Picone's identity*

$$\operatorname{div}\left[\xi\left(\frac{w}{u}\right)(u\nabla w - w\nabla u)\right] = \xi\left(\frac{w}{u}\right)(u\Delta w - w\Delta u) + \xi'\left(\frac{w}{u}\right)u^2 \left|\nabla\left(\frac{w}{u}\right)\right|^2 .$$

Chapter 9

Numerical Computations

Easy to use software packages, like *Mathematica*, provide an effective tool for solving differential equations. In this chapter some computational methods are described in general, and not too much tied to *Mathematica*, as there are other excellent software choices. (However, the author is a *Mathematica* enthusiast, and hopefully the readers will share in the excitement.) Initial value problems (including the case of systems), and boundary value problems, both linear and nonlinear, are discussed. The chapter concludes with the topic of direction fields.

9.1 The Capabilities of Software Systems, Like *Mathematica*

Mathematica uses the command DSolve to solve differential equations analytically (by a formula). This is not always possible, but *Mathematica* does seem to know the solution methods that we studied in Chapters 1 and 2. For example, to solve the equation

$$y' = 2y - \sin^2 x, \quad y(0) = 0.3\,, \tag{1.1}$$

we enter the commands

```
sol = DSolve[{y'[x] == 2 y[x] - Sin[x]^2, y[0] == .3}, y[x], x]
z[x_] = y[x] /. sol[[1]]
Plot[z[x], {x, 0, 1}]
```

Mathematica returns the solution, $y(x) = -0.125\cos 2x + 0.175e^{2x} + 0.125\sin 2x + 0.25$, and plots its graph, which is given in Figure 9.1.

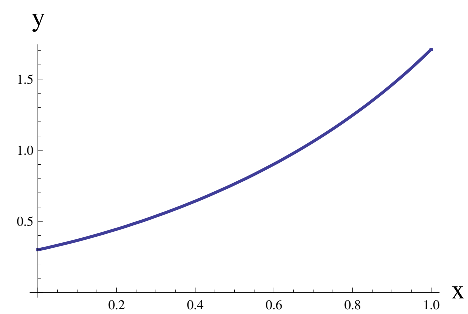


Figure 9.1: The solution of the equation (1.1)

If you are new to *Mathematica*, do not worry about its syntax now. Try to solve other equations by making the obvious modifications to the above commands.

If one needs the general solution of this equation, the command is

DSolve[y′[x] == 2y[x]-Sin[x]^2, y[x], x]

Mathematica returns:

$$y(x) \to e^{2x}c[1] + \frac{1}{8}(-\text{Cos}[2x] + \text{Sin}[2x] + 2)\,.$$

Observe that $c[1]$ is *Mathematica*'s way to write an arbitrary constant c, and that the answer is returned as a "replacement rule" (and that was the reason for an extra command in the preceding example). Equations of second (and higher) order are solved similarly. To solve the following resonant problem

$$y'' + 4y = 8\sin 2t, \quad y(0) = 0, \quad y'(0) = -2\,,$$

we enter

**DSolve[{y″[t]+4y[t] == 8 Sin[2t], y[0]==0, y′[0]==-2 }, y[t], t]
// Simplify**

and *Mathematica* returns the solution $y(t) = -2t\cos 2t$, which involves unbounded oscillations.

When we try to use the DSolve command to solve the nonlinear equation

$$y' = 2y^3 - \sin^2 x\,,$$

Mathematica thinks for a while, and then it throws this equation back at us. It cannot solve it, and most likely, nobody can. However, we can use Euler's method to compute a numerical approximation of the solution if an initial condition is provided; for example, we can find a numerical solution of

$$y' = 2y^3 - \sin^2 x\,, \quad y(0) = 0.3\,. \tag{1.2}$$

Mathematica can also compute the numerical approximation of this solution. Instead of Euler's method it uses a much more sophisticated method. The command is NDSolve. We enter the following commands:

```
sol = NDSolve[{y'[x] == 2 y[x]^3 - Sin[x]^2, y[0] == .3}, y, {x, 0, 3}]
z[x_] = y[x] /. sol[[1]]
Plot[z[x], {x, 0, 1}, AxesLabel → {"x", "y"}]
```

Mathematica produced the graph of the solution, which is given in Figure 9.2. *Mathematica* returns the solution as an *interpolation function*, which means that after computing the values of the solution at a sequence of points, it joins the points on the graph by a smooth curve. The solution function (it is $z(x)$ in our implementation), and its derivatives, can be evaluated at any point. The computed solution is *practically indistinguishable* from the exact solution. When one uses the NDSolve command to solve the problem (1.1), and then plots the solution, the resulting graph is practically identical to the graph of the exact solution given in Figure 9.1.

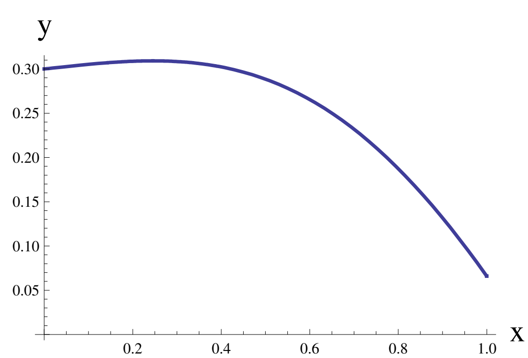


Figure 9.2: The solution curve of the equation (1.2)

The NDSolve command can also be used for systems of differential equa-

```
In[21]:= Clear["`*"]
         sol =
          NDSolve[{{x'[t] == -y[t] + y[t]^2, y'[t] == x[t]}, {x[0] == 0.2, y[0] == 0.3}}, {x, y}, {t, 0, 20}]
Out[22]= {{x → InterpolatingFunction[{{0., 20.}}, <>], y → InterpolatingFunction[{{0., 20.}}, <>]}}

In[24]:= ParametricPlot[{x[t] /. sol[[1, 1]], y[t] /. sol[[1, 2]]}, {t, 0, 20}]
```

Out[24]=

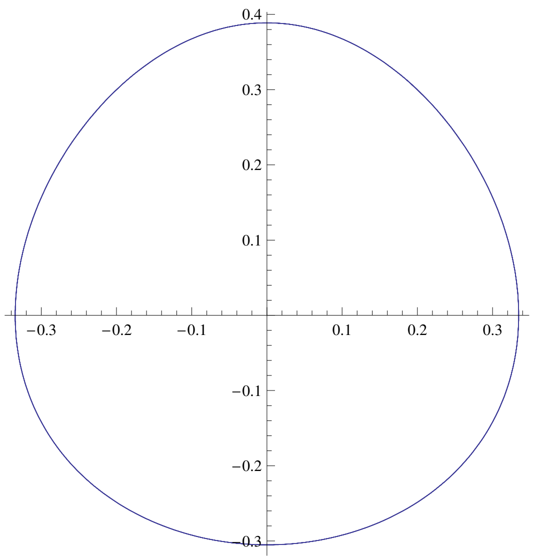


Figure 9.3: The solution of the system (1.3)

tions. For example, let $x = x(t)$ and $y = y(t)$ be solutions of

$$
\begin{aligned}
x' &= -y + y^2, \quad x(0) = 0.2 \\
y' &= x, \qquad y(0) = 0.3\,.
\end{aligned}
\tag{1.3}
$$

Once the solution is computed, the solution components $x = x(t)$, $y = y(t)$, define a parametric curve in the xy-plane, which we draw. The commands, and the output, are given in Figure 9.3. (The first command tells *Mathematica*: "forget everything." This is a good practice with heavy usage.) If you play with other initial conditions, in which $|x(0)|$ and $|y(0)|$ are small, you will discover that the *rest point* $(0, 0)$ is a *center*, meaning that the solutions near $(0, 0)$ are closed loops.

9.2 Solving Boundary Value Problems

Given the functions $a(x)$ and $f(x)$, we wish to find the solution $y = y(x)$ of the following boundary value problem, on some interval $[a, b]$,

$$y'' + a(x)y = f(x), \quad a < x < b \tag{2.1}$$
$$y(a) = y(b) = 0\,.$$

The general solution of the equation in (2.1) is, of course,

$$y(x) = Y(x) + c_1 y_1(x) + c_2 y_2(x)\,, \tag{2.2}$$

where $Y(x)$ is any particular solution, and $y_1(x)$, $y_2(x)$ are two solutions of the corresponding homogeneous equation

$$y'' + a(x)y = 0\,, \tag{2.3}$$

which are not constant multiples of one another. To compute $y_1(x)$, we use the NDSolve command to solve the homogeneous equation (2.3), with the initial conditions

$$y_1(a) = 0, \quad y_1'(a) = 1\,. \tag{2.4}$$

To compute $y_2(x)$, we solve (2.3), with the initial conditions

$$y_2(b) = 0, \quad y_2'(b) = -1\,. \tag{2.5}$$

(*Mathematica* has no problem solving differential equations "backward" on (a, b).) Observe that the values of $y_1'(a)$ and $y_2'(b)$ could have been replaced by any other non-zero numbers. To find a particular solution $Y(x)$, we may solve the equation in (2.1), with any initial conditions, say $Y(a) = 0$, $Y'(a) = 1$. We have computed the general solution (2.2). It remains to pick the constants c_1 and c_2 to satisfy the boundary conditions. Using (2.4),

$$y(a) = Y(a) + c_1 y_1(a) + c_2 y_2(a) = Y(a) + c_2 y_2(a) = 0\,,$$

so that $c_2 = -\dfrac{Y(a)}{y_2(a)}$. We assume here that $y_2(a) \neq 0$, otherwise our problem (2.1) is not solvable for general $f(x)$. Similarly, using (2.5),

$$y(b) = Y(b) + c_1 y_1(b) + c_2 y_2(b) = Y(b) + c_1 y_1(b) = 0\,,$$

giving $c_1 = -\dfrac{Y(b)}{y_1(b)}$, assuming that $y_1(b) \neq 0$. The solution of our problem (2.1) is then

$$y(x) = Y(x) - \frac{Y(b)}{y_1(b)}\, y_1(x) - \frac{Y(a)}{y_2(a)}\, y_2(x)\,.$$

```
Clear["`*"]
lin :=
 Module[{s1, s2, s3, y1, y2, Y},
  s1 = NDSolve[{ y''[x] + a[x] y[x] == 0, y[0] == 0, y'[0] == 1}, y, {x, 0, 1}];
  s2 = NDSolve[{ y''[x] + a[x] y[x] == 0, y[1] == 0, y'[1] == -1}, y, {x, 0, 1}];
  s3 = NDSolve[{ y''[x] + a[x] y[x] == f[x], y[0] == 0, y'[0] == 1}, y, {x, 0, 1}];
y1[x_] = y[x] /. s1[[1]];
  y2[x_] = y[x] /. s2[[1]];
  Y[x_] = y[x] /. s3[[1]];
  z[x_] := Y[x] - Y[1]/y1[1] y1[x] - Y[0]/y2[0] y2[x] ;
 ]
```

Figure 9.4: The solution module for the problem (2.1)

```
a[x_] = ⅇ^x;
f[x_] = -3 x + 1;
lin
Plot[z[x], {x, 0, 1}, AxesLabel → {"x", "z"}]
```

Figure 9.5: Solving the problem (2.6)

Mathematica's *subroutine*, or *module*, to produce this solution, called **lin**, is given in Figure 9.4. We took $a = 0$, and $b = 1$.

For example, entering the commands in Figure 9.5, produces the graph of the solution for the boundary value problem

$$y'' + e^x y = -3x + 1, \quad 0 < x < 1, \quad y(0) = y(1) = 0\,, \tag{2.6}$$

which is given in Figure 9.6.

9.3 Solving Nonlinear Boundary Value Problems

Review of Newton's Method

Suppose that we wish to solve the equation

$$f(x) = 0\,, \tag{3.1}$$

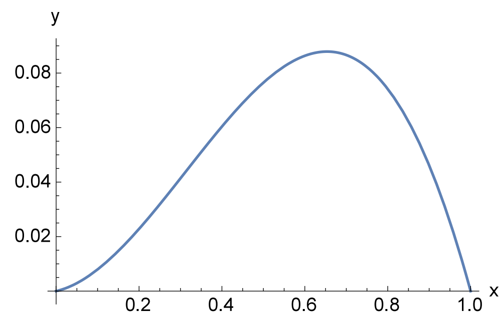


Figure 9.6: Solution of the problem (2.6)

with a given function $f(x)$. For example, in case $f(x) = e^{2x} - x - 2$, the equation

$$e^{2x} - x - 2 = 0$$

has a solution on the interval (0,1) (because $f(0) = -1 < 0$, and $f(1) = e^2 - 1 > 0$), but this solution cannot be expressed by a formula. Newton's method produces a sequence of iterates $\{x_n\}$ to approximate a solution of (3.1). If the iterate x_n is already computed, we use the linear approximation

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n), \quad \text{for } x \text{ close to } x_n \,.$$

Then we replace the equation (3.1) by

$$f(x_n) + f'(x_n)(x - x_n) = 0\,,$$

solve this linear equation for x, and declare its solution x to be our next approximation, so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, \ldots, \quad \text{beginning with some } x_0 \,.$$

Newton's method does not always converge, but when it does, the convergence is usually super fast. To explain that, let us denote by x^* the (true) solution of (3.1). Then $|x_n - x^*|$ gives the *error of the approximation on the n-th step*. Under some mild conditions on $f(x)$, it can be shown that

$$|x_{n+1} - x^*| < c|x_n - x^*|^2\,,$$

with some constant $c > 0$. Let us suppose that $c = 1$ and $|x_0 - x^*| = 0.1$. Then the errors of approximation are estimated as follows: $|x_1 - x^*| < |x_0 - x^*|^2 = 0.1^2 = 0.01$, $|x_2 - x^*| < |x_1 - x^*|^2 < 0.01^2 = 0.0001$, $|x_3 - x^*| < |x_2 - x^*|^2 < 0.0001^2 = 0.00000001$. We see that x_3 is practically the exact solution!

A Class of Nonlinear Boundary Value Problems

We wish to solve the *nonlinear boundary value problem*

$$\begin{aligned} y'' + g(y) &= e(x), \quad a < x < b \\ y(a) &= y(b) = 0\,, \end{aligned} \tag{3.2}$$

with given functions $g(y)$ and $e(x)$.

We shall use Newton's method to produce a sequence of iterates $\{y_n(x)\}$ to approximate one of the solutions of (3.2). (The problem (3.2) may have multiple solutions.) We begin with some initial guess $y_0(x)$. If the iterate $y_n(x)$ is already computed, we use the linear approximation

$$g(y) \approx g(y_n) + g'(y_n)(y - y_n)\,,$$

and replace (3.2) with the *linear problem*

$$\begin{aligned} y'' + g(y_n(x)) + g'(y_n(x))(y - y_n(x)) &= e(x), \quad a < x < b \\ y(a) &= y(b) = 0\,. \end{aligned} \tag{3.3}$$

The solution of this problem we declare to be our next approximation, $y_{n+1}(x)$. We rewrite (3.3) as

$$\begin{aligned} y'' + a(x)y &= f(x), \quad a < x < b \\ y(a) &= y(b) = 0\,, \end{aligned}$$

with the known functions

$$\begin{aligned} a(x) &= g'(y_n(x))\,, \\ f(x) &= -g(y_n(x)) + g'(y_n(x))y_n(x) + e(x)\,, \end{aligned}$$

and call on the procedure **lin** from the preceding section to solve (3.3), and produce $y_{n+1}(x)$. If the initial guess $y_0(x)$ is chosen not too far from one of the actual solutions, then four or five iterations of Newton's method will usually produce an excellent approximation!

Example We solved the nonlinear boundary value problem

$$\begin{aligned} y'' + y^3 &= 2\sin 4\pi x - x, \quad 0 < x < 1 \\ y(0) &= y(1) = 0\,. \end{aligned} \tag{3.4}$$

The commands are given in Figure 9.7. (The procedure **lin** has been executed before these commands.) We started with $y_0(x) = 1$ (yold[x]= 1 in *Mathematica*'s code). We did five iterations of Newton's method. The solution (the function z[x]) is plotted in Figure 9.8.

The resulting solution is very accurate, and we verified it by the following independent calculation. We used *Mathematica* to calculate the slope of this solution at zero, z′[0] $\approx$ 0.00756827, and then we solved the equation in (3.4), together with the initial conditions $y(0) = 0$, $y'(0) = 0.00756827$ (using the NDSolve command). The graph of this solution $y(x)$ is identical to the one in Figure 9.8.

```
e[x_] = 2 Sin[4 π x] - x;
yold[x_] = 1;
g[y_] = y^3;
st = 5;
For[i = 1, i ≤ st, i++,
 a[x_] =  g'[yold[x]];
 f[x_] = e[x] -  g[yold[x]] +  g'[yold[x]] yold[x] ;
 lin;
 yold[x_] = z[x];
]
```

Figure 9.7: Solving the problem (3.4)

9.4 Direction Fields

The equation (here $y = y(x)$)

$$y' = \cos 2y + 2\sin 2x \tag{4.1}$$

cannot be solved analytically (like most equations). If we add an initial condition, we can find the corresponding solution, by using the NDSolve command. But this is just one solution. Can we visualize a bigger picture?

The right hand side of the equation (4.1) gives us the slope of the solution passing through the point (x, y) (for example, if $\cos 2y + 2\sin 2x > 0$, then the solution $y(x)$ is increasing at (x, y)). The vector $< 1, \cos 2y + 2\sin 2x >$ is called the *direction vector*. If the solution is increasing at (x, y), this vector points up, and the faster is the rate of increase, the larger is the amplitude of the direction vector. If we plot the direction vectors at many points, the

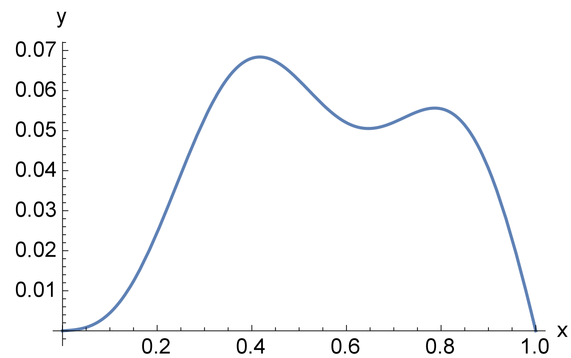


Figure 9.8: Solution of the nonlinear problem (3.4)

result is called the *direction field*, which can tell us at a glance how various solutions are behaving. In Figure 9.9, the direction field for the equation (4.1) is plotted using *Mathematica*'s command

$$\text{VectorPlot}[\{1,\text{Cos}[2\text{y}]+2\text{Sin}[2\text{x}]\ \},\{\text{x},0,6.5\},\{\text{y},0,5\ \}].$$

The reader should also try *Mathematica*'s command

$$\text{StreamPlot}[\{1,\text{Cos}[2\text{y}]+2\text{Sin}[2\text{x}]\ \},\{\text{x},0,6.5\},\{\text{y},0,5\ \}]\,,$$

which draws a number of solution curves of (4.1).

How will the solution of (4.1), with the initial condition $y(0) = 1$, behave? Imagine a particle placed at the initial point $(0, 1)$ (see Figure 9.9). The direction field, or the "wind," will take it a little down, but soon the direction of motion will be up. After a while, a strong downdraft will take the particle much lower, but eventually it will be going up again. In Figure 9.10, we give the actual solution of

$$y' = \cos 2y\, + 2\sin 2x, \quad y(0) = 1\,, \tag{4.2}$$

produced using the NDSolve command. It confirms the behavior suggested by the direction field.

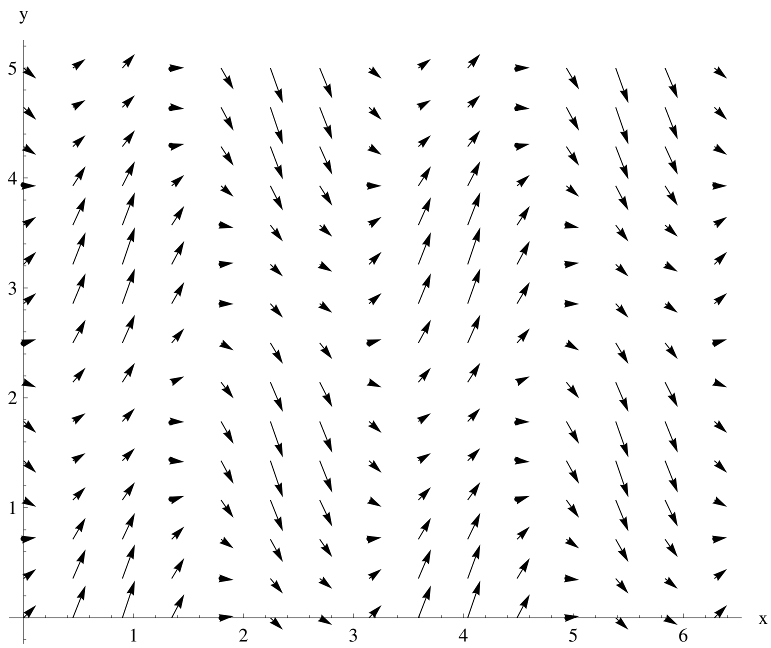


Figure 9.9: The direction field for the equation (4.1)

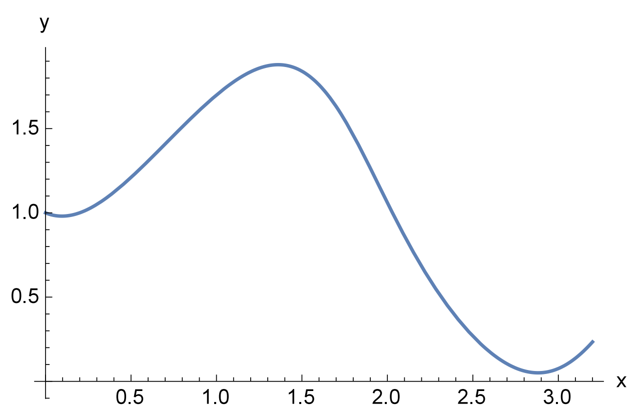


Figure 9.10: The solution of the initial value problem (4.2)

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.1 The Chain Rule and Its Descendants

The *chain rule*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

allows us to differentiate the composition of two functions $f(x)$ and $g(x)$. In particular, if $f(x) = x^r$ with a constant r, then $f'(x) = rx^{r-1}$, and we conclude

$$\frac{d}{dx}\left[g(x)\right]^r = r\left[g(x)\right]^{r-1} g'(x)\,, \tag{1.3}$$

the *generalized power rule*. In case $f(x) = e^x$, we have

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)\,. \tag{1.4}$$

In case $f(x) = \ln x$, we get

$$\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}\,. \tag{1.5}$$

These are *children* of the chain rule. These formulas should be memorized separately, even though they can derived from the chain rule. If $g(x) = ax$, with a constant a, then by (1.4)

$$\frac{d}{dx}e^{ax} = ae^{ax}\,.$$

This grandchild of the chain rule should be also memorized separately. For example, $\frac{d}{dx}e^{-x} = -e^{-x}$.

The chain rule also lets us justify the following integration formulas (a and b are constants):

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\,,$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + c\,,$$

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c\,.$$

The situation is similar for functions of two or more variables. If $u = u(x,y)$, while $x = x(t)$ and $y = y(t)$, then u is really a function of t and

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}\,, \tag{1.6}$$

where u_x and u_y are the partial derivatives. If $y = y(x)$, it follows that

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx}\,.$$

The equation (c is a constant)

$$u(x, y) = c$$

defines implicitly a function $y = y(x)$. Differentiating this equation, we have

$$u_x + u_y \frac{dy}{dx} = 0\,,$$

which gives the formula for implicit differentiation:

$$\frac{dy}{dx} = -\frac{u_x(x, y)}{u_y(x, y)}\,.$$

If $u = u(\xi, \eta)$, while $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, then (1.6) is adjusted to give

$$\frac{\partial u}{\partial x} = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}\,, \quad \frac{\partial u}{\partial y} = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y}\,.$$

If $u = u(x)$, and $x = x(t, s)$, then

$$\frac{\partial u}{\partial t} = u' \frac{\partial x}{\partial t}\,, \quad \frac{\partial u}{\partial s} = u' \frac{\partial x}{\partial s}\,.$$

For example, if $u(t, s) = f(t - 2s)$, where f is some function of one variable, then $u_t = f'(t - 2s)$, $u_{tt} = f''(t - 2s)$. $u_s = -2f'(t - 2s)$, $u_{ss} = 4f''(t - 2s)$, so that $u(t, s)$ satisfies the following wave equation

$$u_{tt} - 4u_{ss} = 0\,.$$

.2 Partial Fractions

This method is needed for both computing integrals, and inverse Laplace transforms. We have

$$\frac{x + 1}{x^2 + x - 2} = \frac{x + 1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}\,.$$

The first step is to factor the denominator. We now look for the constants A and B so that the original fraction is equal to the sum of two simpler ones. Adding the fractions on the right, we need

$$\frac{x + 1}{(x - 1)(x + 2)} = \frac{A(x + 2) + B(x - 1)}{(x - 1)(x + 2)}\,.$$

Both fractions have the same denominator. We need to arrange for the numerators to be the same:

$$A(x+2)+B(x-1)=x+1\,;$$
$$(A+B)x+2A-B=x+1\,.$$

Equating the coefficients of the two linear polynomials, gives

$$A+B=1$$
$$2A-B=1\,.$$

We calculate $A=\frac{2}{3}$, $B=\frac{1}{3}$. Conclusion:

$$\frac{x+1}{x^2+x-2}=\frac{2/3}{x-1}+\frac{1/3}{x+2}\,.$$

Our next example

$$\frac{s^2-1}{s^3+s^2+s}=\frac{s^2-1}{s(s^2+s+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+s+1}$$

involves a quadratic factor in the denominator that cannot be factored (an *irreducible quadratic*). Adding the fractions on the right, we need

$$A(s^2+s+1)+s(Bs+C)=s^2-1\,.$$

Equating the coefficients of the two quadratic polynomials, we get

$$\begin{aligned} A+B&=1\\ A+\quad C&=0\\ A&=-1\,, \end{aligned}$$

so that $A=-1$, $B=2$, $C=1$. Conclusion:

$$\frac{s^2-1}{s^3+s^2+s}=-\frac{1}{s}+\frac{2s+1}{s^2+s+1}\,.$$

The denominator of the next example, $\dfrac{s-1}{(s+3)^2\,(s^2+3)}$ involves a product of a square of a linear factor and an irreducible quadratic. The way to proceed is:

$$\frac{s-1}{(s+3)^2\,(s^2+3)}=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{Cs+D}{s^2+3}\,.$$

As before, we calculate $A=-\frac{1}{12}$, $B=-\frac{1}{3}$, $C=D=\frac{1}{12}$. Conclusion:

$$\frac{s-1}{(s+3)^2\,(s^2+3)}=-\frac{1}{12(s+3)}-\frac{1}{3(s+3)^2}+\frac{s+1}{12\,(s^2+3)}\,.$$

.3 Eigenvalues and Eigenvectors

The vector $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is very special for the matrix $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. We have

$$B\,z = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2z\,,$$

so that $Bz = 2z$. We say that z is an *eigenvector* of B, corresponding to an *eigenvalue* 2. In general, we say that a vector $x \neq 0$ is an eigenvector of a square matrix A, corresponding to an *eigenvalue* λ if

$$Ax = \lambda x\,. \tag{3.7}$$

If A is 2×2, then in components an eigenvector must satisfy $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. In case A is 3×3, then we need $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

If $c \neq 0$ is any constant, then

$$A\,(cx) = cAx = c\lambda x = \lambda\,(cx)\ ,$$

which implies that cx is also an eigenvector of the matrix A, corresponding to the *eigenvalue* λ. In particular, $c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gives us the eigenvectors of the matrix B above, corresponding to the eigenvalue $\lambda = 2$.

We now rewrite (3.7) in the form

$$(A - \lambda I)\,x = 0\,, \tag{3.8}$$

where I is the identity matrix. This is a homogeneous system of linear equations. To have non-zero solutions, its determinant must be zero:

$$|A - \lambda I| = 0\,. \tag{3.9}$$

This is a polynomial equation for λ, called the *characteristic equation*. If the matrix A is 2×2, this is a quadratic equation, and it has two roots λ_1 and λ_2. In case A is 3×3, this is a cubic equation, and it has three roots λ_1, λ_2 and λ_3, and so on for larger A. To calculate the eigenvectors corresponding to λ_1, we solve the system

$$(A - \lambda_1 I)\,x = 0\,,$$

and proceed similarly for other eigenvalues.

Example 1 Consider $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

The characteristic equation

$$|B - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

has the roots $\lambda_1 = 2$ and $\lambda_2 = 4$. (Because: $3 - \lambda = \pm 1$, $\lambda = 3 \pm 1$.) We already know that $c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors for $\lambda_1 = 2$, so let us compute the eigenvectors for $\lambda_2 = 4$. We need to solve the system $(A - 4I)\,x = 0$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, which is

$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0\,. \end{aligned}$$

The second equation is superfluous, and the first one gives $x_1 = x_2$. If we let $x_2 = 1$, then $x_1 = 1$ so that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and more generally $c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives us the eigenvectors corresponding to $\lambda_2 = 4$.

If an eigenvalue λ has multiplicity two (it is a double root of the characteristic equation) then it may have either two linearly independent eigenvectors, or only one. In the case there are two, x_1 and x_2, then $Ax_1 = \lambda x_1$ and $Ax_2 = \lambda x_2$, and for any constants c_1 and c_2

$$A\left(c_1x_1 + c_2x_2\right) = c_1Ax_1 + c_2Ax_2 = c_1\lambda x_1 + c_2\lambda x_2 = \lambda\left(c_1x_1 + c_2x_2\right),$$

so that $c_1x_1 + c_2x_2$ is an eigenvector corresponding to λ.

Example 2 $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

The characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

is a cubic equation, so we need to guess a root. $\lambda_1 = 1$ is a root. We then factor

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = (\lambda - 1)\left(\lambda^2 - 5\lambda + 4\right).$$

Setting the second factor to zero, we find the other two roots $\lambda_2 = 1$ and $\lambda_3 = 4$. Turning to the eigenvectors, let us begin with the simple eigenvalue $\lambda_3 = 4$. We need to solve the system $(A - 4I)\,x = 0$ for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, or

$$
\begin{aligned}
-2x_1 + x_2 + x_3 &= 0 \\
x_1 - 2x_2 + x_3 &= 0 \\
x_1 + x_2 - 2x_3 &= 0\,.
\end{aligned}
$$

The third equation is superfluous, because adding the first two equations gives the negative of the third. We are left with

$$
\begin{aligned}
-2x_1 + x_2 + x_3 &= 0 \\
x_1 - 2x_2 + x_3 &= 0\,.
\end{aligned}
$$

There are more variables to play with, than equations to satisfy. We are free to set $x_3 = 1$, and then solve the system for x_1 and x_2, obtaining $x_1 = 1$ and $x_2 = 1$. Conclusion: $c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are the eigenvectors corresponding to $\lambda_3 = 4$.

To find the eigenvectors of the double eigenvalue $\lambda_1 = 1$, one needs to solve the system $(A - I)\,x = 0$, or

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_2 + x_3 &= 0\,.
\end{aligned}
$$

Discarding both the second and the third equations, we are left with

$$x_1 + x_2 + x_3 = 0\,.$$

Now both x_2 and x_3 are *free variables*. Letting $x_3 = 1$ and $x_2 = 0$, we calculate $x_1 = -1$, so that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector. Letting $x_3 = 0$ and

$x_2 = 1$, gives $x_1 = -1$, so that $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector. Conclusion: the linear combination, or the *span*, of these eigenvectors

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

with arbitrary constants c_1 and c_2, gives us *all eigenvectors, corresponding to* $\lambda_1 = 1$, or the *eigenspace* of $\lambda_1 = 1$.

For the matrix $A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$, the eigenvalues are equal $\lambda_1 = \lambda_2 = -3$, but there is only one linearly independent eigenvector: $c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

.4 Matrix Functions and the Norm

If $A(t)$ is an $m \times n$ matrix with the entries a_{ij}, $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$, with the entries depending on t, it is customary to write $A(t) = [a_{ij}(t)]$. The transpose matrix is then $A^T(t) = [a_{ji}]$. The derivative matrix $A'(t) = \lim_{h \to 0} \frac{A(t+h)-A(t)}{h} = [\lim_{h \to 0} \frac{a_{ij}(t+h)-a_{ij}(t)}{h}] = [a'_{ij}]$ is computed by differentiating all of the entries. Correspondingly, $\int A(t)\,dt = [\int a_{ij}(t)\,dt]$. Clearly

$$\frac{d}{dt} A^T(t) = \left(A'(t)\right)^T .$$

If it is admissible to multiply the matrices $A(t)$ and $B(t)$, then using the product rule from calculus, one justifies the product rule for matrices

$$\frac{d}{dt} [A(t)B(t)] = A'(t)B(t) + A(t)B'(t) .$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be any constant vector. Consider

$$y = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

The vector y is the rotation of the vector x by an angle t, counterclockwise. Changing t to $-t$, one concludes that the vector $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the result of rotation of $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by an angle t clockwise.

If $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ are two n-dimensional vectors, then the scalar (inner) product is defined as

$$(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \,.$$

The norm (length) of an n-dimensional vector x is defined by

$$||x||^2 = (x, x) = \sum_{i=1}^{n} x_i^2 \,.$$

The Cauchy-Schwartz inequality states:

$$(x, y) \le ||x|| \, ||y|| \,.$$

If A is an $n \times n$ matrix, then

$$(Ax, y) = \left(x, A^T y\right) \,.$$

Let A be an $n \times n$ matrix, given by its columns $A = [C_1 \, C_2 \, \ldots C_n]$. ($C_1$ is the first column of A, etc.) Define the norm $||A||$ of A, as follows

$$||A||^2 = \sum_{i=1}^{n} ||C_i||^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \,.$$

Clearly

$$|a_{ij}| \le ||A|| \,, \text{ for all } i \text{ and } j \,. \tag{4.10}$$

If $x = (x_1, x_2, \ldots, x_n)^T$, we claim that

$$||Ax|| \le ||A|| \, ||x|| \,. \tag{4.11}$$

Indeed, using the Cauchy-Schwartz inequality

$$\begin{aligned} ||Ax||^2 = \textstyle\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j\right)^2 &\le \textstyle\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \, \sum_{j=1}^{n} x_j^2\right) \\ &= ||x||^2 \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = ||A||^2 \, ||x||^2 \,. \end{aligned}$$

Let B be an $n \times n$ matrix, given by its columns $A = [K_1 \, K_2 \, \ldots K_n]$. Recall that $AB = [AK_1 \, AK_2 \, \ldots AK_n]$. ($AK_1$ is the first column of the product AB, etc.) Then, using (4.11),

$$||AB||^2 = \sum_{i=1}^{n} ||AK_i||^2 \le ||A||^2 \sum_{i=1}^{n} ||K_i||^2 = ||A||^2 ||B||^2 \,,$$

which implies that

$$||AB|| \leq ||A|| \, ||B|| \, .$$

Similar inequalities hold for arbitrary number of matrices, which are not necessarily square matrices. For example, if a product ABC of three matrices is defined, then

$$||ABC|| \leq ||A|| \, ||B|| \, ||C|| \, .$$

Similarly one proves the inequalities like

$$||A + B + C|| \leq ||A|| + ||B|| + ||C|| \, , \tag{4.12}$$

for an arbitrary number of matrices of the same type. The inequalities (4.10) and (4.12) imply that the exponential of any square matrix A

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \, A^k$$

is convergent (in each component). Since integral of a matrix function $B(t)$ is the limit of its Riemann sum, it follows that

$$|| \int_{t_1}^{t_2} B(t) \, dt || \leq \int_{t_1}^{t_2} ||B(t)|| \, dt \, .$$