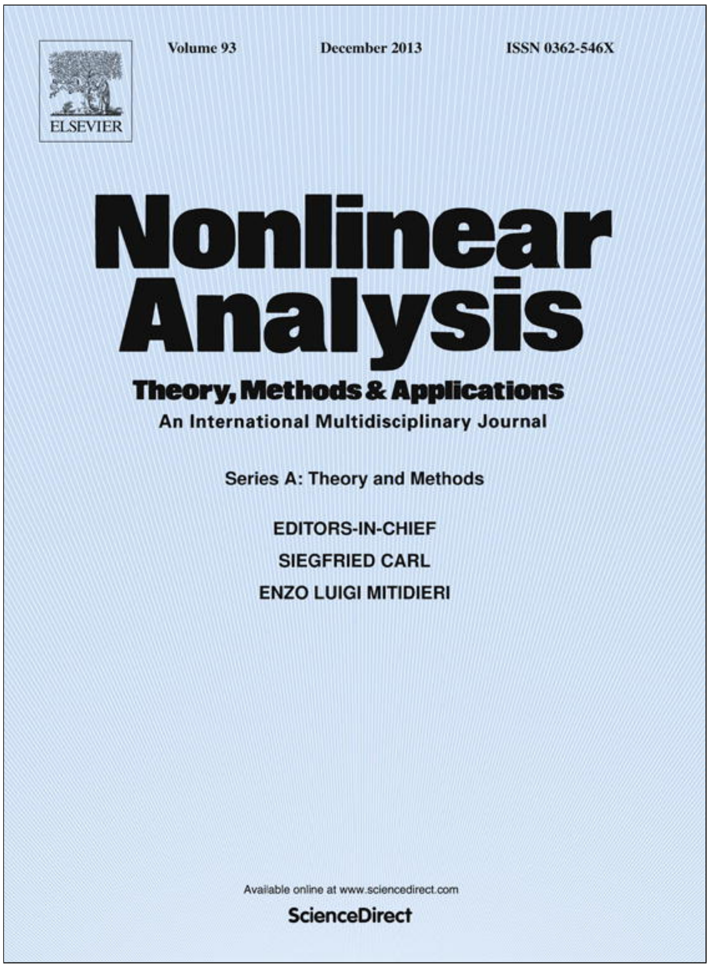
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# Exact multiplicity and numerical computation of solutions for two classes of non-autonomous problems with concave–convex nonlinearities

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ABSTRACT

We establish the exact multiplicity of positive solutions, and the global solution structure for two classes of problems on circular domains. The first class involves non-autonomous concave–convex problems on a ball in $R^n$, and the second one deals with concave–convex problems on a "thin" annulus in $R^n$. We illustrate our results by numerical computations, using a novel algorithm, which involves continuation in a global parameter.



## 1. Introduction

In recent years bifurcation theory methods were applied to the study of the exact multiplicity of positive solutions, and the global solution structure of Dirichlet problems

$$\Delta u + \lambda f(x, u) = 0 \quad \text{in } D, \; u = 0 \text{ on } \partial D, \tag{1.1}$$

depending on a positive parameter $\lambda$. Let us briefly review the bifurcation theory approach, and more details can be found in the author's book [1], or in T. Ouyang and J. Shi [2]. If at some solution $(\lambda_0, u_0)$ the corresponding linearized problem

$$\Delta w + \lambda f_u(x, u)w = 0 \quad \text{in } D, \; w = 0 \text{ on } \partial D \tag{1.2}$$

admits only the trivial solution, then we can continue the solutions of (1.1) in $\lambda$, by using the Implicit Function Theorem, see e.g., L. Nirenberg [3]. If, on the other hand, the problem (1.2) has non-trivial solutions then the Implicit Function Theorem cannot be used, instead one tries to show that the Crandall–Rabinowitz [4] bifurcation theorem applies. The crucial condition one needs to verify is

$$\int_D f_{uu}(x, u)w^3 \, dx \neq 0. \tag{1.3}$$

Then the Crandall–Rabinowitz theorem guarantees the existence of a solution curve through the critical point $(\lambda_0, u_0)$, and if a turn occurs at $(\lambda_0, u_0)$, its direction is governed by the sign of (see e.g., the exposition in P. Korman [1], or in [5,2])

$$I = \frac{\int_D f_{uu}(x, u_0)w^3 \, dx}{\int_D f(x, u_0)w \, dx}. \tag{1.4}$$

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If $I > 0$ ($I < 0$) the direction of the turn is to the left (right) in the $(\lambda, u)$ "plane". If one can show that a turn to the left occurs at any critical point, then there is at most one critical point. Often, there is exactly one critical point, which provides us with the exact shape of solution curve, and the exact multiplicity count for solutions.

Under some conditions, the sign of $I$ can be computed with the help of a Picone's identity, see the Theorem 2.1 below. This method turned out to be effective for non-autonomous equations with concave–convex nonlinearities, i.e., when $f(x, u)$ is concave in $u$ for $0 < u < u_0$, and convex in $u$ for $u > u_0$, for some $u_0$. For example, on a unit ball $B \in R^n$ consider the problem

$$\Delta u + \lambda \left(a(|x|)u^p + b(|x|)u^q\right) = 0 \quad \text{in } B,\ u = 0 \text{ on } \partial B, \tag{1.5}$$

with constants $0 < p < 1 < q < \frac{n+2}{n-2}$, and given positive functions $a(r)$ and $b(r)$. There has been a great interest in such problems, following the publication of A. Ambrosetti, H. Brezis and G. Cerami [6]. In the case of constant $a(r)$ and $b(r)$, an exact multiplicity result for positive solutions of (1.5) was given by P. Korman [7], and by M. Tang [8]. It turns out that the solution curve here has only one turn to the left, similarly to the one in Fig. 1. This implies that for some $\lambda_0$, the problem (1.5) has exactly two positive solutions for $\lambda \in (0, \lambda_0)$, exactly one positive solution for $\lambda = \lambda_0$, and no positive solutions for $\lambda > \lambda_0$. In one dimension, but for the case of $p$-Laplacian, similar results appeared in [9,10].

In this paper we obtain an exact multiplicity result for a class of non-autonomous equations on a ball in $R^n$, which includes (1.5). A crucial step in order to apply the Theorem 2.1 was to show that any non-trivial solution of the corresponding linearized problem does not change sign. We also make an application to a class of "thin" annular domains, for which the above mentioned positivity property of the linearized problem we proved in [11]. We obtain a similar exact multiplicity result.

We develop a numerical algorithm to illustrate our results. Instead of performing continuation in $\lambda$, which is a local parameter, we continue in the maximum value of the solution, which often can be proved to be a global parameter. When specified to the much easier autonomous case, this algorithm is faster and more stable than the standard shoot-and-scale method (and the latter method is restricted to the autonomous case). This numerical algorithm is a major part of the present work.

## 2. The direction of bifurcation

We begin with a well-known Calculus formula, known as *Picone's identity*. Here $x \in R^n$, and $u = u(x)$, $w = w(x)$. Its proof is straightforward.

**Lemma 2.1.** *Assume $\xi(t) \in C^1(R)$, $u, w \in C^2$, and $u(x) > 0$. Then*

$$\operatorname{div}\left[\xi\left(\frac{w}{u}\right)(u\nabla w - w\nabla u)\right] = \xi\left(\frac{w}{u}\right)(u\Delta w - w\Delta u) + \xi'\left(\frac{w}{u}\right)u^2\left|\nabla\left(\frac{w}{u}\right)\right|^2. \tag{2.1}$$

We now consider a semilinear Dirichlet problem

$$\Delta u + \lambda f(x, u) = 0 \quad \text{in } D,\ u = 0 \text{ on } \partial D, \tag{2.2}$$

and the corresponding linearized problem

$$\Delta w + \lambda f_u(x, u)w = 0 \quad \text{in } D,\ w = 0 \text{ on } \partial D. \tag{2.3}$$

We call a solution $u(x)$ of (2.2) *singular*, if the problem (2.3) has non-trivial solutions. The value of $\int_D f_{uu}(x, u)w^3\,dx$ is important at a singular solution. If this integral is non-zero, the singular solution is *non-degenerate*, i.e., it persists under small perturbations, and in case the Crandall–Rabinowitz theorem applies, the sign of this integral governs the direction of bifurcation, see e.g., [1] for an exposition.

**Theorem 2.1.** *Let $u(x)$ be a positive solution of (2.2), and assume that the linearized problem (2.3) has a non-trivial solution, and moreover $w(x) > 0$ on D. If we have, for some $c > 0$,*

$$u^2 f_{uu}(x, u) \ge c\,(uf_u(x, u) - f(x, u)), \quad \textit{for all } u > 0,\ \textit{and } x \in D, \tag{2.4}$$

*then*

$$\int_D f_{uu}(x, u)w^3\,dx > 0. \tag{2.5}$$

*If, on the other hand, for some $c > 0$,*

$$u^2 f_{uu}(x, u) \le -c\,(uf_u(x, u) - f(x, u)), \quad \textit{for all } u > 0,\ \textit{and } x \in D, \tag{2.6}$$

*then*

$$\int_D f_{uu}(x, u)w^3\,dx < 0. \tag{2.7}$$

**Proof.** We multiply the Eq. (2.3) by $\frac{w^2}{u}$, and subtract from that the Eq. (2.2) multiplied by $\frac{w^3}{u^2}$, then integrate

$$\lambda \int_D \left[ \frac{f_u(x,u)}{u} - \frac{f(x,u)}{u^2} \right] w^3 \, dx = \int_D \frac{w^2}{u^2} \left( w\Delta u - u\Delta w \right) \, dx.$$

We now apply Picone's identity (2.1), with $\xi(t) = t^2$, and use the divergence theorem and our boundary conditions

$$\int_D \frac{w^2}{u^2} \left( w\Delta u - u\Delta w \right) \, dx = 2 \int_D uw \left| \nabla \left( \frac{w}{u} \right) \right|^2 \, dx > 0.$$

If the condition (2.5) holds, then

$$\int_D f_{uu}(x,u) w^3 \, dx \geq c \int_D \left[ \frac{f_u(x,u)}{u} - \frac{f(x,u)}{u^2} \right] w^3 \, dx > 0.$$

Similarly, the condition (2.6) implies

$$-\int_D f_{uu}(x,u) w^3 \, dx \geq c \int_D \left[ \frac{f_u(x,u)}{u} - \frac{f(x,u)}{u^2} \right] w^3 \, dx > 0,$$

concluding the proof. ◇

**Remark.** A similar result was proved in J. Shi [12], in case $f = f(u)$ As was pointed out in [12], this result has limited usefulness for autonomous problems.

**Example.** $f(x,u) = a(x)u^p + b(x)u^q$, with given positive functions $a(x)$ and $b(x)$, and positive constants $p$ and $q$. One computes, with $c = 1$,

$$u^2 f_{uu} - uf_u + f = (p-1)^2 a(x) u^p + (q-1)^2 b(x) u^q > 0 \quad \text{for all } u > 0, \text{ and } x \in D,$$

and so (2.4) holds, and then we conclude (2.5), provided that $w(x) > 0$. The case when $0 < p < 1 < q$ is of particular interest. Then $f(x,u)$ is *concave–convex* in $u$, i.e., concave on $(0, u_0)$ and convex on $(u_0, \infty)$, for some $u_0 > 0$, for each fixed $x$.

## 3. An exact multiplicity result on a ball

We consider positive solutions of

$$\Delta u + \lambda f(|x|, u) = 0 \quad \text{in } B, \; u = 0 \text{ on } \partial B, \tag{3.1}$$

where $B$ is the unit ball $|x| < 1$ in $R^n$. We shall assume that $f(r,u) \in C^2(B \times R_+)$ satisfies

$$f(r,u) > 0, \quad \text{for } 0 < r < 1 \text{ and } u > 0, \tag{3.2}$$

$$f_r(r,u) \leq 0, \quad \text{for } 0 < r < 1 \text{ and } u > 0. \tag{3.3}$$

By B. Gidas, W.-M. Ni and L. Nirenberg [13], any positive solution of (3.1) is radially symmetric, i.e., $u = u(r)$, $r = |x|$, and hence it satisfies

$$u'' + \frac{n-1}{r} u' + \lambda f(r,u) = 0, \qquad u'(0) = u(1) = 0. \tag{3.4}$$

The corresponding linearized problem is

$$Lw \equiv w'' + \frac{n-1}{r} w' + \lambda f_u(r,u) w = 0, \qquad w'(0) = w(1) = 0. \tag{3.5}$$

**Lemma 3.1.** *Assume that $n \geq 4$, and $f(r,u)$ satisfies (3.2) and (3.3). Assume also*

$$(n-2) u f_u(r,u) - n f(r,u) - r f_r(r,u) \leq 0 \quad \text{for } u > 0, \text{ and } 0 \leq r < 1, \tag{3.6}$$

$$(n-2) u f_u(r,u) + (n-4) f(r,u) \geq 0 \quad \text{for } u > 0, \text{ and } 0 \leq r < 1. \tag{3.7}$$

*Then any non-trivial solution of (3.5) is of one sign, i.e., we may assume that $w(r) > 0$ on $[0, 1)$.*

**Proof.** We proceed similarly to the Lemma 4.7 in T. Ouyang and J. Shi [2]. We use two comparison functions: $v_1(r) = ru_r(r) + (n-2)u(r)$ and $v_2(r) = r^{2-n} v_1(r)$. One computes

$$Lv_1(r) = \lambda \left[ (n-2) u f_u(r,u) - n f(r,u) \right] - \lambda r f_r(r,u) \leq 0, \tag{3.8}$$

for $u > 0$, and $0 \le r < 1$, by the assumption (3.6). Also

$$Lv_2(r) = \lambda r^{2-n} \left[(n-2)uf_u(r,u) + (n-4)f(r,u)\right] - \lambda r^{3-n} f_r(r,u) \ge 0, \tag{3.9}$$

for $u > 0$, and $0 \le r < 1$, by our conditions (3.3) and (3.7). Compute

$$v_1'(r) = ru_{rr}(r) + (n-1)u(r) = -\lambda r f(r,u) < 0,$$

in view of (3.2), and so the function $v_1(r)$ is decreasing. Since $v_1(0) = (n-2)u(0) > 0$, and $v_1(1) = u'(1) < 0$ (by Hopf's boundary lemma), the function $v_1(r)$ has a unique root on $(0, 1)$, which we denote by $\eta$. We shall show that $w(r)$ cannot vanish on $[0, \eta)$, and on $[\eta, 1)$.

Assume that $w(r)$ vanishes on $[0, \eta)$, and let $\xi < \eta$ be its first root. We may assume that $w(0) > 0$, and then $w'(\xi) < 0$. Combining (3.5) and (3.8), we get

$$\left[r^{n-1}\left(w'v_1 - wv_1'\right)\right]' \ge 0.$$

Integrating this over $(0, \xi)$

$$\xi^{n-1} w'(\xi) v_1(\xi) \ge 0,$$

but the quantity on the left is negative, a contradiction.

Assume now that $w(r)$ vanishes on $[\eta, 1)$, and let $\xi \ge \eta$ be its largest root. We may assume that $w(r) > 0$ on $(\xi, 1)$, and then $w'(\xi) > 0$, $w'(1) < 0$. Combining (3.5) and (3.9), we get

$$\left[r^{n-1}\left(w'v_2 - wv_2'\right)\right]' \le 0.$$

Integrating this over $(\xi, 1)$

$$-\xi^{n-1} w'(\xi) v_2(\xi) + w'(1) v_2(1) \le 0,$$

but the quantity on the left is positive, since the first term is non-negative, and the second one is positive (recall that $v_2(r) < 0$ on $(\eta, 1]$), a contradiction. $\diamond$

**Example.** Assume that $n \ge 4$, and consider $f(r,u) = a(r)u^p + b(r)u^q$, with constants $0 < p < 1 < q$, and the functions $a(r), b(r) \in C^1[0,1) \cap C[0,1]$ are assumed to satisfy

$$a(r) > 0, \quad b(r) > 0, \quad a'(r) \le 0, \quad b'(r) \le 0 \quad \text{for } 0 \le r < 1, \tag{3.10}$$

$$na(r) + ra'(r) \ge (n-2)pa(r) \quad \text{for } 0 \le r < 1, \tag{3.11}$$

$$nb(r) + rb'(r) \ge (n-2)qb(r) \quad \text{for } 0 \le r < 1. \tag{3.12}$$

Then $f(r,u)$ satisfies the conditions of the theorem. Notice that our conditions imply that $q \le \frac{n}{n-2}$, i.e., $q$ is subcritical.

We have the following exact multiplicity result.

**Theorem 3.1.** *For the problem* $(n \ge 4)$

$$u'' + \frac{n-1}{r}u' + \lambda\left(a(r)u^p + b(r)u^q\right) = 0, \qquad u'(0) = u(1) = 0 \tag{3.13}$$

*assume that the conditions of (3.10)–(3.12) hold. Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (3.13) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve $u(r, \lambda)$, which for $\lambda < \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^-(r, 0) = 0$, and $\lim_{\lambda \to 0} u^+(0, \lambda) = \infty$.*

**Proof.** Existence of positive solutions for small $\lambda$ follows by the method of super- and sub-solutions. Indeed, consider first the problem

$$u'' + \frac{n-1}{r}u' + \lambda a(r)u^p = 0, \qquad u'(0) = u(1) = 0. \tag{3.14}$$

Let $\varphi_1(r)$ denote the principal eigenfunction of the Laplacian on the unit ball $B$, with the Dirichlet boundary condition, and by $\psi_1$ the principal eigenfunction on a slightly larger ball. Then $\lambda\epsilon\varphi_1(r)$, and $\lambda M\psi_1$ form an ordered sub-solution–supersolution pair for (3.14), provided that we choose $\epsilon$ small and $M$ large. The same pair also works for

$$u'' + \frac{n-1}{r}u' + \lambda\left(a(r)u^p + \delta b(r)u^q\right) = 0, \qquad u'(0) = u(1) = 0 \tag{3.15}$$

for $\delta$ small enough, giving us a positive solution of (3.15). A suitable multiple of that solution is a solution of (3.13). This solution tends to zero, as $\lambda \to 0$.

We now continue the solutions forward in $\lambda$, by using either the Implicit Function Theorem, or the bifurcation theorem of Crandall–Rabinowitz [4], see [1] or [2] for more details. By the a priori estimates of B. Gidas and J. Spruck [14], this curve cannot go to infinity at some $\lambda > 0$ (observe that our conditions imply that $q < \frac{n}{n-2}$). This curve cannot be continued forward in $\lambda$ indefinitely. Indeed, multiplying the Eq. (3.1) by $\varphi_1(r)$ and integrating, we conclude

$$\lambda < \lambda_1 \sup_{x\in D, u>0} \frac{u}{f(x,u)},$$

which implies that $\lambda$ is bounded.

It follows that the solution curve will reach a singular point $(\lambda_0, u_0)$, at which the linearized problem (3.5) has a non-trivial solution, and where the Crandall–Rabinowitz [4] theorem applies, see [1] or [2] for more details. By Theorem 2.1 and Lemma 3.1, the quantity $I$ defined in (1.4) is positive, and hence a turn to the left occurs at this, and any other singular points. It follows that $(\lambda_0, u_0)$ is the only singular point, and after the turn at $(\lambda_0, u_0)$ the curve continues without any more turns, tending to infinity as $\lambda \to 0$. $\diamond$

## 4. A class of concave–convex equations on a thin annulus

On an annulus $\Omega = \{x \mid 0 < A < |x| < B\}$ in $R^n$, $n \geq 2$, we consider the problem

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

We study positive radially symmetric solutions of (4.1), depending on a positive parameter $\lambda$. It is known that the problem (4.1) may have positive non-radial solutions, in contrast to the case when domain is a ball in $R^n$, and all positive solutions are necessarily radially symmetric. This problem arises in many applications, and there is a large literature on the subject, including W.-M. Ni and R. Nussbaum [15], S.S. Lin [16]. To get an exact multiplicity result, we shall restrict our attention to the case of "thin" annulus, which we define next. Set $c_n = (2n-3)^{\frac{1}{n-2}}$ for $n \geq 3$, and $c_2 = e^2$. We shall assume that

$$B \leq c_n A. \tag{4.2}$$

The special role of "thin" annulus was recognized first by W.-M. Ni and R. Nussbaum [15], but they had a more restrictive condition, involving $c_n = (n-1)^{\frac{1}{n-2}}$ for $n \geq 3$, and $c_2 = e$. Their condition appeared later in S.S. Lin [16]. The condition (4.2) was used in P. Korman [11].

Radial solutions of (4.1) satisfy

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0 \quad \text{for } A < r < B, \qquad u(A) = u(B) = 0. \tag{4.3}$$

We make a standard change of variables. In case $n \geq 3$, we let $s = r^{2-n}$ and $u(s) = U(r)$, transforming (4.3) into the problem

$$u'' + \alpha(s) f(u) = 0, \quad \text{for } a < s < b, \qquad u(a) = u(b) = 0, \tag{4.4}$$

where $\alpha(s) = (n-2)^{-2} s^{-2k}$, with $k = 1 + \frac{1}{n-2}$, $a = B^{2-n}$ and $b = A^{2-n}$. In case $n = 2$, we set $s = -\log r$, and $u(s) = U(r)$, obtaining again the problem (4.4), this time with $\alpha(s) = e^{-2s}$, and $a = -\log B$, $b = -\log A$. The corresponding linearized problem is

$$w'' + \alpha(s) f'(u) w = 0, \quad \text{for } a < s < b, \qquad w(a) = w(b) = 0. \tag{4.5}$$

We proved the following theorem in [11], see also the exposition in [1].

**Theorem 4.1.** *Assume that the annulus is thin, i.e., the condition (4.2) holds, and the function $f(u) \in C^1(R_+) \cap C(\bar{R}_+)$ satisfies $f(u) > 0$ for $u > 0$. Then any non-trivial solution of (4.5) is of one sign.*

This result, together with the Theorem 2.1 provides us with the following exact multiplicity result, whose proof is similar to that of Theorem 3.1.

**Theorem 4.2.** *On an annulus $\Omega = \{x \mid A < |x| < B\} \subset R^n$ consider the problem*

$$\Delta u + \lambda \left(u^p + u^q\right) = 0 \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{4.6}$$

*with constants $0 < p < 1 < q$. Assume that the annulus is thin, i.e., the condition (4.2) holds. Then there is a critical $\lambda_0 > 0$, such that for $\lambda > \lambda_0$ the problem (4.6) has no positive solutions, it has exactly one positive solution for $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda < \lambda_0$. Moreover, all positive solutions lie on a single smooth solution curve $u(r,\lambda)$, which for $\lambda < \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^-(r,0) = 0$, and $\lim_{\lambda \to 0} u^+(0,\lambda) = \infty$.*

## 5. Numerical computation of solution curves

In this section we present computations of the global curves of positive solutions $u = u(r)$ for the problem

$$\Delta u + \lambda f(|x|, u) = 0 \quad \text{in } B,\ u = 0 \text{ on } \partial B, \tag{5.1}$$

where $B$ is the unit ball $|x| < 1$ in $R^n$, $n \geq 1$. We shall assume that $f(r, u) \in C^2(B \times R_+) \cap C(\bar{B} \times \bar{R}_+)$ satisfies (3.3), and then by B. Gidas, W.-M. Ni and L. Nirenberg [13] any positive solution of (5.1) is radially symmetric, i.e., $u = u(r)$, $r = |x|$, with $u'(r) < 0$, and hence it satisfies

$$u'' + \frac{n-1}{r}u' + \lambda f(r, u) = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = u(1) = 0. \tag{5.2}$$

The standard approach to numerical computation of solutions involves curve following, i.e., continuation in $\lambda$ by using the predictor–corrector type methods, see e.g., E.L. Allgower and K. Georg [17]. These methods are well developed, but not easy to implement, as the solution curve $u = u(x, \lambda)$ may consist of several parts, each having multiple turns. Here $\lambda$ is a local parameter, but not a global one, because of the possible turning points.

The quantity $\alpha = u(0)$ gives the maximum value of any positive solution. There are a number of situations, where $\alpha = u(0)$ is a global parameter, i.e., the value of $\alpha$ uniquely identifies the solution pair $(\lambda, u(r))$ (see the discussion below). We shall do a continuation in $\alpha$, i.e., we shall compute the solution curve of (5.2) in the form $\lambda = \lambda(\alpha)$. We begin with a simple lemma.

**Lemma 5.1.** *The solution of the linear problem*

$$u'' + \frac{n-1}{r}u' + f(r) = 0 \quad \textit{for } 0 < r < 1, \qquad u(0) = \alpha, \qquad u'(0) = 0$$

*can be represented in the form*

$$u(r) = \alpha + \frac{1}{n-2}r^{-n+2}\int_0^r \left(z^{n-2} - r^{n-2}\right) zf(z)\,dz, \quad \textit{for } n \neq 2,$$

$$u(r) = \alpha + \int_0^r (\ln z - \ln r)\ zf(z)\,dz, \quad \textit{for } n = 2.$$

**Proof.** Integrating

$$\left(r^{n-1}u'\right)' = -r^{n-1}f(r)$$

over the interval $(0, z)$, we express

$$u'(z) = -\frac{1}{z^{n-1}}\int_0^z t^{n-1}f(t)\,dt.$$

Integrating over the interval $(0, r)$, we have

$$u(r) = \alpha - \int_0^r \frac{1}{z^{n-1}}\int_0^z t^{n-1}f(t)\,dt\,dz.$$

Integrating by parts in the last integral (with $u = \int_0^z t^{n-1}f(t)\,dt$, $dv = \frac{1}{z^{n-1}}\,dz$), we conclude the proof. ◇

If we solve the initial value problem

$$u'' + \frac{n-1}{r}u' + \lambda f(r, u) = 0, \qquad u(0) = \alpha, \qquad u'(0) = 0, \tag{5.3}$$

then we need to find $\lambda$, so that $u(1) = 0$, in order to obtain a solution of (5.2). By Lemma 5.1, we rewrite the Eq. (5.3) in the integral form (for $n \neq 2$)

$$u(r) = \alpha + \frac{\lambda}{n-2}r^{-n+2}\int_0^r \left(z^{n-2} - r^{n-2}\right) zf(z, u(z))\,dz, \quad \text{for } n \neq 2,$$

and then the equation for $\lambda$ is

$$F(\lambda) \equiv u(1) = \alpha + \frac{\lambda}{n-2}\int_0^1 \left(z^{n-2} - 1\right) zf(z, u(z))\,dz = 0. \tag{5.4}$$

We solve this equation by using Newton's method

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)}.$$

We have

$$F(\lambda_n) = \alpha + \frac{\lambda_n}{n-2} \int_0^1 \left(z^{n-2} - 1\right) zf(z, u(z))\, dz,$$

$$F'(\lambda_n) = \frac{1}{n-2} \int_0^1 \left(z^{n-2} - 1\right) zf(z, u(z))\, dz + \frac{\lambda_n}{n-2} \int_0^1 \left(z^{n-2} - 1\right) zf_u(z, u(z)) u_\lambda\, dz,$$

where $u = u(r, \lambda_n)$ and $u_\lambda = u_\lambda(r, \lambda_n)$ are respectively the solutions of

$$u'' + \frac{n-1}{r} u' + \lambda_n f(r, u) = 0, \qquad u(0) = \alpha, \qquad u'(0) = 0; \tag{5.5}$$

$$u_\lambda'' + \frac{n-1}{r} u_\lambda' + \lambda_n f_u(r, u(r, \lambda_n)) u_\lambda + f(r, u(r, \lambda_n)) = 0, \qquad u_\lambda(0) = 0, \qquad u_\lambda'(0) = 0. \tag{5.6}$$

(As we vary $\lambda$, when solving (5.4), we keep $u(0) = \alpha$ fixed, that is the reason why $u_\lambda(0) = 0$.) This method is very easy to implement. It requires repeated solutions of the initial value problems (5.5) and (5.6) (using the NDSolve command in *Mathematica*).

In case $n = 2$, we have

$$F(\lambda) = \alpha + \lambda \int_0^1 z \ln z f(z, u(z))\, dz,$$

$$F'(\lambda) = \int_0^1 z \ln z f(z, u(z))\, dz + \lambda \int_0^1 z \ln z f_u(z, u(z)) u_\lambda\, dz,$$

and the rest is as before.

**Example.** We have solved the problem

$$u'' + \frac{n-1}{r} u' + \lambda \left(u^{1/2} + (1 - 0.2r^2) u^{3/2}\right) = 0 \quad \text{for } 0 < r < 1, \tag{5.7}$$
$$u'(0) = u(1) = 0.$$

The Theorem 3.1 applies here. The global curve of positive solutions, for $n = 4$, is presented in Fig. 1. For any point $(\lambda, \alpha)$ on this curve, the actual solution $u(r)$ is easily computed by shooting (using the NDSolve command in *Mathematica*), i.e., by solving (5.3). In Fig. 2 we present the solution $u(r)$ for $\lambda \approx 6.30263$, when $u(0) = 7.5$. (This solution lies on the upper branch of our solution curve.)

We now discuss under what conditions $\alpha = u(0)$ is a global parameter, i.e., the value of $\alpha$ uniquely identifies the solution pair $(\lambda, u(r))$ for the problem (5.2). This property also has theoretical significance, as it rules out any secondary bifurcations. In case $n = 1$, $u(0)$ is a global parameter, assuming the condition (3.3) holds ($f_r \le 0$), see P. Korman [18], or P. Korman and J. Shi [19]. It is natural to ask if the same result holds in case $n > 1$. We have no proof, and this question appears much harder in case $n > 1$, although our computations suggest that the result is still true. (If $u(0)$ were not a global parameter, the computed solution curve would have to be discontinuous, which never happened in our computations.) We have the following simple result (which does not require that $f_r \le 0$).

**Theorem 5.1.** *Assume that $f(r, u) \in C([0, 1] \times \bar{R}_+)$ satisfies*

$$f(r, u) \ge 0 \quad \text{for } 0 \le r < 1,\ u > 0, \tag{5.8}$$

$$\frac{f(r, u)}{u} \text{ is decreasing in } u, \quad \text{for } 0 \le r < 1,\ u > 0. \tag{5.9}$$

*Then $\alpha = u(0)$ is a global parameter for positive solutions of (5.2), i.e., for any $\alpha$ there is at most one solution pair $(\lambda, u(r) > 0)$ of (5.2), with $u(0) = \alpha$.*

**Proof.** Assume, on the contrary, that there are two solution pairs $(\lambda, u(r))$ and $(\mu, v(r))$ of (5.2), with $\mu > \lambda$, and $u(0) = v(0)$. By (5.8),

$$v'' + \frac{n-1}{r} v' + \lambda f(r, v) \le 0 \quad \text{for } 0 \le r < 1. \tag{5.10}$$

Combining this with (5.2),

$$\left[r^{n-1}\left(u'v - uv'\right)\right]' + \lambda r^{n-1} uv \left[\frac{f(r, u)}{u} - \frac{f(r, v)}{v}\right] \ge 0. \tag{5.11}$$

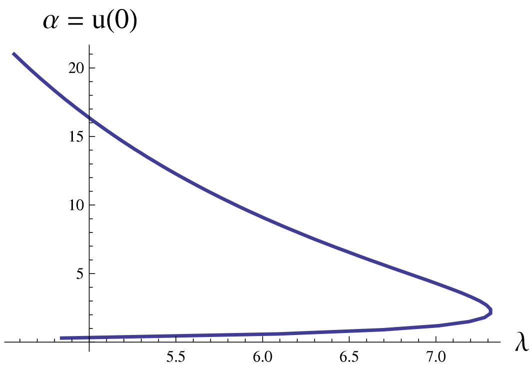


**Fig. 1.** The global solution curve for the problem (5.7), $n = 4$.

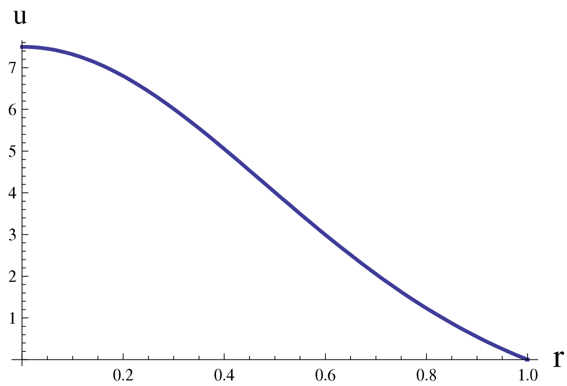


**Fig. 2.** The solution $u(r)$, corresponding to $\alpha = u(0) = 7.5$, and $\lambda \approx 6.30263$.

Observe that

$$nv''(0) < -\lambda f(0, v(0)) = nu''(0).$$

It follows that $v(r) < u(r)$ for small $r > 0$. Let $0 < \xi \le 1$ be the first point of intersection of $u(r)$ and $v(r)$, i.e., $v(r) < u(r)$ for $r \in (0, \xi)$. Integrating (5.11) over $(0, \xi)$,

$$\xi^{n-1} u(\xi) \left(u'(\xi) - v'(\xi)\right) + \lambda \int_0^{\xi} r^{n-1} uv \left[\frac{f(r, u)}{u} - \frac{f(r, v)}{v}\right] dr \ge 0,$$

which results in a contradiction, because the first term on the left is non-positive, and the second one is negative. ◇

**Example.** Consider $f(r, u) = u(a(r) - u)$, corresponding to the logistic population model. Assume that $a(r) > 0$ on $[0, 1]$, and it satisfies

$$a'(r) < 0, \quad \text{and} \quad a'' + \frac{n-1}{r} a' < 0 \quad \text{for } 0 \le r < 1. \tag{5.12}$$

Since $f_r < 0$, it follows by [13] that $u'(r) < 0$ for $r \in (0, 1)$. Then $u''(0) \le 0$, and so $u(0) \le a(0)$. We need to verify that $u(r) \le a(r)$, so that (5.8) holds, and the theorem applies. Assuming the contrary, we have $u(r) > a(r)$ on some interval $(\xi, \eta)$, with $0 \le \xi < \eta < 1$, and $u(\xi) = a(\xi)$, $u(\eta) = a(\eta)$. We then have

$$u'' + \frac{n-1}{r} u' > 0 \quad \text{for } r \in (\xi, \eta).$$

Letting $w = u(r) - a(r)$, we have

$$\left(r^{n-1} w'\right)' > 0, \qquad w(r) > 0 \quad \text{for } r \in (\xi, \eta), \qquad w(\xi) = w(\eta) = 0.$$

Integrating over $(\xi, \eta)$, we have a contradiction. We conclude that $u(0)$ is a global parameter.

Another situation when $u(0)$ is a global parameter occurs for positive solutions of

$$u'' + \frac{n-1}{r} u' + \lambda r^{\alpha} f(u) = 0 \quad \text{for } 0 < r < 1, \qquad u'(0) = u(1) = 0. \tag{5.13}$$

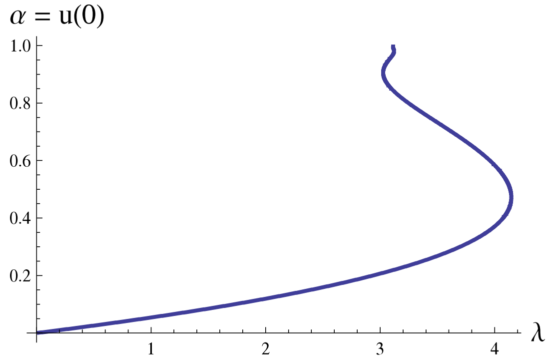


**Fig. 3.** The global solution curve for the problem (5.16), $n = 3$.

Here $\alpha$ is any real number, $f(u) \in C(R)$. If we change the variables

$$r = \frac{1}{\lambda^{\frac{1}{\alpha+2}}} t,$$

then $u = u(t)$ satisfies

$$u'' + \frac{n-1}{t} u' + t^{\alpha} f(u) = 0 \qquad u'(0) = 0. \tag{5.14}$$

Its first root occurs at $t = \lambda^{\frac{1}{\alpha+2}}$ (corresponding to $r = 1$). Let $v(r)$ solve

$$v'' + \frac{n-1}{r} v' + \mu r^{\alpha} f(v) = 0 \quad \text{for } 0 < r < 1, \qquad v'(0) = v(1) = 0, \tag{5.15}$$

with $\mu \neq \lambda$. Let us assume that $u(0) = v(0)$. Then the substitution $r = \frac{1}{\mu^{\frac{1}{\alpha+2}}} t$ takes (5.15) also into (5.14), and $u(0) = v(0)$. By uniqueness for initial value problems $v(t) \equiv u(t)$, but the first root of $v(t)$ occurs at $t = \mu^{\frac{1}{\alpha+2}} \neq \lambda^{\frac{1}{\alpha+2}}$, a contradiction.

We now describe the shoot-and-scale method for computing the bifurcation diagrams for (5.13) (and this is the only class of non-autonomous equations, for which this method works). Given the value of $\alpha = u(0)$, solve the initial problem (5.14), together with $u(0) = \alpha$. Compute $t_0$, the first root of this solution. Then set

$$\lambda = t_0^{\alpha+2}.$$

The point $(\lambda, \alpha)$ belong to the solution curve. Repeating these calculations for a mesh of $\alpha$'s, we approximate the solution curve of (5.13).

**Example.** The problem

$$u'' + \frac{n-1}{r} u' + \lambda \frac{r^2}{(1-u)^2} = 0 \quad \text{for } 0 < r < 1, \tag{5.16}$$
$$u'(0) = u(1) = 0, \quad 0 < u(r) < 1$$

arises in modeling electrostatic micro-electromechanical systems (MEMS), see [20–22]. We have solved this problem using continuation in $u(0)$ for $n = 3$, and the solution curve is given in Fig. 3. We have also solved this problem by the shoot-and-scale method, and obtained the same result. Previously, a similar picture was given in N. Ghoussoub and Y. Guo [21]. It follows from the results of Z. Guo and J. Wei [22], that the solution curve of (5.16) makes infinitely many turns. In Fig. 3 we followed the solution curve until $u(0) = 0.9995$. It follows that infinitely many turns must occur in the remaining 0.05% of the solution curve's range.

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