Solutions Manual
Lectures on Differential Equations
Philip L. Korman

Chapter 1

Section 1.3.1, Page 10

I.1   The initial guess is $x\dfrac{e^{5x}}{5}$. Its derivative

$$\frac{d}{dx}\left(x\frac{e^{5x}}{5}\right) = xe^{5x} + \frac{1}{5}e^{5x}$$

has an extra term $\frac{1}{5}e^{5x}$. The derivative of $-\dfrac{e^{5x}}{25}$ will cancel this term. Answer. $x\dfrac{e^{5x}}{5} - \dfrac{e^{5x}}{25} + c.$

I.2   The initial guess is $x\,\dfrac{\sin 2x}{2}$. Its derivative

$$\frac{d}{dx}\left(x\,\frac{\sin 2x}{2}\right) = x\cos 2x + \frac{1}{2}\sin 2x$$

has an extra term $\frac{1}{2}\sin 2x$. The derivative of $\dfrac{\cos 2x}{4}$ will cancel this term. Answer. $x\,\dfrac{\sin 2x}{2} + \dfrac{\cos 2x}{4} + c.$

I.3   The initial guess is $-(2x+1)\dfrac{\cos 3x}{3}$. The extra term $-\dfrac{2}{3}\cos 3x$ will be cancelled by the derivative of $\dfrac{2}{9}\sin 3x$. Answer. $-(2x+1)\dfrac{\cos 3x}{3}+\dfrac{2}{9}\sin 3x+c.$

I.4   The initial guess is $-2xe^{-x/2}$.   Answer. $e^{-x/2}(-4-2x)+c.$

I.5   The initial guess is $-x^2e^{-x}$. The first correction is $-x^2e^{-x} - 2xe^{-x}$. Answer. $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c.$

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I.6   The initial guess is $\frac{1}{2}x^2 \sin 2x$. The first correction is $\frac{1}{2}x^2 \sin 2x +$ $\frac{1}{2}x \cos 2x$. Answer. $\frac{1}{2}x \cos 2x + \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\sin 2x + c$.

I.7   The initial guess is $\left(x^2 + 1\right)^{\frac{1}{2}}$, and it gives the correct answer.

I.9   The initial guess is $\dfrac{1}{x^2 + 1} - \dfrac{1}{x^2 + 9}$. (Write the fraction with the smaller denominator $x^2 + 1$ before the fraction with the larger denominator $x^2 + 9$.) Adjust to

$$\frac{1}{(x^2 + 1)(x^2 + 9)} = \frac{1}{8}\left[\frac{1}{x^2 + 1} - \frac{1}{x^2 + 9}\right].$$

I.10   The initial guess is $\dfrac{1}{x^2 + 1} - \dfrac{1}{x^2 + 2}$. Adjust to

$$\frac{x}{(x^2 + 1)(x^2 + 2)} = \frac{x}{x^2 + 1} - \frac{x}{x^2 + 2}.$$

I.11 $\dfrac{1}{x^3 + 4x} = \dfrac{1}{x(x^2 + 4)} = \dfrac{1}{4}\left[\dfrac{1}{x} - \dfrac{x}{x^2 + 4}\right].$

I.12   Substitution $u = \ln x$ gives $\displaystyle\int u^5\, du = \dfrac{u^6}{6} + c = \dfrac{\ln^6 x}{6} + c.$

I.14 $\displaystyle\int_0^\pi x \sin nx\, dx = \left[-x\dfrac{\cos nx}{n} + \dfrac{\sin nx}{n^2}\right]\Big|_0^\pi = \dfrac{\pi}{n}(-1)^{n+1}.$

I.15   Look for the antiderivative in the form

$$\int e^{2x} \sin 3x\, dx = Ae^{2x}\sin 3x + Be^{2x}\cos 3x + c.$$

We need to choose the constants A and B so that

$$\frac{d}{dx}\left(Ae^{2x}\sin 3x + Be^{2x}\cos 3x\right) = e^{2x}\sin 3x.$$

Calculate

$$\frac{d}{dx}\left(Ae^{2x}\sin 3x + Be^{2x}\cos 3x\right) = 2Ae^{2x}\sin 3x + 3Ae^{2x}\cos 3x + 2Be^{2x}\cos 3x - 3Be^{2x}\sin 3x$$

$$= (2A - 3B)\,e^{2x}\sin 3x + (3A + 2B)\,e^{2x}\cos 3x.$$

Then we need

$$(2A - 3B)\,e^{2x}\sin 3x + (3A + 2B)\,e^{2x}\cos 3x = e^{2x}\sin 3x.$$

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This will hold, provided

$$2A - 3B = 1$$
$$3A + 2B = 0\,.$$

Solving this system of equations gives $A = \dfrac{2}{13}$ and $B = -\dfrac{3}{13}$. We conclude that

$$\int e^{2x} \sin 3x \, dx = \frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x + c\,.$$

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II.1   The integrating factor $\mu = e^{-\int \sin x \, dx} = e^{\cos x}$. Then

$$\frac{d}{dx}\left[e^{\cos x}\, y\right] = e^{\cos x} \sin x\,,$$

$$e^{\cos x}\, y = -e^{\cos x} + c\,,$$

$$y = -1 + c\, e^{-\cos x}\,.$$

II.2   Here $\mu = e^{\int \frac{1}{x}\, dx} = e^{\ln x} = x$. Then

$$\frac{d}{dx}\left[x\, y\right] = x \cos x\,,$$

$$x\, y = x \sin x + \cos x + c\,,$$

$$y = \frac{c}{x} + \sin x + \frac{\cos x}{x}\,.$$

II.3   Divide the equation by x, to put it into the form for which the method of integrating factor $\mu(x)$ was developed. Obtain $y' + \dfrac{2}{x}y = \dfrac{e^{-x}}{x}$, and $\mu = e^{\int \frac{2}{x}\, dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$. Then

$$\frac{d}{dx}\left[x^2\, y\right] = xe^{-x}\,,$$

$$x^2\, y = -xe^{-x} - e^{-x} + c\,,$$

$$y = \frac{c}{x^2} - \frac{(x+1)e^{-x}}{x^2}\,.$$

II.4   Divide the equation by x^4, to put it into the form for which the method was developed. Obtain $y' + \dfrac{3}{x}y = \dfrac{e^x}{x^2}$, and $\mu = e^{\int \frac{3}{x}\, dx} = e^{3\ln x} = e^{\ln(x^3)} = x^3$. Then

$$\frac{d}{dx}\left[x^3\, y\right] = xe^x\,,$$

$$x^3 y = \int x e^x \, dx = x e^x - e^x + c \, ,$$

$$y = \frac{c}{x^3} + \frac{(x-1)e^x}{x^3} \, .$$

II.5 $y' - 2xy = 2x^3$. $\mu = e^{-\int 2x \, dx} = e^{-x^2}$. Then

$$\frac{d}{dx}\left[e^{-x^2} y \right] = 2x^3 e^{-x^2} \, ,$$

$$e^{-x^2} y = 2 \int x^3 e^{-x^2} \, dx = e^{-x^2}\left(-x^2 - 1 \right) + c \, .$$

The integral was computed by a substitution $z = x^2$, followed by integration by parts. Solve for y:

$$y = c e^{x^2} - x^2 - 1 \, .$$

II.6 $y' - \dfrac{2}{x} y = e^{\frac{1}{x}}$. $\mu = e^{-\int \frac{2}{x} \, dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = \dfrac{1}{x^2}$. Then

$$\frac{d}{dx}\left[\frac{1}{x^2} y \right] = \frac{1}{x^2} e^{\frac{1}{x}} \, ,$$

$$\frac{1}{x^2} y = \int \frac{1}{x^2} e^{\frac{1}{x}} \, dx = -e^{\frac{1}{x}} + c \, ,$$

$$y = c x^2 - x^2 e^{\frac{1}{x}} \, .$$

II.7 Here $\mu = e^{2x}$. Then

$$\frac{d}{dx}\left[e^{2x} y \right] = e^{2x} \sin 3x \, ,$$

$$e^{2x} y = \int e^{2x} \sin 3x \, dx = \frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x + c \, ,$$

$$y = \frac{2}{13} \sin 3x - \frac{3}{13} \cos 3x + c \, .$$

(One looks for the integral in the form $\int e^{2x} \sin 3x \, dx = A e^{2x} \sin 3x + B e^{2x} \cos 3x$, and the constants A and B are found by differentiation.)

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III.1 With $\mu = e^{-2x}$, write the equation as

$$\frac{d}{dx}\left[e^{-2x} y \right] = e^{-2x} e^x = e^{-x} \, .$$

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The general solution is $y = ce^{2x} - e^x$, and $c = 3$.

III.2   Here $\mu = x$. Obtain

$$\frac{d}{dx}[xy] = x \cos x \, ,$$

$$x\,y = \int x \cos x \, dx = x \sin x + \cos x + c \, .$$

The general solution is $y = \dfrac{c + x \sin x + \cos x}{x}$, and $c = 0$. The solution becomes $y = \dfrac{x \sin x + \cos x}{x}$. It can be continued indefinitely to the right of the initial point $x = \dfrac{\pi}{2}$, while to the left of $x = \dfrac{\pi}{2}$ one can continue the solution only to $x = 0$. The solution is valid on $(0, \infty)$.

III.3   Divide the equation by x: $y' + \dfrac{2}{x}y = \dfrac{\sin x}{x^2}$, then $\mu = e^{2 \ln x} = x^2$. The general solution is $y = \dfrac{c - \cos x}{x^2}$, and $c = -\dfrac{\pi^2}{4}$.

III.4   Divide the equation by x: $y' + (1 + \dfrac{2}{x})y = \dfrac{1}{x}$, $\mu = e^{x + 2 \ln x} = x^2 e^x$. The general solution is $y = \dfrac{ce^{-x} + x - 1}{x^2}$, and $c = 3e^{-2}$. It can be continued indefinitely to the left of the initial point $x = -2$, while to the right of $x = -2$ one can continue the solution only to $x = 0$. The solution is valid on $(-\infty, 0)$.

III.5   Divide the equation by x: $y' - y = \dfrac{e^x}{x}$. The general solution is $y = ce^x + e^x \ln|x|$, and $c = 1$. Beginning at $x = -1$, this solution may be continued on $(-\infty, 0)$.

III.6   Divide the equation by $t + 2$: $y' + \dfrac{1}{t+2}y = \dfrac{5}{t+2}$, $\mu = e^{\int \frac{1}{t+2} dt} = e^{\ln(t+2)} = t + 2$. This leads to $y = \dfrac{5t + c}{t+2}$. Using $y(1) = 1$, get $c = -2$, so that $y = \dfrac{5t - 2}{t + 2}$. Starting from the initial point $t = 1$, the solution can be continued on the interval $(-2, \infty)$.

III.7 $y' - \dfrac{2}{t}y = t^3 \cos t$, $\mu = t^{-2}$. The general solution is $y = t^3 \sin t + t^2 \cos t + ct^2$, and $c = -\dfrac{\pi}{2}$.

III.8   Divide the equation by $t \ln t$: $\dfrac{dr}{dt} + \dfrac{1}{t \ln t} y = 5 \dfrac{e^t}{\ln t}$. $\mu = e^{\int \frac{1}{t \ln t} dt} = e^{\ln(\ln t)} = \ln t$. The general solution is $r = \dfrac{5 e^t + c}{\ln t}$, and $c = -5e^2$. It is valid on $(1, \infty)$, because $\ln t$ in the denominator vanishes at $t = 1$.

III.9   Combine the xy' and y' terms, then solve for y' to write the equation as
$$y' + \frac{2}{(x-1)} y = \frac{1}{(x-1)^3} \, .$$

Here $\mu = (x-1)^2$, and the general solution is $y = \dfrac{\ln|x-1| + c}{(x-1)^2}$, with $c = -\ln 3$. Obtain: $y = \dfrac{\ln|x-1| - 3}{(x-1)^2}$. Starting at $x = -2$, one encounters zero denominator at $x = 1$. The solution is valid on $(-\infty, 1)$. On that interval $x < 1$, so that $x - 1 < 0$, and $|x - 1| = -(x-1) = 1 - x$. Then the solution can writen as $y = \dfrac{\ln(1-x) - 3}{(x-1)^2}$.

III.10   Obtain a linear equation for $x = x(y)$
$$\frac{dx}{dy} - x = y^2 \, .$$

Calculate $\mu(y) = e^{-y}$, and then
$$\frac{d}{dy}\left(e^{-y} x\right) = y^2 e^{-y} \, ,$$

$$e^{-y} x = \int y^2 e^{-y} \, dy = -y^2 e^{-y} - 2y e^{-y} - 2e^{-y} + c \, .$$

The general solution is $x = -y^2 - 2y - 2 + c e^y$. The initial condition for the inverse function is $x(0) = 2$, giving $c = 4$.

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IV.2   The general solution is $y = \pm \sqrt{2e^x + c}$. Choose "minus" and $c = -1$, to satisfy the initial condition.

IV.3   By factoring the expression in the bracket, simplify
$$\frac{dy}{y} = (x + \frac{1}{x}) \, dx \, ,$$

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giving $y = e^{\frac{x^2}{2} + \ln|x| + c}$. Also, there is a solution $y = 0$. If we write the solution in the form $y = c|x|e^{\frac{x^2}{2}}$, then $y = 0$ is included in this family (when $c = 0$).

IV.4   The factor $\sqrt{4 - y^2} = 0$ if $y(x) = -2$ or $y(x) = 2$, both solutions of this differential equation. Assuming that $\sqrt{4 - y^2} \neq 0$, we separate the variables

$$\int \frac{1}{\sqrt{4 - y^2}}\, dy = \int x^2\, dx\,,$$

$$\sin^{-1} \frac{y}{2} = \frac{x^3}{3} + c\,,$$

$$y = 2\sin\left(\frac{x^3}{3} + c\right)\,.$$

IV.5   In the equation

$$\frac{dy}{dt} = ty^2 \left(1 + t^2\right)^{-1/2}$$

we separate the variables

$$\int \frac{dy}{y^2} = \int t \left(1 + t^2\right)^{-1/2}\, dt\,,$$

$$-\frac{1}{y} = \left(1 + t^2\right)^{1/2} + c\,,$$

$$y(t) = -\frac{1}{(1 + t^2)^{1/2} + c}\,.$$

The initial condition

$$y(0) = -\frac{1}{1 + c} = 2$$

implies that $c = -\frac{3}{2}$.

IV.6   The key is to factor

$$y - xy + x - 1 = -y(x - 1) + x - 1 = (x - 1)(-y + 1)\,.$$

Then

$$x^2\, dy = (x - 1)(y - 1)\, dx\,,$$

$$\int \frac{dy}{y - 1} = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx\,,$$

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$$\ln|y-1| = \ln|x| + \frac{1}{x} + c\,,$$

$$|y-1| = c|x|e^{\frac{1}{x}}\,. \quad (e^c \text{ is the new } c)$$

Writing $|y-1| = \pm(y-1)$, and $|x| = \pm x$, we can absorb $\pm$ into c, to get

$$y - 1 = cxe^{\frac{1}{x}}\,.$$

Now solve for y, and determine $c = -\frac{1}{e}$ from the initial condition.

IV.7   The function $y = 1$ is a solution of our equation

$$x^2y^2\frac{dy}{dx} = y - 1\,.$$

Assuming that $y \neq 1$, we separate the variables

$$\int \frac{y^2}{y-1}\,dy = \int \frac{1}{x^2}\,dx\,.$$

The integral on the left is evaluated by performing division of the polynomials y^2 and $y - 1$, or as follows

$$\int \frac{y^2}{y-1}\,dy = \int \frac{y^2-1+1}{y-1}\,dy = \int \left(y+1+\frac{1}{y-1}\right)dy = \frac{1}{2}y^2+y+\ln|y-1|\,.$$

We obtained the solution in the implicit form

$$\frac{1}{2}y^2 + y + \ln|y-1| = -\frac{1}{x} + c\,,$$

which can be solved for x as a function of y.

IV.8   Because the integral $\int e^{x^2}\,dx$ cannot be evaluated in elementary functions, write the solution as $y = c\int_a^x e^{t^2}\,dt$. Choose $a = 2$, because the initial condition is prescribed at $x = 2$, and then $c = 1$.

IV.9   Variables will separate after the factoring $xy^2 + xy = xy(y+1)$. Obtain:

$$\int \frac{dy}{y(y+1)} = \int x\,dx\,,$$

$$\int \left[\frac{1}{y} - \frac{1}{y+1}\right]dy = \frac{x^2}{2} + \ln c\,,$$

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$$\ln\frac{y}{y+1} = \frac{x^2}{2} + \ln c\,,$$

$$\frac{y}{y+1} = ce^{\frac{x^2}{2}}\,.$$

Now solve for y:

$$y = ce^{\frac{x^2}{2}}(y+1) = ce^{\frac{x^2}{2}}y + ce^{\frac{x^2}{2}}\,,$$

$$y = \frac{ce^{\frac{x^2}{2}}}{1 - ce^{\frac{x^2}{2}}}\,.$$

The initial condition gives

$$y(0) = \frac{c}{1-c} = 2\,.$$

Solve for c:

$$c = 2(1-c) = 2 - 2c\,,$$

giving $c = \dfrac{2}{3}$.

IV.10 $\quad \dfrac{dy}{dx} = 2x(y^2 + 4)$. The general solution is $y = 2\tan(2x^2 + c)$. The initial condition requires that $\tan c = -1$. There are infinitely many choices for c, but they all lead to the same solution as $c = -\frac{\pi}{4}$ because of the periodicity of the tangent.

IV.11 $\quad$ Completing the square, write our equation in the form

$$\frac{dy}{dt} = -\frac{1}{4}(2y-1)^2\,.$$

The function $y(t) = \dfrac{1}{2}$ is a solution of this equation. Assuming that $y \neq \dfrac{1}{2}$, we separate the variables

$$-4\int \frac{dy}{(2y-1)^2} = \int dt\,,$$

$$\frac{2}{2y-1} = t + c\,.$$

Taking the reciprocals of both sides, obtain $y - \dfrac{1}{2} = \dfrac{1}{t+c}$, or $y = \dfrac{1}{2} + \dfrac{1}{t+c}$.

IV.12 $\quad \dfrac{1}{y(y-1)} = \dfrac{dx}{x}$, $\quad \displaystyle\int \left[\dfrac{1}{y-1} - \dfrac{1}{y}\right] dy = \int \dfrac{dx}{x}$, $\quad \ln|y-1| - \ln y =$
$\ln|x| + \ln c$, or $\left|\dfrac{y-1}{y}\right| = c|x|$. Also, $y = 0$ and $y = 1$ are solutions.

IV.13   The general solution was obtained in the preceding problem. Near
the initial point $x = 1$ and $y = 2$, defined by the initial condition $y(1) = 2$,
we may drop the absolute value signs and obtain $\dfrac{y-1}{y} = cx$. Solve for y:
$y = \dfrac{1}{1-cx}$, and determine $c = \dfrac{1}{2}$.

IV.14   Setting $z = x + y$, gives $y' = z' - 1$, $z(0) = y(0) = 1$. Obtain

$$z' = z^2 + 1, \quad z(0) = 1.$$

Separating the variables, $z(x) = \tan(x + c)$. From the initial condition,
$z(0) = \tan c = 1$, so that $c = \dfrac{\pi}{4}$. (By periodicity of $\tan x$, there are infinitely
many other choices for c, but they all lead to the same solution.) Conclude:
$z(x) = \tan(x + \dfrac{\pi}{4})$, and $y = -x + z = -x + \tan(x + \dfrac{\pi}{4})$.

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Writing the following equations in the form $y' = f(\dfrac{y}{x})$, we justify that
they are homogeneous, and then set $v = \dfrac{y}{x}$ to reduce them to separable
equations.

I.1 $\quad \dfrac{dy}{dx} = \dfrac{y}{x} + 2$, $\quad v = \dfrac{y}{x}$, $\quad v + x\dfrac{dv}{dx} = v + 2$, $\quad \displaystyle\int dv = 2 \int \dfrac{dx}{x}$, $\quad v = 2\ln x + c$,
$y = x(2\ln x + c)$.

I.2 $\quad \dfrac{dy}{dx} = \dfrac{x+y}{x} = \dfrac{y}{x} + 1$, $\quad v = \dfrac{y}{x}$, $\quad v + x\dfrac{dv}{dx} = v + 1$, $\quad v = \ln|x| + c$,
$y = x(\ln|x| + c)$.

I.3 $\quad \dfrac{dy}{dx} = 1 - \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$, $\quad v = \dfrac{y}{x}$, $\quad v + x\dfrac{dv}{dx} = 1 - 2v + v^2$, $\quad x\dfrac{dv}{dx} = (v-1)^2$.
One solution of the last equation is $v = 1$, giving $y = x$. If $v \neq 1$, obtain
$\dfrac{dv}{(v-1)^2} = \dfrac{dx}{x}$,

$$-\dfrac{1}{v-1} = \ln|x| + c,$$

giving $\quad v = 1 - \dfrac{1}{\ln x + c}$, $\quad y = x\left(1 - \dfrac{1}{\ln x + c}\right)$.

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I.4   This equation is not homogeneous. (We will solve it later as a Bernoulli's equation.)

I.5 $v + x\dfrac{dv}{dx} = v^2 + v$. The general solution is $y = -\dfrac{x}{\ln|x| + c}$. Calculate $c = -1$. Near the initial point $x = 1$, $\ln|x| = \ln x$, so that $y = \dfrac{x}{1 - \ln x}$.

I.6   This time $c = 1$, and near the initial point $x = -1$, the absolute value sign is needed.

I.7 $v + x\dfrac{dv}{dx} = v^2 + 2v$. Separating the variables

$$\int \frac{dv}{v(v+1)} = \int \frac{dx}{x},$$

$$\ln v - \ln(v+1) = \ln x + \ln c,$$

$$\frac{v}{v+1} = cx.$$

Solve this equation for v, starting by clearing the denominator

$$v = cx(v+1) = cxv + cx,$$

$$(1 - cx)v = cx,$$

$$v = \frac{cx}{1 - cx}.$$

Then $y = xv = \dfrac{cx^2}{1 - cx}$, and to satisfy the initial condition $y(1) = 2$ need

$$\frac{c}{1-c} = 2.$$

Then $c = 2(1 - c)$, giving $c = \frac{2}{3}$.

I.8 $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$, $v + x\dfrac{dv}{dx} = v + \tan v$. Separating the variables

$$\int \frac{\cos v}{\sin v}\, dv = \int \frac{dx}{x},$$

$$\ln|\sin v| = \ln|x| + \ln c,$$

$$|\sin v| = c|x|.$$

We may drop the absolute value signs by redefining c. Then $\sin\dfrac{y}{x} = cx$, or $y = x\sin^{-1} cx$.

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I.9 $\quad \dfrac{dy}{dx} = \dfrac{x}{x+y} + \dfrac{y}{x}, \quad v + x\dfrac{dv}{dx} = \dfrac{x}{x+xv} + v, \quad \displaystyle\int (1+v)\,dx = \int \dfrac{dx}{x},$

$$\dfrac{1}{2}v^2 + v = \ln|x| + c.$$

Solving this quadratic equation for v gives $v = -1 \pm \sqrt{1 + 2\ln|x| + c}$ (with a new c), or $y = -x \pm x\sqrt{1 + 2\ln|x| + c}$.

I.10 $\quad \dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x}, \quad v + x\dfrac{dv}{dx} = \dfrac{1}{v} + v, \quad \dfrac{1}{2}v^2 = \ln|x| + c, \quad y = \pm x\sqrt{2\ln|x| + c}.$
To satisfy the initial condition $y(1) = -2$, select "minus", and $c = 4$. Obtain $y = -x\sqrt{2\ln x + 4}$ (using that $\ln|x| = \ln x$ near the initial point $x = 1$).

I.11 $\quad \dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^{-1/2} + \dfrac{y}{x}, \quad v + x\dfrac{dv}{dx} = v^{-1/2} + v, \quad 2\sqrt{v} = \ln x + c,$

$$2\sqrt{\dfrac{y}{x}} = \ln x + c, \quad y = x\,\dfrac{(\ln x + c)^2}{4}.$$

I.12 $\quad$ Begin by solving for y', $\quad y' = \dfrac{y^3}{x^3 + xy^2}$. Let $\dfrac{y}{x} = v$ or $y = xv$.

$$v + x\dfrac{dv}{dx} = \dfrac{x^3 v^3}{x^3 + x^3 v^2} = \dfrac{v^3}{1 - v^2},$$

$$x\dfrac{dv}{dx} = \dfrac{v^3}{1 + v^2} - v = -\dfrac{v}{1 + v^2},$$

$$\int \dfrac{1 + v^2}{v}\,dv = -\int \dfrac{dx}{x},$$

$$\ln\left|\dfrac{y}{x}\right| + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = -\ln|x| + c,$$

$$\ln|y| + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = c.$$

Also, $y = 0$.

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II.1 $\quad y(t) = 0$ is a solution. Assuming that $y \neq 0$, divide the equation by y^2

$$y^{-2}y' = 3y^{-1} - 1,$$

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and set $v = y^{-1}$. Then $v' = -y^{-2}y'$, and one obtains a linear equation

$$-v' = 3v - 1 \, .$$

Its general solution is $v = \dfrac{1}{3} + ce^{-3t}$, and then $y = \dfrac{1}{v} = \dfrac{3}{1 + ce^{-3t}}$ (with a new c).

II.2   Divide the equation by y^2, and set $v = y^{-1}$. Then $y^{-2}y' = -v'$, and one obtains a linear equation

$$-v' - \dfrac{1}{x}v = 1 \, .$$

Its general solution is $v = -\dfrac{x}{2} + \dfrac{c}{x}$. Then $y = \dfrac{1}{v} = \dfrac{2x}{c - x^2}$ (with a new c). The initial condition implies that $c = 2$.

II.3   Divide the equation by y^2, and set $v = y^{-1}$. Then $y^{-2}y' = -v'$, and one obtains a linear equation

$$xv' - v = x \, .$$

Its general solution is $v = x\,(c + \ln x)$. Then $y = \dfrac{1}{v} = \dfrac{1}{x(c + \ln x)}$. Using the initial condition, $c = \frac{1}{2}$.

II.4   Divide the equation by y^3

$$y^{-3}y' + y^{-2} = x \, ,$$

and set $v = y^{-2} = \dfrac{1}{y^2}$. Then $v' = -2y^{-3}y'$, $y^{-3}y' = -\dfrac{1}{2}v'$, and one obtains a linear equation

$$-\dfrac{1}{2}v' + v = x \, .$$

Its general solution is $v = x + \dfrac{1}{2} + ce^{2x}$. Then $y = \pm\dfrac{1}{\sqrt{v}} = \pm\dfrac{1}{\sqrt{x + \frac{1}{2} + ce^{2x}}}$.

The initial condition $y(0) = -1$ requires us to select "minus" and $c = \dfrac{1}{2}$.

II.5 $y' = y + 2xy^{-1}$. Divide the equation by y^{-1}, which is the same as multiplying by y

$$yy' = y^2 + 2x \, .$$

Setting $v = y^2$, $v' = 2yy'$, produces a linear equation

$$\dfrac{1}{2}v' = v + x \, ,$$

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with the general solution $v = ce^{2x} - x - \dfrac{1}{2}$. Then $y = \pm\sqrt{v} = \pm\sqrt{ce^{2x} - x - \dfrac{1}{2}}$.

II.6 $\quad y' + xy^{\frac{1}{3}} = 3y$. There is a solution $y = 0$. If $y \neq 0$, divide the equation by $y^{\frac{1}{3}}$

$$y^{-\frac{1}{3}}y' + x = 3y^{\frac{2}{3}},$$

and set $v = y^{\frac{2}{3}}$, so that $y = \pm v^{\frac{3}{2}}$. Calculate $v' = \dfrac{2}{3}y^{-\frac{1}{3}}y'$, and obtain a linear equation

$$\frac{3}{2}v' - 3v = -x.$$

Its general solution is $v = \dfrac{x}{3} + \dfrac{1}{6} + ce^{2x}$, and then $y = \pm\left(\dfrac{x}{3} + \dfrac{1}{6} + ce^{2x}\right)^{\frac{3}{2}}$.

II.7 $\quad y' + xy^{\frac{1}{3}} = 3y$. There is a solution $y = 0$. If $y \neq 0$, divide the equation by y^2, then set $v = y^{-1} = \dfrac{1}{y}$, $v' = -y^{-2}y'$

$$y^{-2}y' + y^{-1} = -x,$$

$$-v' + v = -x.$$

Obtain $v = ce^x - x - 1$, and $y = \dfrac{1}{ce^x - x - 1}$.

II.8 $\quad$ Divide the equation by y^3, $y^{-3}y' + xy^{-2} = 1$. Set $v = y^{-2}$, $v' = -2y^{-3}y'$, and obtain a linear equation

$$v' - 2xv = -2, \quad v(1) = y^{-2}(1) = e^2.$$

Its integrating factor is $\mu = e^{-x^2}$, and then

$$\frac{d}{dx}\left[e^{-x^2}v\right] = -2e^{-x^2}.$$

The integral of e^{-x^2} cannot be computed in the elementary functions, so that we use definite integration, beginning at $x = 1$ (where the initial condition is known)

$$e^{-x^2}v = -2\int_1^x e^{-t^2}\,dt + c,$$

$$v = -2e^{x^2}\int_1^x e^{-t^2}\,dt + ce^{x^2}.$$

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From the initial condition, $v(1) = c\,e = e^2$, so that $c = e$, and then

$$v = -2e^{x^2} \int_1^x e^{-t^2} \, dt + e^{x^2+1} \, .$$

It follows that

$$y = \pm \frac{1}{\sqrt{v}} = \pm \frac{1}{\sqrt{-2e^{x^2} \int_1^x e^{-t^2} \, dt + e^{x^2+1}}} \, .$$

The original initial condition, $y(1) = -\dfrac{1}{e}$, requires us to select "minus".

II.9 $y' = y + 2xy^{-1}$. Multiply the equation by y, $yy' = y^2 + 2x$, and set $v = y^2$, to obtain a linear equation

$$\frac{1}{2}v' = v + 2x \, .$$

Its general solution is $v = ce^{2x} - 2x - 1$, and then $y = \pm\sqrt{ce^{2x} - 2x - 1}$.

II.10   Multiply the equation by y, $yy' = y^2 + xe^{2x}$, and set $v = y^2$, to obtain a linear equation

$$\frac{1}{2}v' = v + xe^{2x} \, .$$

Its general solution is $v = e^{2x}\left(x^2 + c\right)$, and then $y = \pm e^x \sqrt{x^2 + c}$.

II.11   Write the equation as

$$\frac{1}{x}\frac{dx}{dt} = a - b\ln x \, ,$$

then set $y(t) = \ln x(t)$, $y' = \frac{1}{x}\frac{dx}{dt}$. Obtain a *linear equation*

$$y' + by = a \, ,$$

giving $y = \frac{a}{b} + ce^{-bt}$. Then $x = e^y = e^{a/b}e^{ce^{-bt}}$.

II.12   Divide the equation by e^y

$$xe^{-y}y' - x + 2e^{-y} = 0 \, .$$

Setting here $v = e^{-y}$, with $v' = -e^{-y}y'$, obtain a linear equation for $v = v(x)$

$$-xv' + 2v = x \, .$$

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Solving it gives (here $\mu = e^{-2\ln x} = e^{\ln(x^{-2})} = \frac{1}{x^2}$)

$$v = x + cx^2 \,,$$

and then $y = -\ln v = -\ln\left(x + cx^2\right).$

II.13   This equation has a solution $y = 0$. Assuming that $y \neq 0$, take the reciprocals of both sides, to obtain Bernoulli's equation for the inverse function $x = x(y)$

$$x'(y) = \frac{1}{y}x + x^2 \,.$$

Divide this equation by x^2

$$x^{-2}(y)x'(y) = \frac{1}{y}\,x^{-1}(y) + 1\,,$$

and set $v(y) = x^{-1}(y)$. With $v'(y) = -x^{-2}(y)x'(y)$, obtain a linear equation

$$-v' = \frac{1}{y}\,v + 1\,.$$

Its general solution is $v = -\dfrac{y}{2} + \dfrac{c}{y}$, and then $x = \dfrac{1}{v} = \dfrac{2y}{c - y^2}$ (with $2c = c$).

Answer: $x = \dfrac{2y}{c - y^2}$, and $y = 0$.

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III.1   Set $y' = t$, then $x = t^3 + t$, and

$$dy = y'\,dx = t\left(3t^2 + 1\right)\,dt\,,$$

$$\frac{dy}{dt} = 3t^3 + t\,,$$

$$y = \frac{3}{4}t^4 + \frac{1}{2}t^2 + c\,.$$

III.2   Set $y' = t$, then $y = \ln(1 + t^2)$. The possibility of $y' = 0$ leads to a solution $y = 0$. If $y' \neq 0$, we can write

$$dx = \frac{dy}{y'} = \frac{\frac{2t}{1+t^2}}{t}\,dt = \frac{2}{1 + t^2}\,dt\,,$$

$$\frac{dx}{dt} = \frac{2}{1 + t^2}\,,$$

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$$x = 2\tan^{-1} t + c\,.$$

III.3   Set $y' = t$, then $x = t + \sin t$, and

$$dy = y'\,dx = t\,(1 + \cos t)\,dt\,,$$

$$\frac{dy}{dt} = t + t\cos t\,,$$

$$y = \frac{1}{2}t^2 + t\sin t + \cos t + c\,.$$

The initial condition requires that the curve $(x(t), y(t))$ passes through $(0, 0)$. The equation

$$x = t + \sin t = 0$$

has $t = 0$ as its only solution. In order to get $y = 0$ at $t = 0$, one needs to choose $c = -1$.

III.4   Divide the equation by y^2, and set $v = y^{-1}$. Then $y^{-2}y' = -v'$, and one obtains a linear equation

$$-v' = 3v - 1\,.$$

Its general solution is $v = ce^{-3t} + \dfrac{1}{3}$, and then $y = \dfrac{1}{v} = \dfrac{3}{1 + 3ce^{-3t}}$. The initial condition implies that $y = \dfrac{3}{1 + 2e^{-3t}}$.

III.5   Let $y(t)$ be the weight of salt in the tank at a time t.

$$\text{salt in} = 0\,,$$

$$\text{salt out (per minute)} = 5\,\frac{y}{100} = 0.05y\,.$$

Obtain the equation

$$y' = -0.05y\,,\quad y(0) = 10\,,$$

with the solution $y(t) = 10e^{-0.05t}$. In particular, $y(60) \approx 0.5$.

III.6   Let $y(t)$ be the weight of salt in the tank at a time t.

$$\text{salt in (per minute)} = 0.3\,,$$

$$\text{salt out (per minute)} = 3\,\frac{y}{100} = 0.03y\,.$$

Obtain the equation

$$y' = 0.3 - 0.03y\,,\quad y(0) = 10\,,$$

with the solution $y(t) = 10$.

III.7 Let $y(t)$ be the weight of poison in the stomach at a time t.

$$\text{poison in} = 0,$$

$$\text{poison out (per minute)} = 0.5\frac{y}{3} = \frac{1}{6}y.$$

Obtain the equation

$$y' = -\frac{1}{6}y, \quad y(0) = 300,$$

with the solution $y(t) = 300e^{-\frac{1}{6}t}$. In particular, $y(t) = 50$ at $t = 6\ln 6 \approx 10.75$.

III.9 The tangent line at the point $(x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$. It intersects the x-axis at $\frac{x_0}{2}$, so that $y = 0$ when $x = x_0/2$, giving

$$0 = f(x_0) + f'(x_0)\left(-\frac{1}{2}x_0\right),$$

$$x_0 f'(x_0) = 2f(x_0).$$

Replacing the arbitrary point x_0 by x, obtain a separable equation, which is easy to solve

$$x f'(x) = 2f(x),$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{2}{x}\,dx,$$

$$\ln f(x) = 2\ln x + \ln c,$$

$$f(x) = cx^2.$$

III.10 The tangent line at the point $(x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$. At the point x_1 where it intersects the x-axis obtain

$$0 = f(x_0) + f'(x_0)(x_1 - x_0),$$

so that the horizontal side of our triangle is

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}.$$

The vertical side of the triangle is $f(x_0)$. We are given that the sum of the sides is b, so that

$$f(x_0) - \frac{f(x_0)}{f'(x_0)} = b\,,$$

$$f(x_0)f'(x_0) - f(x_0) = bf'(x_0)\,.$$

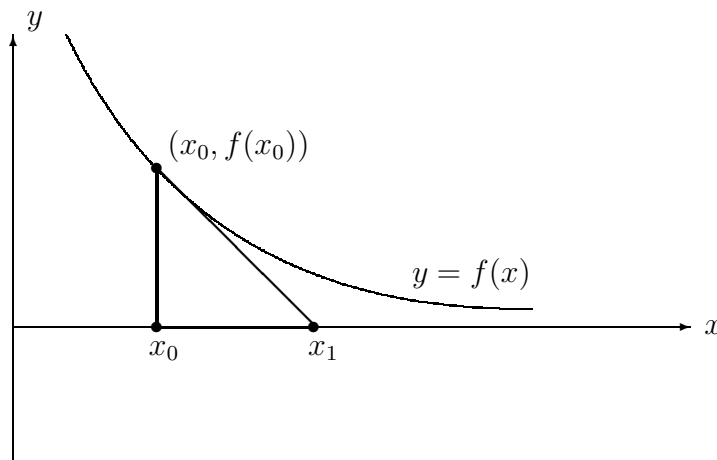Replace the arbitrary point x_0 by x, and the function $f(x)$ by $y(x)$:

$$yy' - y = by'\,.$$

The quickest way to solve this equation is to divide by y, then integrate

$$y' - 1 = b\frac{y'}{y}\,,$$

$$y - x = b\ln y + c\,.$$

We obtained the solution in implicit form (it can be solved for x).

The triangle formed by the tangent line, the line $x = x_0$, and the x-axis

III.11   The tangent line at the point $(x_0, f(x_0))$ is $y = f(x_0)+f'(x_0)(x-x_0)$. It intersects the y-axis at the point $(0, f(x_0)-x_0f'(x_0))$. Calculate the square of the distance between the points $(x_0, f(x_0))$ and $(0, f(x_0) - x_0 f'(x_0))$, and set it equal to 1

$$x_0^2 + x_0^2 f'^2(x_0) = 1\,.$$

We now replace the arbitrary point x_0 by x, and the function $f(x)$ by $y(x)$, obtaining

$$x^2 + x^2 y'^2(x) = 1 \,,$$

$$y' = -\frac{\sqrt{1-x^2}}{x} \,,$$

$$y = -\int \frac{\sqrt{1-x^2}}{x} \, dx = -\sqrt{1-x^2} - \ln x + \ln\left[1 + \sqrt{1-x^2}\right] + c \,.$$

The integral was computed by a substitution $x = \sin\theta$.

III.12   The tangent line at the point $(x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$. It intersects the x-axis at the point $(x_1, 0)$ in the picture above, which is the point $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$. The distance from this point to the origin is $x_0 - \dfrac{f(x_0)}{f'(x_0)}$. The distance from this point to the point $(x_0, f(x_0))$ is $\sqrt{\dfrac{f^2(x_0)}{f'^2(x_0)} + f^2(x_0)}$. We are given that

$$x_0 - \frac{f(x_0)}{f'(x_0)} = \sqrt{\frac{f^2(x_0)}{f'^2(x_0)} + f^2(x_0)} \,.$$

Square both sides, expand the square on the left, then cancel a pair of terms

$$x_0^2 - \frac{2x_0 f(x_0)}{f'(x_0)} = f^2(x_0) \,.$$

We now replace the arbitrary point x_0 by x, and the function $f(x)$ by $y(x)$, obtaining

$$x^2 - \frac{2xy}{y'} = y^2 \,,$$

$$x^2 - y^2 = \frac{2xy}{y'} \,,$$

$$y' = \frac{2xy}{x^2 - y^2} \,.$$

The last equation is homogeneous. Set $v = \dfrac{y}{x}$, $y = xv$, $y' = v + x\dfrac{dv}{dx}$, to obtain

$$v + x\frac{dv}{dx} = \frac{2v}{1 - v^2} \,,$$

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$$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{v+v^3}{1-v^2},$$

$$\int \frac{1-v^2}{v+v^3}\,dv = \int \frac{dx}{x}.$$

Using partial fractions

$$\frac{1-v^2}{v+v^3} = \frac{1-v^2}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{v^2+1}.$$

We then continue with the integration

$$\ln v - \ln(1+v^2) = \ln x + \ln c,$$

$$\ln \frac{v}{1+v^2} = \ln cx,$$

$$\frac{v}{1+v^2} = cx,$$

$$\frac{y/x}{1+(y/x)^2} = cx,$$

$$\frac{y}{x^2+y^2} = c,$$

$$x^2 + y^2 = cy, \quad \text{with a new } c = \frac{1}{c}.$$

This is a family of circles centered along the y-axis.

III.13   After guessing a particular solution $y = e^x$, a substitution $y(x) = e^x + z(x)$ produces the equation

$$z' = z^2$$

with the general solution $z(x) = -\dfrac{1}{x+c}$, giving $y(x) = e^x - \dfrac{1}{x+c}$.

Answer: $y = e^x - \dfrac{1}{x+c}$, and $y = e^x$.

III.14   After guessing a particular solution $y = 1$, a substitution $y(x) = 1 + z(x)$ produces Bernoulli's equation

$$z' + 2z - e^x z^2 = 0$$

with the general solution $z(x) = \dfrac{1}{e^x + c\,e^{2x}}$, giving $y(x) = 1 + \dfrac{1}{e^x + c\,e^{2x}}$.

Answer: $y = 1 + \dfrac{1}{e^x + c\,e^{2x}}$, and $y = 1$.

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III.16 Differentiate the equation and use the fundamental theorem of calculus

$$y' = y + 1, \quad y(1) = 2.$$

Solving this linear equation, $y = \dfrac{3}{e}e^x - 1$.

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IV.1 Here $M(x,y) = 2x + 3x^2 y$, $N(x,y) = x^3 - 3y^2$. Calculate $M_y = 3x^2 = N_x$. The equation is exact. To find $\psi(x,y)$, one needs to solve the equations

$$\psi_x = 2x + 3x^2 y$$
$$\psi_y = x^3 - 3y^2.$$

Taking the antiderivative in x in the first equation gives $\psi(x,y) = x^2 + x^3 y + h(y)$, where $h(y)$ is (at the moment) an arbitrary function of y. Substituting $\psi(x,y)$ into the second equation obtain

$$x^3 + h'(y) = x^3 - 3y^2,$$

so that $h'(y) = -3y^2$, and $h(y) = -y^3$. (The constant of integration was taken to be zero, because an arbitrary constant will appear at the next step.) It follows that $\psi(x,y) = x^2 + x^3 y - y^3$, and $x^2 + x^3 y - y^3 = c$ gives solution in implicit form.

IV.4 Clearing the denominators

$$x\,dx + 2y^3\,dy = 0$$

gives a separable equation $2y^3\,dy = -x\,dx$.

IV.5 Here $M(x,y) = 6xy - \cos y$, $N(x,y) = 3x^2 + x\sin y + 1$. Calculate $M_y = \sin y = N_x$. The equation is exact. To find $\psi(x,y)$, one needs to solve the equations

$$\psi_x = 6xy - \cos y$$
$$\psi_y = 3x^2 + x\sin y + 1.$$

From the first equation $\psi(x,y) = 3x^2 y - x\cos y + h(y)$, where $h(y)$ is an arbitrary function of y. Substituting $\psi(x,y)$ into the second equation gives

$$3x^2 + x\sin y + h'(y) = 3x^2 + x\sin y + 1,$$

so that $h'(y) = 1$, and $h(y) = y$. It follows that $\psi(x,y) = 3x^2 y - x\cos y + y$, and $3x^2 y - x\cos y + y = c$ gives solution in implicit form.

IV.6   Here $M(x, y) = 2x - y$, $N(x, y) = 2y - x$. Calculate $M_y = -1 = N_x$. The equation is exact. To find $\psi(x, y)$, one needs to solve the equations

$$\psi_x = 2x - y$$
$$\psi_y = 2y - x .$$

From the first equation $\psi(x, y) = x^2 - xy + h(y)$, where $h(y)$ is an arbitrary function of y. Substituting $\psi(x, y)$ into the second equation gives

$$-x + h'(y) = 2y - x ,$$

so that $h'(y) = 2y$, and $h(y) = y^2$. It follows that $\psi(x, y) = x^2 - xy + y^2$, and $x^2 - xy + y^2 = c$ gives solution in implicit form. Substituting $x = 1$ and $y = 2$ obtain $c = 3$, so that $x^2 - xy + y^2 = 3$.

IV.7   Here $M(x, y) = 2x + 2x\sqrt{x^2 - y}$, $N(x, y) = -\sqrt{x^2 - y}$. Calculate $M_y = -x\left(x^2 - y\right)^{-\frac{1}{2}} = N_x$, so that the equation is exact. To find $\psi(x, y)$, one needs to solve the equations

$$\psi_x = 2x + 2x\sqrt{x^2 - y}$$
$$\psi_y = -\sqrt{x^2 - y} .$$

It is easier to start with the second equation, taking the antiderivative in y to get

$$\psi(x, y) = \frac{2}{3}\left(x^2 - y\right)^{\frac{3}{2}} + h(x) ,$$

where $h(x)$ is an arbitrary function of x. Substituting $\psi(x, y)$ into the first equation gives

$$2x\left(x^2 - y\right)^{\frac{1}{2}} + h'(x) = 2x + 2x\left(x^2 - y\right)^{\frac{1}{2}} ,$$

so that $h'(x) = 2x$, $h(x) = x^2$, and $\left(x^2 - y\right)^{\frac{3}{2}} + x^2 = c$ gives the solution.

IV.8   Here $M(x, y) = ye^{xy}\sin 2x + 2e^{xy}\cos 2x + 2x$, $N(x, y) = xe^{xy}\sin 2x - 2$. One verifies that $M_y = N_x$, and the equation is exact. To find $\psi(x, y)$, one needs to solve the equations

$$\psi_x = ye^{xy}\sin 2x + 2e^{xy}\cos 2x + 2x$$
$$\psi_y = xe^{xy}\sin 2x - 2 .$$

This time it is advantageous to begin integration with the second equation, obtaining

$$\psi(x, y) = e^{xy} \sin 2x - 2y + h(x) \,,$$

where $h(x)$ is an arbitrary function of x. Substituting $\psi(x, y)$ into the first equation gives

$$ye^{xy} \sin 2x + 2e^{xy} \cos 2x + h'(x) = ye^{xy} \sin 2x + 2e^{xy} \cos 2x + 2x \,,$$

so that $h'(x) = 2x$, $h(x) = x^2$, and $e^{xy} \sin 2x - 2y + x^2 = c$ gives the general solution. The initial condition says that the point $x = 0$, $y = -2$ lies on this curve, so that $c = 4$.

IV.9   Here $M(x, y) = ye^{xy} + 2x$, $N(x, y) = bxe^{xy}$. The condition $M_y = N_x$ requires that

$$e^{xy} + xye^{xy} = be^{xy} + bxye^{xy} \,, \quad \text{for all } x \text{ and } y \,,$$

which makes it necessary that $b = 1$. When $b = 1$ the equation becomes

$$(ye^{xy} + 2x) \, dx + xe^{xy} \, dy = 0$$

and it is exact. One calculates $\psi = e^{xy} + x^2$. The solution $e^{xy} + x^2 = c$ can be solved for y.

IV.11   Here $M(x, y) = x - 3y$, $N(x, y) = x + y$. The equation is not exact, because $M_y = -3$, while $N_x = 1$. Writing this equation in the form

$$\frac{dy}{dx} = -\frac{x - 3y}{x + y}$$

one obtains a simple homogeneous equation.

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V.1   Here $f(x, y) = (y - 1)^{1/3}$, and $f_y(x, y) = \dfrac{1}{3} (y - 1)^{-2/3}$. The function f_y is not continuous at $y = 1$ (it is not even defined at $y = 1$). The existence and uniqueness theorem does not apply.

V.2   Here $f(x, y) = \dfrac{x}{y^2 - 2x}$. In order for $f(x, y)$ and $f_y(x, y)$ to be continuous at the point $(2, y_0)$, one needs the denominator of $f(x, y)$ to be non-zero at that point, so that $y_0^2 - 4 \neq 0$, or $y_0 \neq 2$.

V.3 If $x \geq 0$, then $y = \dfrac{x|x|}{4} = \dfrac{x^2}{4}$. Calculate $y' = \dfrac{x}{2}$, and $\sqrt{|y|} = \sqrt{\dfrac{x^2}{4}} = \dfrac{x}{2}$.

If $x < 0$, then $y = \dfrac{x|x|}{4} = -\dfrac{x^2}{4}$. Calculate $y' = -\dfrac{x}{2}$, and $\sqrt{|y|} = \sqrt{\dfrac{x^2}{4}} = -\dfrac{x}{2}$.

Observe that $y = \dfrac{x|x|}{4}$ is differentiable at $x = 0$.

Another solution is $y = 0$.

Section 1.8.1, Page 45

1.   Let $v(x) = xu(x)$. We are given that

$$v(x) \leq K + \int_1^x \frac{v(t)}{t}\, dt\,.$$

By the Bellman-Gronwall lemma

$$v(x) \leq K e^{\int_1^x \frac{1}{t}\, dt} = K e^{\ln x} = Kx\,.$$

It follows that $xu(x) \leq Kx$, or $u(x) \leq K$.

2.   Observe that for any $K \geq 0$

$$u(x) \leq K + \int_{x_0}^x a(t)u(t)\, dt\,.$$

By the Bellman-Gronwall lemma

$$u(x) \leq K e^{\int_{x_0}^x a(t)\, dt}\,.$$

Letting $K \to 0$, obtain $u(x) \leq 0$. We are given that $u(x) \geq 0$, which implies that $u(x) \equiv 0$.

3. Set $b(t) = a(t)u(t)$. Then

$$u(x) \leq \int_{x_0}^x b(t)u(t)\, dt\,.$$

By the preceding problem $u(x) = 0$.

Chapter 2

Section 2.3.3, Page 60

I.1   One could reduce order by letting $y' = v$, $y'' = v'$. Alternatively, write this equation as

$$\frac{d}{dx}y'^2 = 1\,,$$

and integrate

$$y'^2 = x + c_1\,,$$

$$y' = \pm(x + c_1)^{1/2}\,,$$

$$y = \pm\frac{2}{3}(x + c_1)^{3/2} + c_2\,.$$

I.2   Let $y' = v$, $y'' = v'$ to obtain a linear first order equation

$$xv' + v = x\,,$$

$$(xv)' = x\,,$$

$$xv = \frac{1}{2}x^2 + c_1\,,$$

$$y' = v = \frac{1}{2}x + \frac{c_1}{x}\,,$$

$$y = \frac{1}{4}x^2 + c_1 \ln x + c_2\,.$$

I.3   Let $y' = v$, $y'' = v'$ to obtain a linear first order equation

$$v' + v = x^2\,.$$

Its general solution is $y' = v(x) = x^2 - 2x + 2 + c_1 e^{-x}$. Then $y = \frac{x^3}{3} - x^2 + 2x + c_1 e^{-x} + c_2$ (with a new c_1).

I.4   Letting $y' = v$, $y'' = v'$ obtain a Bernoulli's equation

$$xv' + 2v = v^2\,.$$

(This equation is also separable.) Divide by v^2, then set $w = v^{-1}$, with $w' = -v^{-2}v'$. Obtain

$$xv^{-2}v' + 2v^{-1} = 1\,,$$

$$-xw' + 2w = 1\,, \quad w(1) = 1\,.$$

(Observe that $w(1) = \dfrac{1}{v(1)} = \dfrac{1}{y'(1)} = 1$.)

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The solution of the last equation is $w = \dfrac{1}{2}\left(x^2 + 1\right)$. Then $y' = v = \dfrac{1}{w} = \dfrac{2}{x^2 + 1}$. Integrating, obtain $y = 2\tan^{-1} x + c$, and then $c = -\dfrac{\pi}{2}$, because $y(1) = 0$.

I.5   Let $y' = v$, $y'' = v'$ to obtain a separable first order equation

$$v' = -2xv^2 \,,$$

$$y' = v = \frac{1}{x^2 + c_1}\,.$$

We need to determine c_1 now, because the integral of $\dfrac{1}{x^2 + c_1}$ depends on the sign of c_1. From the second initial condition

$$y'(0) = \frac{1}{c_1} = -4$$

get $c_1 = -\dfrac{1}{4}$. Then

$$y' = \frac{4}{4x^2 - 1} = 2\left[\frac{1}{2x - 1} - \frac{1}{2x + 1}\right],$$

$$y = \ln|2x - 1| - \ln|2x + 1| + c_2 \,.$$

From the first initial condition $c_2 = 0$.

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II.1   The substitution $y' = v(y)$, $y'' = v'v$ produces a separable first order equation

$$yv\frac{dv}{dy} = 3v^3 \,.$$

The possibility that $y' = v = 0$ produces a family of solutions $y = c$. If $v \neq 0$, obtain

$$\frac{dv}{v^2} = 3\frac{dy}{y}\,,$$

$$-\frac{1}{v} = 3\ln y + c_1\,,$$

$$v = -\frac{1}{3\ln y + c_1}\,,$$

$$\frac{dy}{dx} = -\frac{1}{3\ln y + c_1}\,.$$

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This is again a separable first order equation. Obtain

$$\int (3 \ln y + c_1) \, dy = -\int dx \,,$$

$$3 \left(y \ln y - y\right) + c_1 y = -x + c_2 \,.$$

II.2   The substitution $y' = v(y)$, $y'' = v'v$ produces a separable first order equation

$$yv\frac{dv}{dy} = -v^2 \,.$$

The possibility that $y' = v = 0$ produces a family of solutions $y = c$. If $v \neq 0$, obtain

$$\frac{dv}{v} = -\frac{dy}{y} \,,$$

$$\ln v = -\ln y + \ln c_1 = \ln \frac{c_1}{y} \,,$$

$$\frac{dy}{dx} = v = \frac{c_1}{y} \,,$$

$$\int y \, dy = c_1 \int dx \,,$$

$$y^2 = c_1 x + c_2 \ \ (\text{with a new } c_1) \,.$$

This time the family of solutions $y = c$ need not be presented separately, because it is included in the second family when $c_1 = 0$ and $c_2 > 0$.

II.3   One could use the substitution $y' = v(y)$, as in the preceding problems. Alternatively, write this equation in the form

$$\frac{d}{dx}y' = \frac{d}{dx}y^2 \,,$$

and integrate both sides:

$$y'(x) = y^2(x) + c_1 \,.$$

Setting here $x = 0$

$$y'(0) = y^2(0) + c_1 \,,$$

and using the initial conditions, one determines that $c_1 = 1$. Then

$$\frac{dy}{dx} = y^2 + 1 \,,$$

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$$\int \frac{dy}{y^2 + 1} = \int dx \,,$$

or $y = \tan x + c_2$, and $c_2 = 0$ by the first initial condition.

II.4 Again, rather than using the substitution $y' = v(y)$, it is easier to write the equation as

$$\frac{d}{dx}y' = \frac{d}{dx}\left(y^3 + y\right) \,,$$

and integrate both sides:

$$y' = y^3 + y + c_1 \,.$$

Setting here $x = 0$, and using the initial conditions, one determines that $c_1 = 0$. Then

$$\frac{dy}{dx} = y^3 + y \,,$$

$$\int \frac{1}{y^3 + y}\, dy = x + c_2 \,.$$

Use partial fractions

$$\frac{1}{y^3 + y} = \frac{1}{y(y^2 + 1)} = \frac{1}{y} - \frac{y}{y^2 + 1}$$

to obtain

$$\ln y - \frac{1}{2}\ln(y^2 + 1) = x + c_2 \,,$$

$$2\ln y - \ln(y^2 + 1) = 2x + c_2 \quad \text{(with a new } c_2\text{)} \,,$$

$$\ln \frac{y^2}{y^2 + 1} = 2x + c_2 \,,$$

$$\frac{y^2}{y^2 + 1} = c_2 e^{2x} \quad \text{(with a new } c_2\text{)} \,.$$

Solve this, first for y^2, and then for y

$$y^2 = c_2 e^{2x}(y^2 + 1) = c_2 e^{2x}y^2 + c_2 e^{2x} \,,$$

$$(1 - c_2 e^{2x})y^2 = c_2 e^{2x} \,,$$

$$y = \pm\sqrt{\frac{c_2 e^{2x}}{1 - c_2 e^{2x}}} \,.$$

To satisfy the first initial condition, one needs to take "plus", and $c_2 = \frac{1}{2}$.

III.1   The roots of the characteristic equation

$$r^2 + 4r + 3 = (r+1)(r+3) = 0$$

are $r = -3$ and $r = -1$. The solution is $y = c_1 e^{-3t} + c_2 e^{-t}$.

III.2   The roots of the characteristic equation

$$r^2 - 3r = 0$$

are $r = 0$ and $r = 3$. The solution is $y = c_1 + c_2 e^{3t}$.

III.3   The roots of the characteristic equation

$$2r^2 + r - 1 = 0$$

are $r = -1$ and $r = \dfrac{1}{2}$. The solution is $y = c_1 e^{-t} + c_2 e^{\frac{1}{2}t}$.

III.4   The roots of the characteristic equation

$$r^2 - 3 = 0$$

are $r = \pm\sqrt{3}$. The solution is $y = c_1 e^{-\sqrt{3}t} + c_2 e^{\sqrt{3}t}$.

III.5   The roots of the characteristic equation

$$3r^2 - 5r - 2 = 0$$

are $r = -\dfrac{1}{3}$ and $r = 2$. The solution is $y = c_1 e^{-\frac{1}{3}t} + c_2 e^{2t}$.

III.6   The roots of the characteristic equation

$$r^2 - 9 = 0$$

are $r = \pm 3$. The general solution is $y = c_1 e^{-3t} + c_2 e^{3t}$. The initial conditions imply

$$y(0) = c_1 + c_2 = 3$$
$$y'(0) = -3c_1 + 3c_2 = 3\,.$$

Solving this linear system for c_1 and c_2 gives $c_1 = 1$, $c_2 = 2$.

III.7   The roots of the characteristic equation

$$r^2 + 5r = 0$$

are $r = -5$ and $r = 0$. The general solution is $y = c_1 e^{-5t} + c_2$. The initial conditions imply

$$y(0) = c_1 + c_2 = -1$$

$$y'(0) = -5c_1 = -10.$$

Solving this linear system for c_1 and c_2 gives $c_1 = 2$, $c_2 = -3$.

III.8   The roots of the characteristic equation

$$r^2 + r - 6 = 0$$

are $r = -3$ and $r = 3$. The general solution is $y = c_1 e^{-3t} + c_2 e^{2t}$. The initial conditions imply

$$y(0) = c_1 + c_2 = -2$$

$$y'(0) = -3c_1 + 2c_2 = 3.$$

Solving this linear system for c_1 and c_2 gives $c_1 = -\dfrac{7}{5}$, $c_2 = -\dfrac{3}{5}$.

III.11   The roots of the characteristic equation

$$3r^2 - 2r - 1 = 0$$

are $r = -\dfrac{1}{3}$ and $r = 1$. The general solution is $y = c_1 e^{-\frac{1}{3}t} + c_2 e^t$. The initial conditions imply

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = -\frac{1}{3}c_1 + c_2 = a.$$

Solving this linear system for c_1 and c_2 gives $c_1 = -\dfrac{3}{4}(a-1)$, $c_2 = \dfrac{1}{4}(3a+1)$. The solution is then

$$y = -\frac{3}{4}(a-1)e^{-\frac{1}{3}t} + \frac{1}{4}(3a+1)e^t.$$

In order for this solution to remain bounded as $t \to \infty$ one needs $3a + 1 = 0$, or $a = -\dfrac{1}{3}$.

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IV.1   The characteristic equation

$$r^2 + 6r + 9 = 0$$

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has a double root $r = 3$. The general solution is $y = c_1 e^{3t} + c_2 t e^{3t}$.

IV.2 The characteristic equation

$$4r^2 - 4r + 1 = 0$$

has a double root $r = \dfrac{1}{2}$. The general solution is $y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$.

IV.3 The characteristic equation

$$r^2 - 2r + 1 = 0$$

has a double root $r = 1$. The general solution is $y = c_1 e^t + c_2 t e^t$. The initial conditions imply

$$y(0) = c_1 = 0$$

$$y'(0) = c_1 + c_2 = -2 .$$

Solving this linear system for c_1 and c_2 gives $c_1 = 0$, $c_2 = -2$.

IV.4 The characteristic equation

$$9r^2 - 6r + 1 = 0$$

has a double root $r = \dfrac{1}{3}$. The general solution is $y = c_1 e^{\frac{1}{3}t} + c_2 t e^{\frac{1}{3}t}$. The initial conditions imply

$$y(0) = c_1 = 1$$

$$y'(0) = \frac{1}{3}c_1 + c_2 = -2 .$$

Solving this linear system for c_1 and c_2 gives $c_1 = 1$, $c_2 = -\dfrac{7}{3}$.

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V.1 (i) $e^{i\pi} = \cos \pi + i \sin \pi = -1$.

(ii) $e^{-i\pi/2} = \cos(-\pi/2) + i \sin(-\pi/2) = \cos(\pi/2) - i \sin(\pi/2) = -i$.

(iii) $e^{i\frac{3\pi}{4}} = \cos\dfrac{3\pi}{4} + i \sin\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$.

(iv) $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$.

(v) $\sqrt{2} e^{i\frac{9\pi}{4}} = \sqrt{2}\left(\cos\dfrac{9\pi}{4} + i \sin\dfrac{9\pi}{4} \right) = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2} \right) = 1 + i$.

(vi) $\left(\cos \dfrac{\pi}{5} + i \sin \dfrac{\pi}{5}\right)^5 = \left(e^{i\frac{\pi}{5}}\right)^5 = e^{i\pi} = -1.$

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VI.1   The characteristic equation

$$r^2 + 4r + 8 = 0$$

has complex roots $r = -2 \pm 2i$. The general solution is $y = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t.$

VI.2   The characteristic equation

$$r^2 + 16 = 0$$

has purely imaginary roots $r = \pm 4i$. The general solution is $y = c_1 \cos 4t + c_2 \sin 4t.$

VI.3   The characteristic equation

$$r^2 - 4r + 5 = 0$$

has complex roots $r = 2 \pm i$. The general solution is $y = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$. The initial conditions imply

$$y(0) = c_1 = 1$$

$$y'(0) = 2c_1 + c_2 = -2\,.$$

Solving this linear system for c_1 and c_2 gives $c_1 = 1$, $c_2 = -4$.

VI.4   The characteristic equation

$$r^2 + 4 = 0$$

has purely imaginary roots $r = \pm 2i$. The general solution is $y = c_1 \cos 2t + c_2 \sin 2t$. The initial conditions imply

$$y(0) = c_1 = -2$$

$$y'(0) = 2c_2 = 0\,,$$

so that $y = -2 \cos 2t.$

VI.5   The characteristic equation

$$9r^2 + 1 = 0$$

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has purely imaginary roots $r = \pm \frac{1}{3}$. The general solution is $y = c_1 \cos \frac{1}{3} t + c_2 \sin \frac{1}{3} t$. The initial conditions imply

$$y(0) = c_1 = 0$$

$$y'(0) = \frac{1}{3} c_2 = 5 \,,$$

so that $y = 15 \sin \frac{1}{3} t$.

VI.7   The characteristic equation

$$4r^2 + 8r + 5 = 0$$

has the roots $r = -1 \pm \frac{1}{2} i$. The general solution is $y = c_1 e^{-t} \cos \frac{1}{2} t + c_2 e^{-t} \sin \frac{1}{2} t$. The initial conditions imply

$$y(\pi) = c_2 e^{-\pi} = 0$$

$$y'(\pi) = -c_1 \frac{1}{2} e^{-\pi} - c_2 e^{-\pi} = 4 \,,$$

so that $y = -8 e^{\pi} e^{-t} \cos \frac{1}{2} t$.

VI.8   The characteristic equation

$$r^2 + 1 = 0$$

has the roots $r = \pm i$. The general solution is $y = c_1 \cos t + c_2 \sin t$. The initial conditions imply

$$y(\frac{\pi}{4}) = c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2} = 0$$

$$y'(\frac{\pi}{4}) = -c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2} = -1 \,.$$

From the first equation, $c_2 = -c_1$. Then from the second equation, $\sqrt{2} c_1 = 1$, $c_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. The solution is $y = \frac{\sqrt{2}}{2} \cos t - \frac{\sqrt{2}}{2} \sin t$, which can also be written as $y = -\sin(t - \pi/4)$.

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VII.1   The characteristic equation

$$r^2 + br + c = 0$$

has the roots $r = -\dfrac{b}{2} \pm \dfrac{\sqrt{b^2 - 4c}}{2}$.

Case (i) $b^2 - 4c > 0$. Both roots are real. Clearly, $r_1 = -\dfrac{b}{2} - \dfrac{\sqrt{b^2 - 4c}}{2} < 0$. The other root is also negative $r_2 = -\dfrac{b}{2} + \dfrac{\sqrt{b^2 - 4c}}{2} < -\dfrac{b}{2} + \dfrac{\sqrt{b^2}}{2} = 0$. The solution $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \to 0$, as $t \to \infty$.

Case (ii) $b^2 - 4c = 0$. Both roots are equal to $-\dfrac{b}{2} < 0$. The solution $y = c_1 e^{-\frac{b}{2}t} + c_2 t e^{-\frac{b}{2}t} \to 0$, as $t \to \infty$.

Case (iii) $b^2 - 4c < 0$. The roots are complex $-\dfrac{b}{2} \pm iq$. The solution $y = c_1 e^{-\frac{b}{2}t} \cos qt + c_2 e^{-\frac{b}{2}t} \sin qt \to 0$, as $t \to \infty$.

VII.2   The characteristic equation

$$r^2 + br - c = 0$$

has the roots $r = -\dfrac{b}{2} \pm \dfrac{\sqrt{b^2 + 4c}}{2}$. Both roots are real. Clearly, $r_1 = -\dfrac{b}{2} - \dfrac{\sqrt{b^2 - 4c}}{2} < 0$. The other root $r_2 = -\dfrac{b}{2} + \dfrac{\sqrt{b^2 + 4c}}{2}$ is positive. The general solution is $y = y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. If some solution is bounded as $t \to \infty$, one must have $c_2 = 0$, and then this solution tends to zero, as $t \to \infty$.

VII.3   The function te^{-t} **is a solution of**

$$(*) \qquad\qquad\qquad ay'' + by' + cy = 0$$

when the characteristic equation

$$r^2 + br + c = 0$$

has a double root $r = -1$. The quadratic equation cannot have any more roots, and therefore the equation $(*)$ cannot have a solution $y = e^{3t}$ (which would correspond to the root $r = 3$ of the characteristic equation).

VII.4   Observe that $\dfrac{d}{dt}\left(\dfrac{ty'}{y}\right) = \dfrac{ty''y + y'y - ty'^2}{y^2}$, and then this equation can be written in the form

$$\frac{d}{dt}\left(\frac{ty'}{y}\right) = 0\,.$$

Integrate

$$\frac{ty'}{y} = c_1\,.$$

Then

$$\frac{y'}{y} = \frac{c_1}{t}\,,$$

$$\ln y = c_1 \ln t + \ln c_2\,,$$

$$y = c_2 t^{c_1}\,.$$

Section 2.5.4, Page 70

I.1 (i) $W(t) = \begin{vmatrix} e^{3t} & e^{-\frac{1}{2}t} \\ 3e^{3t} & -\frac{1}{2}e^{-\frac{1}{2}t} \end{vmatrix} = e^{3t}\left(-\dfrac{1}{2}e^{-\frac{1}{2}t}\right) - 3e^{3t}\,e^{-\frac{1}{2}t} = -\dfrac{7}{2}e^{\frac{5}{2}t}.$

(ii) $W(t) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{2t}\left(e^{2t} + 2te^{2t}\right) - 2e^{2t}\,te^{2t} = e^{4t}.$

(iii) $W(t) = \begin{vmatrix} e^t \cos 3t & e^t \sin 3t \\ e^t \cos 3t - 3e^t \sin 3t & e^t \sin 3t + 3e^t \cos 3t \end{vmatrix} = 3e^{2t}\left(\cos^2 3t + \sin^2 3t\right) =$ $3e^{2t}.$

(iv) $W(t) = \begin{vmatrix} \cosh 4t & \sinh 4t \\ 4\sinh 4t & 4\cosh 4t \end{vmatrix} = 4\left(\cosh^2 4t - \sinh^2 4t\right) = 4.$

I.2   We are given that

$$\begin{vmatrix} t^2 & g \\ 2t & g' \end{vmatrix} = t^5 e^t\,,$$

$$t^2 g' - 2tg = t^5 e^t\,.$$

To find g one needs to solve a linear first order equation:

$$g' - \frac{2}{t}g = t^3 e^t\,,$$

$$\frac{d}{dt}\left[\frac{1}{t^2}g\right] = te^t\,, \qquad (\mu = \frac{1}{t^2})$$

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$$\frac{1}{t^2}\,g = \int te^t\,dt = te^t - e^t + c\,,$$

$$g = t^3 e^t - t^2 e^t + ct^2\,.$$

I.3 We are given that

$$\left|\begin{array}{cc} e^{-t} & g \\ -e^{-t} & g' \end{array}\right| = t\,, \quad g(0) = 0\,,$$

$$e^{-t}g' + e^{-t}g = t\,, \quad g(0) = 0\,.$$

To find g one needs to solve a linear first order equation:

$$g' + g = te^t\,, \quad g(0) = 0\,,$$

$$\frac{d}{dt}\left[e^t\,g\right] = te^{2t}\,, \quad (\mu = e^t)$$

$$e^t\,g = \int te^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + c\,,$$

$$g = \frac{1}{2}te^t - \frac{1}{4}e^t + ce^{-t}\,, \quad c = \frac{1}{4}\,.$$

I.4 Given that
$$W(f, g)(t) = fg' - f'g = 0\,,$$

write
$$fg' = f'g\,,$$

$$\frac{g'}{g} = \frac{f'}{f}\,,$$

and integrate both sides
$$\ln g = \ln f + \ln c\,,$$

$$g = cf\,.$$

I.5 Apply the Theorem 2.4.2. Here $p(t) = 0$ for all t. Therefore,

$$W\left(y_1(t), y_2(t)\right)(t) = c\,.$$

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II.1   The general solution is $y = c_1 \cosh 2t + c_2 \sinh 2t$. The initial conditions give
$$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 = 0 \,,$$
$$y'(0) = 2c_1 \sinh 0 + 2c_2 \cosh 0 = 2c_2 = -\frac{1}{3} \,,$$
so that $c_1 = 0$, and $c_2 = -\frac{1}{6}$, giving $y = -\frac{1}{6} \sinh 2t$.

II.2   The general solution is $y = c_1 \cosh 3t + c_2 \sinh 3t$. The initial conditions give
$$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 = 2 \,,$$
$$y'(0) = 3c_1 \sinh 0 + 3c_2 \cosh 0 = 3c_2 = 0 \,,$$
so that $c_1 = 2$, and $c_2 = 0$, giving $y = 2 \cosh 3t$.

II.3   The general solution is $y = c_1 \cosh t + c_2 \sinh t$. The initial conditions give
$$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 = -3 \,,$$
$$y'(0) = c_1 \sinh 0 + c_2 \cosh 0 = c_2 = 5 \,,$$
so that $y = -3 \cosh t + 5 \sinh t$.

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III.1   The general solution centered at $\pi/8$ is $y = c_1 \cos(t - \pi/8) + c_2 \sin(t - \pi/8)$. The initial conditions give
$$y(\pi/8) = c_1 = 0 \,,$$
$$y'(\pi/8) = c_2 = 3 \,.$$

III.2   The general solution centered at $\pi/4$ is $y = c_1 \cos 2(t - \pi/4) + c_2 \sin 2(t - \pi/4)$. The initial conditions give
$$y(\pi/4) = c_1 = 0 \,,$$
$$y'(\pi/4) = 2c_2 = 4 \,,$$
so that $c_2 = 2$.

III.3   The general solution centered at 1 is $y = c_1 e^{-(t-1)} + c_2 e^{3(t-1)}$. The initial conditions give
$$y(1) = c_1 + c_2 = 1 \,,$$
$$y'(1) = -c_1 + 3c_2 = 7 \,.$$

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Calculate $c_1 = -1$, $c_2 = 2$.

III.4 The general solution centered at 2 may be written as $y = c_1 \cosh 3(t - 2) + c_2 \sinh 3(t - 2)$. The initial conditions give

$$y(2) = c_1 = -1\,,$$

$$y'(2) = 3c_2 = 15\,,$$

so that $c_2 = 5$.

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IV.1 We need to find $y_2(t)$, the second solution in the fundamental set. It will be convenient to write $y = y(t)$ instead of $y_2(t)$. By the Theorem 2.4.2

$$W(t) = \begin{vmatrix} e^t & y \\ e^t & y' \end{vmatrix} = ce^{-\int(-2)\,dt} = ce^{2t}\,,$$

$$e^t y' - e^t y = e^{2t}\,.$$

We set here $c = 1$, because we need just one solution to fill the role of $y_2(t)$. Now solve the linear first order equation for y:

$$y' - y = e^t\,,$$

$$\frac{d}{dt}\left[e^{-t}y\right] = 1\,,$$

$$y(t) = te^t\,.$$

Again, we set the constant of integration to zero, because we need just one solution, which is not a multiple of $y_1 = e^t$. We found $y_2(t) = te^t$. The general solution is $y = c_1 e^t + c_2 te^t$.

IV.2 We begin with dividing this equation by t^2

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$$

to put it into the form required by the Theorem 2.4.2. Here $p(t) = -\dfrac{2}{t}$. We need to find $y_2(t)$, the second solution in the fundamental set. Again, we write $y = y(t)$ instead of $y_2(t)$. Using the Theorem 2.4.2

$$W(t) = \begin{vmatrix} t & y \\ 1 & y' \end{vmatrix} = ce^{-\int(-\frac{2}{t})\,dt} = ct^2\,,$$

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$$ty' - y = t^2 \, .$$

We set here $c = 1$, because we need just one solution to fill the role of $y_2(t)$. The general solution of this linear first order equation is $y = t^2 + c_1 t$. Again we set here $c_1 = 0$, because we need just one solution, which is not a multiple of $y_1 = t$. We found $y_2(t) = t^2$. The general solution is $y = c_1 t + c_2 t^2$.

IV.3 We begin with dividing this equation by $1 + t^2$

$$y'' - \frac{2t}{1 + t^2} y' + \frac{2}{1 + t^2} y = 0$$

to put it into the form required by the Theorem 2.4.2. Here $p(t) = -\dfrac{2t}{1 + t^2}$. We need to find $y_2(t)$, the second solution in the fundamental set. Again, we write $y = y(t)$ instead of $y_2(t)$. Using the Theorem 2.4.2

$$W(t) = \begin{vmatrix} t & y \\ 1 & y' \end{vmatrix} = ce^{-\int \left(-\frac{2t}{1+t^2}\right) dt} = ce^{\ln(1+t^2)} = c\left(1 + t^2\right),$$

$$ty' - y = 1 + t^2 \, .$$

Again, we set here $c = 1$. The general solution of this linear first order equation is $y = t^2 - 1 + c_1 t$. We set here $c_1 = 0$, to obtain $y_2(t) = t^2 - 1$. The general solution is $y = c_1 t + c_2 (t^2 - 1)$.

IV.4 Divide this equation by $t - 2$

$$y'' - \frac{t}{t - 2} y' + \frac{2}{t - 2} y = 0$$

to put it into the form required by the Theorem 2.4.2. Here

$$p(t) = -\frac{t}{t - 2} = -\frac{t - 2 + 2}{t - 2} = -1 - \frac{2}{t - 2} \, .$$

Again, we write $y = y(t)$ instead of $y_2(t)$. Using the Theorem 2.4.2

$$W(t) = \begin{vmatrix} e^t & y \\ e^t & y' \end{vmatrix} = ce^{-\int \left(-1 - \frac{2}{t-2}\right) dt} = ce^{t + \ln(t-2)^2} = c\, e^t (t - 2)^2,$$

$$e^t y' - e^t = e^t (t - 2)^2 \, ,$$
$$y' - y = (t - 2)^2 \, .$$

Again, we set here $c = 1$. The general solution of this linear first order equation is $y = -t^2 + 2t - 2 + c_1 e^t$. We set here $c_1 = 0$, to obtain $y_2(t) = -t^2 + 2t - 2$. The general solution is $y = c_1 e^t + c_2 (-t^2 + 2t - 2)$.

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V.1 Calculate the Wronskian of $y_1 = t$ and $y_2 = \sin t$ to be $W(t) = t \cos t - \sin t$, and observe that $W(0) = 0$. If $y_1 = t$ and $y_2 = \sin t$ were solutions of some equation of the form

$$y'' + p(t)y' + g(t)y = 0,$$

then by the Corollary 2.4.2 we would have $W(t) = 0$ for all t, which is clearly not the case.

V.2 The Wronskian of the solutions is $W(1, \cos t) = -\sin t$. By the Theorem 2.4.2

$$-\sin t = ce^{-\int p(t)\, dt}.$$

Choose $c = -1$, then take logs, and differentiate both sides:

$$\ln \sin t = -\int p(t)\, dt,$$

$$\cot t = -p(t).$$

With $p(t) = -\cot t$, the equation becomes

$$y'' - \cot t y' + g(t)y = 0.$$

Since $y(t) = 1$ is a solution of this equation, it follows that $g(t) = 0$. Finally, if one chooses a different value for c, the resulting equation is the same.

V.3 Substituting $y = tv(t)$ into the Legendre's equation obtain

$$t(t^2 - 1)v''(t) + (4t^2 - 2)v'(t) = 0.$$

Setting $z(t) = v'(t)$, obtain a separable first order equation

$$t(t^2 - 1)z'(t) + (4t^2 - 2)z(t) = 0.$$

(This equation is also linear.) Its solution is $z(t) = \dfrac{c}{t^2(1 - t^2)}$. Set here $c = 1$, then integrate $v'(t) = \dfrac{1}{t^2(1 - t^2)}$ as in the text, obtaining

$$v(t) = \int \frac{1}{t^2(1 - t^2)}\, dt = -\frac{1}{t} - \frac{1}{2}\ln(1 - t) + \frac{1}{2}\ln(1 + t).$$

Then

$$y_2(t) = tv(t) = -1 - \frac{1}{2}t\ln(1 - t) + \frac{1}{2}t\ln(1 + t).$$

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Section 2.10.2, Page 83

I.1   According to the Prescription 1, look for a particular solution in the form $y = A \cos t + B \sin t$. Substituting this function into the equation, and combining the like terms gives

$$(-A - 3B) \cos t + (3A - B) \sin t = 2 \sin t.$$

We need to solve a linear system

$$-A - 3B = 0$$
$$3A - B = 2.$$

Obtain $A = \dfrac{3}{5}$, $B = -\dfrac{1}{5}$. The particular solution we obtained is $Y = \dfrac{3}{5} \cos t - \dfrac{1}{5} \sin t$. The general solution of the corresponding homogeneous equation

$$2y'' - 3y' + y = 0$$

is $c_1 e^{t/2} + c_2 e^t$. The general solution of the original non-homogeneous equation is $y = \dfrac{3}{5} \cos t - \dfrac{1}{5} \sin t + c_1 e^{t/2} + c_2 e^t$.

I.2   According to the Prescription 1, look for a particular solution in the form $y = A \cos 2t + B \sin 2t$. Substituting this function into the equation, and combining the like terms gives

$$(A + 8B) \cos 2t + (-8A + B) \sin 2t = 2 \cos 2t - 3 \sin 2t.$$

We need to solve a linear system

$$A + 8B = 2$$
$$-8A + B = -3.$$

Obtain $A = \dfrac{2}{5}$, $B = \dfrac{1}{5}$. The particular solution we obtained is $Y = \dfrac{2}{5} \cos t + \dfrac{1}{5} \sin t$. The general solution of the corresponding homogeneous equation

$$y'' + 4y' + 5y = 0$$

is $c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$. The general solution of the original non-homogeneous equation is $y = \dfrac{2}{5} \cos t + \dfrac{1}{5} \sin t + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$.

I.4   We use a short-cut to the Prescription 1, looking for a particular solution in the form $y = A\cos\nu t$. Substitution into the equation gives

$$A(9 - \nu^2)\cos\nu t = 2\cos\nu t\,,$$

so that $A = \dfrac{2}{9 - \nu^2}$, and $Y = \dfrac{2}{9 - \nu^2}\cos\nu t$.

I.5   On the right we see a linear polynomial multiplying $\cos t$. According to the Prescription 1, look for a particular solution in the form $y = (At + B)\cos t + (Ct + D)\sin t$. Substituting this function into the equation, and combining the like terms gives

$$2Ct\cos t - 2At\sin t + (2A + 2C + 2D)\cos t + (-2A - 2B + 2C)\sin t = 2t\cos t\,.$$

It follows that

$$2C = 2$$
$$-2A = 0$$
$$2A + 2C + 2D = 0$$
$$-2A - 2B + 2C = 0\,.$$

Obtain: $C = 1$, $A = 0$, $D = -1$, $B = 1$. The particular solution we obtained is $Y = \cos t + (t-1)\sin t$. The general solution of the corresponding homogeneous equation

$$y'' + 2y' + y = 0$$

is $c_1 e^{-t} + c_2 t e^{-t}$. The general solution of the original non-homogeneous equation is $y = \cos t + (t - 1)\sin t + c_1 e^{-t} + c_2 t e^{-t}$.

I.6   According to the Prescription 2, look for a particular solution in the form $y = At + B$. Substituting this function into the equation, and combining the like terms gives

$$At - 2A + B = t + 2\,,$$

so that $A = 1$, $B = 4$. The particular solution we obtained is $Y = t + 4$.

I.7   According to the Prescription 2, look for a particular solution in the form $y = At^2 + Bt + C$. Substitution of this function into the equation, and combining the like terms gives

$$4At^2 + 4Bt + 2A + 4C = t^2 - 3t + 1\,,$$

so that $A = \dfrac{1}{4}$, $B = -\dfrac{3}{4}$, $C = \dfrac{1}{8}$. The particular solution we obtained is $Y = \dfrac{1}{4}t^2 - \dfrac{3}{4}t + \dfrac{1}{8}$.

I.8   According to the Prescription 3, look for a particular solution in the form $y = Ae^{5t}$. Substituting this function into the equation, and combining the like terms get

$$16Ae^{5t} = e^{5t} .$$

It follows that $A = \dfrac{1}{16}$, and $Y = \dfrac{1}{16}e^{5t}$.

I.10   According to the Prescription 3, look for a particular solution in the form $y = \left(At^2 + Bt + C\right)e^{-2t}$. Substituting this function into the equation, and combining the like terms get

$$\left[5At^2 + (-14A + 5B)\,t + 4A - 7B + 5C\right]e^{-2t} = \left(5t^2 + t - 1\right)e^{-2t} .$$

We need to solve a linear system

$$5A = 5$$
$$-14A + 5B = 1$$
$$4A - 7B + 5C = -1 .$$

It follows that $A = 1$, $B = 3$, $C = \dfrac{16}{5}$, and then $Y = \left(t^2 + 3t + \dfrac{16}{5}\right)e^{-2t}$.

I.12   We search for a particular solution Y in the form $Y = Y_1 + Y_2$, where Y_1 is a particular solution of

$$y'' + y = 2e^{4t} ,$$

while Y_2 is a particular solution of

$$y'' + y = t^2 .$$

Using the Prescription 3, calculate $Y_1 = \dfrac{2}{17}e^{4t}$, and by the Prescription 2, $Y_2 = t^2 - 2$. Then $Y(t) = \dfrac{2}{17}e^{4t} + t^2 - 2$.

I.13   Because the sine and cosine functions have different arguments, the Prescription 1 does not work directly. We need to search for a particular solution Y in the form $Y = Y_1 + Y_2$, where Y_1 is a particular solution of

$$y'' - y' = 2\sin t ,$$

while Y_2 is a particular solution of

$$y'' - y' = -\cos 2t .$$

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The Prescription 1 applies to each of these equations, giving $Y_1 = \cos t - \sin t$, and $Y_2 = \dfrac{1}{5}\cos 2t + \dfrac{1}{10}\sin 2t$. Then $Y = \cos t - \sin t + \dfrac{1}{5}\cos 2t + \dfrac{1}{10}\sin 2t$.

I.14 $Y(x) = x^2$ is a particular solution. The general solution of the corresponding homogeneous equation

$$y' - x^2 y = 0$$

is $y = ce^{\frac{x^3}{3}}$. Similar theory applies to linear first order equations, therefore the general solution is $y = x^2 + ce^{\frac{x^3}{3}}$.

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II.1   Because $\cos t$ and $\sin t$ are solutions of the corresponding homogeneous equation, we look for a particular solution in the form $At\cos t + Bt\sin t$. Substituting this function into the equation gives

$$-2A\sin t + 2B\cos t = 2\cos t .$$

It follows that $A = 0$, $B = 1$, and $Y = t\sin t$.

II.2   Because the function e^{2t} solves the corresponding homogeneous equation, we look for a particular solution in the form Ate^{2t}. Substituting this function into the equation gives

$$5Ae^{2t} = -e^{2t} .$$

It follows that $A = -\dfrac{1}{5}$, and $Y = -\dfrac{1}{5}te^{2t}$.

II.4   According to the Prescription 3, one searches for a particular solution in the form $y = (At + B)e^t$. However, both pieces constituting this function (te^t and e^t) are solutions of the corresponding homogeneous equation. Multiply this prescription by t, and consider $y = t(At+B)e^t = (At^2+Bt)e^t$. The second piece, Bte^t is still a solution of the corresponding homogeneous equation. We multiply by t again, and consider $t(At^2 + Bt)e^t = At^3e^t + Bt^2e^t$. Substituting this function into the equation gives

$$6Ate^t + 2Be^t = te^t .$$

It follows that $A = \dfrac{1}{6}$, $B = 0$, and then $Y = \dfrac{1}{6}t^3e^{2t}$.

II.5   We search for a particular solution in the form $Y = Y_1 + Y_2$. Here Y_1 is a particular solution of

$$y'' - 4y' = 2 .$$

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The Prescription 2 tells us to try $y = A$. However, constants are solutions of the corresponding homogeneous equation. We multiply the guess by t, $y = At$, and calculate $A = -\dfrac{1}{2}$, so that $Y_1 = -\dfrac{1}{2}t$. Y_2 is a particular solution of

$$y'' - 4y' = -\cos t.$$

By the Prescription 1, $Y_2 = \dfrac{1}{17}\cos t + \dfrac{4}{17}\sin t$, so that $Y(t) = -\dfrac{1}{2}t + \dfrac{1}{17}\cos t + \dfrac{4}{17}\sin t$.

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IV.1 The functions $y_1(t) = e^{-3t}$ and $y_2(t) = e^{2t}$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = 5e^{-t}$, and the right hand side $f(t) = 5e^{2t}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{5e^{2t}e^{2t}}{5e^{-t}} = -e^{5t},$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{5e^{2t}e^{-3t}}{5e^{-t}} = 1.$$

Integration gives $u_1(t) = -\dfrac{1}{5}e^{5t}$, $u_2(t) = t$. We set the constants of integration to be zero, because we need just one particular solution. We now "assemble" a particular solution $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -e^{5t}e^{-3t} + te^{2t} = -e^{2t} + te^{2t}$. The general solution is then $y = -\dfrac{1}{5}e^{2t} + te^{2t} + c_1e^{-3t} + c_2e^{2t} = te^{2t} + c_1e^{-3t} + c_2e^{2t}$, with a new c_2.

IV.2 The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = e^{2t}$, and the right hand side $f(t) = \dfrac{e^t}{1 + t^2}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{t}{1 + t^2},$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{1}{1 + t^2}.$$

Integration gives $u_1(t) = -\dfrac{1}{2}\ln(1 + t^2)$, $u_2(t) = \tan^{-1} t$. We set the constants of integration to be zero, because we need just one particular solution. Then $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\dfrac{1}{2}e^t\ln(1 + t^2) + te^t\tan^{-1} t$.

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IV.3   The functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = 1$, and the right hand side $f(t) = \sin t$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\sin t \sin t = -\frac{1}{2} + \frac{1}{2}\cos 2t\,,$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \sin t \cos t\,.$$

Integration gives $u_1(t) = -\frac{1}{2}t + \frac{1}{4}\sin 2t$, $u_2(t) = \frac{1}{2}\sin^2 t$. We set the constants of integration to be zero, because we need just one particular solution. Then $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \left(-\frac{1}{2}t + \frac{1}{4}\sin 2t\right)\cos t + \frac{1}{2}\sin^2 t \sin t$.

Expanding $\sin 2t = 2\sin t \cos t$, one can simplify $Y(t) = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$. The general solution is

$$y(t) = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t + c_1\cos t + c_2\sin t = -\frac{1}{2}t\cos t + c_1\cos t + c_2\sin t\,,$$

with a new constant c_2.

IV.6   The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = e^{-4t}$, and $f(t) = \dfrac{e^{-2t}}{t^2}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{1}{t}\,,$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{1}{t^2}\,.$$

Obtain $u_1(t) = -\ln t$, $u_2(t) = -\dfrac{1}{t}$, $Y(t) = -e^{-2t}(1 + \ln t)$. The general solution:

$$y(t) = -e^{-2t}(1 + \ln t) + c_1e^{-2t} + c_2te^{-2t} = -e^{-2t}\ln t + c_1e^{-2t} + c_2te^{-2t}\,,$$

with a new constant c_1.

IV.8   The functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = 1$, and the right hand side $f(t) = \sec t$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\sec t \sin t = -\frac{\sin t}{\cos t}\,,$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \sec t \, \cos t = 1 \,.$$

Integration gives $u_1(t) = \ln|\cos t|$, $u_2(t) = t$. We set the constants of integration to be zero, because we need just one particular solution. Then $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \cos t \ln|\cos t| + t \sin t$.

IV.9   The functions $y_1(t) = e^{-3t}$ and $y_2(t) = 1$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = 3e^{-3t}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{6t}{3e^{-3t}} = -2te^{3t} \,,$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{6te^{-3t}}{3e^{-3t}} = 2t \,.$$

Integration gives $u_1(t) = -\frac{2}{3}te^{3t} + \frac{2}{9}e^{3t}$, $u_2(t) = t^2$. Then $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{2}{3}t + \frac{2}{9} + t^2$, and the general solution is $y(t) = -\frac{2}{3}t + \frac{2}{9} + t^2 + c_1 e^{-3t} + c_2 = -\frac{2}{3}t + t^2 + c_1 e^{-3t} + c_2$, with a new c_2.

IV.10   The functions $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = 3e^t$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{e^{-t}e^{2t}}{3e^t} = -\frac{1}{3} \,,$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{e^{-t}e^{-t}}{3e^t} = \frac{1}{3}e^{-3t} \,.$$

Integration gives $u_1(t) = -\frac{1}{3}t$, $u_2(t) = -\frac{1}{9}e^{-3t}$. Then $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{1}{3}te^{-t} - \frac{1}{9}e^{-t}$, and the general solution is $y(t) = -\frac{1}{3}te^{-t} - \frac{1}{9}e^{-t} + c_1 e^{-t} + c_2 e^{2t} = -\frac{1}{3}te^{-t} + c_1 e^{-t} + c_2 e^{2t}$, with a different c_1. From the initial conditions

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = -\frac{1}{3} - c_1 + 2c_2 = 0 \,,$$

giving $c_1 = \frac{5}{9}$ and $c_2 = \frac{4}{9}$.

IV.12   The functions $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is

$W(t) = 3e^{-t}$. Our formulas for u_1' and u_2' were developed on the assumption that the leading coefficient of the equation is one. Accordingly, we divide the equation by 2:

$$y'' + y' - 2y = \frac{1}{2}e^{-2t},$$

so that $f(t) = \frac{1}{2}e^{-2t}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = -\frac{\frac{1}{2}e^{-2t}e^t}{3e^{-t}} = -\frac{1}{6},$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = \frac{\frac{1}{2}e^{-2t}e^{-2t}}{3e^{-t}} = \frac{1}{6}e^{-3t}.$$

Integration gives $u_1(t) = -\frac{1}{6}t$, $u_2(t) = -\frac{1}{2}e^{-3t}$. Then $Y(t) = -\frac{1}{6}te^{-2t} + \frac{1}{6}e^{-2t}$, and the general solution can be written as $= -\frac{1}{6}te^{-2t} + c_1e^{-2t} + c_2e^t$ (after redefining c_1).

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V.1   The functions $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ form a fundamental solution set of the corresponding homogeneous equation. Their Wronskian is $W(t) = -3$. Our formulas for u_1' and u_2' were developed on the assumption that the leading coefficient of the equation is one. Accordingly, we divide the equation by t^2:

$$y'' - \frac{2}{t^2}y = t - \frac{1}{t^2},$$

so that $f(t) = t - \frac{1}{t^2}$. Then

$$u_1'(t) = -\frac{f(t)y_2(t)}{W(t)} = \frac{1}{3}\left(t - \frac{1}{t^2}\right)t^{-1} = \frac{1}{3} - \frac{1}{3}t^{-3},$$

$$u_2'(t) = \frac{f(t)y_1(t)}{W(t)} = -\frac{1}{3}\left(t - \frac{1}{t^2}\right)t^2 = -\frac{1}{3}t^3 + \frac{1}{3}.$$

Integration gives $u_1(t) = \frac{1}{3}t + \frac{1}{6}t^{-2}$, $u_2(t) = -\frac{1}{12}t^4 + \frac{1}{3}t$. Then $Y(t) = \left(\frac{1}{3}t + \frac{1}{6}t^{-2}\right)t^2 + \left(-\frac{1}{12}t^4 + \frac{1}{3}t\right)t^{-1} = \frac{1}{4}t^3 + \frac{1}{2}$.

V.3   The functions $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ form a fundamental solution set of the corresponding homogeneous equation. (These

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are the standard Bessel's functions $Y_{1/2}(x)$ and $J_{1/2}(x)$ respectively.) Their Wronskian is $W(t) = \dfrac{1}{x}$. Our formulas for $u_1'(x)$ and $u_2'(x)$ were developed on the assumption that the leading coefficient of the equation is one. Accordingly, we divide the equation by x^2:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = x^{-1/2},$$

so that $f(x) = x^{-1/2}$. Then

$$u_1'(x) = -\frac{f(x)y_2(x)}{W(x)} = -x\,x^{-1/2}\,x^{-1/2}\sin x = -\sin x,$$

$$u_2'(x) = \frac{f(x)y_1(x)}{W(x)} = x\,x^{-1/2}\,x^{-1/2}\cos x = \cos x,$$

so that $u_1(x) = \cos x$, $u_2(x) = \sin x$ and $Y(x) = x^{-1/2}$. The general solution is $y = x^{-1/2} + c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x$.

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VII.1   We need to solve

$$y'' + 4y = 0 \quad y(0) = -1,\ y'(0) = 2.$$

Obtain $y = -\cos 2t + \sin 2t$.

VII.2   Calculate $\omega = \sqrt{\dfrac{k}{m}} = 3$. We need to solve

$$y'' + 9y = 0 \quad y(0) = -3,\ y'(0) = 2.$$

(The "up" direction corresponds to negative y's.) Obtain $y = -3\cos 3t + \dfrac{2}{3}\sin 3t$. The amplitude is $A = \sqrt{3^2 + \left(\dfrac{2}{3}\right)^2} = \dfrac{\sqrt{85}}{3}$.

VII.3   We need to solve

$$y'' + 9y = 2\cos \nu t,$$

which in case $\nu \neq 3$ is easily done using the Prescription 1.

VII.4   This is the case of resonance. One needs to use either the method of variation of parameters, or the modified Prescription 1 to find a particular solution.

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VII.5   The corresponding characteristic equation

$$r^2 + \alpha r + 9 = 0$$

has for $\alpha < 6$ a pair of complex roots with negative real parts, producing damped oscillations. In case $\alpha \geq 6$ both roots are real, and there are no oscillations.

VII.6   This problem shows that any amount of dissipation ($\alpha > 0$) destroys resonance.

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I.1   On the interval $0 < t < \pi$ we need to solve

$$y'' + 9y = 0 \,, \quad y(0) = 0 \,, \quad y'(0) = -2 \,.$$

The solution is $y(t) = -\dfrac{2}{3}\sin 3t$. Evaluate $y(\pi) = 0$, $y'(\pi) = 2$. For $t > \pi$ we need to solve

$$y'' + 9y = t \,, \quad y(\pi) = 0 \,, \quad y'(\pi) = 2 \,.$$

Obtain: $y(t) = \dfrac{t}{9} + \dfrac{\pi}{9}\cos 3t - \dfrac{17}{27}\sin 3t$.

I.2   On the interval $0 < t < \pi$ we need to solve

$$y'' + y = 0 \,, \quad y(0) = 2 \,, \quad y'(0) = 0 \,.$$

The solution is $y(t) = 2\cos t$. Evaluate $y(\pi) = -2$, $y'(\pi) = 0$. For $t > \pi$ we need to solve

$$y'' + y = t \,, \quad y(\pi) = -2 \,, \quad y'(\pi) = 0 \,.$$

Obtain: $y(t) = t + (\pi + 2)\cos t + \sin t$. We conclude that

$$y(t) = \begin{cases} 2\cos t \,, & \text{if } t \leq \pi \\[2mm] t + (\pi + 2)\cos t + \sin t \,, & \text{if } t > \pi \end{cases} .$$

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II.1   The characteristic equation

$$r(r - 1) - 2r + 2 = 0$$

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has roots $r_1 = 1$, $r_2 = 2$. The general solution is $y = c_1 t + c_2 t^2$.

II.2   The characteristic equation

$$r(r-1) + r + 4 = 0$$

has roots $r = \pm 2i$. The general solution is $y = c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)$.

II.3   The characteristic equation

$$r(r-1) + 5r + 4 = 0$$

has roots $r_1 = r_2 = -2$. The general solution is $y = c_1 t^{-2} + c_2 t^{-2} \ln t$.

II.4   The characteristic equation

$$r(r-1) + 5r + 5 = 0$$

has roots $r = -2 \pm i$. The general solution is $y = c_1 t^{-2} \cos(\ln t) + c_2 t^{-2} \sin(\ln t)$.

II.5   The characteristic equation

$$r(r-1) - 3 = 0$$

has roots $r_1 = 0$, $r_2 = 4$. The general solution is $y = c_1 + c_2 t^4$.

II.6   Write this equation as

$$4t^2 y'' + y = 0 \,.$$

The characteristic equation

$$4r(r-1) + 1 = 0$$

has a repeated root $r_1 = r_2 = \dfrac{1}{2}$. The general solution is $y = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$.

II.7   The characteristic equation

$$2r(r-1) + 5r + 1 = 0$$

$$2r^2 + 3r + 1 = 0$$

has roots $r_1 = -\dfrac{1}{2}$, $r_2 = -1$. The general solution is $y = c_1 t^{-\frac{1}{2}} + c_2 t^{-1}$.

II.8 The characteristic equation

$$9r(r-1) - 3r + 4 = 0$$

$$(3r-2)^2 = 0$$

has roots $r_1 = r_2 = \dfrac{2}{3}$. The general solution is $y = c_1 t^{\frac{2}{3}} + c_2 t^{\frac{2}{3}} \ln t$.

II.9 The characteristic equation

$$4r(r-1) + 4r + 1 = 0$$

has roots $r = \pm\dfrac{1}{2}i$. The general solution is $y = c_1 \cos\left(\dfrac{1}{2}\ln x\right) + c_2 \sin\left(\dfrac{1}{2}\ln x\right)$.

II.10 Looking for a particular solution in the form $Y = \dfrac{A}{t}$, one calculates $A = \dfrac{1}{4}$, so that $Y = \dfrac{1}{4t}$. Multiply the corresponding homogeneous equation by t^2:

$$t^2 y'' + 3ty' + 5y = 0\,,$$

to obtain Euler's equation. Its characteristic equation

$$r(r-1) + 3r + 5 = 0$$

has roots $r = -1 \pm 2i$, and the general solution of the homogeneous equation is $y = c_1 t^{-1} \cos(2\ln t) + c_2 t^{-1} \sin(2\ln t)$. The general solution of the original equation is $y = \dfrac{1}{4t} + c_1 t^{-1} \cos(2\ln t) + c_2 t^{-1} \sin(2\ln t)$.

II.11 Multiplication of the corresponding homogeneous equation by t^2 produces Euler's equation

$$t^2 y'' + 3ty' + 5y = 0\,.$$

Its characteristic equation

$$r(r-1) + 3r + 5 = 0$$

has roots $r = -1 \pm 2i$, and the fundamental solution set of the homogeneous equation consists of $y_1(t) = t^{-1}\cos(2\ln t)$ and $y_2(t) = t^{-1}\sin(2\ln t)$. Calculate their Wronskian $W = W(y_1(t), y_2(t)) = \dfrac{2}{t^3}$. Then

$$u_1' = -\frac{f(t)y_2(t)}{W} = -\frac{\ln t \, \sin(2\ln t)}{2t}\,,$$

$$u_2' = \frac{f(t)y_1(t)}{W} = \frac{\ln t \, \cos(2\ln t)}{2t} \, .$$

Integration gives

$$u_1(t) = \frac{1}{4}\cos(2\ln t)\ln t - \frac{1}{8}\sin(2\ln t)\,, \quad u_2(t) = \frac{1}{4}\sin(2\ln t)\ln t + \frac{1}{8}\cos(2\ln t)\,,$$

by using the substitution $u = \ln t$ in both integrals. Then the particular solution is

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \frac{\ln t}{4t}\,.$$

The general solution of the original non-homogeneous equation is

$$y = \frac{\ln t}{4t} + c_1 t^{-1}\cos(2\ln t) + c_2 t^{-1}\sin(2\ln t)\,.$$

II.12   Substitution of $y = t^r$ into the equation gives

$$t^r\left[r(r-1)(r-2) + r(r-1) - 2r + 2\right] = 0\,,$$

which leads to a cubic characteristic equation

$$r(r-1)(r-2) + r(r-1) - 2r + 2\,.$$

To solve it, we factor

$$r(r-1)(r-2) + r(r-1) - 2(r-1) = (r-1)\left[r(r-2) + r - 2\right] = 0\,,$$

so that the roots are $r_1 = 1$, $r_2 = -1$, and $r_3 = 2$. The general solution is $y = c_1 t^{-1} + c_2 t + c_3 t^2$.

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III.1   The characteristic equation

$$r(r-1) + r + 4 = 0$$

has roots $\pm 2i$. The general solution which is valid for all $t \neq 0$ is $y = c_1\cos(2\ln|t|) + c_2\sin(2\ln|t|)$.

III.2   The characteristic equation

$$2r(r-1) - r + 1 = 0$$

$$2r^2 - 3r + 1 = 0$$

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has roots $r = \dfrac{1}{2}$ and $r = 1$. The solution $y = \sqrt{t}$ is valid for $t > 0$, while the solution $y = \sqrt{|t|}$ is valid for $t \neq 0$. The second solution, $y = t$ is valid for all t, and $y = |t|$ is valid for $t \neq 0$. The general solution which is valid for all $t \neq 0$ can be written in two forms: $y = c_1 \sqrt{|t|} + c_2 |t|$, or $y = c_1 \sqrt{|t|} + c_2 t$.

III.3 The characteristic equation

$$4r(r - 1) - 4r + 13 = 0$$

has roots $r = 1 \pm \dfrac{3}{2} i$. The general solution valid for $t \neq 0$ is $y = c_1 |t| \cos\left(\dfrac{3}{2} \ln |t|\right) + c_2 |t| \sin\left(\dfrac{3}{2} \ln |t|\right)$.

III.4 The characteristic equation

$$9r(r - 1) + 3r + 1 = 0$$

has double root $r = \dfrac{1}{3}$. The general solution valid for $t \neq 0$ is $y = c_1 |t|^{1/3} + c_2 |t|^{1/3} \ln |t|$.

III.5 The characteristic equation

$$2r(r - 1) + r = 0$$

has roots $r = 0$ and $r = \dfrac{1}{2}$. The general solution valid for $t \neq 0$ is $y = c_1 + c_2 \sqrt{|t|}$.

III.6 Look for a particular solution in the form $y = At^2 + Bt + C$. Substitution into the equation gives

$$3At^2 + C = t^2 - 3,$$

so that $A = \dfrac{1}{3}$, $C = -3$. B is arbitrary, and we set $B = 0$. Obtain $Y = \dfrac{1}{3} t^2 - 3$. The general solution of the correponding homogeneous equation

$$2t^2 y'' - ty' + y = t^2 - 3$$

is $y = c_1 \sqrt{|t|} + c_2 |t|$. The general solution of the original equation is $y = t^2 - 3 + c_1 \sqrt{|t|} + c_2 |t|$.

III.8   Substitution of $y = (t+1)^r$ into the equation gives

$$(t+1)^r \left[2r(r-1) - 3r + 2\right] = 0\,,$$

which leads to a characteristic equation

$$2r(r-1) - 3r + 2 = 0\,.$$

The roots are $r_1 = \dfrac{1}{2}$, $r_2 = 2$. The general solution, which is valid for $t \neq -1$, is $y = c_1|t+1|^{1/2} + c_2(t+1)^2$.

III.9   Setting $t = 0$ in our integro-differential equation

$$4y'(t) + \int_0^t \frac{y(s)}{(s+1)^2}\, ds = 0\,,$$

obtain $y'(0) = 0$. Differentiating this equation, and using the fundamental theorem of calculus, obtain

$$4y''(t) + \frac{y(t)}{(t+1)^2} = 0\,,$$

$$4(t+1)^2 y'' + y = 0\,, \quad y'(0) = 0\,.$$

Substitution of $y = (t+1)^r$ into the last equation gives

$$(t+1)^r \left[4r(r-1) + 1\right] = 0\,,$$

which leads to a characteristic equation

$$4r(r-1) + 1 = 0\,.$$

The roots are $r_1 = r_2 = \dfrac{1}{2}$. The general solution, which is valid for $t > -1$, is $y = c_1(t+1)^{1/2} + c_2(t+1)^{1/2}\ln(t+1)$. The condition $y'(0) = 0$ implies that $c_2 = -\dfrac{1}{2}c_1$, so that $y = c_1(t+1)^{1/2} - \dfrac{1}{2}c_1(t+1)^{1/2}\ln(t+1) = \dfrac{c_1}{2}\left[2(t+1)^{1/2} - (t+1)^{1/2}\ln(t+1)\right]$.

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IV.1   The characteristic equation

$$r(r-1) - 2r + 2 = 0$$

has roots $r = 1$ and $r = 2$. The general solution is $y = c_1 t + c_2 t^2$. The initial conditions imply that $c_1 = -1$ and $c_2 = 3$.

IV.2   The characteristic equation

$$r(r - 1) - 3r + 4 = 0$$

has a repeated root $r_1 = r_2 = 2$. Because the initial conditions are given at $t = -1$, the general solution is taken in the form $y = c_1 t^2 + c_2 t^2 \ln |t|$, with $y'(t) = 2c_1 t + c_2 (2t \ln |t| + t)$. The initial conditions give

$$y(-1) = c_1 = 1 \,,$$

$$y'(-1) = -2c_1 - c_2 = 2 \,.$$

It follows that $c_1 = 1$, $c_2 = -4$, and $y = t^2 - 4t^2 \ln |t|$.

IV.4   The characteristic equation

$$r(r - 1) - r + 5 = 0$$

has a pair of complex roots $r = 1 \pm 2i$. The general solution is $y = c_1 t \cos(2 \ln t) + c_2 t \sin(2 \ln t)$, and the initial conditions give $c_1 = 0$ and $c_2 = 1$.

IV.5   The characteristic equation

$$r(r - 1) + r + 4 = 0$$

has a pair of purely imaginary roots $r = \pm 2i$. Because the initial conditions are given at a negative $t = -1$, the general solution is taken in the form $y = c_1 \cos(2 \ln |t|) + c_2 \sin(2 \ln |t|)$, with $y'(t) = -\dfrac{2c_1}{t} \sin(2 \ln |t|) + \dfrac{2c_2}{t} \cos(2 \ln |t|)$. The initial conditions give

$$y(-1) = c_1 = 0 \,,$$

$$y'(-1) = -2c_2 = 4 \,.$$

It follows that $y = -2 \sin(2 \ln |t|)$.

IV.7   The characteristic equation

$$r(r - 1) + r = 0$$

has a repeated root $r_1 = r_2 = 0$. Because the initial conditions are given at a negative $t = -3$, the general solution is taken in the form $y = c_1 + c_2 \ln |t|$, with $y'(t) = \dfrac{c_2}{t}$. The initial conditions give

$$y(-3) = c_1 + c_2 \ln 3 = 0,$$

$$y'(-3) = -\frac{1}{3}c_2 = 1.$$

It follows that $c_2 = -3$, $c_1 = 3 \ln 3$, and $y = 3 \ln 3 - 3 \ln |t|$.

IV.8   The characteristic equation

$$2r(r - 1) - r + 1 = 0$$

has roots $r = 1$ and $r = \dfrac{1}{2}$. The general solution which is valid for all $t \neq 0$ is $y = c_1|t| + c_2|t|^{\frac{1}{2}}$. For $t < 0$, this solution becomes

$$y(t) = c_1 t + c_2(-t)^{\frac{1}{2}},$$

with a new c_1. Calculate $y'(t) = c_1 - \dfrac{1}{2}c_2(-t)^{-\frac{1}{2}}$. The initial conditions give

$$y(-1) = -c_1 + c_2 = 0$$

$$y'(-1) = c_1 - \frac{1}{2}c_2 = \frac{1}{2}.$$

It follows that $c_1 = c_2 = 1$. The solution is $y(t) = t + (-t)^{\frac{1}{2}} = t + |t|^{\frac{1}{2}}$, valid on the interval $(-\infty, 0)$.

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V.1   The solutions are the three complex cubic roots of 1. A quicker alternative solution is to factor

$$(r - 1)\left(r^2 + r + 1\right) = 0.$$

The first factor gives the root $r_1 = 1$, while setting the quadratic to zero gives $r_2 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ and $r_3 = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$.

V.2   Guessing that $r = -3$ is a root, one factors

$$r^3 + 27 = (r + 3)(r^2 - 3r + 9).$$

Solving
$$r^2 - 3r + 9 = 0$$

gives the other two roots. Alternatively, one could calculate the three complex roots of -27.

V.3 Factor
$$r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r^2 + 4) = 0\,,$$

and set each factor to zero to get $r_1 = 2$, $r_2 = -2$, and the other two roots $r = \pm 2i$.

V.4 After guessing that $r = 1$ is a root, use long division to factor
$$r^3 - 3r^2 + r + 1 = (r - 1)\left(r^2 - 2r - 1\right) = 0\,,$$

and obtain the other two roots $r = 1 \pm \sqrt{2}$.

V.6 Factor
$$r^3 + 2r^2 + r + 2 = r^2(r + 2) + r + 2 = (r + 2)(r^2 + 1) = 0\,.$$

Obtain $r = -2$, and $r = \pm i$.

V.7 Factor
$$3r^4 + 5r^3 + r^2 - r = r\left(3r^3 + 5r + r - 1\right) = r(r + 1)^2(3r - 1) = 0\,.$$

(For the second step one guesses that $r = -1$ is a root.) The roots are: 0, -1, -1, $\dfrac{1}{3}$.

V.9 Write
$$-4 = 4e^{i\pi} = 4e^{i(\pi + 2\pi m)}\,.$$

Then the four solutions of the equation $r^4 + 4 = 0$ are
$$r = (-1)^{1/4} = 4^{1/4}e^{i\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)}\,, \quad m = 0, 1, 2, 3\,.$$

Observe that $4^{1/4} = \left(2^2\right)^{1/4} = 2^{1/2} = \sqrt{2}$. Then, in case $m = 0$, obtain

$$4^{1/4}e^{i\frac{\pi}{4}} = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i\,,$$

and the other three roots are calculated similarly.

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V.10   Write the equation as

$$\left(r^2 + 4\right)^2 = 0\,.$$

The roots are $r = \pm 2i$, each one is a double root.

V.11   This is a "biquadratic equation", which can be reduced to a quadratic equation by setting $r^2 = x$. Or one may just factor:

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4) = 0\,,$$

obtaining the roots $\pm i$ and $\pm 2i$.

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VI.1   The characteristic equation

$$r^3 - 1 = 0$$

is factored as

$$(r - 1)(r^2 + r + 1) = 0\,.$$

One of the roots is $r = 1$. The other two roots are obtained by solving the quadratic equation

$$r^2 + r + 1 = 0\,,$$

giving $r = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$. The general solution is $y = c_1 e^t + c_2 e^{-t/2} \cos \dfrac{\sqrt{3}}{2}t + c_3 e^{-t/2} \sin \dfrac{\sqrt{3}}{2}t$.

VI.2   The characteristic equation is

$$r^3 - 5r^2 + 8r - 4 = 0\,.$$

One guesses the root $r = 1$. By long division

$$r^3 - 5r^2 + 8r - 4 = (r - 1)(r^2 - 4r + 4) = 0\,,$$

so that the other two roots are both equal to 2. The general solution is then $y = c_1 e^t + c_2 e^{2t} + c_3 t e^{2t}$.

VI.3   The characteristic equation is

$$r^3 - 3r^2 + r + 1 = 0\,.$$

One guesses the root $r = 1$. By long division

$$r^3 - 3r^2 + r + 1 = (r-1)(r^2 - 2r - 1) = 0,$$

so that the other two roots are equal to $1 \pm \sqrt{2}$. The general solution is then $y = c_1 e^t + c_2 e^{(1-\sqrt{2})t} + c_3 e^{(1+\sqrt{2})t}$.

VI.4   The characteristic equation is solved by factoring

$$r^3 - 3r^2 + r - 3 = r^2(r-3) + r - 3 = (r-3)(r^2 + 1) = 0.$$

The roots are $r = 3$, and $r = \pm 1$. The general solution is then $y = c_1 e^{3t} + c_2 \cos t + c_3 \sin t$.

VI.5   The characteristic equation

$$r^4 - 8r^2 + 16 = (r^2 - 4)^2 = 0$$

has two double roots: ± 2. The general solution is $y = c_1 e^{-2t} + c_2 e^{2t} + c_3 t e^{-2t} + c_4 t e^{2t}$.

VI.6   The characteristic equation

$$r^4 + 8r^2 + 16 = (r^2 + 4)^2 = 0$$

has two double roots: $\pm 2i$. The general solution is $y = c_1 \cos 2t + c_2 \sin 2t + c_3 t \cos 2t + c_4 t \sin 2t$.

VI.7   The roots of the characteristic equation

$$r^4 + 1 = 0$$

$r = \sqrt[4]{-1}$ are the four complex fourth roots of -1, namely: $\dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}} i$, $-\dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}} i$. The general solution is $y = c_1 e^{\frac{t}{\sqrt{2}}} \cos \dfrac{t}{\sqrt{2}} + c_2 e^{\frac{t}{\sqrt{2}}} \sin \dfrac{t}{\sqrt{2}} + c_3 e^{-\frac{t}{\sqrt{2}}} \cos \dfrac{t}{\sqrt{2}} + c_4 e^{-\frac{t}{\sqrt{2}}} \sin \dfrac{t}{\sqrt{2}}$.

VI.8   It is easy to guess that $Y = -t^2$ is a particular solution of

$$y''' - y = t^2$$

because the third derivative of this function is zero. The characteristic equation of the corresponding homogeneous equation is solved by factoring:

$$r^3 - 1 = (r-1)(r^2 + r + 1) = 0.$$

VI.9   The characteristic equation is solved by factoring:

$$r^6 - r^2 = r^2(r^4 - 1) = r^2(r^2 - 1)(r^2 + 1) = 0 \, .$$

The six roots are $0, 0, \pm 1, \pm i$. The general solution is $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^t + c_5 \cos t + c_6 \sin t$.

VI.10   After guessing that $r = 1$ is a root, the characteristic equation is solved by factoring:

$$2r^3 - 5r^2 + 4r - 1 = (r - 1)^2(2r - 1) = 0 \, .$$

The roots are $1, 1, \dfrac{1}{2}$. The general solution is $y = c_1 e^{\frac{1}{2}t} + c_2 e^t + c_3 t e^t$.

VI.11   One factors the characteristic equation

$$r^5 - 3r^4 + 3r^3 - 3r^2 + 2r = r\left(r^4 - 3r^3 + 3r^2 - 3r + 2\right) = 0 \, .$$

Then one guesses that the quartic has a root $r = 1$, so that

$$r^4 - 3r^3 + 3r^2 - 3r + 2 = (r - 1)(r^3 - 2r^2 + r - 2) \, ,$$

and the last cubic has a root $r = 2$, so that

$$r^3 - 2r^2 + r - 2 = (r - 2)(r^2 + 1) \, .$$

Putting these formulas together, the characteristic equation is factored as

$$r(r - 1)(r - 2)(r^2 + 1) = 0 \, .$$

The five roots are $0, 1, 2, \pm i$. The general solution is $y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 \cos t + c_5 \sin t$.

VI.12   One searches for a particular solution in the form $y = A \sin t$, obtaining $Y = \dfrac{1}{2} \sin t$. The characteristic equation of the corresponding homogeneous equation is solved by factoring:

$$r^8 - r^6 = r^6(r^2 - 1) = 0 \, .$$

The eight roots are $r = 0$ of multiplicity six, and $r = \pm 1$. The general solution is $y = \dfrac{1}{2} \sin t + c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 t^5 + c_7 e^{-t} + c_8 e^t$.

VI.13   One searches for a particular solution in the form $y = At^2 + Bt + C$, obtaining $Y = t^2 - \dfrac{1}{4}$. The characteristic equation of the corresponding homogeneous equation

$$r^4 + 4 = 0$$

has roots $r = 1 \pm i$ and $r = -1 \pm i$, which were determined in the preceding problem set. The general solution is

$$y = t^2 - \frac{1}{4} + c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t .$$

VI.14   One searches for a particular solution in the form $y = Ae^{-t}$, obtaining $Y = e^{-t}$. The characteristic equation of the corresponding homogeneous equation

$$r^4 - 2r^3 - 8r + 16 = (r - 2)^2(r^2 + 2r + 4) = 0$$

is solved by factoring, after guessing that $r = 2$ is a root. The roots are $r = 2, 2, -1 \pm \sqrt{3}$. The general solution is

$$y = e^{-t} + c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-t} \cos \sqrt{3}t + c_4 e^{-t} \sin \sqrt{3}t .$$

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VII.1 The characteristic equation

$$r^3 + 4r = r(r^2 + 4) = 0$$

has roots $r = 0$ and $r = \pm 2$. The general solution is $y = c_1 + c_2 \cos 2t + c_3 \sin 2t$. The initial conditions give

$$y(0) = c_1 + c_2 = 1 ,$$

$$y'(0) = 2c_3 = -1 ,$$

$$y''(0) = -4c_2 = 2 .$$

It follows that $c_2 = c_3 = -\dfrac{1}{2}$, and $c_1 = \dfrac{3}{2}$. Obtain $y = \dfrac{3}{2} - \dfrac{1}{2}\cos 2t - \dfrac{1}{2}\sin 2t$.

VII.2   The characteristic equation

$$r^4 + 4 = 0$$

has roots $-1 \pm i$ and $1 \pm i$, found in the problem V.9 earlier in this set. The general solution is $y = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$. The initial conditions give

$$y(0) = c_1 + c_3 = 1\,,$$

$$y'(0) = c_1 + c_2 - c_3 + c_4 = -1\,,$$

$$y''(0) = 2c_2 - 2c_4 = 2\,,$$

$$y'''(0) = -2c_1 + 2c_2 + 2c_3 + 2c_4 = 3\,.$$

Divide the last of these equations by 2

$$-c_1 + c_2 + c_3 + c_4 = \frac{3}{2}\,,$$

and add the result to the second equation, to obtain

$$2c_2 + 2c_4 = \frac{1}{2}\,.$$

Add this to the third equation, to obtain

$$4c_2 = \frac{5}{2}\,,$$

so that $c_2 = \frac{5}{8}$, and then $c_4 = -\frac{3}{8}$. Using these values, the second of the original four equations becomes

$$y(0) = c_1 - c_3 = -\frac{5}{4}\,.$$

Solving this equation together with the first of the original four equations gives $c_1 = -\frac{1}{8}$ and $c_3 = \frac{9}{8}$.

VII.4   Look for a particular solution in the form $y = At^4$, and calculate $Y = \frac{1}{24}t^4$. The characteristic equation of the corresponding homogeneous equation

$$r^5 + r^4 = r^4(r+1) = 0$$

has a root $r = 0$ repeated four times, and the root $= -1$. The general solution of this equation is $y = \frac{1}{24}t^4 + c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{-t}$. The initial conditions give

$$y(0) = c_1 + c_5 = 1\,,$$

$$y'(0) = c_2 - c_5 = -1 \,,$$

$$y''(0) = 2c_3 + c_5 = 1 \,,$$

$$y'''(0) = 6c_4 - c_5 = -1 \,,$$

$$y''''(0) = c_5 + 1 = 2 \,.$$

From the last equation, $c_5 = 1$, and then $c_1 = c_2 = c_3 = c_4 = 0$, giving $y = \frac{1}{24}t^4 + e^{-t}$.

VII.6   The characteristic equation

$$r^4 - 3r^2 - 4 = (r^2 - 4)(r^2 + 1) = 0$$

is solved by factoring as a biquadratic equation. The roots are $r = \pm 2$ and $r = \pm i$. Because the initial conditions are given at $t = 0$, it is convenient to use the general solution in the form

$$y(t) = c_1 \cosh 2t + c_2 \sinh 2t + c_3 \cos t + c_4 \sin t \,.$$

The initial conditions give

$$y(0) = c_1 + c_3 = 1 \,,$$

$$y'(0) = 2c_2 + c_4 = -1 \,,$$

$$y''(0) = 4c_1 - c_3 = 4 \,,$$

$$y'''(0) = 8c_2 - c_4 = 1 \,.$$

Solving the first of these equations together with the third one gives $c_1 = 1$ and $c_3 = 0$. Solving the second equation together with the fourth one gives $c_2 = 0$ and $c_4 = -1$. Obtain $y = \cosh 2t - \sin t$.

VII.7   The characteristic equation

$$r^5 - r = r(r^2 - 1)(r^2 + 1) = 0$$

has roots $r = \pm 1$, $r = 0$, $r = \pm i$. Because the initial conditions are given at $t = 0$, when it comes to the roots $r = \pm 1$, it is better to use the functions $\cosh t$ and $\sinh t$ in the fundamental set (rather than e^t and e^{-t}). Then the general solution is $y = c_1 + c_2 \cos t + c_3 \sin t + c_4 \cosh t + c_5 \sinh t$. Using the initial conditions, one quickly calculates $c_1 = c_4 = 1$, $c_2 = c_3 = c_5 = 0$.

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VIII.2 The solution $y = 1$ is produced by the root $r = 0$ of the characteristic equation, while $y = e^{-2t}$ corresponds to $r = -2$, and $y = \sin t$ to $r = \pm i$. The characteristic equation of the lowest possible order with these four roots is

$$r(r+2)(r^2+1) = 0\,,$$

or expanding

$$r^4 + 2r^3 + r^2 + 2r = 0\,.$$

The differential equation with this characteristic equation is

$$y'''' + 2y''' + y'' + 2y' = 0\,.$$

VIII.3 Substitution of $y = (t+1)^r$ into our equation produces a characteristic equation

$$r(r-1) - 4r + 6 = 0\,.$$

The roots are $r = 2$ and $r = 3$. The general solution is $y = c_1 (t+1)^2 + c_2 (t+1)^3$.

VIII.4 Substitution $v = y''$ will produce a linear first order equation for v. However, it is easier to write this equation in the form

$$(ty'')' = 1\,,$$

and integrate

$$ty'' = t + c_1\,,$$

$$y'' = 1 + \frac{c_1}{t}\,,$$

$$y' = t + c_1 \ln t + c_2\,,$$

$$y = \frac{t^2}{2} + c_1 (t \ln t - 1) + c_2 t + c_3\,.$$

Chapter 3

Section 3.3.1, Page 142

I.1 Begin with the series $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$, and replace x by x^2:

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,.$$

I.2 In the geometric series $\dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} t^n$, set $t = -x^2$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n}\,.$$

I.3 Begin with the series $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, replace x by $-3x$, then multiply through by x:

$$xe^{-3x} = x\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\,\frac{3^n x^{n+1}}{n!}\,.$$

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II.1(ii) We have $f^{(n)}(1) = e$ for all n. Therefore, the Taylor series is

$$e^x = e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}\,.$$

II.1(iii) Use the geometric series

$$\frac{1}{x} = \frac{1}{1+x-1} = \frac{1}{1-[-(x-1)]} = \sum_{n=0}^{\infty}(-1)^n\,(x-1)^n\,.$$

II.2 Decompose the series into a sum of its even and odd powers, then observe that all odd coefficients are equal to zero, and all even ones are equal to 2:

$$\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2}\,x^n = \sum_{n=1}^{\infty}\frac{1+(-1)^{2n}}{(2n)^2}\,x^{2n} + \sum_{n=1}^{\infty}\frac{1+(-1)^{2n-1}}{(2n-1)^2}\,x^{2n-1}$$

$$= \sum_{n=1}^{\infty}\frac{2}{(2n)^2}\,x^{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,x^{2n}\,.$$

II.4 Replace $n \to n-1$, and observe that the new series should begin at $n = 1$.

II.5 Replace $n \to n-2$, and observe that the new series should begin at $n = 2$, to conclude that $\displaystyle\sum_{n=0}^{\infty} a_n\,x^{n+2} = \sum_{n=2}^{\infty} a_{n-2}\,x^n$.

II.6 Replace $n \to n-2$, and observe that the new series begins at $n = 2$.

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II.7 We use the formula for the n-th derivative $(fg)^{(n)}$. Here $f(x) = x^2 + x$, $f'(x) = 2x + 1$, $f''(x) = 2$, while $f'''(x)$ and all of the derivatives of higher order are zero. Therefore, only the last three terms in the formula for $(fg)^{(n)}$ are non-zero:

$$(fg)^{(n)} = \frac{n(n-1)}{2}f''(x)g^{(n-2)}(x) + nf'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x)$$

$$= n(n-1)g^{(n-2)}(x) + n(2x+1)g^{(n-1)}(x) + (x^2 + x)g^{(n)}(x) \,.$$

II.8 Here $f(x) = x^2 + x$, $g(x) = e^{2x}$, and $g^{(k)}(x) = 2^k e^{2x}$, for any derivative k. Obtain

$$\left[(x^2 + x)e^{2x}\right]^{(n)} = \frac{n(n-1)}{2}f''(x)g^{(n-2)}(x) + nf'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x)$$

$$= e^{2x}\left[n(n-1)2^{n-2} + n(2x+1)2^{n-1} + (x^2 + x)2^n\right] \,.$$

II.9   Here $f(x) = x$, $f' = 1$, $f'' = 0$, and all of the derivatives of higher order are zero. Therefore, only the last two terms in the formula for $(fg)^{(n)}$ are non-zero. With $g(x) = y'(x)$, obtain

$$\left[xy'\right]^{(n)} = nf'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x) = ny'^{(n-1)} + xy'^{(n)} = ny^{(n)} + xy^{(n+1)} \,.$$

II.10   Here $f(x) = x^2 + 1$, $f' = 2x$, $f'' = 2$, $f''' = 0$, and all of the derivatives of higher order are zero. Therefore, only the last three terms in the formula for $(fg)^{(n)}$ are non-zero. With $g(x) = y''(x)$, obtain

$$\left[(x^2 + 1)y''\right]^{(n)} = \frac{n(n-1)}{2}f''g^{(n-2)}(x) + nf'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x)$$

$$= n(n-1)y''^{(n-2)} + n2xy''^{(n-1)} + \left(x^2 + 1\right)y''^{(n)}$$

$$= n(n-1)y^{(n)} + 2nxy^{(n+1)} + \left(x^2 + 1\right)y^{(n+2)} \,.$$

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III.1   Differentiate both sides of this equation n times, and use that $\left[xy'\right]^{(n)} = ny^{(n)} + xy^{(n+1)}$. Obtain

$$y^{(n+2)}(x) - ny^{(n)}(x) - xy^{(n+1)}(x) - y^{(n)}(x) = 0 \,.$$

Setting here $x = 0$, obtain the recurrence relation

$$y^{(n+2)}(0) = (n+1)y^{(n)}(0).$$

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$. Using $n = 0$ in the recurrence relation, obtain

$$y''(0) = y(0) = 1.$$

When $n = 2$, the recurrence relation gives

$$y''''(0) = 3y''(0) = 1 \cdot 3.$$

When $n = 4$, the recurrence relation gives

$$y^{(6)}(0) = 5y''''(0) = 1 \cdot 3 \cdot 5,$$

and in general

$$y^{(2n)}(0) = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Then

$$y_1(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n}.$$

On the last step we separated the term with a given value of $y(0) = 1$, from the terms involving the values of $y^{(2n)}(0)$ that we just computed. Using these values, we continue

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} x^{2n} = 1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero at $x = 0$. Using $n = 1$ in the recurrence relation, obtain

$$y'''(0) = 2y'(0) = 2.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(0) = 4y'''(0) = 2 \cdot 4.$$

When $n = 5$, the recurrence relation gives

$$y^{(7)}(0) = 6y'''''(0) = 2 \cdot 4 \cdot 6,$$

and in general

$$y^{(2n+1)}(0) = 2 \cdot 4 \cdot 6 \cdots (2n).$$

Then

$$y_2(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$

$$= x + \sum_{n=1}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}.$$

On the last step we separated the term corresponding to the given value of $y'(0) = 1$, from the terms involving the values of $y^{(2n+1)}(0)$ that we just computed. Using these values, we continue

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{(2n+1)!} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}.$$

The general solution is

$$y = c_1 y_1(x) + c_2 y_2(x) = c_1 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + c_2 \sum_{n=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}.$$

III.2   Differentiate both sides of this equation n times

$$y^{(n+2)}(x) - ny^{(n)}(x) - xy^{(n+1)}(x) + 2y^{(n)}(x) = 0,$$

and set here $x = 0$, to obtain the recurrence relation

$$y^{(n+2)}(0) = (n-2)y^{(n)}(0).$$

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$. Using $n = 0$ in the recurrence relation, obtain

$$y''(0) = -2y(0) = -2.$$

When $n = 2$, the recurrence relation gives

$$y''''(0) = 0 \,.$$

But then all higher even order derivatives are also zero, as follows from the recurrence relation. We conclude that

$$y_1(1) = 1 + \frac{y''(0)}{2} x^2 = 1 - x^2 \,.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero at $x = 0$. Using $n = 1$ in the recurrence relation, obtain

$$y'''(0) = -y'(0) = -1 \,.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(0) = y'''(0) = -1 \,.$$

In the book we stopped the calculation of $y_2(x)$ at this point, obtaining

$$y_2(x) = x + \frac{y'''(0)}{3!} x^3 + \frac{y^{(5)}(0)}{5!} x^5 + \cdots = x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \cdots \,.$$

One could continue, and obtain a general formula $y^{(2n+1)}(0) = -1 \cdot 3 \cdot 5 \cdots (2n - 3)$. However, for many equations getting a general formula may be too complicated, but one can "crank out" as many terms as one wishes.

III.3 Differentiate both sides of this equation n times

$$n(n - 1)y^{(n)} + 2nxy^{(n+1)} + \left(x^2 + 1\right) y^{(n+2)} + ny^{(n)} + xy^{(n+1)} + y^{(n)} = 0 \,.$$

Set here $x = 0$. Several terms vanish, and we get the recurrence relation:

$$y^{(n+2)}(0) = -(n^2 + 1)y^{(n)}(0) \,.$$

We shall calculate the first three non-zero terms for both $y_1(x)$ and $y_2(x)$.

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$. Using $n = 0$ in the recurrence relation, obtain

$$y''(0) = -y(0) = -1 \,.$$

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When $n = 2$, the recurrence relation gives

$$y''''(0) = -5y''(0) = 5.$$

Obtain

$$y_1(x) = 1 + \frac{y''(0)}{2}x^2 + \frac{y''''(0)}{4!}x^4 + \cdots = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \cdots.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero at $x = 0$. Using $n = 1$ in the recurrence relation, obtain

$$y'''(0) = -2y'(0) = -2.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(0) = -10\,y'''(0) = 20.$$

Obtain

$$y_2(x) = y'(0)x + \frac{y'''(0)}{3!}x^3 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots = x - \frac{1}{3}x^3 + \frac{1}{6}x^5 - \cdots.$$

III.4   Differentiate both sides of this equation n times

$$n(n-1)y^{(n)} + 2nxy^{(n+1)} + \left(x^2 + 1\right)y^{(n+2)} + 3ny^{(n)} + 3xy^{(n+1)} + y^{(n)} = 0.$$

Set here $x = 0$. Several terms vanish, and we get the recurrence relation:

$$y^{(n+2)}(0) = -(n+1)^2 y^{(n)}(0).$$

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$, $y^{(2n+1)}(0) = 0$. Using $n = 0$ in the recurrence relation, obtain

$$y''(0) = -y(0) = -1.$$

Setting $n = 2$ in the recurrence relation gives

$$y''''(0) = -3^2 y''(0) = -3^2(-1) = (-1)^2\,1^2 \cdot 3^2.$$

Similarly, when $n = 4$,

$$y^{(6)}(0) = -5^2 y''''(0) = (-1)^3\,1^2 \cdot 3^2 \cdot 5^2,$$

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and in general

$$y^{(2n)}(0) = (-1)^n 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2 \,.$$

Obtain

$$y_1(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} x^{2n} \,.$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero at $x = 0$, $y^{(2n)}(0) = 0$. Using $n = 1$ in the recurrence relation, obtain

$$y'''(0) = -2^2 y'(0) = -2^2 \,.$$

Setting $n = 3$ in the recurrence relation gives

$$y^{(5)}(0) = -4^2 y'''(0) = (-1)^2 2^2 \cdot 4^2 \,.$$

Similarly, when $n = 5$,

$$y^{(7)}(0) = -6^2 y''''(0) = (-1)^3 \cdot 2^2 \cdot 4^2 \cdot 6^2 \,,$$

and in general

$$y^{(2n+1)}(0) = (-1)^n \cdot 2^2 \cdot 4^2 \cdots (2n)^2 \,.$$

Obtain

$$y_2(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1} \,.$$

III.5   Differentiate both sides of this equation n times

$$n(n-1)y^{(n)} + 2nxy^{(n+1)} + \left(x^2 + 1\right) y^{(n+2)} - ny^{(n)} - xy^{(n+1)} + y^{(n)} = 0 \,.$$

Set here $x = 0$. Several terms vanish, and we get the recurrence relation:

$$y^{(n+2)}(0) = -(n-1)^2 y^{(n)}(0) .$$

To compute $y_1(x)$, we use the initial conditions $y(0) = 1$ and $y'(0) = 0$. It follows from the recurrence relation that all of the derivatives of odd order are zero at $x = 0$, $y^{(2n+1)}(0) = 0$. Using $n = 0$ in the recurrence relation, obtain

$$y''(0) = -y(0) = -1 .$$

When $n = 2$, the recurrence relation gives

$$y''''(0) = -1^2\, y''(0) = (-1)^2 1^2 .$$

When $n = 4$,

$$y^{(6)}(0) = -3^2 y''''(0) = (-1)^3 1^2 \cdot 3^2 ,$$

and in general

$$y^{(2n)}(0) = (-1)^n 1^2 \cdot 3^2 \cdots (2n-3)^2 .$$

Obtain

$$y_1(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{y^{(2n)}(0)}{(2n)!} x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1^2 \cdot 3^2 \cdots (2n-3)^2}{(2n)!} x^{2n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-3)}{2^n n! (2n-1)} x^{2n} .$$

To compute $y_2(x)$, we use the initial conditions $y(0) = 0$ and $y'(0) = 1$. It follows from the recurrence relation that all of the derivatives of even order are zero at $x = 0$. Using $n = 1$ in the recurrence relation, obtain

$$y'''(0) = 0 .$$

It follows from the recurrence relation that all of the derivatives of odd order, beginning with the third order, are also zero at $x = 0$. It follows that

$$y_2(x) = y_2(0) + y_2'(0)x = 1 + x .$$

III.6   Differentiate both sides of this equation n times

$$y^{(n+2)} - ny^{(n-1)} - xy^{(n)} = 0 .$$

Setting $x = 2$ in the equation, express

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$(*)$ $\qquad\qquad\qquad y''(2) = 2y(2).$

Set here $x = 2$. Obtain the recurrence relation

$$y^{(n+2)}(2) = 2y^{(n)}(2) + ny^{(n-1)}(2)\,.$$

This recurrence relation is too complex to give a compact formula for $y^{(n)}(2)$. We shall calculate the terms up to the fifth power for both $y_1(x)$ and $y_2(x)$.

To compute $y_1(x)$, we use the initial conditions $y(2) = 1$ and $y'(2) = 0$. From $(*)$

$$y''(2) = 2y(2) = 2\,.$$

When $n = 1$, the recurrence relation gives

$$y'''(2) = 2y'(2) + y(2) = 1\,.$$

When $n = 2$, the recurrence relation gives

$$y''''(2) = 2y''(2) + 2y'(2) = 4\,.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(2) = 2y'''(2) + 3y''(2) = 8\,.$$

Obtain

$$y_1(x) = y(2) + y'(2)(x-2) + \frac{y''(2)}{2}(x-2)^2 + \frac{y'''(2)}{3!}(x-2)^3$$

$$+\frac{y''''(2)}{4!}(x-2)^4 + \frac{y^{(5)}(2)}{5!}(x-2)^5 \cdots$$

$$= 1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \frac{1}{15}(x-2)^5 \cdots\,.$$

To compute $y_2(x)$, we use the initial conditions $y(2) = 0$ and $y'(2) = 1$. From $(*)$

$$y''(2) = 2y(2) = 0\,.$$

When $n = 1$, the recurrence relation gives

$$y'''(2) = 2y'(2) + y(2) = 2\,.$$

When $n = 2$, the recurrence relation gives

$$y''''(2) = 2y''(2) + 2y'(2) = 2\,.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(2) = 2y'''(2) + 3y''(2) = 4\,.$$

Obtain

$$y_2(x) = y(2) + y'(2)(x-2) + \frac{y''(2)}{2}(x-2)^2 + \frac{y'''(2)}{3!}(x-2)^3$$

$$+\frac{y''''(2)}{4!}(x-2)^4 + \frac{y^{(5)}(2)}{5!}(x-2)^5 \cdots$$

$$= x - 2 + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \frac{1}{30}(x-2)^5 + \cdots\,.$$

The general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

III.7   Differentiate both sides of this equation n times

$$y^{(n+2)} - ny^{(n)} - xy^{(n+1)} - y^{(n)} = 0\,.$$

Set here $x = 1$. Obtain the recurrence relation

$$y^{(n+2)}(1) = y^{(n+1)}(1) + (n+1)y^{(n)}(1)\,.$$

We shall calculate the first four non-zero terms for both $y_1(x)$ and $y_2(x)$.

To compute $y_1(x)$, we use the initial conditions $y(1) = 1$ and $y'(1) = 0$.
When $n = 0$, the recurrence relation gives

$$y''(1) = y'(1) + y(1) = 1\,.$$

When $n = 1$, the recurrence relation gives

$$y'''(1) = y''(1) + 2y'(1) = 1\,.$$

When $n = 2$, the recurrence relation gives

$$y''''(1) = y'''(1) + 3y''(1) = 4\,.$$

Obtain

$$y_1(x) = y(1)+y'(1)(x-1)+\frac{y''(1)}{2}(x-1)^2+\frac{y'''(1)}{3!}(x-1)^3+\frac{y''''(1)}{4!}(x-1)^4+\cdots$$

$$= 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\,.$$

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To compute $y_2(x)$, we use the initial conditions $y(1) = 0$ and $y'(1) = 1$. When $n = 0$, the recurrence relation gives

$$y''(1) = y'(1) + y(1) = 1 .$$

When $n = 1$, the recurrence relation gives

$$y'''(1) = y''(1) + 2y'(1) = 3 .$$

When $n = 2$, the recurrence relation gives

$$y''''(1) = y'''(1) + 3y''(1) = 6 .$$

Obtain

$$y_2(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \frac{y''''(1)}{4!}(x-1)^4 + \cdots$$

$$= (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots .$$

III.8   Differentiate both sides of this equation n times

$$y^{(n+2)} + ny^{(n)} + (x+2)y^{(n+1)} + y^{(n)} = 0 .$$

Set here $x = -2$. Obtain the recurrence relation

$$y^{(n+2)}(-2) = -(n+1)y^{(n)}(-2) .$$

To calculate $y_1(x)$, we use the initial conditions $y(-2) = 1$ and $y'(-2) = 0$. When $n = 0$, the recurrence relation gives

$$y''(-2) = -y(-2) = -1 .$$

When $n = 1$, the recurrence relation gives

$$y'''(-2) = -2y'(-2) = 0 ,$$

and similarly all odd order derivatives are zero, $y^{(2n+1)}(-2) = 0$. When $n = 2$, the recurrence relation gives

$$y''''(-2) = -3y''(-2) = (-1)^2 1 \cdot 3 .$$

When $n = 4$, the recurrence relation gives

$$y^{(6)}(-2) = -5y''''(-2) = (-1)^3 1 \cdot 3 \cdot 5 ,$$

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and in general

$$y^{(2n)}(-2) = (-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1) \,.$$

Obtain

$$y_1(x) = \sum_{n=0}^{\infty} \frac{y^{(2n)}(-2)}{(2n)!} (x+2)^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n n!} (x+2)^{2n} \,.$$

To calculate $y_2(x)$, we use the initial conditions $y(-2) = 0$ and $y'(-2) = 1$. When $n = 0$, the recurrence relation gives

$$y''(-2) = -y(-2) = 0 \,.$$

Similarly all even order derivatives are zero, $y^{(2n)}(-2) = 0$. When $n = 1$, the recurrence relation gives

$$y'''(-2) = -2y'(-2) = -2 \,.$$

When $n = 3$, the recurrence relation gives

$$y^{(5)}(-2) = -4y'''(-2) = (-1)^2 2 \cdot 4 \,,$$

When $n = 5$, the recurrence relation gives

$$y^{(7)}(-2) = -6y^{(5)}(-2) = (-1)^3 1 \cdot 2 \cdot 4 \cdot 6 \,,$$

and in general

$$y^{(2n+1)}(-2) = (-1)^n 1 \cdot 2 \cdot 4 \cdots (2n) \,.$$

Obtain

$$y_2(x) = \sum_{n=0}^{\infty} \frac{y^{(2n+1)}(-2)}{(2n+1)!} (x+2)^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} (x+2)^{2n+1} \,.$$

III.9   Differentiate both sides of this equation n times

$$y^{(n+2)} + ny^{(n)} + (x+1)y^{(n+1)} - y^{(n)} = 0 \,.$$

Set here $x = -1$. Obtain the recurrence relation

$$y^{(n+2)}(-1) = -(n-1)y^{(n)}(-1) \,.$$

To calculate $y_1(x)$, we use the initial conditions $y(-1) = 1$ and $y'(-1) = 0$. When $n = 0$, the recurrence relation gives

$$y''(-1) = y(-1) = 1 \,.$$

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When $n = 1$, the recurrence relation gives

$$y'''(-1) = 0\,,$$

and then all odd order derivatives are zero, $y^{(2n+1)}(-1) = 0$. When $n = 2$, the recurrence relation gives

$$y''''(-1) = -y''(-1) = -1\,.$$

When $n = 4$, the recurrence relation gives

$$y^{(6)}(-1) = -3y''''(-1) = (-1)^2 1 \cdot 3\,.$$

When $n = 6$, the recurrence relation gives

$$y^{(8)}(-1) = -5y^{(6)}(-1) = (-1)^3 1 \cdot 3 \cdot 5\,,$$

and in general

$$y^{(2n)}(-1) = (-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n - 3)\,.$$

Obtain

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{y^{(2n)}(-1)}{(2n)!}(x+1)^{2n} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{(2n)!}(x+1)^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n n!(2n-1)}(x+1)^{2n}\,.$$

To calculate $y_2(x)$, we use the initial conditions $y(-1) = 0$ and $y'(-1) = 1$. When $n = 0$, the recurrence relation gives

$$y''(-1) = y(-1) = 0\,,$$

and similarly all derivatives of even order are zero, $y^{(2n)}(-1) = 0$. When $n = 1$, the recurrence relation gives

$$y'''(-1) = 0\,,$$

and then all odd order derivatives are zero, $y^{(2n+1)}(-1) = 0$, starting with the third derivative. It follows that

$$y_2(x) = y(-1) + y'(-1)(x + 1) = x + 1\,.$$

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IV.1   Express from the equation $y'' = xy' - 2y$, and $y''' = xy'' - y'$. Then $y''(0) = -2y(0) = -2$, and $y'''(0) = -y'(0) = -2$. Obtain

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots = 1 + 2x - x^2 - \frac{1}{3}x^3 + \cdots .$$

IV.2   Express from the equation $y'' = 2xy$, $y''' = 2y + 2xy'$, and $y'''' = 4y' + 2xy''$. Using the initial conditions, calculate $y''(2) = 4y(2) = 4$, $y'''(2) = 2y(2) + 4y'(2) = 2$, and $y''''(2) = 4y'(2) + 4y''(2) = 16$. Obtain

$$y(x) = y(2) + y'(2)(x-2) + \frac{y''(2)}{2}(x-2)^2 + \frac{y'''(2)}{3!}(x-2)^3 + \frac{y''''(2)}{4!}(x-2)^4 + \cdots$$

$$= 1 + 2(x-2)^2 + \frac{1}{3}(x-2)^3 + \frac{2}{3}(x-2)^4 + \cdots .$$

IV.3   Express from the equation $y'' = -xy$, $y''' = -y - xy'$, and $y'''' = -2y' - xy''$. Using the initial conditions, calculate $y''(-1) = y(-1) = 2$, $y'''(-1) = -y(-1) + y'(-1) = -5$, and $y''''(-1) = -2y'(-1) + y''(-1) = 8$. Obtain

$$y(x) = y(-1) + y'(-1)(x+1) + \frac{y''(-1)}{2}(x+1)^2 + \frac{y'''(-1)}{3!}(x+1)^3 + \frac{y''''(-1)}{4!}(x+1)^4 + \cdots$$

$$= 2 - 3(x+1) + (x+1)^2 - \frac{5}{6}(x+1)^3 + \frac{1}{3}(x+1)^4 \cdots .$$

IV.4   Differentiate the equation

$$2xy'' + (1+x^2)y''' - 2y' - 2xy'' + 2y' = 0 ,$$

which simplifies to

$$y''' = 0 .$$

Integration gives $y = c_1x^2 + c_2x + c_3$. We were given only two initial conditions: $y(0) = 1$, $y'(0) = -2$. However, setting $x = 0$ in the original equation gives

$$y''(0) = -2y(0) = -2 .$$

It follows that $y = 1 - 2x - x^2$.

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V.1   It is convenient to multiply this mildly singular equation by x:

$$(\ast) \qquad\qquad 2x^2y'' + xy' + x^2y = 0.$$

Look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

with $a_0 = 1$. Using that

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots,$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + \cdots,$$

a substitution of $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation $(\ast)$, gives

$$\sum_{n=2}^{\infty} 2a_n n(n-1) x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

The third series is not "lined up" with the other two. We therefore shift the index of summation, replacing n by $n-2$ in that series, obtaining

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

The equation becomes

$$\sum_{n=1}^{\infty} 2a_n n(n-1) x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

We can begin the first of these series at $n = 2$ (because at $n = 1$ the coefficient is zero). The second series is the only one containing the linear term involving x. We split this term off, and then all series start at $n = 2$:

$$a_1 x + \sum_{n=2}^{\infty} 2a_n n(n-1) x^n + \sum_{n=2}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

In order for a series to be zero for all x, all of its coefficients must be zero:

$$a_1 = 0,$$

$$2a_n n(n-1) + a_n n + a_{n-2} = 0 \,,$$

giving the recurrence relation

$$a_n = -\frac{1}{n(2n-1)} \, a_{n-2} \,.$$

Because $a_1 = 0$, the recurrence relation shows that all coefficients with odd indeces are zero, $a_{2n+1} = 0$. We now calculate the even coefficients a_{2n}:

$$a_2 = -\frac{1}{2 \cdot 3} \, a_0 = -\frac{1}{2 \cdot 3} \,,$$

$$a_4 = -\frac{1}{4 \cdot 7} \, a_2 = (-1)^2 \frac{1}{2 \cdot 4 \cdot 3 \cdot 7} \,,$$

$$a_6 = -\frac{1}{6 \cdot 11} \, a_6 = (-1)^3 \frac{1}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} = (-1)^3 \frac{1}{2^3 3! \cdot 3 \cdot 7 \cdot 11} \,,$$

and in general

$$a_{2n} = (-1)^n \frac{1}{2^n n! \cdot 3 \cdot 7 \cdot (4n-1)} \,.$$

Obtain

$$y(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \, \frac{x^{2n}}{2^n n! \, 3 \cdot 7 \cdot 11 \cdots (4n-1)} \,.$$

V.2   It is convenient to multiply this mildly singular equation by x:

$$x^2 y'' + xy' - xy = 0 \,.$$

Look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \,,$$

with $a_0 = 1$. Calculate

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \,,$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + \cdots \,.$$

Substitution of $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation, gives

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \,.$$

The third series is not "lined up" with the other two. We therefore shift the index of summation, replacing n by $n-1$ in that series, obtaining

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n \,.$$

Then the equation becomes

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \,.$$

Our goal is to combine the three series into a single one, so that we can set all of the resulting coefficients to zero. The x term is present in the second and the third series, but not in the first. However, we can start the first series at $n = 1$, because at $n = 1$ the coefficient is zero. The equation becomes

$$\sum_{n=1}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \,.$$

Now for all $n \geq 1$, the x^n term is present in all three series, so that we can combine these series into one series. We therefore just set the sum of the coefficients to zero

$$a_n n(n-1) + a_n n - a_{n-1} = 0 \,.$$

Solve for a_n, to get the recurrence relation

$$a_n = \frac{1}{n^2} a_{n-1} \,, \quad n \geq 1 \,.$$

Starting with $a_0 = 1$, compute $a_1 = \dfrac{1}{1^2}$, then $a_2 = \dfrac{1}{2^2} a_1 = \dfrac{1}{1^2 \cdot 2^2}$, $a_3 = \dfrac{1}{3^2} a_3 = \dfrac{1}{1^2 \cdot 2^2 \cdot 3^2} = \dfrac{1}{(3!)^2}$, and, in general, $a_n = \dfrac{1}{(n!)^2}$. The result is

$$y = 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} x^n = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n .$$

V.3 It is convenient to multiply this mildly singular equation by x:

$$x^2 y'' + 2xy' + xy = 0.$$

Look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

beginning with $a_0 = 1$. Calculate

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots,$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + \cdots.$$

Substitution of $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation, gives

$$\sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

The third series is not "lined up" with the other two. We therefore shift the index of summation, replacing n by $n-1$ in that series, obtaining

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Then the equation becomes

$$\sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Our goal is to combine the three series into a single one, so that we can set all of the resulting coefficients to zero. The x term is present in the second and the third series, but not in the first. However, we can start the first series at $n = 1$, because at $n = 1$ the coefficient is zero. The equation becomes

$$\sum_{n=1}^{\infty} a_n n(n-1) x^n + \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Now for all $n \geq 1$, the x^n term is present in all three series, so that we can combine these series into one series. We therefore just set the sum of the coefficients to zero

$$a_n n(n-1) + 2a_n n + a_{n-1} = 0 .$$

Solve for a_n, to get the recurrence relation

$$a_n = -\frac{1}{n(n+1)} a_{n-1} , \quad n \geq 1 .$$

Starting with $a_0 = 1$, compute $a_1 = -\frac{1}{1 \cdot 2}$, then $a_2 = \frac{1}{2 \cdot 3} a_1 = (-1)^2 \frac{1}{1 \cdot 2 \cdot 2 \cdot 3}$, $a_3 = -\frac{1}{3 \cdot 4} a_3 = (-1)^3 \frac{1}{1 \cdot 2 \cdot \cdot 2 \cdot 3 \cdot 4} = (-1)^3 \frac{1}{3! \, 4!}$, and, in general, $a_n = (-1)^n \frac{1}{n! \, (n+1)!}$. The result is

$$y = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n! \, (n+1)!} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \, (n+1)!} x^n .$$

V.5 It is convenient to multiply this mildly singular equation by x:

$$x^2 y'' + x y' + x^2 y = 0 .$$

Look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

with $a_0 = 1$. Using that $y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ $y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + \cdots$, a substitution of $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation, gives

$$\sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 .$$

The third series is not "lined up" with the other two. We therefore shift the index of summation, replacing n by $n-2$ in that series, obtaining

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n .$$

The equation becomes

$$\sum_{n=1}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0 \,.$$

We can begin the first of these series at $n = 2$ (because at $n = 1$ the coefficient is zero). The second series is the only one containing the linear term involving x. We split this term off, and then all series start at $n = 2$:

$$a_1 x + \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=2}^{\infty} a_n n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0 \,.$$

In order for a series to be zero for all x, all of its coefficients must be zero:

$$a_1 = 0 \,,$$

$$a_n n(n-1) + a_n n + a_{n-2} = 0 \,,$$

giving the recurrence relation

$$a_n = -\frac{1}{n^2}\, a_{n-2} \,.$$

Because $a_1 = 0$, the recurrence relation shows that all coefficients with odd indeces are zero, $a_{2n+1} = 0$. We now calculate the even coefficients a_{2n}:

$$a_2 = -\frac{1}{2^2}\, a_0 = -\frac{1}{2^2} \,,$$

$$a_4 = -\frac{1}{4^2}\, a_2 = (-1)^2 \frac{1}{2^2 \cdot 4^2} \,,$$

$$a_6 = -\frac{1}{6^2}\, a_6 = (-1)^3 \frac{1}{2^2 \cdot 4^2 \cdot 6^2} = (-1)^3 \frac{1}{2^6 (3!)^2} \,,$$

and in general

$$a_n = (-1)^n \frac{1}{2^{2n}(n!)^2} \,.$$

Obtain

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(n!)^2} x^{2n} \,.$$

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VI.1   Searching for a solution of this mildly singular equation in the form $y = \sum_{n=0}^{\infty} a_n x^n$, we are led to the recurrence relation

(∗) $$a_n = -\frac{1}{n(n-5)} a_{n-1}.$$

If we start with $a_0 = 1$, then at $n = 5$ the denominator is zero, and the computation stops! To avoid the trouble at $n = 5$, we look for the solution in the form $y = \sum_{n=5}^{\infty} a_n x^n$. Substituting this series into the equation

$$x^2 y'' - 4xy' + xy = 0$$

(which is the original equation, multiplied by x), gives

$$\sum_{n=5}^{\infty} a_n n(n-1) x^n - \sum_{n=5}^{\infty} 4 a_n n x^n + \sum_{n=5}^{\infty} a_n x^{n+1} = 0\,.$$

The coefficient in x^5, which is $a_5(5 \cdot 4 - 4 \cdot 5) = 0$, is zero for any choice of a_5. We can then begin the first two series at $n = 6$:

$$\sum_{n=6}^{\infty} a_n n(n-1) x^n - \sum_{n=6}^{\infty} 4 a_n n x^n + \sum_{n=5}^{\infty} a_n x^{n+1} = 0\,.$$

Shifting here $n \to n-1$ in the last series, we see that the recurrence relation (∗) holds for $n \geq 6$. We choose $a_5 = 1$, and use the recurrence relation (∗) to calculate a_6, a_7, etc. Calculate $a_6 = -\frac{1}{6} a_5 = -\frac{1}{6}$, $a_7 = -\frac{1}{7 \cdot 2} a_6 = (-1)^2 \frac{1}{7 \cdot 6 \cdot 1 \cdot 2}$,

$$a_8 = -\frac{1}{8 \cdot 3} a_7 = (-1)^3 \frac{1}{8 \cdot 7 \cdot 6 \cdot 1 \cdot 2 \cdot 3} = (-1)^3 \frac{120}{8! 3!}\,,$$

and in general
$$a_n = (-1)^{n-5} \frac{120}{n!(n-5)!}\,.$$

We calculated a solution

$$y = x^5 + 120 \sum_{n=6}^{\infty} \frac{(-1)^{n-5}}{n!(n-5)!} x^n = 120 \sum_{n=5}^{\infty} \frac{(-1)^{n-5}}{n!(n-5)!} x^n\,.$$

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VI.3   Searching for a solution of this mildly singular equation in the form $y = \sum_{n=0}^{\infty} a_n x^n$, we are led to the recurrence relation

(∗) $$a_n = -\frac{1}{n(n-1)} a_{n-1}.$$

If we start with $a_0 = 1$, then immediately at $n = 1$ the denominator is zero, and the computation stops! We look for solution in the form $y = \sum_{n=1}^{\infty} a_n x^n$. Substituting this series into the equation

$$x^2 y'' + xy = 0$$

(which is the original equation, multiplied by x), gives

$$\sum_{n=1}^{\infty} a_n n(n-1) x^n + \sum_{n=1}^{\infty} a_n x^{n+1} = 0.$$

The constant term is zero in both series. The coefficient in x is also zero in both series for any choice of a_1. We can then begin the first series at $n = 2$, and make a shift $n \to n-1$ in the second series:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=2}^{\infty} a_{n-1} x^n = 0.$$

Combining the series, and setting the coefficients to zero, shows that the recurrence relation (∗) holds for $n \geq 2$. We choose $a_1 = 1$, and use the recurrence relation (∗) to calculate a_2, a_3, etc. Calculate $a_2 = -\frac{1}{2} a_1 = -\frac{1}{2}$, $a_3 = -\frac{1}{3 \cdot 2} a_3 = (-1)^2 \frac{1}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2}$,

$$a_4 = -\frac{1}{4 \cdot 3} a_7 = (-1)^3 \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2 \cdot 3} = (-1)^3 \frac{1}{4! 3!},$$

and in general

$$a_n = (-1)^{n-1} \frac{1}{n!(n-1)!}.$$

We calculated a solution

$$y = x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n!(n-1)!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!(n-1)!} x^n.$$

(On the last step we separated the linear term, for which $a_1 = 1$ was chosen from the other terms for which a_n's were calculated. Then we noticed that both terms can be combined in a single series.)

VI.4   Expressing y'' from the equation, calculate

$$E' = 2y'y'' + 2yy' = 2y'\left(-\frac{1}{x}y' - y\right) + 2yy' = -\frac{2}{x}y'^2 < 0\,,$$

and then the energy $E(x) = y'^2(x) + y^2(x)$ is decreasing for all $x > 0$. At each critical point $y'(x) = 0$ and therefore $y^2(x)$ is decreasing along the critical points. It follows that $|y(x)|$ is decreasing along the critical points.

VI.5   Because the energy $E(x) = y'^2(x) + y^2(x)$ is decreasing for all $x > 0$, it follows that $y'^2(x)$ is decreasing along the roots of $y(x)$. We conclude that $|y'(x)|$ is decreasing along the roots of $y(x)$.

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To solve a *moderately singular equation* of the form

(∗) $x^2 y''(x) + x p(x) y'(x) + q(x) y(x) = 0$

one begins by solving the characteristic equation

$$r(r-1) + p(0)r + q(0) = 0\,.$$

If $r_1 \neq r_2$ are its real roots, then a substitution $y(x) = x^{r_1} v(x)$ produces a mildly singular equation for $v(x)$, leading to the first solution of (∗), $y_1(x) = x^{r_1} v(x)$. The second solution of the form $y_2(x) = x^{r_2} v(x)$ (with a different $v(x)$) is produced similarly.

VII.1   For this moderately singular equation $p(x) = 1$, $q(x) = x^2 - \frac{1}{4}$. The characteristic equation

$$r(r-1) + r - \frac{1}{4} = 0$$

has roots $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$.

Case 1. $r = \frac{1}{2}$. We know that the substitution $y = x^{\frac{1}{2}} v$ will produce a mildly singular equation for $v(x)$. Substituting this y into our equation gives

$$xv'' + 2v' + xv = 0\,.$$

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Multiply this mildly singular equation by x, for convenience,

$$x^2 v'' + 2xv' + x^2 v = 0\,,$$

and look for a solution in the form $v(x) = \sum_{n=0}^{\infty} a_n x^n$, beginning with $a_0 = 1$.
Substitution into the last equation gives

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + 2\sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0\,.$$

To line up the powers, shift $n \to n-2$ in the last series. Obtain

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0\,.$$

The linear term is present only in the second series. We split it off, and then combine all three series

$$2a_1 x + \sum_{n=2}^{\infty} \left[a_n n(n-1) + 2a_n n + a_{n-2}\right] x^n = 0\,.$$

It follows that

$$a_1 = 0\,,$$

and

$$a_n n(n-1) + 2a_n n + a_{n-2} = 0\,,$$

which leads to the recurrence relation

$$a_n = -\frac{1}{n(n+1)} a_{n-2}\,.$$

The recurrence relation shows that $a_3 = 0$, $a_5 = 0$, and similarly all odd coefficients are zero, $a_{2n+1} = 0$. Beginning with $a_0 = 1$, we now calculate the even coefficients. Obtain: $a_2 = -\frac{1}{2 \cdot 3} a_0 = -\frac{1}{2 \cdot 3}$, $a_4 = -\frac{1}{4 \cdot 5} a_2 = (-1)^2 \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} = (-1)^2 \frac{1}{5!}$, and in general $a_{2n} = (-1)^n \frac{1}{(2n+1)!}$. We conclude that

$$v(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,,$$

and then

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] = x^{-1/2}\left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = x^{-1/2} \sin x\,.$$

Case 2.　$r = -\dfrac{1}{2}$. Substituting this $y = x^{-\frac{1}{2}}v$ into our equation gives

$$v'' + v = 0 \,.$$

We take $v(x) = \cos x$, and obtain $y_2(x) = x^{-1/2}\cos x$. (Other choices for $v(x)$ would lead to the same general solution $y = c_1 x^{-1/2}\sin x + c_2 x^{-1/2}\cos x$.)

Observe that the solution $y_1(x) = x^{-1/2}\sin x$ could also be derived the same way. Bessel's equation of order $\dfrac{1}{2}$ is very special. It can be solved without using infinite series.

VII.3　For this moderately singular equation $p(x) = 1$, $q(x) = -1 - x$. The characteristic equation

$$2r(r-1) + 3r - 1 = 0$$

has roots $r_1 = \dfrac{1}{2}$ and $r_2 = -1$.

Case 1.　$r = \dfrac{1}{2}$. We know that the substitution $y = x^{\frac{1}{2}}v$ will produce a mildly singular equation for $v(x)$. Substituting this y into our equation gives

$$2xv' + 5v' - v = 0 \,.$$

Multiply this equation by x, for convenience,

(1) $$2x^2 v'' + 5xv' - xv = 0 \,,$$

and look for a solution in the form $v(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, beginning with $a_0 = 1$. Substitution into the equation gives

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^n + 5\sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \,.$$

To line up the powers, shift $n \to n - 1$ in the last series. The first series we may begin at $n = 1$, instead of $n = 2$, because its coefficient at $n = 1$ is zero. Then

$$\sum_{n=1}^{\infty} 2a_n n(n-1)x^n + 5\sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \,.$$

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Combine these series into a single series, and set its coefficients to zero

$$2a_n n(n-1) + 5a_n n - a_{n-1} = 0 \,,$$

which gives us the recurrence relation

$$a_n = \frac{1}{2n(n+\frac{3}{2})} a_{n-1} \,.$$

Starting with $a_0 = 1$, compute $a_1 = \dfrac{1}{2 \cdot 1 \cdot (1+\frac{3}{2})}$, $a_2 = \dfrac{1}{2 \cdot 2 \cdot (2+\frac{3}{2})} a_1 = \dfrac{1}{2^2 2!(1+\frac{3}{2})(2+\frac{3}{2})}$, and in general $a_n = \dfrac{1}{n! \, 2^n \, (1+\frac{3}{2})(2+\frac{3}{2}) \cdots (n+\frac{3}{2})}$. It follows that $v = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n! \, 2^n \, (1+\frac{3}{2})(2+\frac{3}{2}) \cdots (n+\frac{3}{2})}$, and

$$y_1 = x^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \, 2^n \, (1+\frac{3}{2})(2+\frac{3}{2}) \cdots (n+\frac{3}{2})} \right] \,.$$

Case 2. $r = -1$. Set $y = x^{-1} v$. Substituting this y into our equation and simplifying gives

$$2xv'' - v' - v = 0 \,.$$

Multiply this equation by x, for convenience,

$$2x^2 v'' - xv' - xv = 0 \,.$$

We look for a solution in the form $v = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Substituting $v(x)$ into the last equation, obtain

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \,.$$

We start the first series at $n = 1$, and make a shift $n \to n-1$ in the third series:

$$\sum_{n=1}^{\infty} 2a_n n(n-1)x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \,.$$

Setting the coefficient of x^n to zero,

$$2a_n n(n-1) - a_n n - a_{n-1} = 0 \,,$$

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gives us the recurrence relation

$$a_n = \frac{1}{n(2n-3)} \, a_{n-1} \, .$$

Starting with $a_0 = 1$, compute $a_1 = -\dfrac{1}{1 \cdot 1}$, $a_2 = \dfrac{1}{1 \cdot 1} a_1 = -\dfrac{1}{1 \cdot 1}$, $a_3 = \dfrac{1}{2 \cdot 3} a_2 = -\dfrac{1}{2!1 \cdot 3}$, $a_4 = \dfrac{1}{3 \cdot 3} a_3 = -\dfrac{1}{3! \cdot 1 \cdot 3 \cdot 5}$, and in general

$$a_n = -\frac{1}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)} \, .$$

The second solution is then

$$y_2(x) = x^{-1} \left[1 - \sum_{n=1}^{\infty} \frac{1}{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)} \, x^n \right] \, .$$

The general solution is, of course, $y(x) = c_1 y_1 + c_2 y_2$.

VII.5   The characteristic equation

$$9r(r-1) + 2 = 0$$

has roots $r_1 = \dfrac{1}{3}$ and $r_2 = \dfrac{2}{3}$.

Case 1. $r = \dfrac{1}{3}$. Set $y = x^{\frac{1}{3}} v$. Substituting this y into our equation and simplifying gives

$$9xv'' + 6v' + v = 0 \, .$$

Multiply this equation by x, for convenience,

$$9x^2 v'' + 6xv' + xv = 0 \, .$$

We look for a solution in the form $v = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Substituting $v(x)$ into the last equation, obtain

$$\sum_{n=2}^{\infty} 9a_n n(n-1)x^n + \sum_{n=1}^{\infty} 6a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \, .$$

We make a shift $n \to n - 1$ in the third series, and begin the first series at $n = 1$:

$$\sum_{n=1}^{\infty} 9a_n n(n-1)x^n + \sum_{n=1}^{\infty} 6a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \, .$$

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Combining these series into a single series, and setting its coefficients to zero, we obtain the recurrence relation

$$a_n = -\frac{1}{3n(3n-1)} a_{n-1}.$$

Beginning with $a_0 = 1$, calculate $a_1 = -\frac{1}{3 \cdot 1 \cdot 2} a_0 = -\frac{1}{3 \cdot 1 \cdot 2}$, $a_2 = -\frac{1}{3 \cdot 2 \cdot 5} a_2 = (-1)^2 \frac{1}{3^2 \cdot 1 \cdot 2 \cdot 2 \cdot 5}$, $a_3 = -\frac{1}{3 \cdot 3 \cdot 8} a_2 = (-1)^3 \frac{1}{3^3 \cdot 1 \cdot 2 \cdot 3 \cdot 2 \cdot 5 \cdot 8}$ and in general

$$a_n = (-1)^n \frac{1}{3^n \, n! \cdot 2 \cdot 5 \cdot 8 \cdots (3n-1)}.$$

It follows that $v(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n \, n! \cdot 2 \cdot 5 \cdot 8 \cdots (3n-1)} x^n$, and

$$y_1(x) = x^{\frac{1}{3}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n \, n! \cdot 2 \cdot 5 \cdot 8 \cdots (3n-1)} x^n \right].$$

Case 2. $r = \frac{2}{3}$. Set $y = x^{\frac{2}{3}} v$. Substituting this y into our equation and simplifying gives

$$9xv'' + 12v' + v = 0.$$

Multiply this equation by x, for convenience,

$$9x^2 v'' + 12xv' + xv = 0.$$

We look for a solution in the form $v = \sum_{n=0}^{\infty} a_n x^n$. Substituting $v(x)$ into the last equation, obtain

$$\sum_{n=2}^{\infty} 9a_n n(n-1)x^n + \sum_{n=1}^{\infty} 12a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

We make a shift $n \to n-1$ in the third series, and begin the first series at $n = 1$:

$$\sum_{n=1}^{\infty} 9a_n n(n-1)x^n + \sum_{n=1}^{\infty} 12a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Combining these series into a single series, and setting its coefficients to zero, we obtain the recurrence relation

$$a_n = -\frac{1}{3n(3n+1)} a_{n-1}.$$

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Beginning with $a_0 = 1$, calculate $a_1 = -\dfrac{1}{3 \cdot 1 \cdot 4} a_0 = -\dfrac{1}{3 \cdot 1 \cdot 4}$, $a_2 = -\dfrac{1}{3 \cdot 2 \cdot 7} a_2 = (-1)^2 \dfrac{1}{3^2 \cdot 1 \cdot 2 \cdot 4 \cdot 7}$, $a_3 = -\dfrac{1}{3 \cdot 3 \cdot 10} a_2 = (-1)^3 \dfrac{1}{3^3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 7 \cdot 10}$ and in general

$$a_n = (-1)^n \frac{1}{3^n \, n! \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)}.$$

It follows that $v(x) = 1 + \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n \, n! \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)} x^n$, and

$$y_2(x) = x^{\frac{2}{3}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n \, n! \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)} x^n \right].$$

Chapter 4

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I.4 $\mathcal{L}\left(e^{2(t-1)}\right) = \mathcal{L}\left(e^{-2} e^{2t}\right) = e^{-2} \mathcal{L}\left(e^{2t}\right) = \dfrac{1}{e^2(s-2)}$.

I.5   Begin with $\mathcal{L}\left(\cos 3t\right) = \dfrac{s}{s^2+9}$. By the shift formula

$$\mathcal{L}\left(e^{2t} \cos 3t\right) = \frac{s-2}{(s-2)^2 + 9}.$$

I.6 $\mathcal{L}\left(\dfrac{t^3 - 3t}{t}\right) = \mathcal{L}\left(t^2 - 3\right) = \dfrac{2}{s^3} - \dfrac{3}{s}$.

I.7   Begin with $\mathcal{L}\left(t^4\right) = \dfrac{4!}{s^5} = \dfrac{24}{s^5}$. Then by the shift formula

$$\mathcal{L}\left(e^{-3t} t^4\right) = \frac{24}{(s+3)^5}.$$

I.8   Use that $\sin^2 2t = \dfrac{1}{2} - \dfrac{1}{2} \cos 4t$. Then

$$\mathcal{L}\left(\sin^2 2t\right) = \left(\frac{1}{2} - \frac{1}{2} \cos 4t\right) = \frac{1}{2s} - \frac{s}{2(s^2 + 16)}.$$

I.9   Use that $\sin 2t \cos 2t = \dfrac{1}{2}\sin 4t$. Then

$$\mathcal{L}\left(\sin 2t \cos 2t\right) = \frac{2}{s^2 + 16}.$$

I.10   Use the formula $\sinh t \cosh t = \dfrac{1}{2}\sinh 2t$, which follows easily from the definitions of $\sinh t$ and $\cosh t$. Then

$$\mathcal{L}\left(\sinh t \cosh t\right) = \frac{1}{s^2 - 4}.$$

I.11   Split the integral into two pieces

$$\mathcal{L}\left(|t - 2|\right) = \int_0^\infty e^{-st}\,|t - 2|\,dt = -\int_0^2 e^{-st}\,(t - 2)\,dt + \int_2^\infty e^{-st}\,(t - 2)\,dt$$

$$= \frac{e^{-2s} + 2s - 1}{s^2} + \frac{e^{-2s}}{s^2} = \frac{2e^{-2s} + 2s - 1}{s^2}.$$

(If you are using *Mathematica*, enter LaplaceTransform[$|t - 2|, t, s$].)

I.12 $F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt = \int_1^3 e^{-st} t\,dt = \dfrac{e^{-s}(s + 1)}{s^2} - \dfrac{e^{-3s}(3s + 1)}{s^2}.$

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II.3   Factor the denominator, then use partial fractions

$$\frac{1}{s^2 + s} = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1} \to 1 - e^{-t}.$$

II.4   Factor the denominator, then use partial fractions

$$\frac{1}{s^2 - 3s} = \frac{1}{s(s - 3)} = \frac{1}{3}\left[\frac{1}{s - 3} - \frac{1}{s}\right] \to \frac{1}{3}e^{3t} - \frac{1}{3}.$$

II.5   Factor the denominator (by guessing one of its roots), then use partial fractions

$$\frac{1}{s^3 - 7s + 6} = \frac{1}{(s - 1)(s - 2)(s + 3)} = -\frac{1/4}{s - 1} + \frac{1/5}{s - 2} + \frac{1/20}{s + 3}$$

$$\to -\frac{1}{4}e^t + \frac{1}{5}e^{2t} + \frac{1}{20}e^{-3t}.$$

II.6 $\dfrac{1}{s^3 + s} = \dfrac{1}{s(s^2 + 1)} = \dfrac{1}{s} - \dfrac{s}{s^2 + 1} \to 1 - \cos t.$

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II.7 $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{1}{3}\left[\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right] \to \dfrac{1}{3}\sin t - \dfrac{1}{6}\sin 2t.$

II.8 $\dfrac{s}{s^4+5s^2+4} = \dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{1}{3}\left[\dfrac{s}{s^2+1} - \dfrac{s}{s^2+4}\right] \to \dfrac{1}{3}\cos t -$

$\dfrac{1}{3}\cos 2t.$

II.9   Complete the square, and use the shift formula

$$\frac{1}{s^2+2s+10} = \frac{1}{(s+1)^2+9} \to \frac{1}{3}e^{-t}\sin 3t\,,$$

because $\dfrac{1}{s^2+9} \to \dfrac{1}{3}\sin 3t.$

II.10   Factor the denominator, then use partial fractions

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\left[\frac{1}{s-1} - \frac{1}{s+2}\right] \to \frac{1}{3}e^t - \frac{1}{3}e^{-2t}\,.$$

II.11   Complete the square in the denominator, then produce the same shift in the numerator

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\,.$$

To invert the first term, drop the shift first (drop $+\dfrac{1}{2}$)

$$\frac{s}{s^2+\frac{3}{4}} \to \cos\frac{\sqrt{3}}{2}t\,.$$

Then by the shift formula

$$\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} \to e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t\,.$$

Similarly

$$\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} \to \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\,,$$

because

$$\frac{1}{s^2+\frac{3}{4}} \to \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\,.$$

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II.12 Factor the denominator, then use partial fractions

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1} \to \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

II.13 $\quad\dfrac{s+3}{4s^2+1} = \dfrac{1}{4}\left[\dfrac{s}{s^2+\frac{1}{4}} + 3\dfrac{1}{s^2+\frac{1}{4}}\right] \to \dfrac{1}{4}\cos\dfrac{t}{2} + \dfrac{3}{2}\sin\dfrac{t}{2}.$

II.14 Complete the square in the denominator, then produce the same shift in the numerator

$$\frac{s}{4s^2-4s+5} = \frac{s}{(2s-1)^2+4} = \frac{1}{4}\frac{s}{(s-\frac{1}{2})^2+1}$$

$$= \frac{1}{4}\left[\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+1} + \frac{\frac{1}{2}}{(s-\frac{1}{2})^2+1}\right] \to \frac{1}{4}e^{\frac{1}{2}t}\cos t + \frac{1}{8}e^{\frac{1}{2}t}\sin t.$$

II.15 Factor the denominator, then use partial fractions

$$\frac{s+2}{s^3-3s^2+2s} = \frac{s+2}{s(s-1)(s-2)},$$

$$\frac{s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}.$$

Multiply both sides by s, then set $s = 0$ to get $A = 1$. Multiply both sides by $s-1$, then set $s = 1$ to get $B = -3$. Multiply both sides by $s-2$, then set $s = 2$ to get $C = 2$. We conclude

$$\frac{s+2}{s^3-3s^2+2s} = \frac{1}{s} - \frac{3}{s-1} + \frac{2}{s-2} \to 1 - 3e^t + 2e^{2t}.$$

II.16 Factor the denominator, then use partial fractions

$$\frac{s^3-s}{s^4+5s^2+4} = \frac{s^3-s}{(s^2+1)(s^2+4)} = -\frac{2s}{3(s^2+1)} + \frac{5s}{3(s^2+4)} \to -\frac{2}{3}\cos t + \frac{5}{3}\cos 2t.$$

II.17 Factor the denominator, use partial fractions, then complete the square

$$\frac{s^2+2}{s^3-2s^2+2s} = \frac{s^2+2}{s(s^2-2s+2)} = \frac{1}{s} + \frac{2}{s^2-2s+2}$$

$$= \frac{1}{s} + \frac{2}{(s-1)^2+1} \to 1 + 2e^t\sin t.$$

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III.1   Apply the Laplace transform to both sides of the equation

$$-y'(0) - sy(0) + s^2Y(s) + 3\left(-y(0) + sY(s)\right) + 2Y(s) = 0 \,.$$

Use the initial conditions

$$-2 + s + s^2Y(s) + 3\left(1 + sY(s)\right) + 2Y(s) = 0 \,,$$

then simplify and solve for $Y(s)$:

$$\left(s^2 + 3s + 2\right)Y(s) = -s - 1 \,,$$

$$Y(s) = -\frac{s+1}{s^2 + 3s + 2} = -\frac{1}{s+2} \rightarrow -e^{-2t} \,.$$

III.2   Apply the Laplace transform to both sides of the equation

$$-y'(0) - sy(0) + s^2Y(s) + 2\left(-y(0) + sY(s)\right) + 5Y(s) = 0 \,,$$

$$2 - s + s^2Y(s) + 2\left(-1 + sY(s)\right) + 5Y(s) = 0 \,,$$

then solve for $Y(s)$:
$$\left(s^2 + 2s + 5\right)Y(s) = s \,,$$

$$Y(s) = \frac{s}{s^2 + 2s + 5} \,.$$

Invert $Y(s)$:

$$\frac{s}{s^2 + 2s + 5} = \frac{s+1}{(s+1)^2 + 4} - \frac{1}{(s+1)^2 + 4} \rightarrow e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t \,.$$

III.3   Apply the Laplace transform to both sides of the equation, then solve for $Y(s)$

$$-y'(0) - sy(0) + s^2Y(s) + Y(s) = \frac{2}{s^2 + 4} \,,$$

$$-1 + s^2Y(s) + Y(s) = \frac{2}{s^2 + 4} \,,$$

$$Y(s) = \frac{1}{s^2 + 1} + \frac{2}{(s^2 + 1)(s^2 + 4)} \,.$$

Using partial fractions, invert $Y(s)$:

$$Y(s) = \frac{1}{s^2+1} + \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] = \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} \rightarrow \frac{5}{3}\sin t - \frac{1}{3}\sin 2t\,.$$

III.4 Apply the Laplace transform to both sides of the equation, then solve for $Y(s)$

$$-y'(0) - sy(0) + s^2Y(s) + 2sY(s) + 2Y(s) = \frac{1}{s-1}\,,$$

$$-1 + s^2Y(s) + 2sY(s) + 2Y(s) = \frac{1}{s-1}\,,$$

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{(s-1)(s^2+2s+2)}\,.$$

Using partial fractions on the second term, invert $Y(s)$:

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{5(s-1)} - \frac{s+3}{5(s^2+2s+2)} \rightarrow \frac{1}{5}e^t - \frac{1}{5}e^{-t}\left(\cos t - 3\sin t\right)\,,$$

in view of

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1} \rightarrow e^{-t}\sin t\,,$$

$$\frac{s+3}{s^2+2s+2} = \frac{s+1+2}{(s+1)^2+1} \rightarrow e^{-t}\cos t + 2e^{-t}\sin t\,.$$

III.5 Apply the Laplace transform to both sides of the equation, and use the initial conditions, then solve for $Y(s)$

$$-y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + s^4Y(s) - Y(s) = 0\,,$$

$$-s^2 + s^4Y(s) - Y(s) = 0\,,$$

$$Y(s) = \frac{s^2}{s^4-1} = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1} \rightarrow \frac{1}{2}\sinh t + \frac{1}{2}\sin t\,.$$

III.6 Apply the Laplace transform to both sides of the equation, use the initial conditions, then solve for $Y(s)$, and invert

$$-y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + s^4Y(s) - Y(s) = 0\,,$$

$$-8 - 2s^2 + s^4Y(s) - 16Y(s) = 0\,,$$

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$$Y(s) = \frac{2s^2 + 8}{s^4 - 16} = \frac{2(s^2 + 4)}{(s^2 - 4)(s^2 + 4)} = \frac{2}{s^2 - 4} \rightarrow \sinh 2t \,.$$

III.7 Apply the Laplace transform to both sides of the equation, use the initial conditions, then solve for $Y(s)$

$$-y''(0) - sy'(0) - s^2 y(0) + s^3 Y(s) + 3\left(-y'(0) - sy(0) + s^2 Y(s)\right) + 2\left(-y(0) + sY(s)\right) = 0 \,,$$

$$Y(s) = -\frac{1}{s^3 + 3s^2 + 2s} \,.$$

Using partial fractions, invert

$$Y(s) = -\frac{1}{s(s+1)(s+2)} = -\frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+2)} \rightarrow -\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \,.$$

III.8 Apply the Laplace transform to both sides of the equation, use the initial conditions, then solve for $Y(s)$

$$s^3 Y(s) + 3s^2 Y(s) + 3sY(s) + Y(s) = \frac{1}{s+1} \,,$$

$$(s+1)^3 Y(s) = \frac{1}{s+1} \,,$$

$$Y(s) = \frac{1}{(s+1)^4} \,.$$

Starting with $\dfrac{1}{s^4} \rightarrow \dfrac{t^3}{3!}$, and using the shift formula, invert $y(t) = \dfrac{t^3 e^{-t}}{6}$.

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IV.2 Define the function $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{x}\, dx$. Observe that $\mathcal{L}\left(\sin tx\right) = \dfrac{x}{s^2 + x^2}$ (here x plays the role of the constant a), $s > 0$. Then

$$F(s) = \int_0^\infty \frac{1}{x^2 + s^2}\, dx = \frac{1}{s} tan^{-1} \frac{x}{s} \Big|_0^\infty = \frac{\pi}{2}\frac{1}{s} \,.$$

It follows that $f(t) = \dfrac{\pi}{2}$ for all t, and in particular $f(1) = \dfrac{\pi}{2}$.

IV.3 Apply the Laplace transform to each of the equations

$$-4 + sX(s) = 2X(s) - Y(s)$$

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$$2 + sY(s) = -X(s) + 2Y(s),$$

giving

$$(s-2)X(s) + Y(s) = 4$$

$$X(s) + (s-2)Y(s) = -2.$$

This is a linear system for $X(s)$ and $Y(s)$. Solving it (say, by Cramer's rule) gives $X(s) = \dfrac{4s-6}{s^2-4s+3}$, $Y(s) = -\dfrac{2s}{s^2-4s+3}$. Using partial fractions, one inverts $X(s)$ and $Y(s)$ to conclude $x(t) = e^t + 3e^{3t}$, $y(t) = e^t - 3e^{3t}$. (If you are using *Mathematica* the command is InverseLaplaceTransform$[\dfrac{4s-6}{s^2-4s+3}$, s, $t]$.)

IV.4   Apply the Laplace transform to each of the equations

$$sX(s) = 2X(s) - 3Y(s) + \frac{1}{s^2}$$

$$-1 + sY(s) = -2X(s) + Y(s),$$

giving

$$(s-2)X(s) + 3Y(s) = \frac{1}{s^2}$$

$$2X(s) + (s-1)Y(s) = 1.$$

Solving this linear system for $X(s)$ and $Y(s)$, then using partial fractions obtain

$$X(s) = -\frac{3s^2 - s - 1}{s^2(s^2 - 3s - 4)} = \frac{1}{4s^2} - \frac{7}{16s} + \frac{1}{s+1} + \frac{9}{16(s-4)}$$

$$\rightarrow \frac{t}{4} - \frac{7}{16} + e^{-t} - \frac{9}{16}e^{4t},$$

$$Y(s) = \frac{s^3 - 2s^2 - 2}{s^2(s^2 - 3s - 4)} = \frac{1}{2s^2} - \frac{3}{8s} + \frac{1}{s+1} + \frac{3}{8(s-4)}$$

$$\rightarrow \frac{t}{2} - \frac{3}{8} + e^{-t} + \frac{3}{8}e^{4t}.$$

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V.1   At $t = 1$ the function $f(t)$ jumps up by one unit, and at $t = 5$ it jumps down by one unit. It follows that $f(t) = u_1(t) - u_5(t)$, and then $F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-5s}}{s}$.

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V.2 The function $g(t)$ starts out equal to 1, and at $t = 5$ it jumps down by three units. It follows that $g(t) = 1 - 3u_5(t)$, and then $G(s) = \dfrac{1}{s} - 3\dfrac{e^{-5s}}{s}$.

V.3   The function $h(t)$ starts out equal to -2, at $t = 3$ it jumps up by six units, and at $t = 7$ it jumps down by four units. It follows that $h(t) = -2 + 6u_3(t) - 4u_7(t)$, and then $H(s) = -\dfrac{2}{s} + 6\dfrac{e^{-3s}}{s} - 4\dfrac{e^{-7s}}{s}$.

V.4 $\mathcal{L}(t) = \dfrac{1}{s^2}$, and then $\mathcal{L}(u_4(t)(t-4)) = \dfrac{e^{-4s}}{s^2}$.

V.6   This function is equal to 0 for $t \in (0,2)$, to 1 for $t \in (2,3)$, to -1 for $t \in (3,6)$, and to 3 for $t > 6$.

V.7 Begin with $\dfrac{1}{s^2} \to t$. Then by the second shift formula

$$\frac{1}{s^2}\left(2e^{-s} - 3e^{-4s}\right) \to 2u_1(t)(t-1) - 3u_4(t)(t-4) .$$

V.8   Begin with $\dfrac{3s-1}{s^2+4} \to 3\cos 2t - \dfrac{1}{2}\sin 2t$. Then by the second shift formula

$$e^{-2s}\frac{3s-1}{s^2+4} \to u_2(t)\left[3\cos 2(t-2) - \frac{1}{2}\sin 2(t-2)\right] .$$

V.9 Begin with

$$\frac{1}{s^2+s-6} = \frac{1}{(s-2)(s+3)} = \frac{1}{5}\left[\frac{1}{s-2} - \frac{1}{s+3}\right] \to \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t} .$$

Then by the second shift formula

$$e^{-s}\frac{1}{s^2+s-6} \to u_1(t)\left(\frac{1}{5}e^{2t-2} - \frac{1}{5}e^{-3t+3}\right) .$$

V.10   Completing the square

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} \to \frac{1}{2}e^{-t}\sin 2t .$$

Then by the second shift formula

$$e^{-\frac{\pi}{2}s}\frac{1}{s^2+2s+5} \to \frac{1}{2}u_{\pi/2}(t)e^{-(t-\pi/2)}\sin 2(t-\pi/2) .$$

$$= -\frac{1}{2}u_{\pi/2}(t)e^{-t+\pi/2}\sin 2t .$$

V.11 Apply the Laplace transform to both sides of the equation, then solve for $Y(s)$

$$-2s + s^2Y(s) + Y(s) = 4\frac{e^{-s}}{s} - \frac{e^{-5s}}{s} ,$$

$$Y(s) = \frac{2s}{s^2 + 1} + 4e^{-s}\frac{1}{s(s^2 + 1)} - e^{-5s}\frac{1}{s(s^2 + 1)} .$$

Using partial fractions

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{1}{s^2 + 1} \to 1 - \cos t .$$

Then by the second shift formula

$$Y(s) \to 2\cos t + 4u_1(t)\left[1 - \cos(t - 1)\right] - u_5(t)\left[1 - \cos(t - 5)\right] .$$

V.12 Apply the Laplace transform to both sides of the equation, then solve for $Y(s)$

$$1 + s^2Y + 3sY + 2Y = \frac{e^{-2s}}{s} ,$$

$$Y(s) = -\frac{1}{s^2 + 3s + 2} + e^{-2s}\frac{1}{s(s^2 + 3s + 2)} .$$

Using partial fractions

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 1)(s + 2)} = \frac{1}{s + 1} - \frac{1}{s + 2} \to e^{-t} - e^{-2t} ,$$

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{s(s + 1)(s + 2)} = \frac{1}{2(s + 2)} - \frac{1}{s + 1} + \frac{1}{2s}$$

$$\to \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2} .$$

By the second shift formula

$$e^{-2s}\frac{1}{s(s^2 + 3s + 2)} \to u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right] .$$

Combining both pieces

$$Y(s) \to e^{-2t} - e^{-t} + u_2(t)\left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right] .$$

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V.13   Apply the Laplace transform to both sides of the equation. The Laplace transform of the right hand side is computed using the second shift formula:

$$\mathcal{L}\left(u_\pi(t)\,\sin t\right) = -\mathcal{L}\left(u_\pi(t)\,\sin(t-\pi)\right) = -e^{-\pi s}\,\frac{1}{s^2+1}\,.$$

Obtain

$$-1 + s + s^2 Y(s) + 4Y(s) = -e^{-\pi s}\,\frac{1}{s^2+1}\,.$$

$$Y(s) = \frac{1}{s^2+4} - \frac{s}{s^2+4} - e^{-\pi s}\,\frac{1}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \rightarrow \frac{1}{3}\sin t - \frac{1}{6}\sin 2t\,.$$

Using the second shift formula and trig identities

$$Y(s) \rightarrow -\cos 2t + \frac{1}{2}\sin 2t - u_\pi(t)\left(\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin 2(t-\pi)\right)$$

$$= -\cos 2t + \frac{1}{2}\sin 2t + u_\pi(t)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right)\,.$$

V.14   Write $g(t) = (1 - u_\pi(t))\,t + \pi u_\pi(t) = t - u_\pi(t)(t-\pi)$, so that $G(s) = \frac{1}{s^2} - \frac{e^{-\pi s}}{s^2}$. Apply the Laplace transform

$$s^2 Y + Y = \frac{1}{s^2} - \frac{e^{-\pi s}}{s^2}\,,$$

$$Y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-\pi s}}{s^2(s^2+1)}\,.$$

By the second shift formula

$$e^{-\pi s}\frac{1}{s^2(s^2+1)} = e^{-\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] \rightarrow u_\pi(t)\,(t-\pi) - u_\pi(t)\sin(t-\pi)$$

$$= u_\pi(t)\,(t-\pi) + u_\pi(t)\sin t\,.$$

Combining the pieces

$$Y(s) \rightarrow t - \sin t - u_\pi(t)\,(t - \pi + \sin t)\,.$$

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VI.3 Write

$$\frac{s+1}{s+3} = \frac{s+3-2}{s+3} = 1 - 2\frac{1}{s+3} \rightarrow \delta(t) - 2e^{-3t}\,.$$

VI.4 One needs to perform the division of two polynomials, to obtain a remainder term with a linear numerator. Alternatively, one can write

$$\frac{s^2+1}{s^2+2s+2} = \frac{s^2+2s+2-(2s+1)}{s^2+2s+2} = 1 - \frac{2s+1}{s^2+2s+2}$$

$$= 1 - \frac{2(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \rightarrow \delta(t) - 2e^{-t}\cos t + e^{-t}\sin t\,.$$

VI.5 Apply the Laplace transform

$$-2 + s^2 Y + Y = e^{-\pi s}\,,$$

$$Y(s) = \frac{2}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1} \rightarrow 2\sin t + u_\pi(t)\sin(t-\pi) = 2\sin t - u_\pi(t)\sin t\,.$$

VI.6 Apply the Laplace transform

$$s^2 Y + 2sY + 10Y = e^{-\pi s}\,,$$

$$Y(s) = e^{-\pi s}\frac{1}{s^2+2s+10} = e^{-\pi s}\frac{1}{(s+1)^2+9}$$

$$\rightarrow \frac{1}{3}u_\pi(t)e^{-(t-\pi)}\sin 3(t-\pi) = -\frac{1}{3}u_\pi(t)e^{-t+\pi}\sin 3t\,.$$

VI.7 Apply the Laplace transform

$$4s^2 Y(s) + Y(s) = 1\,,$$

$$Y(s) = \frac{1}{4s^2+1} = \frac{1}{4}\frac{1}{s^2+\frac{1}{4}} \rightarrow \frac{1}{2}\sin\frac{1}{2}t\,.$$

VI.8 Apply the Laplace transform

$$-4 + 4s^2 Y + 4sY + 5Y = e^{-2\pi s}\,,$$

$$Y(s) = \frac{4}{4s^2 + 4s + 5} + e^{-2\pi s} \frac{1}{4s^2 + 4s + 5}.$$

Invert

$$\frac{1}{4s^2 + 4s + 5} = \frac{1}{(2s+1)^2 + 4} = \frac{1}{4} \frac{1}{(s + \frac{1}{2})^2 + 1} \rightarrow \frac{1}{4} e^{-\frac{1}{2}t} \sin t.$$

By the second shift formula

$$e^{-2\pi s} \frac{1}{4s^2 + 4s + 5} \rightarrow \frac{1}{4} u_{2\pi}(t) \, e^{-\frac{1}{2}(t-2\pi)} \sin(t - 2\pi) = \frac{1}{4} u_{2\pi}(t) \, e^{-\frac{1}{2}(t-2\pi)} \sin t.$$

Combining the pieces

$$Y(s) \rightarrow e^{-\frac{1}{2}t} \sin t + \frac{1}{4} u_{2\pi}(t) \, e^{-\frac{1}{2}(t-2\pi)} \sin t.$$

VI.9   By a property of the delta function, for any $t_0 > 0$

$$\mathcal{L}\left(\delta(t - t_0)f(t)\right) = \int_0^\infty e^{-st} \delta(t - t_0)f(t)\, dt = e^{-st_0} f(t_0).$$

VI.10   Apply the Laplace transform, and use the formula derived in the preceding problem

$$s^2 Y + 4Y = e^{-\frac{\pi}{3}s} \cos\frac{\pi}{3} = \frac{1}{2} e^{-\frac{\pi}{3}s},$$

$$Y(s) = \frac{1}{2} e^{-\frac{\pi}{3}s} \frac{1}{s^2 + 4} \rightarrow \frac{1}{4} u_{\pi/3}(t) \sin 2(t - \frac{\pi}{3}).$$

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VII.1 By the definition of the convolution

$$\sin t * 1 = 1 * \sin t = \int_0^t \sin v\, dv = 1 - \cos t.$$

VII.2   By the properties of the delta function

$$f(t) * \delta(t) = \int_0^t \delta(t - v)f(v)\, dv = \int_{-\infty}^\infty \delta(v - t)f(v)\, dv = f(t).$$

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VII.3   By the definition of the convolution

$$t * t = \int_0^t (t - v)v \, dv = t \int_0^t v \, dv - \int_0^t v^2 \, dv = \frac{t^3}{6} \, .$$

Alternatively, $\mathcal{L}(t * t) = \mathcal{L}(t) \cdot \mathcal{L}(t) = \frac{1}{s^2} \cdot \frac{1}{s^2} = \frac{1}{s^4}$. Then

$$t * t = \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{6} \, .$$

VII.5   By the definition of the convolution

$$\cos t * \cos t = \int_0^t \cos(t - v) \cos v \, dv = \int_0^t \left[\cos t \cos v + \sin t \sin v\right] \cos v \, dv$$

$$= \cos t \int_0^t \cos^2 v \, dv + \sin t \int_0^t \sin v \cos v \, dv = \frac{1}{2} t \cos t + \frac{1}{2} \sin t \, .$$

The first integral was evaluated using the trig identity $\cos^2 v = \frac{1}{2} + \frac{1}{2} \cos 2v.$

VII.6(a) $\frac{1}{s^3(s^2 + 1)} \to \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) = \frac{t^2}{2} * \sin t$

$$= \frac{1}{2} \int_0^t (t - v)^2 \sin v \, dv = \frac{1}{2} t^2 \int_0^t \sin v \, dv - t \int_0^t v \sin v \, dv + \frac{1}{2} \int_0^t v^2 \sin v \, dv$$

$$\to \frac{t^2}{2} + \cos t - 1 \, .$$

VII.6(d) $\frac{1}{(s^2 + 9)^2} \to \mathcal{L}^{-1}\left(\frac{1}{s^2 + 9}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2 + 9}\right) = \frac{1}{3} \sin 3t * \frac{1}{3} \sin 3t =$

$\frac{1}{9} \sin 3t * \sin 3t \to \frac{1}{54} \sin 3t - \frac{3}{54} t \cos 3t.$

VII.7   Apply the Laplace transform

$$s^2 Y + 9Y = \frac{s}{s^2 + 9} \, ,$$

$$Y(s) = \frac{s}{(s^2 + 9)^2} = \frac{s}{s^2 + 9} \cdot \frac{1}{s^2 + 9} \to \frac{1}{3} \cos 3t * \sin 3t = \frac{1}{6} t \sin 3t \, .$$

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VII.8   Apply the Laplace transform

$$s^2 Y + 4Y = G(s) \,,$$

$$Y(s) = \frac{G(s)}{s^2 + 4} \to \frac{1}{2} \sin 2t * g(t) = \frac{1}{2} \int_0^t \sin 2(t - v) g(v) \, dv \,.$$

VII.9   Obtain $\dfrac{1}{s^2} F(s) = \dfrac{1}{s^4(s^2 + 1)}$,

$$F(s) = \frac{1}{s^2(s^2 + 1)} = \frac{1}{s^2} - \frac{1}{s^2 + 1} \to t - \sin t \,.$$

VII.10 $\mathcal{L}\left(\int_0^t e^{-(t-v)} \cos v \, dv \right) = \mathcal{L}\left(e^{-t} * \cos t \right)$

$$= \mathcal{L}\left(e^{-t} \right) \mathcal{L}\left(\cos t \right) = \frac{s}{(s + 1)(s^2 + 1)} \,.$$

VII.11   Write this integral equation as

$$y(t) + t * y(t) = \cos 2t \,,$$

and apply the Laplace transform

$$Y + \frac{1}{s^2} Y = \frac{s}{s^2 + 4} \,,$$

$$Y(s) = \frac{s^3}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3} \left[-\frac{s}{s^2 + 1} + \frac{4s}{s^2 + 4} \right] \to -\frac{1}{3} \cos t + \frac{4}{3} \cos 2t \,.$$

VII.12 $\mathcal{L}\left(t * t * t \right) = \dfrac{1}{s^2} \cdot \dfrac{1}{s^2} \cdot \dfrac{1}{s^2} = \dfrac{1}{s^6}$. It follows that

$$t * t * t = \mathcal{L}^{-1}\left(\frac{1}{s^6} \right) = \frac{t^5}{5!} \,.$$

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VIII.1   In the sense of distributions $(|t|)' = 2H(t) - 1$, and then $(|t|)'' = 2\delta(t)$.

VIII.2   Differentiating $n - 1$ times separately for $t < 0$ and for $t \geq 0$, one shows that $f^{(n-1)}(t) = H(t)$. One more differentiation gives $f^{(n)}(t) = \delta(t)$.

Chapter 5

Section 5.3.1, Page 189

I.1 The eigenvalues are $\lambda = -3$, with the corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and $\lambda = 0$, with the corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The general solution

is $x(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

I.2 The eigenvalues are $\lambda = \pm i$. An eigenvector corresponding to $\lambda = i$ is
$\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$, leading to a complex valued solution

$$e^{it} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \begin{bmatrix} \cos t - \sin t \\ 2 \cos t \end{bmatrix} + i \begin{bmatrix} \cos t + \sin t \\ 2 \sin t \end{bmatrix}.$$

Since both the real and the imaginary parts of the complex valued solution
are also solutions, the general solution of our system is

$$x(t) = c_1 \begin{bmatrix} \cos t - \sin t \\ 2 \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + \sin t \\ 2 \sin t \end{bmatrix}.$$

I.3 To solve the cubic characteristic equation

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

one guesses a root $\lambda = 1$. The other two roots are $\lambda = 1$ and $\lambda = 4$. So
that the root $\lambda = 1$ is double. A better way is to factor the characteristic
polynomial while calculating it. Indeed, expanding the determinant of the
characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

in the first row gives

$$(2-\lambda)\left[(2-\lambda)^2 - 1\right] - [2-\lambda - 1] + [1 - 2 + \lambda] = 0.$$

Factor the quadratic in the first square bracket as $(\lambda - 1)(\lambda - 3)$. Then the characteristic equation becomes

$$(2 - \lambda)(\lambda - 1)(\lambda - 3) + 2(\lambda - 1) = 0,$$

$$(\lambda - 1)\left[(2 - \lambda)(\lambda - 3) + 2\right] = 0,$$

$$(\lambda - 1)\left[-\lambda^2 + 5\lambda - 4\right] = 0,$$

so the roots are 1, 1, 4.

The root $\lambda = 1$ has two linearly independent eigenvectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The eigenvector corresponding to $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

I.4   The matrix of this system has a double eigenvalue $\lambda_1 = \lambda_2 = 3$, and only one linearly independent eigenvector $\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have one solution: $x_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system $(A - \lambda_1 I)\eta = \xi$ to determine the generalized eigenvector $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ takes the form

$$\eta_2 = 0.$$

(The second equation is $0 = 0$.) Set $\eta_1 = 1$, to obtain a generalized eigenvector $\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The general solution is then

$$x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

In scalar form this system takes the form

$$x_1' = 3x_1 + x_2$$

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$$x_2' = 3x_2\,.$$

Solve the second equation, $x_2 = c_1 e^{3t}$, then use this x_2 in the first one to get

$$x_1' = 3x_1 + c_1 e^{3t}\,,$$

which is a linear first order equation, easily solved.

I.5 The matrix of this system has a simple eigenvalue $\lambda_1 = -1$, with an eigenvector $\xi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and a double eigenvalue $\lambda_2 = \lambda_3 = 2$, with only one linearly independent eigenvector $\xi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The system $(A - \lambda_1 I)\eta = \xi$ to determine the generalized eigenvector $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$ takes the form

$$-3\eta_1 = 0$$

$$\eta_3 = 0\,.$$

(The third equation is $0 = 0$.) Set $\eta_2 = 1$, to obtain a generalized eigenvector $\eta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \left(t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)\,.$$

I.7 The eigenvalues are $\lambda = -1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\,.$$

The initial condition implies that

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\,,$$

which gives $c_1 = c_2 = \dfrac{1}{2}$.

In scalar form this system takes the form

$$x_1' = x_2$$
$$x_2' = x_1 \,.$$

Differentiating the first equation, and using the second one obtain

$$x_1'' - x_1 = 0 \,, \quad x_1(0) = 2 \,, \quad x_1'(0) = 1 \,.$$

It follows that $x_1 = x_1(t) = \dfrac{1}{2}e^{-t} + \dfrac{3}{2}e^{t}$, and then $x_2(t) = x_1'(t) = -\dfrac{1}{2}e^{-t} + \dfrac{3}{2}e^{t}$, which is a scalar form of the equation above.

I.8 Expanding the determinant of the characteristic equation

$$\begin{vmatrix} -2 - \lambda & 2 & 3 \\ -2 & 3 - \lambda & 2 \\ -4 & 2 & 5 - \lambda \end{vmatrix} = 0$$

in the first row gives

$$(-2 - \lambda)\left[(3 - \lambda)(5 - \lambda) - 4\right] - 2\left[-2(5 - \lambda) + 8\right] + 3\left[-4 + 4(3 - \lambda)\right] = 0 \,.$$

If one factors the quadratic in the first square bracket as $(\lambda - 2)(\lambda - 6)$, then the characteristic equation becomes

$$(-2 - \lambda)(\lambda - 2)(\lambda - 6) + 28 - 16\lambda = 0 \,.$$

Unlike the previous problem I.3, there is no common linear factor. Therefore we completely expand the characteristic equation

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \,,$$

and factor it, observing that $\lambda = 1$ is a root:

$$-(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \,.$$

The eigenvalues are $\lambda = 1$ with an eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\lambda = 2$ with an eigenvector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\lambda = 3$ with an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The general solution is

$$x(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \,.$$

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The initial condition implies that

$$
c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.
$$

One calculates $c_1 = -3$, $c_2 = -2$, $c_3 = 5$, so that

$$
x(t) = -3e^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2e^{2t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 5e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
$$

I.9   The eigenvalues are $\lambda = 1$ with an eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\lambda = 2$ with

an eigenvector $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\lambda = 3$ with an eigenvector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. The general

solution is

$$
x(t) = c_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.
$$

The initial condition implies that

$$
c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}.
$$

One calculates $c_1 = 5$, $c_2 = -2$, $c_3 = 4$, so that

$$
x(t) = 5e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2e^{2t} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + 4e^{3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.
$$

I.10   The general solution was found in the problem I.3 above:

$$
x(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.
$$

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The initial condition requires that

$$
c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}.
$$

One calculates $c_1 = 1$, $c_2 = 1$, and $c_3 = -2$.

I.11   The eigenvalues are $\lambda = 5$, with an eigenvector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and a repeated

eigenvalue $\lambda = -3$, which has only one linearly independent eigenvector

$\xi = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The generalized eigenvector η must satisfy $(A + 3I)\eta = \xi$, or

$$
4\eta_1 + 4\eta_2 = -1
$$

$$
-4\eta_1 - 4\eta_2 = 1
$$

$$
8\eta_3 = 0 \,.
$$

Conclude: $\eta_3 = 0$. After discarding the second equation, we may take $\eta_2 = 0$, and $\eta_1 = -\dfrac{1}{4}$. The general solution is

$$
x(t) = c_1 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \left(t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix} \right).
$$

The initial condition requires that

$$
c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}.
$$

Obtain $c_1 = 1$, $c_2 = 6$, $c_3 = -16$.

I.12   The eigenvalues are $\lambda = \pm 2i$. An eigenvector corresponding to $\lambda = 2i$

is $\begin{bmatrix} i \\ 1 \end{bmatrix}$, leading to a complex valued solution

$$
e^{2it} \begin{bmatrix} i \\ 1 \end{bmatrix} = (\cos 2t + i \sin 2t) \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + i \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.
$$

Since both the real and the imaginary parts of the complex valued solution are also solutions, the general solution of our system is

$$x(t) = c_1 \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} .$$

The initial conditions imply that $c_1 = 1$ and $c_2 = -2$. We conclude that

$$x_1(t) = -2\cos 2t - \sin 2t$$
$$x_2(t) = \cos 2t - 2\sin 2t .$$

Alternatively, our system

$$x_1' = -2x_2$$
$$x_2' = 2x_1$$

can be solved by differentiating the first equation, and then using the second equation

$$x_1'' = -2x_2' = -4x_1 ,$$

leading to the equation

$$x_1'' + 4x_1 = 0 ,$$

with the initial conditions

$$x_1(0) = -2, \, x_1'(0) = -2x_2(0) = -2 .$$

Then $x_1(t) = -2\cos 2t - \sin 2t$, and $x_2(t) = -\dfrac{1}{2}x_1' = \cos 2t - 2\sin 2t$.

I.13   The eigenvalues are $\lambda = 3 \pm 2i$. An eigenvector corresponding to $\lambda = 3 + 2i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$, leading to a complex valued solution

$$e^{(3+2i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{3t} (\cos 2t + i\sin 2t) \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{3t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + ie^{3t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} .$$

Since both the real and the imaginary parts of the complex valued solution are also solutions, the general solution of our system is

$$x(t) = c_1 e^{3t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} .$$

The initial conditions imply that $c_1 = 1$ and $c_2 = 0$.

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I.14   The eigenvalues are $\lambda = 1$, with an eigenvector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and a pair of complex eigenvalues $\pm i$. An eigenvector corresponding to $\lambda = i$ is $\begin{bmatrix} -2 \\ -1-i \\ 2 \end{bmatrix}$, leading to a complex valued solution

$$
e^{it} \begin{bmatrix} -2 \\ -1-i \\ 2 \end{bmatrix} = (\cos t + i\sin t) \begin{bmatrix} -2 \\ -1-i \\ 2 \end{bmatrix}
$$

$$
= \begin{bmatrix} -2\cos t \\ -\cos t + \sin t \\ 2\cos t \end{bmatrix} + i \begin{bmatrix} -2\sin t \\ -\cos t - \sin t \\ 2\sin t \end{bmatrix}.
$$

Taking the real and the imaginary parts of this solution, we obtain two real-valued solutions of our system, and hence its general solution is

$$
x(t) = c_1 e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2\cos t \\ -\cos t + \sin t \\ 2\cos t \end{bmatrix} + c_3 \begin{bmatrix} -2\sin t \\ -\cos t - \sin t \\ 2\sin t \end{bmatrix}.
$$

The initial condition gives

$$
c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.
$$

Calculate: $c_2 = \dfrac{1}{2}$, $c_1 = 1$, and $c_3 = -\dfrac{5}{2}$. The solution is

$$
x(t) = e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2\cos t \\ -\cos t + \sin t \\ 2\cos t \end{bmatrix} - \frac{5}{2} \begin{bmatrix} -2\sin t \\ -\cos t - \sin t \\ 2\sin t \end{bmatrix},
$$

or in scalar form

$$
x_1(t) = -\cos t + 5\sin t
$$
$$
x_2(t) = -e^t + 2\cos t + 3\sin t
$$
$$
x_3(t) = e^t + \cos t - 5\sin t.
$$

I.15   The eigenvalues of this skew-symmetric matrix are $\lambda = 0$, with an

eigenvector $\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$, and a pair of purely imaginary eigenvalues $\pm 5i$. An

eigenvector corresponding to $\lambda = 5i$ is $\begin{bmatrix} 3 \\ -5i \\ 4 \end{bmatrix}$, leading to a complex valued

solution

$$e^{5it} \begin{bmatrix} 3 \\ -5i \\ 4 \end{bmatrix} = (\cos 5t + i\sin 5t) \begin{bmatrix} 3 \\ -5i \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos 5t \\ 5\sin 5t \\ 4\cos 5t \end{bmatrix} + i \begin{bmatrix} 3\sin 5t \\ -5\cos 5t \\ 4\sin 5t \end{bmatrix}.$$

Taking the real and the imaginary parts of this solution, we obtain two
real-valued solutions of our system, and hence its general solution is

$$x(t) = c_1 \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3\cos 5t \\ 5\sin 5t \\ 4\cos 5t \end{bmatrix} + c_3 \begin{bmatrix} 3\sin 5t \\ -5\cos 5t \\ 4\sin 5t \end{bmatrix}.$$

The initial condition gives

$$c_1 \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Calculate: $c_1 = \dfrac{1}{5}$, $c_2 = \dfrac{3}{5}$, and $c_3 = 0$.

I.18   Search for a particular solution in the form $x_1 = Ae^{2t}$, $x_2 = Be^{2t}$.
Substitution into the system gives $A = -3$, $B = 7$. The matrix $\begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}$
of the corresponding homogeneous system has eigenvalues $\lambda = 1$, with a cor-
responding eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\lambda = 3$, with a corresponding eigenvector
$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The general solution is

$$x(t) = e^{2t} \begin{bmatrix} -3 \\ 7 \end{bmatrix} + c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

I.19   In components the system becomes

$$x_1' = x_1 - x_2 + 1$$
$$x_2' = x_1 + x_2 + t\,.$$

Look for a particular solution in the form $x_1(t) = At + B$, $x_2(t) = Ct + D$. Substitution into the above system gives

$$A = (A - C)t + B - D + 1$$
$$C = (A + C)t + B + D + t\,.$$

Equate the t terms and constant terms in each of these two equations, to obtain the following linear system of four equations:

$$0 = A - C$$
$$A = B - D + 1$$
$$0 = A + C + 1$$
$$C = B + D\,.$$

From the first and the third equations $A = C = -\frac{1}{2}$. Then from the second and fourth equations, $B = -1$ and $D = \frac{1}{2}$. The particular solution is

$$y_1(t) = -\frac{1}{2}t - 1\,, \quad y_2(t) = -\frac{1}{2}t + \frac{1}{2}\,.$$

The general solution of the corresponding homogeneous system is

$$x_1 = c_1 e^t \cos t - c_2 e^t \sin t$$
$$x_2 = c_1 e^t \sin t + c_2 e^t \cos t\,.$$

The general solution of the nonhomogeneous system is

$$x_1 = -\tfrac{1}{2}t - 1 + c_1 e^t \cos t - c_2 e^t \sin t$$
$$x_2 = -\tfrac{1}{2}t + \tfrac{1}{2} + c_1 e^t \sin t + c_2 e^t \cos t\,.$$

The constants $c_1 = 5$ and $c_2 = \frac{1}{2}$ are found using the initial conditions.

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II.1   If λ_1 and λ_2 are the eigenvalues of this matrix, then

$$\lambda_1 + \lambda_2 = -2a < 0\,,$$

$$\lambda_1 \lambda_2 = a^2 + b^2 > 0 \,.$$

If the eigenvalues are real, they are both negative. Denoting by ξ_1 and ξ_2 the corresponding eigenvectors, one concludes that the general solution $x(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2 \to 0$, as $t \to \infty$. If the eigenvalues form a complex conjugate pair $\lambda = p \pm iq$, then $p = -a < 0$. Let $\xi + i\eta$ be the eigenvector corresponding to $p + iq$, where $u(t)$ and $v(t)$ are real valued vector functions. Then

$$x(t) = e^{pt}(\cos qt + i \sin qt)(\xi + i\eta)$$
$$= e^{pt}(\cos qt \ \xi - \sin qt \ \eta) + i e^{pt}(\sin qt \ \xi + \cos qt \ \eta)$$

is a complex valued solution, and the general solution

$$x(t) = c_1 e^{pt} (\cos qt \ \xi - \sin qt \ \eta) + c_2 e^{pt}(\sin qt \ \xi + \cos qt \ \eta) \to 0 \,,$$

as $t \to \infty$.

II.2   One obtains the system

$$x_1' = x_2$$

$$x_2' = -cx_1 - bx_2 \,.$$

Its matrix $\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$ has a negative trace and a positive determinant. Similarly to the preceding problem, all solutions tend to zero, as $t \to \infty$.

II.4   (i) The characteristic polynomial of a 3×3 matrix is a cubic, and hence one of its roots is real. That root λ must be zero, in order for $e^{\lambda t}$ to remain bounded, as $t \to \pm\infty$. The root $\lambda = 0$ must be simple, otherwise the solution contains an unbounded factor of t. The other two roots must be purely imaginary $\lambda = \pm i\,q$, for the corresponding solutions to remain bounded as $t \to \pm\infty$. Then the general solution has the form

$$x(t) = c_1 \xi_1 + c_2 \cos qt \ \xi_2 + c_3 \sin qt \ \xi_3 \,,$$

where ξ_1, ξ_2 and ξ_3 are constant, real valued three dimensional vectors.

(ii) Observe that $a_{ji} = -a_{ij}$, and then $a_{ii} = 0$ for any skew-symmetric matrix. Then any 3×3 skew-symmetric matrix is of the form $\begin{bmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{bmatrix}$,

with some real p, q and r. Compute the eigenvalues $\lambda = 0$, $\lambda = \pm i \sqrt{p^2 + q^2 + r^2}$, and use part (i).

II.7   The sum of the eigenvalues is equal to the trace of the matrix, so that

$$\lambda_1 + \lambda_2 = -\frac{1}{2},$$

and the product of the eigenvalues is equal to the determinant of the matrix, giving

$$\lambda_1 \lambda_2 = \frac{1}{2}.$$

From these relations one determines that the eigenvalues are $-\dfrac{1}{4} \pm \dfrac{\sqrt{7}}{4} i$.

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III.1   Calculate $A^2 = O$, the zero matrix, and then $A^k = O$, for all $k \geq 2$. It follows that

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

III.2   Calculate $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = O$, and then $A^k = O$, for all $k \geq 3$. It follows that

$$e^{At} = I + At + \frac{1}{2} A^2 t^2 = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

III.3   Write $A = -3I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then using the preceding problem

$$e^{At} = e^{-3t} e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t} = e^{-3t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

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III.4   Using that the matrix is block diagonal, and

$$e^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

as calculated in the text, compute

$$e^{At} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

III.5   Writing $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = 3I + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, calculate

$$e^{\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} t} = \begin{bmatrix} e^{3t} \cos t & -e^{3t} \sin t \\ e^{3t} \sin t & e^{3t} \cos t \end{bmatrix}.$$

Then use that the matrix A is block diagonal to calculate

$$e^{At} = \begin{bmatrix} e^{3t} \cos t & -e^{3t} \sin t & 0 \\ e^{3t} \sin t & e^{3t} \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

III.6   Begin with

$$e^{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} t} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix},$$

and use that the matrix A is block diagonal to calculate

$$e^{At} = \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}.$$

III.7   Calculate $A^2 = I$, $A^3 = A$, $A^4 = I$, and so on. It follows that

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & t^3 \\ t^3 & 0 \end{bmatrix}$$

$$+\frac{1}{4!}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \frac{1}{5!}\begin{bmatrix} 0 & t^5 \\ t^5 & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots & t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots \\ t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots & 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots \end{bmatrix} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

III.8 Write $J = -2I + S$, where $S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Calculate $S^2 =$

$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $S^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $S^4 = O$, and $S^k = O$ for all $k \geq 4$.

It follows that

$$e^{Jt} = e^{-2t}e^{St} = e^{-2t}\left(I + St + \frac{1}{2!}S^2t^2 + \frac{1}{3!}S^3t^3\right) = e^{-2t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

III.12 For any numbers x and y we have $e^x e^y = e^{x+y}$, so that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}.$$

Calculating the product of the series on the left involves multiplying and adding of numbers. Similarly, $(x + y)^k$ are calculated by multiplying and adding of numbers. Since $AB = BA$, the properties of multiplication and addition for these matrices are the same as for numbers. It follows that

$$e^A e^B = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{m=0}^{\infty} \frac{B^m}{m!} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = e^{A+B}.$$

III.13 Because A commutes with $-A$, it follows that

$$e^A e^{-A} = e^{A-A} = e^O = I,$$

which proves that $\left(e^A\right)^{-1} = e^{-A}$.

III.14   Because A commutes with any multiple of itself, it follows that

$$\left(e^A\right)^2 = e^A e^A = e^{2A},$$

$$\left(e^A\right)^3 = \left(e^A\right)^2 e^A = e^{2A} e^A = e^{3A},$$

and, in general,

$$\left(e^A\right)^m = e^{mA}.$$

III.16 Using that $(A^n)^T = \left(A^T\right)^n$, obtain

$$\left(e^A\right)^T = \sum_{n=0}^{\infty} \frac{(A^n)^T}{n!} = \sum_{n=0}^{\infty} \frac{\left(A^T\right)^n}{n!} = e^{A^T}.$$

Section 5.5.3, Page 209

I.1   Let ξ and η be two linearly independent eigenvectors, corresponding to the repeated eigenvalue λ_1 of the 2×2 matrix A. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} \xi + c_2 e^{\lambda_1 t} \eta = e^{\lambda_1 t} d,$$

denoting $d = c_1 \xi + c_2 \eta$. If d_1 and d_2 are the components of the vector d, then $x_1(t) = e^{\lambda_1 t} d_1$ and $x_2(t) = e^{\lambda_1 t} d_2$, which gives $x_2 = \dfrac{d_2}{d_1} x_1$ the straight lines in the $x_1 x_2$ plane. (If $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ and $\lambda_1 < 0$, then the solution moves from the initial point (x_1^0, x_2^0) on a straight line toward the origin.)

I.2   The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} \xi + c_2 \left(t e^{\lambda_1 t} \xi + e^{\lambda_1 t} \eta \right),$$

where η is the generalized eigenvector. If $\lambda_1 < 0$, then $x(t) \to 0$ as $t \to \infty$, giving a stable node. What makes this node *degenerate* is the absence of another direction along which the approach to the origin happens at a different exponential rate.

I.3   (i) Divide the second equation by the first one, to obtain

$$\frac{dy}{dx} = \frac{mx + ny}{ax + by},$$

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or in differentials

(∗) $(mx + ny)\, dx - (ax + by)\, dy = 0.$

(ii) If the origin $(0, 0)$ is a center, the eigenvalues of the matrix $\begin{bmatrix} a & b \\ m & n \end{bmatrix}$
are purely imaginary, of the form $\lambda = \pm p\, i$, so that $n = -a$ (and also
$-a^2 - bm = p^2 > 0$, which implies that b and m have the opposite signs).
Then the equation (∗) is exact. (Here $P(x, y) = mx + ny$, $Q(x, y) = -ax - by$,
$P_y = n = -a = Q_x$.) Integration of (∗) gives $mx^2 + nxy - by^2 = c$, which is
a family of closed curves around $(0, 0)$.

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II.1   The matrix $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ has the eigenvalues $\lambda = 1$ and $\lambda = 5$, both
positive. One has an unstable node at $(0, 0)$.

II.2   The matrix $\begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}$ has the eigenvalues $\lambda = -3$ and $\lambda = 2$. The
origin is a saddle.

II.3   The matrix $\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = \pm 2i$. The origin is a
center.

II.4   The matrix of this system is $\begin{bmatrix} 2 & 4 \\ -5 & -7 \end{bmatrix}$. It has the eigenvalues
$\lambda = -3$ and $\lambda = -2$, both negative. One has a stable node at $(0, 0)$.

II.5   The matrix of this system is $\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$. It has the eigenvalues $\lambda = -1 \pm 2i$, with negative real parts . One has a stable spiral at $(0, 0)$.

II.6   Write this equation in the equivalent system form

$$\frac{dx}{dt} = 4x + y$$
$$\frac{dy}{dt} = x - y\,.$$

The matrix of this system is $\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$. It has the eigenvalues $\lambda_1 = \frac{1}{2}\left(3 + \sqrt{29}\right) > 0$ and $\lambda_2 = \frac{1}{2}\left(3 - \sqrt{29}\right) < 0$. The origin $(0, 0)$ is a saddle. (**The answer in the book is incorrect.**)

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II.7   The eigenvalues $\pm 3i$ of this matrix are purely imaginary. The origin is a center.

II.8   Write this equation in the equivalent system form

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x \,.$$

The matrix of this system is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It has the eigenvalues $\lambda = \pm 1$. One has a saddle at the origin, which can be confirmed by solving this equation: $y^2 - x^2 = c$.

II.9   Each equation can be solved separately, giving $x_1(t) = c_1 e^{-3t}$ and $x_2(t) = c_2 e^{-3t}$, or $x_2 = \dfrac{c_2}{c_1} x_1$. Each solution trajectory is a straight line segment joining the initial point (c_1, c_2) to the origin. Origin is a stable degenerate node. It is approached by straight lines from all directions.

II.10   Solving the equation gives $y = cx$, so that we a degenerate node at the origin, similarly to the preceeding problem. However, this time we cannot tell the direction on the solution curves, and cannot distinguish between stability and instability.

II.11   The eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ \alpha & 1 \end{bmatrix}$ are $\lambda = 1 \pm \sqrt{\alpha}$. For $\alpha > 1$, one eigenvalue is positive and the other one is negative so that the origin is a saddle. In case $0 < \alpha < 1$, both eigenvalues are positive, resulting in unstable node. Similarly, we have an unstable degenerate node when $\alpha = 0$, and an unstable spiral if $\alpha < 0$.

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Finding *all solutions* of *nonlinear systems* is often challenging. The problems in this section review some basic approaches.

III.1   The rest points are solutions of the system

$$2x + y^2 - 1 = 0$$
$$6x - y^2 + 1 = 0 \,.$$

Adding these equations gives
$$8x = 0 \,,$$

or $x = 0$. Then from the first (or second) equation $y^2 - 1 = 0$, $y = \pm 1$. There are *exactly* two rest points: $(0, 1)$ and $(0, -1)$. Denote $f(x, y) = 2x + y^2 - 1$ and $g(x, y) = 6x - y^2 + 1$.

(i) The linearized system at $(0, 1)$

$$u' = f_x(0, 1)u + f_y(0, 1)v$$
$$v' = g_x(0, 1)u + g_y(0, 1)v$$

takes the form

$$u' = 2u + 2v$$
$$v' = 6u - 2v.$$

Its matrix $\begin{bmatrix} 2 & 2 \\ 6 & -2 \end{bmatrix}$ has the eigenvalues $\lambda = \pm 4$. It follows that the linearized system has saddle at the origin, and then by the Hartman-Grobman Theorem the original system has saddle at $(0, 1)$.

(ii) The linearized system at $(0, -1)$

$$u' = f_x(0, -1)u + f_y(0, -1)v$$
$$v' = g_x(0, -1)u + g_y(0, -1)v$$

takes the form

$$u' = 2u - 2v$$
$$v' = 6u + 2v.$$

Its matrix $\begin{bmatrix} 2 & -2 \\ 6 & 2 \end{bmatrix}$ has the eigenvalues $\lambda = 2 \pm 2\sqrt{3}i$. It follows that the linearized system has an unstable spiral at the origin, and then by the Hartman-Grobman Theorem the original system has an unstable spiral at $(0, -1)$.

III.3   The rest points are solutions of the system

$$y - x = 0$$
$$(x - 2)(y + 1) = 0.$$

From the first equation $y = x$, and then from the second equation $x = -1$ or $x = 2$. The rest points are $(-1, -1)$ and $(2, 2)$. Denote $f(x, y) = y - x$ and $g(x, y) = (x - 2)(y + 1)$.

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(i) The linearized system at $(-1, -1)$

$$u' = f_x(-1, -1)u + f_y(-1, -1)v$$
$$v' = g_x(-1, -1)u + g_y(-1, -1)v$$

takes the form

$$u' = -u + v$$
$$v' = -3v.$$

Its matrix $\begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$ has the eigenvalues $\lambda = -3$ and $\lambda = -1$, both negative. It follows that the linearized system has a stable node at the origin, and then by the Hartman-Grobman Theorem the original system has a stable node at $(-1, -1)$.

(ii) The linearized system at $(2, 2)$

$$u' = f_x(2, 2)u + f_y(2, 2)v$$
$$v' = g_x(2, 2)u + g_y(2, 2)v$$

takes the form

$$u' = -u + v$$
$$v' = 3u.$$

Its matrix $\begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = -\dfrac{1}{2}\left(\sqrt{13} + 1\right) < 0$ and $\lambda = \dfrac{1}{2}\left(\sqrt{13} - 1\right) > 0$. It follows that the linearized system has a saddle at the origin, and then by the Hartman-Grobman Theorem the original system has a saddle at $(2, 2)$.

III.4   The rest points are solutions of the system

$$-3y + x\left(x^2 + y^2\right) = 0$$
$$3x + y\left(x^2 + y^2\right) = 0.$$

If $x = 0$ it follows (from either of the equations) that $y = 0$, and if $y = 0$ then $y = 0$. We has a rest point $(0, 0)$, and in the search for other possible rest points, we may assume that $x \neq 0$ and $y \neq 0$. Then from the first equation

$x^2 + y^2 = 3\dfrac{y}{x}$, and from the second equation $x^2 + y^2 = -3\dfrac{x}{y}$. Setting these expressions equal gives

$$-3\frac{x}{y} = 3\frac{y}{x},$$

$$x^2 + y^2 = 0,$$

resulting in $x = y = 0$. We conclude that $(0,0)$ is the only rest point. The linearized system at $(0,0)$

$$u' = -3v$$
$$v' = 3u$$

has a center at the origin. The Hartman-Grobman Theorem *does not apply*. However, multiply the first of the original equations by x, the second one by y, and add the results to get

$$xx' + yy' = \left(x^2 + y^2\right)^2 .$$

Denoting $\rho(t) = x^2 + y^2$, this implies

$$\frac{1}{2}\rho' = \rho^2 .$$

Integrating, $\rho(t) = \dfrac{\rho(0)}{1 - 2\rho(0)t}$, so that solutions move away from the origin, making it unstable (actually, the solutions tend to infinity in finite time).

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IV.2   Recall that $x(t) = e^{At}c$ gives the general solution of

$$x' = Ax .$$

Choosing the first entry of the vector c to be one, and all other entries zero, conclude that the first column of e^{At} is a solution of this system. Similarly, all other columns are solutions of this system. The columns of e^{At} are linearly independent, because the determinant of this matrix $|e^{At}| = e^{\mathrm{tr}\,(At)} > 0$. It follows that e^{At} is a fundamental solution matrix. It is normalized, because $e^{A0} = e^{O} = I$.

Chapter 6

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Section 6.6.1, Page 249

I.1   (i) Considering the sign of the cubic $x(x+1)(x-2)$ in the equation

$$x'(t) = x(x+1)(x-2),$$

one sees that $x'(t) < 0$ and $x(t)$ is decreasing when $x(t) \in (-\infty, -1)$, $x'(t) > 0$ and $x(t)$ is increasing when $x(t) \in (-1, 0)$, then again $x'(t) < 0$ and $x(t)$ is decreasing when $x(t) \in (0, 2)$, and finally $x'(t) > 0$ and $x(t)$ is increasing when $x(t) > 2$. The rest points are $x = -1$ (unstable), $x = 0$ (stable), and $x = 2$ (unstable).

(ii) If $y(0) = 3$, then $y'(t) > 0$, the solution is increasing and tending to infinity as t increases (actually it tends to infinity, or blows up, in finite time). Backward in time t, or as $t \to -\infty$, this solution tends to the unstable rest point $y = 2$.

(iii) The domain of attraction of the rest point $x = 0$ is $(-1, 2)$, delineated by the two rest points that are closest to $x = 0$, namely $x = -1$ and $x = 2$.

I.3   Considering the sign of the cubic $x^2(2 - x)$ in the equation

$$x'(t) = x^2(2 - x),$$

one sees that $x'(t) \geq 0$ and $x(t)$ is increasing when $x(t) < 2$, while $x'(t) < 0$ and $x(t)$ is decreasing for $x(t) > 2$. It follows that the rest point $x = 2$ is stable, and the rest point $x = 0$ is neither stable nor unstable.

I.4   (i) $x = 0$ is a rest point. For all other values of $x(t)$ the solution is decreasing. It follows that the rest point $x = 0$ is neither stable or unstable.

(ii) If $x(0) > 0$, the solution is decreasing and it satisfies $\lim_{t \to \infty} x(t) = 0$, as can be also seen by solving the equation

$$x' = -x^2,$$

to get $x(t) = \dfrac{x(0)}{1 + x(0)\, t}$.

I.5   (i) Use the Lyapunov function $L(x_1, x_2) = \dfrac{1}{2}x_1^2 + \dfrac{1}{2}x_2^2$. Then expressing x_1' and x_2' from the corresponding equations, obtain

$$\frac{d}{dt}L(x_1(t), x_2(t)) = x_1 x_1' + x_2 x_2' = x_1\left(-2x_1 + x_2 + x_1 x_2\right) + x_2\left(x_1 - 2x_2 + x_1^3\right)$$

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$$= -2x_1^2 - 2x_2^2 + 2x_1x_2 + x_1^2x_2 + x_1^3x_2 \,.$$

Using that $2x_1x_2 \leq x_1^2 + x_2^2$, then switching to the polar coordinates in the x_1x_2 plane, we continue

$$\frac{d}{dt}L(x_1(t), x_2(t)) \leq -x_1^2 - x_2^2 + x_1^2x_2 + x_1^3x_2$$

$$= -r^2 + r^3 \cos^2\theta \sin\theta + r^4 \cos^3\theta \sin\theta < 0 \,,$$

for r small enough.

(ii) The matrix $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ of the corresponding linearized system has the eigenvalues $\lambda = -3$ and $\lambda = -1$, so that the linearized system has a stable node at the origin, and in particular the origin is asymptotically stable.

I.6   Use the Lyapunov function $L(x, y) = \dfrac{1}{5}x^2 + y^2$. Then expressing x' and y' from the corresponding equations, obtain

$$\frac{d}{dt}L(x, y) = \frac{2}{5}x\,x' + 2y\,y' = -\frac{2}{5}x^2\left(x^2 + y^2\right) - y^2\left(x^2 + y^2\right)$$

$$= -\left(\frac{2}{5}x^2 + y^2\right)\left(x^2 + y^2\right) < 0\,,$$

for all $(x, y) \neq (0, 0)$.

I.7   The rest point $y = 1$ is asymptotically stable. Solutions starting with $y(0) \in (0, 1)$ are increasing and tend to $y = 1$ as $t \to \infty$. Solutions starting with $y(0) > 1$ are decreasing and tend to $y = 1$ as $t \to \infty$.

The rest point $y = 0$ is unstable. Solutions starting with $y(0) \in (0, 1)$ are increasing and move away from $y = 0$.

I.8   Use the Lyapunov function $L(x, y) = \dfrac{1}{2}x^4 + y^2$. Then expressing x' and y' from the corresponding equations, obtain

$$\frac{d}{dt}L(x, y) = 2x^3x' + 2yy' = -2x^3y + 2yx^3 = 0\,.$$

It follows that the solutions move on the closed curves

$$\frac{1}{2}x^4 + y^2 = c\,.$$

(Where $c = \dfrac{1}{2}x^4(0) + y^2(0)$.) If the initial point $(x(0), y(0))$ is close to the origin, the solution stays close to the origin for all t, so that the rest point

$(0, 0)$ is stable, but $(0, 0)$ is not asymptotically stable, because solutions do not tend to it.

I.9 (i)   Letting $y = x_1$, and $y' = x_2$ converts our equation into the system

$$x_1' = x_2$$
$$x_2' = -x_1 - f(x_1)x_2,$$

with a rest point $(0, 0)$.

(ii)   Use Lyapunov's function $L(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Then expressing x_1' and x_2' from the corresponding equations, obtain

$$\frac{d}{dt}L(x_1(t), x_2(t)) = -f(x_1)x_2^2 < 0,$$

except when $x_2 = 0$, and possibly when $x_1 = 0$ (in case $f(0) = 0$). However, one sees from the above system that solutions cross the axes only at isolated t's so that the function $L(x_1(t), x_2(t))$ is strictly decreasing. (If $x_2(t_0) = 0$, it follows from the second equation that $x_2'(t_0) \neq 0$.) Since the equation

$$y'' + f(y)y' + y = 0$$

corresponds to the system just considered, we conclude that all solutions of this equation, with $|y(0)|$ and $|y'(0)|$ small, satisfy $\lim_{t \to \infty} y(t) = 0$ and $\lim_{t \to \infty} y'(t) = 0$ (assuming that $f(x_1) > 0$ for all $x_1 \neq 0$).

I.10   To see that $L(x_1, x_2) = \int_0^{x_1} g(s)\, ds + \frac{1}{2}x_2^2$ is a Lyapunov function, observe that $L(0, 0) = 0$, and $L(x_1, x_2) > 0$, for all $(x_1, x_2) \neq (0, 0)$, because the condition $x_1 g(x_1) > 0$ implies that $g(x_1)$ is positive for x_1 positive, and $g(x_1)$ is negative for x_1 negative. Then expressing x_1' and x_2' from the corresponding equations, obtain

$$\frac{d}{dt}L(x_1(t), x_2(t)) = g(x_1)x_1' + x_2 x_2' = g(x_1)x_2 + x_2\left(-g(x_1) - f(x_1)x_2\right)$$

$$= -f(x_1)x_2^2 < 0 \quad (\text{for } x_2 \neq 0).$$

Since the equation

$$y'' + f(y)y' + g(y) = 0$$

corresponds to the system just considered, we conclude that all solutions of this equation, with $|y(0)|$ and $|y'(0)|$ small, satisfy $\lim_{t \to \infty} y(t) = 0$ and

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$\lim\limits_{t\to\infty} y'(t) = 0$ (assuming that the same conditions are imposed on $f(y)$ and $g(y)$).

I.11 (i) The rest points are the solutions of the system

$$-x^3 + 4y\left(z^2 + 1\right) = 0$$
$$-y^5 - x\left(z^2 + 1\right) = 0$$
$$-z - x^4 z^3 = 0\,.$$

Clearly, $(0,0,0)$ is a rest point, and we need to show that there are no others. We claim that for any other rest point $x \neq 0$ and $y \neq 0$. Indeed, if $x = 0$, then from the second equation $y = 0$, and from the third one $z = 0$. Similarly, the assumption that $y = 0$ also lead to the same rest point $(0,0,0)$. From the first equation, $z^2 + 1 = \dfrac{x^3}{4y}$, and from the second equation $z^2 + 1 = -\dfrac{y^5}{x}$. Setting these expressions equal, gives

$$\frac{x^3}{4y} = -\frac{y^5}{x}\,,$$

$$x^4 + 4y^6 = 0\,,$$

so that $x = y = 0$, and there are no other rest points.

To show that $(0,0,0)$ is asymptotically stable, we use a Lyapunov function $L(x,y,z) = \dfrac{1}{2}x^2 + 2y^2 + \dfrac{1}{2}z^2$. Then expressing x', y', and z' from the corresponding equations, obtain

$$\frac{d}{dt}L(x,y) = xx' + 4yy' + zz'$$

$$= x\left[-x^3 + 4y\left(z^2 + 1\right)\right] + 4y\left[-y^5 - x\left(z^2 + 1\right)\right] + z\left[-z - x^4 z^3\right]$$

$$= -x^4 - 4y^6 - z^2 - x^4 z^4 < 0\,,$$

for (x,y,z) close to $(0,0,0)$, as follows by using the spherical coordinates.

(ii) The linearized system at $(0,0,0)$ is

$$x' = 4y$$
$$y' = -x$$
$$z' = -z\,.$$

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The last equation is decoupled (independent) from the other two. Solving it, $z(t) = z(0)e^{-t}$, we see that all solutions tend to the xy-plane. The projection of any solution onto to the xy-plane moves on the solution of the system

$$x' = 4y$$
$$y' = -x\,,$$

which are the ellipses $x^2 + 4y^2 = c$. It follows that any solution of the 3-d linearized system moves on an elliptic cylinder, and it tends to the xy-plane, as $t \to \infty$. We conclude that the rest point $(0, 0, 0)$ is not asymptotically stable. However, the origin is stable: if solution starts near $(0, 0, 0)$, it stays near $(0, 0, 0)$.

I.13. Setting $x = X + 1$ and $y = Y + 1$ obtain

$$X' = -2X + XY$$
$$Y' = -Y + X^2\,.$$

We use a Lyapunov function $L(X, Y) = \dfrac{1}{2}X^2 + Y^2$. Expressing X' and Y' from the corresponding equations, then switching to polar coordinates in the (X, Y) plane, obtain

$$\frac{d}{dt}L(X(t), Y(t)) = XX' + 2YY' = -2X^2 - 2Y^2 + 3X^2Y$$

$$= -2r^2 + 3r^3 \cos^2\theta \sin\theta < 0\,,$$

for r small enough. It follows that for the original system the rest point $(1, 1)$ is asymptotically stable.

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II.1   The null-clines $5 - x - 2y = 0$ and $2 - 3x - y = 0$ do not intersect in the first quadrant of the xy-plane. Above the first null-cline the trajectories move to the left, below it to the right. Above the second null-cline the trajectories move down, and below it up. The null-clines divide the first quarter of the xy-plane into three regions. Drawing the direction of motion in each region, we conclude that the semitrivial solution $(5, 0)$ attracts all other positive solutions.

II.2   The null-clines $2 - x - \dfrac{1}{2}y = 0$ and $3 - x - y = 0$ intersect at the point $(1, 2)$, which is a rest point. Above the first null-cline the trajectories move to the left, below it to the right. Above the second null-cline the trajectories

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move to the south, and below it up (north). The null-clines divide the first quarter of the xy-plane into four regions. Drawing the direction of motion in each region, we conclude that all positive solutions tend to rest point $(1, 2)$.

II.3,4   The null-clines $3 - x - y = 0$ and $4 - 2x - y = 0$ intersect at the point $(1, 2)$, which is a rest point. Considering the direction of motion in each of the four regions of the first quadrant, bounded by these null-clines, one sees that the trajectories move away from the rest point, and toward either one of the semitrivial solution $(3, 0)$ and $(0, 4)$, depending on the initial conditions.

II.5   The second null-cline is the parabola $2 - \dfrac{1}{8}x^2 - y = 0$, but the same analysis applies. The null-clines intersect at the point $(6 - 2\sqrt{7}, 3\sqrt{7} - 6)$, which is a rest point. Again, the null-clines divide the first quadrant of the xy-plane into four regions, and all positive solutions tend to the rest point $(6 - 2\sqrt{7}, 3\sqrt{7} - 6)$.

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III.1   That the ellipse $x = 2\cos t$, $y = \sin t$ is a solution of the system
$$x' = -2y + x\left(1 - \tfrac{1}{4}x^2 - y^2\right)$$
$$y' = \tfrac{1}{2}x + y\left(1 - \tfrac{1}{4}x^2 - y^2\right).$$

can be seen by a direct substitution. Denote $\rho(t) = \dfrac{1}{4}x^2 + y^2$. Then the ellipse $x = 2\cos t$, $y = \sin t$ can be written as $\rho = 1$. Using the corresponding equations, calculate
$$\rho' = \frac{1}{2}xx' + 2yy'$$
$$= \frac{1}{2}x\left[-2y + x\left(1 - \frac{1}{4}x^2 - y^2\right)\right] + 2y\left[\frac{1}{2}x + y\left(1 - \frac{1}{4}x^2 - y^2\right)\right] = 2\rho(1-\rho).$$
So that the integral curves satisfy

$$\rho' = 2\rho(1 - \rho).$$

For points inside the ellipse one has $\rho < 1$, and then $\rho' > 0$, so that the motion is toward this ellipse. For points outside the ellipse one has $\rho > 1$, and then $\rho' < 0$, and again the motion is toward this ellipse. We conclude that the ellipse $x = 2\cos t$, $y = \sin t$ is a stable limit cycle.

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What we have here is a case of *orbital stability*: the trajectories $(x(t), y(t))$ approach the ellipse $\rho = 1$, but not necessarily its parameterization $x = 2\cos t$, $y = \sin t$. Observe that $x = 2\cos(t - \alpha)$, $y = \sin(t - \alpha)$ is another solution of our system (for any α), and it gives another parameterization of the same ellipse.

III.2 (i)   To find the rest point(s) one needs to solve the system

$$x - y - x^3 = 0$$
$$x + y - y^3 = 0\,,$$

or the point(s) of intersection of the curves $x - y - x^3 = 0$ and $x + y - y^3 = 0$. The first of these curves is a cubic $y = x - x^3$, which is easy to draw. The second curve is also a cubic $x = -y + y^3$. A careful graph shows that the curves intersect only at the origin $(0, 0)$. In Figure **??** we present a graph produced by *Mathematica*.
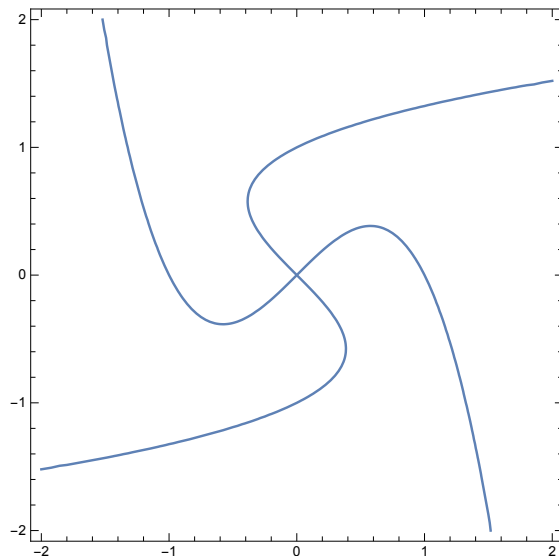

Figure 1: The curves $x - y - x^3 = 0$ and $x + y - y^3 = 0$ intersecting only at the origin

(ii) The linearized system at the origin is

$$x' = x - y$$
$$y' = x + y\,.$$

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(Which can be obtained by simply discarding the nonlinear terms.) The eigenvalues of its matrix are $1 \pm i$, so that the origin $(0,0)$ is an unstable spiral. By the Hartman-Grobman theorem the origin $(0,0)$ is an unstable spiral for the original nonlinear system too.

(iii) It follows from part (ii) that the trajectories of our system cut outside of the circles $x^2 + y^2 = \rho^2$, for ρ small enough. We show next that the trajectories of our system cut inside of the circles $x^2 + y^2 = R^2$, for R large enough. This will imply that the annulus $\rho^2 \le x^2 + y^2 \le R^2$ is a trapping region, and by the Poincare-Bendixson theorem there exists a limit cycle. Indeed, calculate the scalar product of $F = (x - y - x^3, x + y - y^3)$, and the vector (x, y) the circle $x^2 + y^2 = R^2$, then switch to the polar coordinates. Obtain

$$x^2 + y^2 - x^4 - y^4 = x^2 + y^2 - \left(x^2 + y^2\right)^2 + 2x^2 y^2 = R^2 - R^4 + 2R^4 \cos^2 \theta \sin^2 \theta$$

$$\le R^2 - \frac{1}{2} R^4 < 0, \quad \text{for } R \text{ large}.$$

(We used that $2 \cos^2 \theta \sin^2 \theta = \frac{1}{2} \sin^2 2\theta \le \frac{1}{2}$.)

III.3   Denote $f(x, y) = x \left(2 - x - y^3\right)$ and $g(x, y) = y \left(4x - 3y - x^2\right)$. Calculate

$$\frac{\partial}{\partial x} \left[\frac{1}{xy} f(x, y)\right] + \frac{\partial}{\partial y} \left[\frac{1}{xy} g(x, y)\right] = \frac{\partial}{\partial x} \left[\frac{2}{y} - \frac{x}{y} - y^2\right] + \frac{\partial}{\partial y} \left[4 - x - \frac{3y}{x}\right]$$

$$= -\frac{1}{y} - \frac{3}{x} < 0, \quad \text{for } x > 0, y > 0.$$

By the Theorem 6.5.1 it follows that the system has no limit cycles in the first quadrant of the xy-plane.

III.4   Convert the equation

$$x'' + f(x)x' + g(x) = 0$$

into a system for $x_1 = x$ and $x_2 = x'$

$$x_1' = x_2$$
$$x_2' = -g(x_1) - f(x_1)x_2.$$

Calculate

$$\frac{\partial}{\partial x_1} x_2 + \frac{\partial}{\partial x_2} \left[-g(x_1) - f(x_1)x_2\right] = -f(x_1),$$

which keeps the same sign whether $f(x)$ is positive or negative. By the Dulac-Bendixson criterion this system has no limit cycles, and then the original equation has no periodic solutions.

III.6   (i) The rest points of the gradient system

$$x_1' = -V_{x_1}(x_1, x_2)$$
$$x_2' = -V_{x_2}(x_1, x_2),$$

are the solutions of

$$V_{x_1}(x_1, x_2) = 0$$
$$V_{x_2}(x_1, x_2) = 0,$$

which are the critical points of $V(x_1, x_2)$.

(ii)   Differentiating, and using the corresponding equations

$$\frac{d}{dt}V(x_1(t), x_2(t)) = V_{x_1}(x_1, x_2)x_1' + V_{x_2}(x_1, x_2)x_2'$$

$$= -V_{x_1}^2(x_1, x_2) - V_{x_2}^2(x_1, x_2) < 0,$$

showing that $V(x_1(t), x_2(t))$ is a strictly decreasing function of t for any solution $(x_1(t), x_2(t))$, except if $(x_1(t), x_2(t))$ is a rest point.

(iii)   Since $V(x_1, x_2)$ is decreasing along any solution, it follows that no solution can revisit the same point, ruling out the possibility of limit cycles.

(iv)   If (a, b) is a point of strict local minimum of $V(x_1, x_2)$, it follows that $L(x_1, x_2) = V(x_1, x_2) - V(a, b) > 0$ for (x_1, x_2) near (a, b), while $L(a, b) = 0$. It follows that $L(x_1, x_2)$ is a Lyapunov's function, and by part (ii)

$$\frac{d}{dt}L(x_1(t), x_2(t)) = \frac{d}{dt}V(x_1(t), x_2(t)) < 0,$$

proving that (a, b) is an asymptotically stable rest point.

III.7.   (i) For a *Hamiltonian system*

(2) $$\qquad\qquad x_1' = -V_{x_2}(x_1, x_2)$$
$$x_2' = V_{x_1}(x_1, x_2),$$

the function $V(x_1, x_2)$ remains constant along all of the trajectories. Indeed, differentiating and using the corresponding equations

$$\frac{d}{dt}V(x_1(t), x_2(t)) = V_{x_1}(x_1, x_2)x_1' + V_{x_2}(x_1, x_2)x_2'$$

$$= -V_{x_1}(x_1, x_2)V_{x_2}(x_1, x_2) + V_{x_1}(x_1, x_2)V_{x_2}(x_1, x_2) = 0\,.$$

(iii)   If a trajectory $(x_1(t), x_2(t))$ tends to a rest point (x_1^0, x_2^0), then $V(x_1(t), x_2(t))$ tends to $V(x_1^0, x_2^0)$. But $V(x_1, x_2)$ remains constant along all of the trajectories.

(v)   The trajectories of gradient systems move along the vector (V_{x_1}, V_{x_2}), while the trajectories of Hamiltonian systems move along the vector $(-V_{x_2}, V_{x_1})$. These vectors are orthogonal at all points.

III.8   In the Lotka-Volterra predator-prey system

$$x'(t) = a\,x(t) - b\,x(t)\,y(t)$$
$$y'(t) = -c\,y(t) + d\,x(t)\,y(t)$$

Divide the first equation by $x(t) > 0$, the second one by $y(t) > 0$

$$\frac{x'(t)}{x(t)} = a - b\,y(t)$$
$$\frac{y'(t)}{y(t)} = -c + d\,x(t)$$

then let $p = \ln x$, and $q = \ln y$. For the new variables $p(t)$ and $q(t)$ obtain

$$p'(t) = a - b\,e^{q(t)}$$
$$q'(t) = -c + d\,e^{p(t)}\,.$$

The new system is Hamiltonian. Using the Hamiltonian function $V(p, q) = -a\,q + b\,e^q - c\,p + d\,e^p$ it can be written in the form

$$p'(t) = -V_q$$
$$q'(t) = V_p\,.$$

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IV.1   (i) If $x(t)$ is solution of the equation

(∗) $\qquad\qquad\qquad x' = a(t)x + b(t)$

it satisfies it for all t, in particular at $t + p$

$$x'(t + p) = a(t + p)x(t + p) + b(t + p)\,.$$

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By periodicity of $a(t)$ and $b(t)$, this implies that

$$x'(t+p) = a(t)x(t+p) + b(t),$$

so that $x(t+p)$ is also a solution of $(*)$.

(ii)   Given that $x(p) = x(0)$, the solutions $x(t)$ and $x(t+p)$ of $(*)$ coincide at $t = 0$. By the uniqueness of solutions of initial value problems, $x(t+p) = x(t)$ for all t, proving that $x(t)$ is a p-periodic function.

IV.3   Assume that the equation

$(*)$
$$x' + a(t)x = 1$$

has p-periodic solution. Under the condition $\int_0^p a(t)\, dt = 0$, it was proved in the text that the equation

$(**)$
$$z' - a(t)z = 0$$

has a non-trivial (non-zero) p-periodic solution, and we assume that $z(t) > 0$ for all t. Multiply the equation $(*)$ by $z(t)$, the equation $(**)$ by $x(t)$, then add the results:

$$\frac{d}{dt}[x(t)z(t)] = z(t).$$

Integrate both sides over $(0, p)$, and use the p-periodicity

$$x(t)z(t)\big|_0^p = \int_0^p z(t)\, dt,$$

$$0 = \int_0^p z(t)\, dt.$$

The integral on the right is positive, and we have a contradiction, proving that the equation $(*)$ has no p-periodic solution.

IV.4   By the uniqueness of solutions of initial value problems, any non-trivial solution of

$$x' = a(t)x - b(t)x^2$$

satisfies either $x(t) > 0$ or $x(t) < 0$, for all t. Divide the equation by $x(t)$, and integrate over $(0, p)$. Using the periodicity and our condition $\int_0^p a(t)\, dt = 0$, obtain

$$\ln x(t)\big|_0^p = \int_0^p a(t)\, dt - \int_0^p b(t)x(t)\, dt,$$

$$0 = -\int_0^p b(t)x(t)\, dt.$$

This is a contradiction, because the integral on the right is ether negative or positive, proving that the logistic equation above has no non-trivial p-periodic solution.

Chapter 7

Section 7.3.1, Page 269

I.1   Using that $\tan x$ is an odd function, obtain

$$\int_{-1}^{3/2} \tan^{15} x \, dx = \int_{-1}^{1} \tan^{15} x \, dx + \int_{1}^{3/2} \tan^{15} x \, dx = \int_{1}^{3/2} \tan^{15} x \, dx > 0 \,,$$

because $\tan x > 0$ on the interval $(1, 3/2)$ (observe that $3/2 < \pi/2$).

I.2   Write
$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \,,$$
where the first function is even, and the second one is odd.

I.5   The function $f_o(x) = x|x|$ is the odd function, which is equal to $f(x) = x^2$ on the original interval $[0, 1)$.

I.6   The function $f(x) = x|x|^{p-1}$ is odd, therefore its derivative $f'(x)$ is even. For $x > 0$, $f(x) = x^p$, and $f'(x) = px^{p-1}$. The even extension of this function is $p|x|^{p-1}$, therefore $f'(x) = p|x|^{p-1}$.

I.7   Let us denote $F(x) = \int_0^x f(t) \, dt$. Then

$$F(x + 2\pi) = \int_0^{x+2\pi} f(t) \, dt = \int_0^x f(t) \, dt + \int_x^{x+2\pi} f(t) \, dt$$

$$= F(x) + \int_x^{x+2\pi} f(t) \, dt = F(x) + \int_0^{2\pi} f(t) \, dt \,,$$

because the integral of a 2π-periodic function is the same over any interval of length 2π. The function $F(x)$ is 2π-periodic if and only if $\int_0^{2\pi} f(t) \, dt = 0$.

I.8   Evaluate the antiderivative, then use the periodicity of $f(x)$

$$\int_a^{T+a} f'(x) e^{f(x)} \, dx = e^{f(x)} \Big|_a^{T+a} = e^{f(T+a)} - e^{f(a)} = 0 \,.$$

Fourier series for a function $f(x)$ on the interval $(-L, L)$ (or for $2L$-periodic $f(x)$) has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) ,$$

where the coefficients are calculated as follows: $a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx$, $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx$, $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$. Often these coefficients can be calculated without performing the integrations.

II.1 The Fourier series on the interval $(-\pi, \pi)$ has the form $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. Using the trig identities $\sin x \cos x = \frac{1}{2} \sin 2x$, and $\cos^2 2x = \frac{1}{2} + \frac{1}{2} \cos 4x$, one sees that $f(x)$ is its own Fourier series $f(x) = \frac{1}{2} + \frac{1}{2} \cos 4x + \frac{1}{2} \sin 2x$. Here $a_0 = \frac{1}{2}$, $a_4 = \frac{1}{2}$ and $b_2 = \frac{1}{2}$, and the other coefficients are all equal to zero.

II.2 The Fourier series on the interval $(-2\pi, 2\pi)$ has the form $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n}{2} x + b_n \sin \frac{n}{2} x \right)$. As in the preceding problem one sees that $f(x)$ is its own Fourier series $f(x) = \frac{1}{2} + \frac{1}{2} \cos 4x + \frac{1}{2} \sin 2x$. This time $a_0 = \frac{1}{2}$, $a_8 = \frac{1}{2}$ and $b_4 = \frac{1}{2}$, and the other coefficients are all equal to zero.

II.3 The Fourier series on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ has the form $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2nx + b_n \sin 2nx)$. Again, $f(x)$ is its own Fourier series $f(x) = \frac{1}{2} + \frac{1}{2} \cos 4x + \frac{1}{2} \sin 2x$. Here $a_0 = \frac{1}{2}$, $a_2 = \frac{1}{2}$ and $b_1 = \frac{1}{2}$, and the other coefficients are all equal to zero.

II.4 Evaluate the integrals

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x + x^2 \right) dx = \frac{\pi^2}{3} ,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + x^2\right) \cos nx \, dx = \frac{4(-1)^n}{n^2} \,,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(x + x^2\right) \sin nx \, dx = \frac{2(-1)^{n+1}}{n} \,.$$

The last two integrals are evaluated by either guess-and-check, or two integrations by part.

II.5   The Fourier series on the interval $(-\pi, \pi)$ has the form $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. Since here $f(x)$ is odd, it follows that $a_0 = 0$, $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi} \frac{1 - (-1)^n}{n} \,.$$

The even coefficients $a_{2k} = 0$, while the odd coefficients $a_{2k+1} = \frac{4}{\pi} \frac{1}{2k+1}$.

We conclude that $f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x$.

II.6   Here $L = 2$, and the Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{2}x + b_n \sin \frac{n\pi}{2}x \right) \,.$$

Calculate

$$a_0 = \frac{1}{4} \int_{-2}^{2} (1 - |x|) \, dx = \frac{1}{2} \int_0^{2} (1 - x) \, dx = 0 \,,$$

$$a_n = \frac{1}{2} \int_{-2}^{2} (1 - |x|) \cos \frac{n\pi}{2}x \, dx = \int_0^{2} (1 - x) \cos \frac{n\pi}{2}x \, dx = \frac{4}{n^2\pi^2} \left(1 - (-1)^n\right) \,,$$

and finally

$$b_n = \frac{1}{2} \int_{-2}^{2} (1 - |x|) \sin \frac{n\pi}{2}x \, dx = 0 \,,$$

because the integrand is an odd function. One concludes that

$$1 - |x| = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \left(1 - (-1)^n\right) \cos \frac{n\pi}{2}x \,, \quad \text{on the interval } (-2, 2) \,.$$

II.7   Here $L = 1$, and the Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) \,.$$

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Since the function $f(x) = x|x|$ is odd, $a_0 = 0$ and $a_n = 0$ for all n. Compute

$$b_n = \int_{-1}^{1} x|x| \sin n\pi x \, dx = 2 \int_0^1 x^2 \sin n\pi x \, dx = \frac{-2\left(n^2\pi^2 - 2\right)(-1)^n - 4}{n^3\pi^3},$$

using that $\cos n\pi = (-1)^n$.

II.8 Here $L = 1$, and the Fourier series has the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x\right).$$

Calculate

$$a_0 = \frac{1}{2} \int_{-1}^{0} dx = \frac{1}{2},$$

$$a_n = \int_{-1}^{0} \cos n\pi x \, dx = 0,$$

$$b_n = \int_{-1}^{0} \sin n\pi x \, dx = \frac{1}{n\pi}\left[(-1)^n - 1\right].$$

Obtain

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left[(-1)^n - 1\right] \sin n\pi x$$

$$= \frac{1}{2} - \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin(2k-1)\pi x.$$

II.9 Here $L = 2$, and the Fourier series takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{2}x + b_n \sin \frac{n\pi}{2}x\right).$$

Calculate

$$a_0 = \frac{1}{4} \int_{-2}^{0} x \, dx + \frac{1}{4} \int_0^2 (-1) \, dx = -1,$$

$$a_n = \frac{1}{2} \int_{-2}^{0} x \cos \frac{n\pi}{2}x \, dx + \frac{1}{2} \int_0^2 (-1) \cos \frac{n\pi}{2}x \, dx = \frac{2\left[1 - (-1)^n\right]}{n^2\pi^2},$$

$$b_n = \frac{1}{2} \int_{-2}^{0} x \sin \frac{n\pi}{2}x \, dx + \frac{1}{2} \int_0^2 (-1) \sin \frac{n\pi}{2}x \, dx = -\frac{1 + (-1)^n}{n\pi}.$$

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III.1   Here $L = \pi$, and the Fourier cosine series has the form $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$. Using the trig identity $\sin^2 3x = \frac{1}{2} - \frac{1}{2} \cos 6x$, one sees that $f(x)$ is its own Fourier cosine series

$$f(x) = -\frac{1}{2} + \cos 3x + \frac{1}{2} \cos 6x \ \text{ on } (0, \pi).$$

Here $a_0 = -\frac{1}{2}$, $a_3 = 1$, $a_6 = \frac{1}{2}$, and the other coefficients are all equal to zero.

III.2   Here $L = \pi/3$, and the Fourier cosine series has the form $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 3nx$. For the same function as in the preceding problem one obtains the same Fourier cosine series

$$\cos 3x - \sin^2 3x = -\frac{1}{2} + \cos 3x + \frac{1}{2} \cos 6x \ \text{ on } (0, \pi/3).$$

However, in this case $a_0 = -\frac{1}{2}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, and the other coefficients are all equal to zero.

III.3   Here $L = 2$, and the Fourier cosine series has the form $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n}{2} x$. Calculate

$$a_0 = \frac{1}{2} \int_0^2 x \, dx = 1 \,,$$

$$a_n = \frac{2}{2} \int_0^2 x \cos \frac{\pi n}{2} x \, dx = \frac{-4 + 4(-1)^n}{n^2 \pi^2} \,.$$

III.5   Here $L = \pi/2$, and the Fourier cosine series has the form $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nx$. Using trig identities

$$\sin^4 x = \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x$$

$$= \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4x \right) = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \,.$$

Here $a_0 = \dfrac{3}{8}$, $a_1 = -\dfrac{1}{2}$, $a_2 = \dfrac{1}{8}$, and the other coefficients are all equal to zero.

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IV.1   Here $L = \pi$, and the Fourier sine series has the form $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$.
Using a trig identity, we conclude that $f(x)$ is itsOurier sine series on $(0, \pi)$

$$5 \sin x \cos x = \frac{5}{2} \sin 2x \,,$$

where $b_2 = \dfrac{5}{2}$, and the other b_n's are all equal to zero.

IV.2   Here $L = 3$, and the Fourier sine series has the form $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi}{3} x$.
Compute
$$b_n = \frac{2}{3} \int_0^3 \sin \frac{n\pi}{3} x \, dx = \frac{2}{n\pi} \left(1 - (-1)^n \right) \,.$$

IV.3   Here $L = 2$, and the Fourier sine series has the form $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi}{2} x$.
Calculate
$$b_n = \frac{2}{2} \int_0^2 x \sin \frac{n\pi}{2} x \, dx = \frac{4}{n\pi} (-1)^{n+1} \,.$$

IV.5   Here $L = \pi$, and the Fourier sine series has the form $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$.
Using Euler's formula express

$$e^{ix} = \cos x + i \sin x \,,$$

$$e^{3ix} = (\cos x + i \sin x)^3 \,,$$

$$\cos 3x + i \sin 3x = \cos^3 x + 3i \cos^2 x \sin x - 3 \cos x \sin^2 x - i \sin^3 x \,.$$

Equating the imaginary parts

$$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x = 3 \left(1 - \sin^2 x \right) \sin x - \sin^3 x \,,$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x \,.$$

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Solving this equation for $\sin^3 x$, one obtains the desired Fourier sine series

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \,.$$

Here $b_1 = \frac{3}{4}$, $b_3 = -\frac{1}{4}$, and the other b_n's are all equal to zero.

IV.6 Here $L = \pi$, and the Fourier sine series has the form $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.
Calculate

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx$$

$$= -\frac{\cos \frac{n\pi}{2}}{n} + \frac{2}{\pi n^2} \sin \frac{n\pi}{2} + \frac{\cos \frac{n\pi}{2}}{n} + \frac{2}{\pi n^2} \sin \frac{n\pi}{2} = \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \,.$$

IV.7 Here $L = 3$, and the Fourier sine series has the form $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{3} x$.
Calculate

$$b_n = \frac{2}{3} \int_0^3 (x - 1) \sin \frac{n\pi}{3} x \, dx = -\frac{2 + 4(-1)^n}{n\pi} \,.$$

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V.1 If λ is negative, it may be written in the form $\lambda = -\omega^2$, and the general solution of the equation

$$y'' - \omega^2 y = 0$$

may be taken in the form $y = c_1 \cosh \omega x + c_2 \sinh \omega x$. The boundary condition $y'(0) = 0$ implies that $c_2 = 0$, and then $y = c_1 \cosh \omega x$. The boundary condition $y(L) = 0$ gives

$$c_1 \cosh \omega L = 0 \,,$$

so that $c_1 = 0$, and the solution is trivial $(y = 0)$. We conclude that there are no negative eigenvalues. In case $\lambda = 0$, the general solution of the equation

$$y'' = 0$$

is $y = c_1 + c_2 x$. The boundary conditions again imply that $c_1 = c_2 = 0$, and the solution is trivial. We conclude that $\lambda = 0$ is not an eigenvalue. If λ is

positive, it may be written in the form $\lambda = \omega^2$, and the general solution of the equation

$$y'' + \omega^2 y = 0$$

is $y = c_1 \cos \omega x + c_2 \sin \omega x$. The boundary condition $y'(0) = 0$ implies that $c_2 = 0$, and then $y = c_1 \cos \omega x$. The boundary condition $y(L) = 0$ gives

$$c_1 \cos \omega L = 0 .$$

One avoids the trivial solution if ωL is a root of cosine, i.e., if $\omega L = \dfrac{\pi}{2} + n\pi$, or $\omega = \dfrac{\pi(n + \frac{1}{2})}{L}$, and then $\lambda = \dfrac{\pi^2(n + \frac{1}{2})^2}{L^2}$. We conclude that the eigenvalues are $\lambda_n = \dfrac{\pi^2(n + \frac{1}{2})^2}{L^2}$, with the corresponding eigenfunctions $y_n = \cos \dfrac{\pi(n + \frac{1}{2})}{L} x$ (or the constant multiples of y_n).

V.2   Similarly to the preceding problem, we show that there are no eigenvalues for $\lambda \leq 0$. For positive $\lambda = \omega^2$, solving the equation

$$y'' + \omega^2 y = 0 , \quad y(0) = 0 ,$$

together with the first boundary condition, gives $y = c_1 \sin \omega x$. The boundary condition $y'(L) = 0$ requires

$$c_1 \omega \cos \omega L = 0 .$$

One avoids the trivial solution if ωL is a root of cosine, i.e., if $\omega L = \dfrac{\pi}{2} + n\pi$, or $\omega = \dfrac{\pi(n + \frac{1}{2})}{L}$, and then $\lambda = \dfrac{\pi^2(n + \frac{1}{2})^2}{L^2}$. We conclude that the eigenvalues are $\lambda_n = \dfrac{\pi^2(n + \frac{1}{2})^2}{L^2}$, with the corresponding eigenfunctions $y_n = \sin \dfrac{\pi(n + \frac{1}{2})}{L} x$ (or the constant multiples of y_n).

V.4   Denote

$$(*) \qquad \lambda = \int_0^1 y^2(x) \, dx.$$

Then we need to find all non-trivial solutions of

$$y'' + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0$$

that are $\lambda = \lambda_n = n^2\pi^2$ and $y_n = A\sin n\pi x$, where A is a constant. Substitute these eigenvalues and eigenfunctions into $(*)$, then integrate

$$n^2\pi^2 = A^2 \int_0^1 \sin^2 n\pi x\, dx = \frac{1}{2}A^2\,.$$

It follows that $A = \pm\sqrt{2}n\pi$, and the solutions are $y = \pm\sqrt{2}n\pi\,\sin n\pi x$

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The solution of the initial-boundary value problem for the heat equation with the Dirichlet boundary conditions

$$u_t = ku_{xx} \qquad \text{for } 0 < x < L, \text{ and } t > 0$$
$$u(x,0) = f(x) \qquad \text{for } 0 < x < L$$
$$u(0,t) = u(L,t) = 0 \qquad \text{for } t > 0$$

is given by the series $u(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi}{L}x$. Here b_n are the coefficients of the Fourier sine series of $f(x)$, i.e.,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\,,$$

with $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx$. In practice, one may begin with this Fourier sine series, and then insert the exponents $e^{-k\frac{n^2\pi^2}{L^2}t}$.

In case of the Neumann boundary conditions

$$u_x(0,t) = u_x(L,t) = 0 \qquad \text{for } t > 0\,,$$

the solution is given by the series $u(x,t) = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n e^{-k\frac{n^2\pi^2}{L^2}t}\cos\frac{n\pi}{L}x$. Here a_0 and a_n are the coefficients of the Fourier cosine series of $f(x)$.

I.1 Here $k = 2$, $L = \pi$, and

$$\sin x - 3\sin x\cos x = \sin x - \frac{3}{2}\sin 2x\,,$$

so that $b_1 = 1$, $b_2 = -\dfrac{3}{2}$, and all other b_n are zero. Inserting the appropriate exponentials, obtain $u(x,t) = e^{-2t}\sin x - \dfrac{3}{2}e^{-8t}\sin 2x$.

I.2   Here $k = 2$, $L = 2\pi$, and again

$$\sin x - 3 \sin x \cos x = \sin x - \frac{3}{2} \sin 2x,$$

although now $b_2 = 1$, $b_4 = -\frac{3}{2}$, and all other b_n are zero. (The Fourier sine series takes the form $\sum\limits_{n=1}^{\infty} b_n \sin \frac{n}{2} x$.) Inserting the appropriate exponentials, obtain $u(x,t) = e^{-2t} \sin x - \frac{3}{2} e^{-8t} \sin 2x$.

I.3   Here $k = 5$, $L = 2$. Calculate

$$b_n = \frac{2}{2} \int_0^2 x \sin \frac{n\pi}{2} x \, dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Conclude that $u(x,t) = \sum\limits_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} e^{-\frac{5n^2\pi^2}{4}t} \sin \frac{n\pi}{2} x.$

I.4   Here $k = 3$, $L = \pi$. Breaking the integral into two pieces, calculate

$$b_n = \frac{2}{\pi} \int_0^\pi u(x,0) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^\pi (\pi - x) \sin nx \, dx$$

$$= \frac{2 \sin \left(\frac{\pi n}{2} \right)}{\pi n^2} - \frac{\cos \left(\frac{\pi n}{2} \right)}{n} + \frac{2 \sin \left(\frac{\pi n}{2} \right)}{\pi n^2} + \frac{\cos \left(\frac{\pi n}{2} \right)}{n} = \frac{4 \sin \left(\frac{\pi n}{2} \right)}{\pi n^2}.$$

Solution: $u(x,t) = \sum\limits_{n=1}^{\infty} \frac{4 \sin \frac{n\pi}{2}}{\pi n^2} e^{-3n^2 t} \sin nx.$

I.5   Here $k = 1$, $L = 3$. The solution has the form $u(x,t) = \sum\limits_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x$, with

$$b_n = \frac{2}{3} \int_0^3 (x+2) \sin \frac{n\pi}{3} x \, dx = \frac{4 - 10(-1)^n}{n\pi}.$$

I.6   Here $k = 1$, $L = 3$, Neumann boundary conditions. Compute

$$a_0 = \frac{1}{3} \int_0^3 (x+2) \, dx = \frac{7}{2},$$

$$a_n = \frac{2}{3} \int_0^3 (x+2) \cos \frac{n\pi}{3} x \, dx = \frac{6 \left[-1 + (-1)^n \right]}{n^2 \pi^2}.$$

Inserting the appropriate exponentials, obtain the solution:

$$u(x,t) = \frac{7}{2} + \sum_{n=1}^{\infty} \frac{6\left[-1+(-1)^n\right]}{n^2\pi^2} e^{-\frac{n^2\pi^2}{9}t} \cos\frac{n\pi}{3}x \, .$$

I.7   Here $k=2$, $L=\pi$, Neumann boundary conditions. To obtain the Fourier cosine series of $\cos^4 x$, write

$$\cos^4 x = \left(\cos^2 x\right)^2 = \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \, .$$

Inserting the appropriate exponentials, obtain the solution: $u(x,t) = \frac{3}{8} + \frac{1}{2}e^{-8t}\cos 2x + \frac{1}{8}e^{-32t}\cos 4x.$

I.8   Multiply the equation

$$u_t - u = 3u_{xx}$$

by the integrating factor e^{-t}, and call $e^{-t}u(x,t) = v(x,t)$. Then the new unknown function $v(x,t)$ satisfies

$$v_t = 3v_{xx} \quad \text{for } 0 < x < 2, \text{ and } t > 0$$
$$v(x,0) = 1 - x \quad \text{for } 0 < x < 2$$
$$v_x(0,t) = v_x(2,t) = 0 \quad \text{for } t > 0 \, .$$

Here $k=3$, $L=2$, Neumann boundary conditions. Compute

$$a_0 = \frac{1}{2}\int_0^2 (1-x)\,dx = 0\,,$$

$$a_n = \frac{2}{2}\int_0^2 (1-x)\cos\frac{n\pi}{2}x\,dx = \frac{4\left[1-(-1)^n\right]}{n^2\pi^2}\,,$$

If n is even, $a_n = 0$. All odd n are of the form $n = 2k-1$, $k = 1,2,3,\ldots$ So that for odd n, $a_n = a_{2k-1} = \frac{8}{\pi^2(2k-1)^2}$ and the Fourier cosine series of the initial condition is

$$1 - x = \sum_{k=1}^{\infty} \frac{8}{\pi^2(2k-1)^2} \cos\frac{(2k-1)\pi}{2}x\,.$$

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Inserting the appropriate exponentials, obtain

$$v(x,t) = \sum_{k=1}^{\infty} \frac{8}{\pi^2(2k-1)^2} e^{-\frac{3(2k-1)^2\pi^2}{4}t} \cos \frac{(2k-1)\pi}{2} x .$$

Finally, obtain the solution

$$u(x,t) = e^t \, v(x,t) = \sum_{k=1}^{\infty} \frac{8}{\pi^2(2k-1)^2} e^{\left(-\frac{3(2k-1)^2\pi^2}{4}+1\right)t} \cos \frac{(2k-1)\pi}{2} x .$$

I.10 Write $v(x,y) = u(a(x,y), b(x,y))$, where we denoted $a(x,y) = \dfrac{x}{x^2+y^2}$
and $b(x,y) = \dfrac{y}{x^2+y^2}$. Differentiate

$$v_x = u_x a_x + u_y b_x ,$$

$$v_{xx} = u_{xx} a_x^2 + 2u_{xy} a_x b_x + u_{yy} b_x^2 + u_x a_{xx} + u_y b_{xx} .$$

Similarly

$$v_y = u_x a_y + u_y b_y ,$$

$$v_{yy} = u_{xx} a_y^2 + 2u_{xy} a_y b_y + u_{yy} b_y^2 + u_x a_{yy} + u_y b_{yy} .$$

Calculate $a_x = \dfrac{y^2-x^2}{(x^2+y^2)^2} = -b_y$, $a_y = -\dfrac{2xy}{(x^2+y^2)^2} = b_x$. It follows that

$$v_{xx} + v_{yy} = (u_{xx} + u_{yy})\left(a_x^2 + a_y^2\right) + 2u_{xy}\left(a_x b_x + a_y b_y\right)$$

$$+ u_x (a_{xx} + a_{yy}) + u_y (b_{xx} + b_{yy}) .$$

We have $u_{xx} + u_{yy} = 0$, because $u(x,y)$ is harmonic, and $a_x b_x + a_y b_y = 0$, by
above. Further differentiation shows that $a_{xx} + a_{yy} = 0$ and $b_{xx} + b_{yy} = 0$.
It follows that $v_{xx} + v_{yy} = 0$, so that $v(x,y)$ is harmonic.

I.11 $u(x,y) = 5$ is the simple solution.

To solve the Laplace equation on a rectangle

$$u_{xx} + u_{yy} = 0$$

one uses separation of variables, looking for solution in the form $u(x,y) = F(x)G(y)$. Substitution into the Laplace equation gives

$$F''G + FG'' = 0 ,$$

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$$F''G = -FG'',$$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$

The function of x only on the left is equal to the function of y on the right. For that they both must be equal to the same constant. It is convenient to denote this constant by $-\lambda$, if two boundary conditions are available for $F(x)$. If the problem at hand provides two boundary conditions for $G(y)$, it is convenient to call the common constant λ.

I.12 Obtain

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda,$$

giving

$$F'' + \lambda F = 0, \quad F(0) = F(2) = 0,$$

$$G'' - \lambda G = 0, \quad G(3) = 0.$$

Non-trivial solutions of the first of these problems occur when $\lambda = \lambda_n = \frac{n^2\pi^2}{4}$, and they are $F_n(x) = \sin\frac{n\pi}{2}x$ (or constant multiples). The second problem at $\lambda = \lambda_n = \frac{n^2\pi^2}{4}$ becomes

$$G'' - \frac{n^2\pi^2}{4}G = 0, \quad G(3) = 0.$$

The solutions are $G_n = \sinh\frac{n\pi}{2}(y - 3)$ (or constant multiples). The series

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}x \sinh\frac{n\pi}{2}(y - 3)$$

satisfies the Laplace equation and the three homogeneous boundary conditions. We now choose b_n's to satisfy the remaining boundary condition:

$$u(x, 0) = -\sum_{n=1}^{\infty} b_n \sinh\frac{3n\pi}{2} \sin\frac{n\pi}{2}x = 5.$$

We need to represent 5 by its Fourier sine series on $(0, 2)$, so that

$$-b_n \sinh\frac{3n\pi}{2} = \int_0^2 5 \sin\frac{n\pi}{2}x \, dx = \frac{10\left[1 - (-1)^n\right]}{n\pi},$$

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giving
$$b_n = -\frac{10\left[1-(-1)^n\right]}{n\pi \sinh \frac{3n\pi}{2}} \, .$$

The solution is then

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{10\left[1-(-1)^n\right]}{n\pi \sinh \frac{3n\pi}{2}} \sin \frac{n\pi}{2}x \sinh \frac{n\pi}{2}(y-3) \, .$$

I.13   Separation of variables

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \lambda \, ,$$

leads to

$$G'' + \lambda G = 0 \, , \quad G(0) = G(1) = 0 \, ,$$
$$F'' - \lambda F = 0 \, , \quad F(2) = 0 \, ,$$

Non-trivial solutions of the first of these problems occur when $\lambda = \lambda_n = n^2\pi^2$, and they are $G_n(y) = \sin n\pi y$. The second problem at these eigenvalues becomes

$$F'' - n^2\pi^2 F = 0 \, , \quad F(2) = 0 \, ,$$

The solutions are $F_n = \sinh n\pi(x-2)$ (or constant multiples). The series

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh n\pi(x-2) \sin n\pi y$$

satisfies the Laplace equation and the three homogeneous boundary conditions. We now choose b_n's to satisfy the remaining boundary condition:

$$u(0,y) = -\sum_{n=1}^{\infty} b_n \sinh 2n\pi \sin n\pi y = y \, .$$

We need to represent y by its Fourier sine series on $(0,1)$, so that

$$-b_n \sinh 2n\pi = 2\int_0^1 y \sin n\pi y \, dy = \frac{2(-1)^{n+1}}{n\pi} \, ,$$

$$b_n = \frac{2(-1)^n}{n\pi \sinh 2n\pi} \, .$$

Solution: $u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi \sinh 2n\pi} \sinh n\pi(x-2) \sin n\pi y.$

I.14   Obtain
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda \,,$$

giving
$$F'' + \lambda F = 0\,, \quad F(0) = F(\pi) = 0\,,$$
$$G'' - \lambda G = 0\,, \quad G(0) = 0\,.$$

Here $\lambda_n = n^2$, $F_n(x) = \sin nx$, and the second problem
$$G'' - n^2 G = 0\,, \quad G(0) = 0$$

has solutions $G_n(y) = \sinh ny$. The series
$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh ny$$

satisfies the Laplace equation and the three homogeneous boundary conditions. We now choose b_n's to satisfy the remaining boundary condition:
$$u(x,1) = \sum_{n=1}^{\infty} b_n \sinh n \sin nx = 3 \sin 2x\,.$$

The function $3 \sin 2x$ is its own Fourier sine series on $(0, \pi)$. So that we choose $b_2 \sinh 2 = 3$, or $b_2 = \dfrac{3}{\sinh 2}$, and all other b_n's equal to zero. Then solution: $u(x,y) = \dfrac{3}{\sinh 2} \sin 2x \sinh 2y$.

I.15   There are two non-homogeneous boundary conditions. We obtain the solution in the form $u(x,y) = u_1(x,y) + u_2(x,y)$, where $u_1(x,y)$ is the solution of
$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < 2\pi,\text{ and } 0 < y < 2$$
$$u(x,0) = \sin x \quad \text{for } 0 < x < 2\pi$$
$$u(x,2) = 0 \quad \text{for } 0 < x < 2\pi$$
$$u(0,y) = 0 \quad \text{for } 0 < y < 2$$
$$u(2\pi,y) = 0 \quad \text{for } 0 < y < 2\,,$$

which is $u_1(x,y) = -\dfrac{1}{\sinh 2} \sinh(y-2) \sin x$, and $u_2(x,y)$ is the solution of
$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < 2\pi,\text{ and } 0 < y < 2$$
$$u(x,0) = 0 \quad \text{for } 0 < x < 2\pi$$
$$u(x,2) = 0 \quad \text{for } 0 < x < 2\pi$$
$$u(0,y) = 0 \quad \text{for } 0 < y < 2$$
$$u(2\pi,y) = y \quad \text{for } 0 < y < 2\,.$$

One calculates $u_2(x, y) = \sum_{n=1}^{\infty} \dfrac{4(-1)^{n+1}}{n\pi \sinh n\pi^2} \sinh \dfrac{n\pi}{2} x \sin \dfrac{n\pi}{2} y$, so that

$$u(x, y) = -\frac{1}{\sinh 2} \sinh(y - 2) \sin x + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi \sinh n\pi^2} \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y \, .$$

The solution of following initial-boundary value problem for wave equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \quad &&\text{for } 0 < x < L, \text{ and } t > 0 \\
u(0, t) &= u(L, t) = 0 \quad &&\text{for } t > 0 \\
u(x, 0) &= f(x) \quad &&\text{for } 0 < x < L \\
u_t(x, 0) &= g(x) \quad &&\text{for } 0 < x < L
\end{aligned}
$$

may be written as a series

$$u(x, t) = \sum_{n=1}^{\infty} \left(b_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t \right) \sin \frac{n\pi}{L} x \, ,$$

as was shown in the text by using the separation of variables. This series satisfies the wave equation above, and both of the boundary conditions. To satisfy the initial conditions one needs

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x = f(x) \, ,$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi}{L} x = g(x).$$

So that b_n's are the Fourier sine coefficients of $f(x)$ on $(0, L)$, while $B_n \dfrac{n\pi c}{L}$ are the Fourier sine coefficients of $g(x)$ on $(0, L)$.

I.16   Here $c = 2$, $L = \pi$. The solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} (b_n \cos 2n\, t + B_n \sin 2n\, t) \sin nx \, .$$

Calculate

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = \sin 2x \, ,$$

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so that $b_2 = 1$, and all other $b_n = 0$. Similarly

$$u_t(x, 0) = \sum_{n=1}^{\infty} 2n B_n \sin nx = -4 \sin 2x \,.$$

It follows that $B_n = 0$ for all $n \neq 2$, while at $n = 2$, $4B_2 = -4$, giving $B_2 = -1$. The solution: $u(x, t) = \cos 4t \sin 2x - \sin 4t \sin 2x$.

I.17   Here $c = 2$, $L = 1$. The solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} (b_n \cos 2n\pi \, t + B_n \sin 2n\pi \, t) \sin n\pi x \,.$$

Calculate

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = 0 \,,$$

so that $b_n = 0$ for all n. Similarly

$$u_t(x, 0) = \sum_{n=1}^{\infty} 2n\pi B_n \sin nx = x \,,$$

so that $2n\pi B_n$ are the coefficients of the Fourier sine series of x on $(0, 1)$:

$$2n\pi B_n = 2 \int_0^1 x \sin nx = 2 \frac{(-1)^{n+1}}{n\pi} \,,$$

giving

$$B_n = \frac{(-1)^{n+1}}{n^2 \pi^2} \,.$$

Solution: $u(x, t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 \pi^2} \sin 2n\pi \, t \sin n\pi x$.

I.18   Here $c = 2$, $L = 1$. The solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} (b_n \cos 2n\pi \, t + B_n \sin 2n\pi \, t) \sin n\pi x \,.$$

Calculate

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = -3 \,,$$

so that b_n are the coefficients of the Fourier sine series of -3 on $(0, 1)$:

$$b_n = 2 \int_0^1 (-3) \sin nx \, dx = \frac{6}{n\pi} \left[(-1)^n - 1 \right] \,.$$

Similarly

$$u_t(x,0) = \sum_{n=1}^{\infty} 2n\pi B_n \sin nx = x \,,$$

so that $2n\pi B_n$ are the coefficients of the Fourier sine series of x on $(0,1)$:

$$2n\pi B_n = 2 \int_0^1 x \sin nx = 2\frac{(-1)^{n+1}}{n\pi} \,,$$

giving

$$B_n = \frac{(-1)^{n+1}}{n^2\pi^2} \,.$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{6}{n\pi} \left((-1)^n - 1\right) \cos 2n\pi t + \frac{(-1)^{n+1}}{n^2\pi^2} \sin 2n\pi t \right] \sin n\pi x \,.$$

In case of Neumann boundary condition, the solution may be written as a series

$$u(x,t) = a_0 + A_0 t + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c}{L} t + A_n \sin \frac{n\pi c}{L} t \right) \cos \frac{n\pi}{L} x \,,$$

as follows by separation of variables.

I.20   Here $c = 3$, $L = \pi$, Neumann boundary condition. The solution takes the form

$$u(x,t) = a_0 + A_0 t + \sum_{n=1}^{\infty} \left(a_n \cos 3nt + A_n \sin 3nt \right) \cos nx \,.$$

This series satisfies the wave equation and both boundary conditions. The initial conditions imply:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = 4 \,,$$

so that $a_0 = 4$ and $a_n = 0$ for all $n \geq 1$, and

$$u_t(x,0) = A_0 + \sum_{n=1}^{\infty} 3n A_n \cos nx = \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \,.$$

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We see that $A_0 = \dfrac{1}{2}$, while $A_n = 0$ for all $n \neq 2$. At $n = 2$, obtain

$$6A_2 = \frac{1}{2},$$

giving $A_2 = \dfrac{1}{12}$. Solution:

$$u(x, t) = 4 + \frac{1}{2}t + \frac{1}{12}\sin 6t \cos 2x.$$

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II.1   The equation is homogeneous. The first boundary condition is also homogeneous, but the second boundary condition is not homogeneous. We search for solution in the form

$$u(x, t) = U(x) + v(x, t).$$

The role of $U(x)$ is to make the second boundary condition homogeneous, without disturbing the homogeneity of the equation and of the first boundary condition. Therefore we require that

$$U''(x) = 0, \quad U(0) = 0, \; U(1) = 1.$$

The solution of this problem is $U(x) = x$. Then

$$u(x, t) = x + v(x, t),$$

and $v(x, t) = u(x, t) - x$ satisfies

$$\begin{aligned}
v_t &= 5v_{xx}, \quad \text{for } 0 < x < 1, \text{ and } t > 0 \\
v(0, t) &= 0 \quad \text{for } t > 0 \\
v(1, t) &= 0 \quad \text{for } t > 0 \\
v(x, 0) &= -x \quad \text{for } 0 < x < 1.
\end{aligned}$$

We solved such homogeneous problems previously. The solution is given by the series

$$v(x, t) = \sum_{n=1}^{\infty} b_n \, e^{-5n^2\pi^2 t} \, \sin n\pi x,$$

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with b_n chosen to satisfy the initial condition on $(0, 1)$

$$v(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = -x \,,$$

so that

$$b_n = -2 \int_0^1 x \sin n\pi x \, dx = \frac{2(-1)^n}{n\pi} \,.$$

Solution:

$$u(x, t) = x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} e^{-5n^2\pi^2 t} \sin n\pi x \,.$$

II.2   We search for solution in the form

$$u(x, t) = U(x) + v(x, t) \,,$$

and require that

$$U''(x) = 0 \,, \quad U(0) = 0 \,, \ U(\pi) = 1 \,.$$

One calculates $U(x) = x$. Then

$$u(x, t) = \frac{x}{\pi} + v(x, t) \,,$$

and $v(x, t) = u(x, t) - \dfrac{x}{\pi}$ satisfies

$$\begin{aligned}
v_t &= 2v_{xx} && \text{for } 0 < x < \pi, \text{ and } t > 0 \\
v(x, 0) &= 0 && \text{for } 0 < x < \pi \\
v(0, t) &= 0 && \text{for } t > 0 \\
v(\pi, t) &= 0 && \text{for } t > 0 \,.
\end{aligned}$$

Clearly, $v(x, t) = 0$, and the solution is $u(x, t) = \dfrac{x}{\pi}$.

II.3   We search for solution in the form

$$u(x, t) = U(x) + v(x, t) \,,$$

and require that

$$2U''(x) + 4x = 0 \,, \quad U(0) = 0 \,, \ U(1) = 1 \,.$$

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The solution of this problem is $U(x) = \frac{1}{3}(x - x^3)$. So that

$$u(x, t) = \frac{1}{3}(x - x^3) + v(x, t),$$

and $v(x, t) = u(x, t) - \frac{1}{3}(x - x^3)$ satisfies

$$\begin{aligned}
v_t &= 2v_{xx} & \text{for } 0 < x < 1, \text{ and } t > 0 \\
v(x, 0) &= -\tfrac{1}{3}(x - x^3) & \text{for } 0 < x < 1 \\
v(0, t) &= 0 & \text{for } t > 0 \\
v(1, t) &= 0 & \text{for } t > 0.
\end{aligned}$$

The solution is given by the series

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-2n^2 \pi^2 t} \sin n\pi x,$$

with b_n chosen to satisfy the initial condition on $(0, 1)$

$$v(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = -\frac{1}{3}(x - x^3),$$

so that
$$b_n = -\frac{2}{3} \int_0^1 (x - x^3) \sin n\pi x \, dx = \frac{4(-1)^n}{n^3 \pi^3}.$$

Solution:

$$u(x, t) = \frac{1}{3}(x - x^3) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3 \pi^3} e^{-2n^2 \pi^2 t} \sin n\pi x.$$

II.6   Use the result of the preceding problem. Here $f(x, t) = t \sin 3x$, $k = 1$.
Expand $f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin nx$, with $f_3(t) = t$ and $f_n(t) = 0$ for $n \neq 3$.

Then the solution is of the form $u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$, provided that
$u_n(t)$ satisfy
$$u_n' + n^2 u_n = 0, \quad u_n(0) = 0,$$

for $n \neq 3$, and
$$u_3' + 9u_3 = t, \quad u_3(0) = 0.$$

Calculate $u_3(t) = \frac{t}{9} - \frac{1}{81} + \frac{1}{81}e^{-9t}$, $u_n(t) = 0$ for $n \neq 3$, and the solution is

$$u(x,t) = \left(\frac{t}{9} - \frac{1}{81} + \frac{1}{81}e^{-9t}\right)\sin 3x \,.$$

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Real valued function $f(x)$ on $(-L, L)$ can be represented by the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi}{L}x} \,,$$

with the coefficients

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx, \quad n = 0, \pm1, \pm2, \ldots \,.$$

I.1   Here $L = 2$. Using integration by parts and the formula $\int e^{iax}\,dx = \frac{e^{iax}}{ia} + c$, calculate

$$c_n = \frac{1}{4}\int_{-2}^{2} xe^{-i\frac{n\pi}{2}x}\,dx = \frac{2i(-1)^n}{n\pi}\,.$$

Conclude that

$$x = \sum_{n=-\infty}^{\infty} \frac{2i(-1)^n}{n\pi}\, e^{i\frac{n\pi}{2}x}, \quad \text{on } (-2, 2)\,.$$

I.2   Here $L = 1$, the complex Fourier series has the form $\displaystyle\sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x}$.
Calculate

$$c_n = \frac{1}{2}\int_{-1}^{1} e^x \, e^{-in\pi x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{x(1-in\pi)}\,dx = \frac{1}{2}\frac{e^{x(1-in\pi)}}{1-in\pi}\Big|_{-1}^{1}$$

$$= \frac{1}{2}\frac{e^{1-in\pi}}{1-in\pi} - \frac{1}{2}\frac{e^{-1+in\pi}}{1-in\pi} = (-1)^n\frac{(1+in\pi)(e - \frac{1}{e})}{2(1+n^2\pi^2)}\,,$$

because by Euler's formula

$$e^{1-in\pi} = e\,(\cos n\pi - i\sin n\pi) = (-1)^n e\,,$$

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and similarly, $e^{-1+in\pi} = e^{-1}(-1)^n$. Conclude that

$$e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{(1+in\pi)(e-\frac{1}{e})}{2(1+n^2\pi^2)} e^{in\pi x} \,.$$

I.3   Here $L = \pi$, the complex Fourier series has the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, and it can be computed for this function without performing the integrations. Indeed, using a trig identity and Euler's formula

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\frac{e^{i2x}+e^{-i2x}}{2} = -\frac{1}{4}e^{-i2x} + \frac{1}{2} - \frac{1}{4}e^{i2x} \,.$$

Here $c_{-2} = -\frac{1}{4}$, $c_0 = \frac{1}{2}$, and $c_2 = -\frac{1}{4}$, and all other c_n are equal to zero.

I.4   Here $L = \pi/2$, the complex Fourier series has the form $\sum_{n=-\infty}^{\infty} c_n e^{i2nx}$. Using a trig identity and Euler's formula

$$\sin 2x \cos 2x = \frac{1}{2}\sin 4x = \frac{1}{2}\frac{e^{i4x}-e^{-i4x}}{2i} = -\frac{i}{4}\left(e^{i4x}-e^{-i4x}\right) = \frac{i}{4}e^{-i4x} - \frac{i}{4}e^{i4x} \,.$$

Here $c_{-2} = \frac{i}{4}$, $c_2 = -\frac{i}{4}$, and all other c_n are equal to zero.

I.5   (i) Using Euler's formula obtain the following complex conjugate formula

$$\overline{e^{i\theta}} = e^{-i\theta} \,,$$

for any real θ. If a function $f(x)$ is represented by its complex Fourier series on $(-L, L)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \,,$$

then taking the complex conjugates of both sides, obtain

$$\bar{f}(x) = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{-i\frac{n\pi}{L}x} \,.$$

By uniqueness of the complex Fourier series it follows that $\bar{c}_n = c_{-n}$ for all n.

(ii) Observe that

$$\int_{-L}^{L} e^{-i\frac{k\pi}{L}x}\, dx = 0 \,,$$

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for any non-zero integer k. In case $k = 0$, this integral is equal to $2L$.

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To find the steady state temperatures inside the ball of radius R, one needs to solve the following boundary value problem

$$\Delta u = 0, \quad r < R$$
$$u(a, \theta) = f(\theta) \,.$$

One begins by expanding the initial data in its Fourier series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \,,$$

with the coefficients

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta \,, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \,,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \quad (n \geq 1) \,.$$

Then the solution of the boundary value problem is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \,.$$

In case of the exterior problem

$$\Delta u = 0, \quad r > R$$
$$u(a, \theta) = f(\theta) \,,$$

the solution is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \,.$$

II.1 By a trig identity

$$4 \cos^2 \theta = 2 + 2 \cos 2\theta \,,$$

so that $a_0 = 2$, $a_2 = 2$. Here $R = 3$, and the solution is

$$u = 2 + 2 \left(\frac{r}{3}\right)^2 \cos 2\theta = 2 + \frac{2}{9} r^2 \cos 2\theta = 2 + \frac{2}{9} r^2 \left(\cos^2 \theta - \sin^2 \theta\right)$$

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$$= 2 + \frac{2}{9}(x^2 - y^2) \, .$$

II.2 This is an exterior problem, with the same data as in the preceding problem. The solution is

$$u = 2 + 2 \left(\frac{3}{r} \right)^2 \cos 2\theta = 2 + \frac{18}{r^2} \cos 2\theta = 2 + \frac{18}{r^4} r^2 \left(\cos^2 \theta - \sin^2 \theta \right)$$

$$= 2 + 18 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, .$$

II.3 On the boundary circle $r = 2$, one has $y = 2 \sin \theta$, and then

$$f = y^2 = 4 \sin^2 \theta = 2 - 2 \cos 2\theta \, .$$

The solution is

$$u = 2 - 2 \left(\frac{r}{2} \right)^2 \cos 2\theta = 2 - \frac{1}{2} r^2 \left(\cos^2 \theta - \sin^2 \theta \right) = 2 - \frac{1}{2}(x^2 - y^2) \, .$$

II.4 This is an exterior problem, with the same data as in the preceding problem. The solution is

$$u = 2 - 2 \left(\frac{2}{r} \right)^2 \cos 2\theta = 2 - \frac{8}{r^4} \left(r^2 \cos^2 \theta - r^2 \sin^2 \theta \right) = 2 - \frac{8}{(x^2 + y^2)^2}(x^2 - y^2) \, .$$

II.5 To obtain the Fourier series, write

$$\cos^4 \theta = \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} + \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta$$

$$= \frac{1}{4} + \frac{1}{2} \cos 2\theta + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4\theta \right) = \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta \, .$$

Here $R = 1$. The solution is

$$u = \frac{3}{8} + \frac{1}{2} r^2 \cos 2\theta + \frac{1}{8} r^4 \cos 4\theta \, .$$

To present the answer in the Cartesian coordinates, write

$$\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta = 1 - 2 \sin^2 2\theta = 1 - 8 \sin^2 \theta \cos^2 \theta \, ,$$

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$$r^4 \cos 4\theta = r^4 - 8 \, (r \sin \theta)^2 \, (r \cos \theta)^2 = \left(x^2 + y^2 \right)^2 - 8x^2 y^2 \, .$$

Obtain
$$u(x, y) = \frac{3}{8} + \frac{1}{2} \left(x^2 - y^2 \right) + \frac{1}{8} \left(x^2 + y^2 \right)^2 - x^2 y^2 \, .$$

II.6 Calculate
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \theta \, d\theta = \pi \, ,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \theta \cos n\theta \, d\theta = 0 \, ,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \theta \sin n\theta \, d\theta = -\frac{2}{n} \, .$$

The Fourier series for the temperatures on the boundary

$$\theta = \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin n\theta \, .$$

Here $R = 1$. Inserting the factors of r^n into the last series, conclude

$$u(r, \theta) = \pi - \sum_{n=1}^{\infty} \frac{2}{n} r^n \sin n\theta \, .$$

II.7 Calculate
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} (\theta + 2) \, d\theta = \pi + 2 \, ,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} (\theta + 2) \cos n\theta \, d\theta = 0 \, ,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} (\theta + 2) \sin n\theta \, d\theta = -\frac{2}{n} \, .$$

The Fourier series for the temperatures on the boundary

$$\theta + 2 = \pi + 2 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin n\theta \, .$$

Here $R = 3$, exterior problem. Conclude

$$u = \pi + 2 - 2 \sum_{n=1}^{\infty} \frac{3^n}{n} r^{-n} \sin n\theta \, .$$

II.8   Here $R = 2$, $x = 2\cos\theta$, $y = 2\sin\theta$, and then

$$f(\theta) = 4\cos^2\theta - 2\sin\theta = 2 + 2\cos 2\theta - 2\sin\theta \,.$$

The solution is

$$u(r,\theta) = 2 + 2\left(\frac{r}{2}\right)^2 \cos 2\theta - 2\left(\frac{r}{2}\right)\sin\theta$$

$$= 2 + \frac{1}{2}r^2\left(\cos^2\theta - \sin^2\theta\right) - r\sin\theta = 2 + \frac{1}{2}\left(x^2 - y^2\right) - y\,.$$

II.9   We need to solve the following boundary value problem

$$\Delta u = 0, \quad r < 3$$
$$u(3,\theta) = y^2 - x\,.$$

Here $R = 3$, $x = 3\cos\theta$, $y = 3\sin\theta$, and then

$$f(\theta) = 9\sin^2\theta - 3\cos\theta = \frac{9}{2} - 3\cos\theta - \frac{9}{2}\cos 2\theta\,.$$

The solution is

$$u = \frac{9}{2} - 3\left(\frac{r}{3}\right)\cos\theta - \frac{9}{2}\left(\frac{r}{3}\right)^2\cos 2\theta = \frac{9}{2} - x - \frac{1}{2}(x^2 - y^2)\,,$$

using that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

For the *Neumann problem*

$$\Delta u = 0, \quad r < R$$
$$u_r(R,\theta) = f(\theta)$$

one observes that

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

solves the Laplace equation $\Delta u = 0$ for any constants A_n and B_n. The boundary condition requires that

$$\sum_{n=1}^{\infty} nR^{n-1} \left(A_n \cos n\theta + B_n \sin n\theta\right) = f(\theta)\,.$$

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The series on the left has no constant term. Hence, this equality can hold only if the same is true for the Fourier series for $f(\theta)$, i.e.,

$$\int_0^{2\pi} f(\theta)\, d\theta = 0\,.$$

If this condition holds, we can choose A_n and B_n to satisfy the boundary condition. In such a case A_0 is arbitrary, and we obtain infinitely many solutions.

II.10   Here

$$f(\theta) = \sin\theta\cos\theta - 2\sin 3\theta = \frac{1}{2}\sin 2\theta - 2\sin 3\theta$$

satisfies $\int_0^{2\pi} f(\theta)\, d\theta = 0$, so that solutions exist. They have the form

$$u = B_2 r^2 \sin 2\theta - B_3 r^3 \sin 3\theta + c\,.$$

Substituting into the boundary condition, we conclude that

$$u = \frac{1}{12} r^2 \sin 2\theta - \frac{2}{27} r^3 \sin 3\theta + c\,.$$

II.11   Here

$$\int_0^{2\pi} f(\theta)\, d\theta = \int_0^{2\pi} \left(\sin^2\theta - 2\sin 3\theta\right) d\theta = \pi \neq 0\,,$$

and so the Neumann problem has no solution.

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III.1   Assume first that λ is negative, so that we may write $\lambda = -\omega^2$, with $\omega > 0$. The general solution of

$$y'' - \omega^2 y = 0$$

is $y = c_1 e^{-\omega x} + c_2 e^{\omega x}$. The boundary conditions imply

$$c_1(1 - \omega) + c_2(1 + \omega) = 0$$

$$c_1 e^{-\pi\omega}(1 - \omega) + c_2 e^{\pi\omega}(1 + \omega) = 0\,.$$

This system has non-trivial solutions, only if its determinant is zero:

$$\left(1 - \omega^2\right)\left(e^{\pi\omega} - e^{-\pi\omega}\right) = 0 \,.$$

This happens when $\omega = 1$, and then $c_2 = 0$, as follows from either one of the equations for c_1 and c_2. It follows that $\lambda = -1$ is an eigenvalue, and $y = e^{-x}$ is the corresponding eigenfunction.

In case $\lambda = 0$, only the trivial solution is possible. Assume finally that λ is positive, so that we may write $\lambda = \omega^2$, with $\omega > 0$. The general solution of

$$y'' + \omega^2 y = 0$$

is $y = c_1 \cos \omega x + c_2 \sin \omega x$. The boundary conditions imply

$$c_1 + c_2 \omega = 0$$

$$c_1 \left(\cos \pi\omega - \omega \sin \pi\omega\right) + c_2 \left(\omega \cos \pi\omega + \sin \pi\omega\right) = 0 \,.$$

This system has non-trivial solutions, only if its determinant is zero:

$$\left(1 + \omega^2\right) \sin \pi\omega = 0 \,.$$

This happens when $\omega = n$, a positive integer, and then $c_1 = -nc_2$, as follows from the first equation. We obtain the eigenvalues $\lambda_n = n^2$, and the corresponding eigenfunctions $y_n = \sin nx - n \cos nx$.

III.2 Assume first that λ is negative, so that we may write $\lambda = -\omega^2$, with $\omega > 0$. Solution of the equation and the second boundary condition

$$y'' - \omega^2 y = 0 \,, \quad y(\pi) = 0$$

is $y = c \sinh \omega \left(x - \pi\right)$. The first boundary condition implies

$$c \left(\omega \cosh \pi\omega - \sinh \pi\omega\right) = 0 \,.$$

Let ω_0 $(\omega_0 \approx 0.996)$ denote the unique positive solution of the equation

$$\omega = \tanh \pi\omega \,.$$

Then $\lambda = -\omega_0^2$ is an eigenvalue, and $\sinh \omega_0 \left(x - \pi\right)$ is the corresponding eigenfunction.

In case $\lambda = 0$, only the trivial solution is possible. Assume finally that λ is positive, so that we may write $\lambda = \omega^2$, with $\omega > 0$. The general solution of

$$y'' + \omega^2 y = 0$$

is $y = c_1 \cos \omega x + c_2 \sin \omega x$. The boundary conditions imply

$$c_1 + c_2 \omega = 0$$

$$c_1 \cos \pi \omega + c_2 \sin \pi \omega = 0 \,.$$

From the first equation $c_1 = -c_2 \omega$, and then the second equation gives

$$c_2 \left(-\omega \cos \pi \omega + \sin \pi \omega \right) = 0 \,,$$

or

$$\tan \pi \omega = \omega \,.$$

This equation has infinitely many solutions, $0 < \omega_1 < \omega_2 < \omega_3 < \ldots$, as can be seen by drawing the graphs of $y = \omega$ and $y = \tan \omega \pi$ in the ωy-plane. We obtain infinitely many eigenvalues $\lambda_i = \omega_i^2$, and the corresponding eigenfunctions $y_i = -\omega_i \cos \omega_i x + \sin \omega_i x$, $i = 1, 2, 3, \ldots$.

III.3 (i)   The characteristic equation

$$r^2 + ar + \lambda = 0$$

has the roots

$$r = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \lambda} \,.$$

For $\lambda \le \dfrac{a^2}{4}$, the root(s) are real, and no eigenvalues are possible, as is straightforward to see. Turning to the case $\lambda > \dfrac{a^2}{4}$, let $\dfrac{a^2}{4} - \lambda = -\omega^2$, with $\omega > 0$. Then $r = -\dfrac{a}{2} \pm i\omega$, and the general solution is

$$y = c_1 e^{-\frac{a}{2}x} \cos \omega x + c_2 e^{-\frac{a}{2}x} \sin \omega x \,.$$

The boundary condition $y(0) = 0$ implies that $c_1 = 0$, and then the boundary condition $y(L) = 0$ gives

$$c_2 e^{-\frac{a}{2}L} \sin \omega L = 0 \,.$$

This implies $\omega = \dfrac{n\pi}{L}$, with a positive integer n, and then the eigenvalues are $\lambda_n = \dfrac{a^2}{4} + \dfrac{n^2 \pi^2}{L^2}$, with the corresponding eigenfunctions $y_n(x) = e^{-\frac{a}{2}x} \sin \dfrac{n\pi}{L}x$.

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(ii)   Using separation of variables, look for solutions in the form $u(t) = F(x)G(t)$, and obtain

$$F(x)G'(t) = F''(x)G(t) + aF'(x)G(t),$$

$$\frac{F''(x) + aF'(x)}{F(x)} = \frac{G'(t)}{G(t)} = -\lambda,$$

which gives

$$F'' + aF' + \lambda F = 0, \quad F(0) = F(L) = 0,$$

$$G'(t) = -\lambda G(t).$$

Non-trivial solution for the first boundary value problem occur at $\lambda_n = \frac{a^2}{4} + \frac{n^2\pi^2}{L^2}$, and they are $F_n(x) = e^{-\frac{a}{2}x}\sin\frac{n\pi}{L}x$. At $\lambda = \lambda_n$, the second equation gives $G_n(t) = e^{-\left(\frac{a^2}{4} + \frac{n^2\pi^2}{L^2}\right)t}$. The series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{a^2}{4} + \frac{n^2\pi^2}{L^2}\right)t} e^{-\frac{a}{2}x}\sin\frac{n\pi}{L}x$$

satisfies the PDE and both boundary conditions. To satisfy the initial condition, one needs

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{-\frac{a}{2}x}\sin\frac{n\pi}{L}x = f(x),$$

or

$$e^{\frac{a}{2}x}f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x.$$

So that we need to take b_n to be the coefficients of the Fourier sine series of $e^{\frac{a}{2}x}f(x)$, i.e., $b_n = \frac{2}{L}\int_0^L e^{\frac{a}{2}x}f(x)\sin\frac{n\pi}{L}x\,dx$.

III.4 (i)   The characteristic equation of this Euler's equation is

$$r^2 + 2r + \lambda = 0.$$

The roots are $r = -1 \pm \sqrt{1-\lambda}$, and to satisfy the boundary conditions we need them to be complex, i.e., $1 - \lambda = -\omega^2$, with $\omega > 0$. Then the general solution is (here $x > 0$)

$$y = c_1 x^{-1}\cos(\omega\ln x) + c_2 x^{-1}\sin(\omega\ln x).$$

The boundary condition $y(1) = 0$ implies that $c_1 = 0$, and then the boundary condition $y(e) = 0$ gives

$$c_2 e^{-1} \sin \omega = 0.$$

It follows that $\omega = n\pi$, with positive integer n, and then the eigenvalues are $\lambda_n = 1 + n^2\pi^2$, with the corresponding eigenfunctions $y_n(x) = x^{-1} \sin(n\pi \ln x)$.

(ii)   Divide the equation by x^2

$$y'' + \frac{3}{x}y' + \lambda\frac{1}{x^2}y = 0,$$

and then multiply by the integrating factor $\mu = e^{\int \frac{3}{x}\, dx} = x^3$ to put the equation into the self-adjoint form

$$\left(x^3 y'\right)' + \lambda xy = 0.$$

It follows that the eigenfunctions $y_n(x)$ are orthogonal with weight x.

(iii)   Using separation of variables, look for solutions in the form $u(t) = F(x)G(t)$, and obtain

$$F(x)G'(t) = x^2 F''(x)G(t) + 3xF'(x)G(t),$$

$$\frac{x^2 F''(x) + 3xF'(x)}{F(x)} = \frac{G'(t)}{G(t)} = -\lambda,$$

which gives

$$x^2 F'' + 3xF' + \lambda F = 0, \quad F(1) = F(e) = 0,$$

$$G'(t) = -\lambda G(t).$$

Non-trivial solution for the first boundary value problem occur at $\lambda_n = 1 + n^2\pi^2$, and they are $F_n(x) = y_n(x) = x^{-1} \sin(n\pi \ln x)$. At $\lambda = \lambda_n$, the second equation gives $G_n(t) = e^{-\lambda_n t} = e^{-(1+n^2\pi^2)t}$. The solution may be written in the form $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} y_n(x)$, where

$$b_n = \frac{\int_1^e f(x) y_n(x)\, x\, dx}{\int_1^e y_n^2(x)\, x\, dx},$$

because $y_n(x)$ are orthogonal with weight x.

III.5 Assume that $y(x)$ and $z(x)$ are non-trivial solutions of

$$y'''' + \lambda y = 0, \quad y(0) = y'(0) = y(L) = y'(L) = 0,$$

$$z'''' + \mu y = 0, \quad z(0) = z'(0) = z(L) = z'(L) = 0,$$

and $\lambda \neq \mu$. Multiply the first of these equations by z, the second one by y, and subtract the results to obtain

$$y''''z - z''''y + (\lambda - \mu)\, yz = 0,$$

$$\frac{d}{dx}\left(y'''z - y''z' + y'z'' - yz'''\right) + (\lambda - \mu)\, yz = 0.$$

Integrate both sides of the last equation over $(0, L)$. In view of the boundary conditions, the first integral is zero. It follows that

$$(\lambda - \mu) \int_0^L y(x)z(x)\, dx = 0.$$

Since $\lambda \neq \mu$, we conclude that $\int_0^L y(x)z(x)\, dx = 0$, proving that the eigenfunctions $y(x)$ and $z(x)$ are orthogonal.

III.6 We may assume that $\alpha > 0$, $\beta < 0$, and $\gamma > 0$, $\delta > 0$. Multiply the equation by $y(x)$ and integrate over $(0, \pi)$:

$$\lambda \int_0^\pi y^2(x)r(x)\, dx = -\int_0^\pi (p(x)y'(x))'\, y(x)\, dx.$$

Perform an integration by parts:

$$-\int_0^\pi (p(x)y'(x))'\, y(x)\, dx = \int_0^\pi p(x)y'^2(x)\, dx - p(\pi)y'(\pi)y(\pi) + p(0)y'(0)y(0)$$

$$= \int_0^\pi p(x)y'^2(x)\, dx + \frac{\gamma}{\delta}y^2(\pi) - \frac{\alpha}{\beta}y^2(0) > 0.$$

On the last step we used the boundary conditions to express $y'(\pi)$ and $y'(0)$. Since $\int_0^\pi y^2(x)r(x)\, dx > 0$, it follows that $\lambda > 0$.

III.7 Write the equation in the form

$$(ru)'' + \lambda\, (ru) = 0.$$

Letting $y(r) = ru(r)$, obtain

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0.$$

The eigenvalues of the last problem are $\lambda_m = m^2$, and $y_m(r) = \sin m\, r$ are the corresponding eigenfunctions. For the original problem $\lambda_m = m^2$, and $u_m(r) = \dfrac{\sin m\, r}{r}$ are the eigenpairs, $m = 1, 2, 3, \ldots$.

III.10 The solution has the form

$$u(x, y, t) = \sum_{n,m=1}^{\infty} b_{nm} e^{-k(n^2 + m^2)t} \sin nx \sin my\,,$$

where b_{nm} are the coefficients of the double Fourier sine series of $u(x, y, 0)$ on the rectangle $0 < x < \pi$, $0 < y < \pi$. Here $k = 3$, and

$$u(x, y, 0) = \sin x \cos x \sin y = \frac{1}{2} \sin 2x \sin y\,,$$

so that $b_{21} = \dfrac{1}{2}$, and all other b_{nm} are zero. We conclude that

$$u(x, y, t) = \frac{1}{2} e^{-15t} \sin 2x \sin y\,.$$

III.11 (i) The solution has the form

$$u(x, y, t) = \sum_{n,m=1}^{\infty} b_{nm} \sin \frac{n\pi}{3} x \sin \frac{m\pi}{2} y\,,$$

where b_{nm} are the coefficients of the double Fourier sine series of $u(x, y, 0) = xy - y$ on the rectangle $0 < x < 3$, $0 < y < 2$. Calculate

$$b_{nm} = \frac{4}{6} \int_0^2 \int_0^3 (xy - y) \sin \frac{n\pi}{3} x \sin \frac{m\pi}{2} y \, dx \, dy$$

$$= \frac{2}{3} \int_0^3 (x - 1) \sin \frac{n\pi}{3} x \, dx \int_0^2 y \sin \frac{m\pi}{2} y \, dy = \frac{8(-1)^m + 16(-1)^{n+m}}{nm\pi^2}\,.$$

(ii) Follows immediately by separation of variables.

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IV.1 Solving the homogeneous equation with the first boundary condition

$$y'' + y = 0\,, \quad y(a) = 0$$

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gives $y_1(x) = \sin(x - a)$, and solving the same equation with the second boundary condition

$$y'' + y = 0, \quad y(b) = 0$$

gives $y_2(x) = \sin(x - b)$. Their Wronskian

$$W = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \sin(x - a)\cos(x - b) - \cos(x - a)\sin(x - b)$$

$$= \sin\left[(x - a) - (x - b)\right] = \sin(b - a).$$

The Green's function is then

$$G(x, \xi) = \begin{cases} \frac{\sin(x-a)\sin(\xi-b)}{\sin(b-a)} & \text{for } a \leq x \leq \xi \\ \frac{\sin(x-b)\sin(\xi-a)}{\sin(b-a)} & \text{for } \xi \leq x \leq b. \end{cases}$$

IV.2   Solving the homogeneous equation with the first boundary condition

$$y'' + y = 0, \quad y(0) = 0$$

gives $y_1(x) = \sin x$, and solving

$$y'' + y = 0, \quad y'(2) + y(2) = 0$$

gives $y_2(x) = -\sin(x-2) + \cos(x-2)$. Using trig identities, their Wronskian simplifies to

$$W = y_1(x)y_2'(x) - y_1'(x)y_2(x) = -\cos 2 - \sin 2.$$

The Green's function is then

$$G(x, \xi) = \begin{cases} \frac{y_1(x)y_2(\xi)}{K} & \text{for } 0 \leq x \leq \xi \\ \frac{y_2(x)y_1(\xi)}{K} & \text{for } \xi \leq x \leq 2, \end{cases}$$

where $K = -\cos 2 - \sin 2$.

IV.3   The corresponding homogeneous equation

$$x^2 y'' + 4xy' + 2y = 0$$

is Euler's equation, with the general solution $y(x) = c_1 x^{-1} + c_2 x^{-2}$. We then find $y_1(x) = x^{-1} - x^{-2}$, and $y_2(x) = x^{-1} - 2x^{-2}$ (using respectively the conditions $y_1(1) = 0$, and $y_2(2) = 0$). Compute

$$W = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \frac{1}{x^4}.$$

To put the equation

$$x^2 y'' + 4xy + 2y = f(x)$$

into the self-adjoint form, divide it by x^2

$$y'' + \frac{4}{x} y' + \frac{2}{x^2} y = \frac{f(x)}{x^2},$$

and then multiply the new equation by the integrating factor, $\mu = e^{\int \frac{4}{x} dx} = x^4$, obtaining

$$\left(x^4 y' \right)' + 2x^2 y = x^2 f(x).$$

Here $p(x) = x^4$, and $K = p(x)W(x) = 1$. Then the Green's function is

$$G(x, \xi) = \begin{cases} (x^{-1} - x^{-2})(\xi^{-1} - 2\xi^{-2}) & \text{for } 1 \le x \le \xi \\ (\xi^{-1} - \xi^{-2})(x^{-1} - 2x^{-2}) & \text{for } \xi \le x \le 2, \end{cases}$$

and the solution is given by $y(x) = \int_1^2 G(x, \xi)\xi^2 f(\xi) \, d\xi.$

Section 7.12.5, Page 322

I.1   Calculate

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \, e^{-isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \, (\cos sx - i \sin sx) \, dx = \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x) \cos sx \, dx,$$

using that the function $(1 - |x|) \cos sx$ is even, while $(1 - |x|) \sin sx$ is odd. Evaluating the last integral, conclude that $F(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2}(1 - \cos s).$

I.2   Using that the function $e^{-|x|} \cos sx$ is even, while $e^{-|x|} \sin sx$ is odd, calculate

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \, (\cos sx - i \sin sx) \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} \cos sx \, dx = \frac{2}{\sqrt{2\pi}} \frac{1}{s^2 + 1}.$$

The last integral was obtained using the formula (12.5), with $y = 1$.

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I.4   Using that the function $e^{-ax^2}\cos sx$ is even, while $e^{-ax^2}\sin sx$ is odd, calculate

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2} e^{-isx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\left(\cos sx - i\sin sx\right)\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax^2}\cos sx\, dx\,.$$

In the last integral we make a change of variables $x \to z$, by setting $\sqrt{a}\,x = z$, $dx = \frac{1}{\sqrt{a}}\, dz$. Obtain

$$F(s) = \frac{2}{\sqrt{2a\pi}}\int_{0}^{\infty} e^{-z^2}\cos\frac{s}{\sqrt{a}}\, z\, dz = \frac{1}{\sqrt{2a}} e^{-\frac{s^2}{4a}}\,.$$

On the last step we used the formula (12.2).

I.5   We need to calculate two more integrals. One has

$$\int_{-\infty}^{\infty} e^{-a\xi^2}\, d\xi = \sqrt{\frac{\pi}{a}}\,,\quad \text{for any } a > 0\,.$$

This follows by letting $\sqrt{a}\,\xi = x$, and using the integral (12.1) in the text. Also

$(*)$
$$\int_{-\infty}^{\infty} e^{-a\xi^2 + b\xi}\, d\xi = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}\,,\quad \text{for any } a > 0, \text{ and any } b.$$

This follows by completing the square $-a\xi^2 + b\xi = -a\left(\xi - \frac{b}{2a}\right)^2 + \frac{b^2}{4a}$, letting $u = \xi - \frac{b}{2a}$, and using the first integral

$$\int_{-\infty}^{\infty} e^{-a\xi^2 + b\xi}\, d\xi = e^{\frac{b^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(\xi - \frac{b}{2a}\right)^2}\, d\xi = e^{\frac{b^2}{4a}}\int_{-\infty}^{\infty} e^{-au^2}\, du = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}\,.$$

We now use the solution formula (12.8) in the text, square out $(x - \xi)^2$, and simplify

$$u(x, t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} e^{-\xi^2}\, d\xi = \frac{1}{2\sqrt{\pi t}}\, e^{-\frac{x^2}{4t}}\int_{-\infty}^{\infty} e^{-\left(\frac{1}{4t}+1\right)\xi^2 + \frac{x}{2t}\xi}\, d\xi\,.$$

Using the formula $(*)$, evaluate

$$\int_{-\infty}^{\infty} e^{-\left(\frac{1}{4t}+1\right)\xi^2 + \frac{x}{2t}\xi}\, d\xi = \frac{2\sqrt{\pi t}}{\sqrt{1+4t}}\, e^{\frac{x^2}{4t+16t^2}}\,.$$

Using this in the preceding formula, and simplifying, obtain

$$u(x, t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}} .$$

I.7   The solutions are $u(x, y) = c\, y$, with c arbitrary.

I.8   By Poisson's formula

$$u(x, y) = \frac{y}{\pi} \int_{-1}^{1} \frac{d\xi}{y^2 + (\xi - x)^2} = \frac{y}{\pi} \frac{1}{y} \tan^{-1} \frac{\xi - x}{y} \Big|_{\xi=-1}^{\xi=1}$$

$$= \frac{1}{\pi} \left(\tan^{-1} \frac{x+1}{y} - \tan^{-1} \frac{x-1}{y} \right) ,$$

using that arctangent is an odd function.

Chapter 8

Section 8.5.1, Page 342

1   Here $c = 2$. D'Alembert's formula gives

$$u(x, t) = \frac{x - 2t + x + 2t}{2} + \frac{1}{4} \int_{x-2t}^{x+2t} \cos \tau \, d\tau$$

$$= x + \frac{1}{4} [\sin(x + 2t) - \sin(x - 2t)] = x + \frac{1}{2} \cos x \sin 2t ,$$

by using a trig identity.

2   Here $c = 1$. For the point $(3, 1)$, the left characteristic $x - t = 2$ intersects the x-axis at $x = 2$, and the right characteristic $x + t = 4$ intersects the x-axis at $x = 4$. D'Alembert's formula gives

$$u(3, 1) = \frac{2^2 + 4^2}{2} + \frac{1}{2} \int_2^4 \tau \, d\tau = 13 .$$

For the point $(1, 3)$, the left characteristic $x - t = -2$ reaches the t-axis at $t = 2$, then it reflects and reaches the x-axis at $x = 2$. The right characteristic $x + t = 4$ intersects the x-axis at $x = 4$. Obtain

$$u(1, 3) = \frac{-2^2 + 4^2}{2} + \frac{1}{2} \int_2^4 \tau \, d\tau = 9 .$$

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3 Here $c = 2$. The even extension of $f(x) = x$ is $f_e(x) = |x|$, while $g(x) = \cos x$ is already even (so that $g_e(x) = g(x)$).

4 We shall find the solution at an arbitrary point (x_0, y_0). Here $c = 2$. The left characteristic at (x_0, y_0) is $x - 2t = x_0 - 2t_0$. It intersects the x-axis at the point $(x_0 - 2t_0, 0)$. The right characteristic at (x_0, y_0) is $x + 2t = x_0 + 2t_0$. It intersects the the x-axis at the point $(x_0 + 2t_0, 0)$. Obtain

$$u(x_0, t_0) = \frac{1}{4} \iint_\Delta x \, dx dt \, ,$$

where Δ is the characteristic triangle with vertices (x_0, y_0), $(x_0 - 2t_0, 0)$, and $(x_0 + 2t_0, 0)$. The double integral is evaluated as a "type II" domain:

$$\iint_\Delta x \, dx dt = \int_0^{t_0} \left(\int_{2t+x_0-2t_0}^{-2t+x_0+2t_0} x \, dx \right) dt$$

$$= \frac{1}{2} \int_0^{t_0} \left[(-2t + x_0 + 2t_0)^2 - (2t + x_0 - 2t_0)^2 \right] dt$$

$$= -\frac{1}{12} (-2t + x_0 + 2t_0)^3 \Big|_0^{t_0} - \frac{1}{12} (2t + x_0 - 2t_0)^3 \Big|_0^{t_0} = 2x_0 t_0^2 \, .$$

So that $u(x_0, t_0) = \frac{1}{2} x_0 t_0^2$. Replacing $(x_0, t_0) \to (x, t)$, we conclude

$$u(x, t) = \frac{1}{2} x t^2 \, .$$

5 We shall find the solution at an arbitrary point (x_0, y_0). Here $c = 2$. The left characteristic at (x_0, y_0) is $x - 2t = x_0 - 2t_0$. It intersects the x-axis at the point $(x_0 - 2t_0, 0)$. The right characteristic at (x_0, y_0) is $x + 2t = x_0 + 2t_0$. It intersects the the x-axis at the point $(x_0 + 2t_0, 0)$. Obtain

$$u(x_0, t_0) = \frac{1}{4} \int_{x_0 - 2t_0}^{x_0 + 2t_0} \cos \tau \, d\tau + \frac{1}{4} \iint_\Delta (x + 3t) \, dx dt \, ,$$

where Δ is the characteristic triangle with vertices (x_0, y_0), $(x_0 - 2t_0, 0)$, and $(x_0 + 2t_0, 0)$. Using trig identities calculate

$$\frac{1}{4} \int_{x_0 - 2t_0}^{x_0 + 2t_0} \cos \tau \, d\tau = \frac{1}{4} \sin(x_0 + 2t_0) - \frac{1}{4} \sin(x_0 - 2t_0) = \frac{1}{2} \cos x_0 \sin 2t_0 \, .$$

The double integral is evaluated as a "type II" domain:

$$\iint_\Delta x \, dx dt = \int_0^{t_0} \left(\int_{2t+x_0-2t_0}^{-2t+x_0+2t_0} (x + 3t) \, dx \right) dt$$

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$$= \int_0^{t_0} \left(\frac{1}{2}(x+3t)^2 \Big|_{x=2t+x_0-2t_0}^{x=-2t+x_0+2t_0} \right) dt$$

$$= \frac{1}{2} \int_0^{t_0} \left[(x_0 + 2t_0 + t)^2 - (x_0 - 2t_0 + 5t)^2 \right] dt = 2x_0 t_0^2 + 2t_0^3 .$$

It follows that $u(x_0, t_0) = \frac{1}{2} \cos x_0 \sin 2t_0 + \frac{1}{2} x_0 t_0^2 + \frac{1}{2} t_0^3$. Replacing $(x_0, t_0) \rightarrow (x, t)$, we conclude

$$u(x, t) = \frac{1}{2} \cos x \sin 2t + \frac{1}{2} xt^2 + \frac{1}{2} t^3 .$$

6   The functions $f(x) = x$ and $g(x) = \sin x$ are odd, so that $f_o(x) = x$ and $g_o(x) = \sin x$, and by D'Alembert's formula (with $c = 2$)

$$u(x, t) = \frac{x - 2t + x + 2t}{2} + \frac{1}{4} \int_{x-2t}^{x+2t} \sin \tau \, d\tau = x + \frac{1}{2} \sin x \sin 2t .$$

7   We shall find the solution at an arbitrary point (x_0, y_0). Here $c = 2$. The left characteristic at (x_0, y_0) is $x - 2t = x_0 - 2t_0$. It intersects the x-axis at the point $(x_0 - 2t_0, 0)$. If $x_0 - 2t_0 \geq 0$, this point lies in the physical domain (where $x \geq 0$), and if $x_0 - 2t_0 < 0$ then a reflection occurs.

Case (i) $x_0 - 2t_0 \geq 0$. The right characteristic at (x_0, y_0) is $x + 2t = x_0 + 2t_0$. It intersects the x-axis at the point $(x_0 + 2t_0, 0)$. D'Alembert's formula gives

$$u(x_0, t_0) = \frac{(x_0 - 2t_0)^2 + (x_0 + 2t_0)^2}{2} + \frac{1}{4} \int_{x_0 - 2t_0}^{x_0 + 2t_0} \cos \tau \, d\tau$$

$$= x_0^2 + 4t_0^2 + \frac{1}{2} \cos x_0 \sin 2t_0 .$$

Case (ii) $x_0 - 2t_0 < 0$. After a reflection the left characteristics lands at the point $2t_0 - x_0 > 0$ on the x-axis. Conclude

$$u(x_0, t_0) = \frac{-(2t_0 - x_0)^2 + (x_0 + 2t_0)^2}{2} + \frac{1}{4} \int_{2t_0 - x_0}^{x_0 + 2t_0} \cos \tau \, d\tau$$

$$= 4x_0 t_0 + \frac{1}{2} \cos x_0 \sin 2t_0 .$$

Replacing $(x_0, t_0) \rightarrow (x, t)$, we conclude

$$u(x, t) = \begin{cases} x^2 + 4t^2 + \frac{1}{2} \cos x \sin 2t & \text{for } x \geq 2t \\ \\ 4xt + \frac{1}{2} \cos x \sin 2t & \text{for } x < 2t . \end{cases}$$

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8   Here $c = 2$. The left characteristic passing through the point $(3, 1)$ is $x - 2t = 1$. It intersects the x-axis at $x = 1$. The right characteristic passing through the point $(3, 1)$ is $x + 2t = 5$. It intersects the x-axis at $x = 5$. The initial displacement is $f(x) = x+1$, and the initial velocity is $g(x) = 0$. Since both characteristics land in the physical domain $x \geq 0$, the D'Alembert's formula is without change:

$$u(3, 1) = \frac{f(1) + f(5)}{2} = 4 \,.$$

For the point $(1, 3)$ the left characteristic is $x - 2t = -5$. It intersects the x-axis at $x = -5$. When this characteristic is reflected at the y-axis, it comes down at $x = 5$. The right characteristic passing through the point $(1, 3)$ is $x + 2t = 7$. It intersects the x-axis at $x = 7$. Obtain

$$u(1, 3) = \frac{-f(5) + f(7)}{2} = 1 \,.$$

9   We shall find the solution at an arbitrary point (x_0, y_0). Here $c = 1$. The left characteristic at (x_0, y_0) is $x - t = x_0 - t_0$. It intersects the x-axis at the point $(x_0 - t_0, 0)$. If $x_0 - t_0 \geq 0$, this point lies in the physical domain (where $x \geq 0$), and if $x_0 - t_0 < 0$ then a reflection occurs.

Case (i) $x_0 - t_0 \geq 0$. The right characteristic at (x_0, y_0) is $x + t = x_0 + t_0$. It intersects the x-axis at the point $(x_0 + t_0, 0)$. D'Alembert's formula gives

$$u(x_0, t_0) = \frac{(x_0 - t_0)^2 + (x_0 + t_0)^2}{2} + \frac{1}{2} \int_{x_0 - t_0}^{x_0 + t_0} \tau \, d\tau = x_0^2 + x_0 t_0 + t_0^2 \,.$$

Case (ii) $x_0 - t_0 < 0$. After a reflection the left characteristics lands at the point $t_0 - x_0 > 0$ on the x-axis. Conclude

$$u(x_0, t_0) = \frac{-(t_0 - x_0)^2 + (x_0 + t_0)^2}{2} + \frac{1}{2} \int_{t_0 - x_0}^{x_0 + t_0} \tau \, d\tau = 3xt \,.$$

Replacing $(x_0, t_0) \to (x, t)$, we conclude

$$u(x, t) = \begin{cases} x^2 + xt + t^2 & \text{for } x \geq t \\ \\ 3xt & \text{for } x < t \,. \end{cases}$$

10   Here $c = 2$. The left characteristic passing through the point $(3, 1)$ is $x - 2t = 1$. It intersects the x-axis at $x = 1$. The right characteristic passing through the point $(3, 1)$ is $x + 2t = 5$. It intersects the x-axis at

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$x = 5$. The initial displacement is $f(x) = x + 1$, and the initial velocity is $g(x) = 0$. Since both characteristics land in the physical domain $x \geq 0$, the D'Alembert's formula is without change:

$$u(3, 1) = \frac{f(1) + f(5)}{2} = 4\,.$$

For the point $(1, 3)$ the left characteristic is $x - 2t = -5$. It intersects the x-axis at $x = -5$. When this characteristic is reflected at the y-axis, it comes down at $x = 5$. The right characteristic passing through the point $(1, 3)$ is $x + 2t = 7$. It intersects the x-axis at $x = 7$. For Neumann boundary condition there is no sign change after a reflection. Obtain

$$u(1, 3) = \frac{f(5) + f(7)}{2} = 7\,.$$

11   Here $c = 1$. The left characteristic passing through the point $(1/2, 2)$ is $x - t = 3/2$. It has slope 1 in the xt-plane, and it intersects the t-axis at $t = 3/2$. It then reflects, and travels with the slope -1, until it lands on the x-axes at $x = 3/2$. The right characteristic passing through the point $(1/2, 2)$ is $x + t = 5/2$. It has slope -1 in the xt-plane, and it intersects the line $x = 2$ at $t = 1/2$. It then reflects, and travels with the slope 1, until it lands on the x-axes at $x = 3/2$. Here $f(x) = x$. Obtain

$$u(1/2, 2) = \frac{-f(3/2) - f(3/2)}{2} = -3/2\,.$$

Since the characteristics have slopes ± 1, one can trace their reflections purely geometrically, without writing their equations. The left characteristic through $(1/3, 3)$ will reach the t-axes at $t = 8/3$, reflect and travel to the right, then reflect from the line $x = 2$ at $t = 2/3$, and land at the x-axes at $x = 4/3$, with a total of two reflections. The right characteristic through $(1/3, 3)$ will reach the line $x = 2$ at $t = 4/3$, reflect and land at the x-axes at $x = 2/3$, with a total of one reflection. Since the "price" of each reflection is a change of sign, obtain

$$u(1/3, 3) = \frac{-f(2/3) + f(4/3)}{2} = 1/3\,.$$

12   The characteristics follow the same path as in the preceding problem. However, for the Neumann boundary conditions there is no change of sign after a reflection. Obtain

$$u(1/2, 2) = \frac{f(3/2) + f(3/2)}{2} = 3/2\,,$$

$$u(1/3, 3) = \frac{f(2/3) + f(4/3)}{2} = 1 \,.$$

15 Using the equation, then integrating by parts, calculate

$$E'(t) = 2 \int_0^1 u u_t \, dx = 10 \int_0^1 u u_{xx} \, dx = 10 u u_x \big|_0^1 - 10 \int_0^1 u_x^2 \, dx$$

$$= -10 \int_0^1 u_x^2 \, dx \le 0 \,.$$

The term $u u_x \big|_0^1$ is zero, because of the boundary conditions.

16 Using the equation, then integrating by parts, calculate

$$E'(t) = 2 \int_0^1 u u_t \, dx = -10 \int_0^1 u_x^2 \, dx - 2 \int_0^1 u^4 \, dx + 2 \int_0^1 u^2 u_x \, dx$$

$$= -10 \int_0^1 u_x^2 \, dx - 2 \int_0^1 u^4 \, dx \le 0 \,,$$

because (in view of the boundary conditions)

$$\int_0^1 u^2 u_x \, dx = \frac{1}{3} u^3 \big|_0^1 = \frac{1}{3} u^3(1) - \frac{1}{3} u^3(0) = 0 \,.$$

From the initial condition

$$E(0) = \int_0^1 u^2(x, 0) \, dx = 0 \,.$$

It follows that $E(t) = 0$ for all t, so that $u(x, t) = 0$.

17 For any differentiable function $f(z)$, the functions $u_1(x, t) = f(x - 3t)$ and $u_2(x, t) = f(x + 3t)$ are solutions of the wave equation

$$u_{tt} - 9 u_{xx} = 0 \,.$$

Section 8.6.1, Page 351

I.1 Here the solution is prescribed along the x-axis. The equation to find the characteristics is

$$\frac{dy}{dx} = 1 \,.$$

The characteristics are the straight lines $y = x + c$. The one passing through a point (x_0, y_0) is

$$y = x + y_0 - x_0 \,.$$

It intersects the x axis at $x = x_0 - y_0$.

Choosing y as the parameter, the PDE becomes
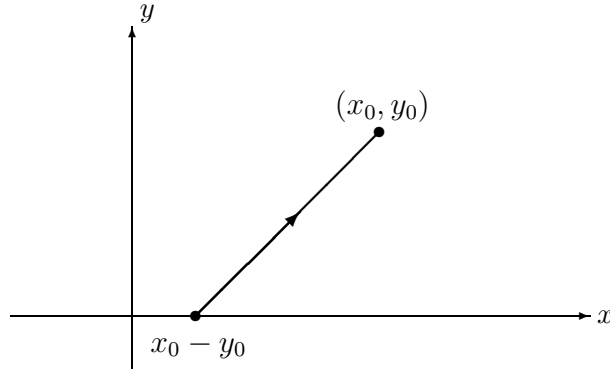
$$\frac{du}{dy} = 1 \, .$$

We integrate this equation along the characteristic line, between the points $(x_0 - y_0, 0)$ and (x_0, y_0), or between the parameter values of $y = 0$ and $y = y_0$

$$\int_0^{y_0} \frac{du}{dy} \, dy = \int_0^{y_0} dy \, ,$$

$$u(x_0, y_0) - u(x_0 - y_0, 0) = y_0 \, ,$$

$$u(x_0, y_0) = u(x_0 - y_0, 0) + y_0 = e^{x_0 - y_0} + y_0 \, .$$

Finally, replace the arbitrary point (x_0, y_0) by (x, y) to conclude that $u(x, y) = e^{x-y} + y$.

Integration along the characteristic line

I.2   Compared with the Example 4 in the text, only the picture changes, from the first quarter to the third quarter. The data is given along the line $y = x$. The characteristics are the solutions of

$$\frac{dy}{dx} = -\frac{y}{x} \, ,$$

which are the hyperbolas $y = \dfrac{c}{x}$. The one passing through the point (x_0, y_0) is

$$y = \frac{x_0 y_0}{x} \, .$$

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Connect any point (x_0, y_0) in the third quarter, along this hyperbola to the data line $y = x$. Then proceed exactly as in the text, to obtain the same solution.

I.3   The characteristics are the solutions of

$$\frac{dy}{dx} = \frac{y}{x},$$

which are the lines $y = cx$. The one passing through the point (x_0, y_0) is $y = \dfrac{y_0}{x_0}x$. It intersects the line $x + y = 1$ (on which the data is given) at the point $\left(\dfrac{x_0}{x_0 + y_0}, \dfrac{y_0}{x_0 + y_0}\right)$. To take x as the parameter, begin by dividing our PDE by x:

$$u_x + \frac{y}{x}u_y + \frac{1}{x}u = 1.$$

Then the PDE becomes

$$u_x + u_y \frac{dy}{dx} + \frac{1}{x}u = 1,$$

$$\frac{du}{dx} + \frac{1}{x}u = 1,$$

or

$$\frac{d}{dx}(xu) = x.$$

We integrate along the characteristic curve, between the points $\left(\dfrac{x_0}{x_0 + y_0}, \dfrac{y_0}{x_0 + y_0}\right)$ and (x_0, y_0)

$$\int_{\frac{x_0}{x_0 + y_0}}^{x_0} \frac{d}{dx}(xu)\ dx = \int_{\frac{x_0}{x_0 + y_0}}^{x_0} x\ dx,$$

$$x_0 u(x_0, y_0) - \frac{x_0}{x_0 + y_0} u\left(\frac{x_0}{x_0 + y_0}, \frac{y_0}{x_0 + y_0}\right) = \frac{1}{2}x_0^2 - \frac{1}{2}\frac{x_0^2}{(x_0 + y_0)^2}.$$

Use that $u\left(\dfrac{x_0}{x_0 + y_0}, \dfrac{y_0}{x_0 + y_0}\right) = 1$, to obtain

$$u(x_0, y_0) = \frac{1}{x_0 + y_0} + \frac{x_0}{2} - \frac{x_0}{2(x_0 + y_0)^2}.$$

Finally, replace the arbitrary point (x_0, y_0) by (x, y) to conclude

$$u(x, y) = \frac{1}{x + y} + \frac{x}{2} - \frac{x}{2(x + y)^2}.$$

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I.4   The data is given along the x-axis. The characteristics are solutions of

$$\frac{dx}{dy} = \sin y \,,$$

which are $x = -\cos y + c$. The one passing through the point (x_0, y_0) is

(3) $x = -\cos y + x_0 + \cos y_0 \,.$

It intersects the x-axis at $x = -1 + x_0 + \cos y_0$. We use y as the parameter. The original equation becomes, along the characteristic curve,

$$\frac{du}{dy} = -\cos y + x_0 + \cos y_0 \,.$$

We integrate along the characteristic curve, between the points $(-1 + x_0 + \cos y_0, 0)$ and (x_0, y_0), or between the parameter values of $y = 0$ (where $x = -1 + x_0 + \cos y_0$) and $y = y_0$ (where $x = x_0$)

$$\int_0^{y_0} \frac{du}{dy}\, dy = \int_0^{y_0} (-\cos y + x_0 + \cos y_0)\ dy \,,$$

$$u(x_0, y_0) - u(-1 + x_0 + \cos y_0, 0) = -\sin y_0 + (x_0 + \cos y_0)\, y_0 \,,$$

$$u(x_0, y_0) = u(-1 + x_0 + \cos y_0, 0) - \sin y_0 + (x_0 + \cos y_0)\, y_0$$

$$= (-1 + x_0 + \cos y_0)^2 - \sin y_0 + (x_0 + \cos y_0)\, y_0 \,.$$

Replacing $(x_0, y_0) \to (x, y)$, obtain the solution

$$u(x, y) = (-1 + x + \cos y)^2 - \sin y + (x + \cos y)\, y \,.$$

I.5   The general solution can be obtained by prescribing arbitrary data, on any non-characteristic line. We shall prescribe that

$$u(x, 0) = f(x) \,,$$

where $f(x)$ is an arbitrary function. The characteristics are the solutions of

$$\frac{dy}{dx} = \frac{1}{2} \,,$$

which are the lines $y = \frac{1}{2} x + c$. The characteristic passing through the point (x_0, y_0) is $y = \frac{1}{2} x + y_0 - \frac{1}{2} x_0$. It intersects the x-axis at the point $x = x_0 - 2 y_0$. To take x as a parameter, we divide the equation by 2

$$u_x + \frac{1}{2} u_x = \frac{x}{2}$$

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and obtain
$$\frac{du}{dx} = \frac{x}{2}$$
along the characteristics. Integrating along the characteristic curve
$$\int_{x_0-2y_0}^{x_0} \frac{du}{dx} \, dx = \int_{x_0-2y_0}^{x_0} \frac{x}{2} \, dx \,,$$
$$u(x_0, y_0) - u(x_0 - 2y_0, 0) = x_0 y_0 - y_0^2 \,.$$
Using that $u(x_0 - 2y_0, 0) = f(x_0 - 2y_0)$, obtain
$$u(x_0, y_0) = f(x_0 - 2y_0) + x_0 y_0 - y_0^2 \,.$$
Finally, replace the arbitrary point (x_0, y_0) by (x, y) to conclude
$$u(x, y) = f(x - 2y) + xy - y^2 \,,$$
where f is an arbitrary function.

I.6   The characteristics are the lines $y = \frac{1}{2}x + c$. The solution is prescribed along the line $y = \frac{1}{2}x$, which is one of the characteristics. The solution cannot be arbitrarily prescribed along a characteristic line, because the value of solution at any point determines the values of solution for all points along a characteristic line.

I.8   The characteristics are the solutions of
$$\frac{dy}{dx} = 2\frac{y}{x} \,,$$
which are the parabolas $y = cx^2$. The one passing through the point (x_0, y_0) is $y = \frac{y_0}{x_0^2} x^2$. It intersects the line $x = 1$ (on which the data is given) at the point $\left(1, \frac{y_0}{x_0^2}\right)$. To take x as the parameter, begin by dividing our PDE by x:
$$u_x + 2\frac{y}{x}u_y + \frac{y}{x^2}u = 0 \,.$$
Along the characteristics this equation becomes
$$\frac{du}{dx} + \frac{y}{x^2}u = 0 \,,$$
$$\frac{du}{dx} + \frac{y_0}{x_0^2}u = 0 \,.$$

Multiplying by the integrating factor $e^{\frac{y_0}{x_0^2}x}$, convert this to

$$\frac{d}{dx}\left[e^{\frac{y_0}{x_0^2}x}u(x,y)\right]=0\,.$$

Integrate in x between $x=1$ and $x=x_0$ (or between the points $\left(1,\frac{y_0}{x_0^2}\right)$ and (x_0,y_0) along the characteristic curve)

$$e^{\frac{y_0}{x_0}}u(x_0,y_0)-e^{\frac{y_0}{x_0^2}}u(1,\frac{y_0}{x_0^2})=0\,,$$

$$u(x_0,y_0)=e^{\frac{y_0}{x_0^2}-\frac{y_0}{x_0}}u(1,\frac{y_0}{x_0^2})=e^{\frac{y_0}{x_0^2}-\frac{y_0}{x_0}}f(\frac{y_0}{x_0^2})\,,$$

$$u(x,y)=e^{\frac{y}{x^2}-\frac{y}{x}}f(\frac{y}{x^2})\,.$$

Section 8.12.1, Page 369

I.1   The function $v(x,y)\ge 0$ is harmonic in the entire plane. By Liouville's theorem $v(x,y)$ is a constant. It follows that $u(x,y)=v(x,y)-12$ is a constant.

I.2   The function $v(x,y)>0$ is harmonic in the entire plane. By Liouville's theorem $v(x,y)$ is a constant. It follows that $u(x,y)=-v(x,y)$ is a constant.

I.3   Assume that a harmonic in the entire plane function $u(x,y)$ is bounded from above by some constant A, so that $u(x,y)\le A$, for all x and y. Then the function $v(x,y)=A-u(x,y)$ is non-negative and harmonic in the entire plane. By Liouville's theorem $v(x,y)$ is a constant. It follows that $u(x,y)=A-v(x,y)$ is a constant too.

I.4   By the strong maximum principle, a harmonic in D function $u(x,y)$ assumes both its minimum and maximum values on the boundary ∂D. Both of the extreme values are equal to 5, because $u(x,y)=5$ on ∂D. It follows that $u(x,y)=5$ in D.

I.5   By Poisson's integral formula the integral

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\,f(\phi)\,d\phi$$

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gives a function harmonic in the disc $r < R$, and equal to $f(\theta)$ on the boundary of that disc $r = R$. If we take $f(\theta)$, then $u(r, \theta) = 1$ in the disc $r < R$, similarly to the preceding problem. It follows that

$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \, d\phi = 2\pi \,.$$

I.6   Calculate $\Delta v = 0$, and observe that $v(x, y) = \dfrac{1}{4}\left(x^2 + y^2\right)$ on ∂D. The harmonic function $v(x, y)$ assumes both its minimum and maximum values on the boundary ∂D. One has

$$\frac{1}{4} \leq \frac{1}{4}\left(x^2 + y^2\right) \leq \frac{1}{2} \text{ on } \partial D \,.$$

It follows that

$$\frac{1}{4} \leq v(x, y) \leq \frac{1}{2} \text{ in } D \,,$$

and, in particular, $\dfrac{1}{4} \leq v(0, 0) \leq \dfrac{1}{2}$. But $v(0, 0) = u(0, 0)$.

I.7   Assume that $u > 1$ somewhere in D. Using that $u = 0$ on ∂D, we conclude that there is point (x_0, y_0) at which $u(x, y)$ achieves its local maximum, and $u(x_0, y_0) > 1$. At the point of maximum, $\Delta u(x_0, y_0) = u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) \leq 0$. We now evaluate the given equation at the point (x_0, y_0):

$$\Delta u(x_0, y_0) + u^2(x_0, y_0)\left(1 - u(x_0, y_0)\right) = 0 \,.$$

The first term on the left is non-positive, while the second one is strictly negative. We have a contradiction, proving that $u(x, y) \leq 1$. One shows similarly that $u(x, y) \geq 0$.

I.8   Similarly to the preceding problem, one shows that $u(x, y)$ cannot assume either positive or negative values inside D. It follows that the only solution is $u(x, y) = 0$, the trivial solution.

I.9   Calculate
$$\Delta u = 16x^2 + 16y^2 \geq 0 \,.$$

By the maximum principle, $u(x, y)$ takes on its maximum value on the circle $x^2 + y^2 = 4$, which is the boundary of the disc $x^2 + y^2 \leq 4$. On that circle $x = 2\cos\theta$, $y = 2\sin\theta$, and

$$u = \left(x^2 + y^2\right)^2 - x^2 + y^2 = 16 - 4\cos 2\theta \,.$$

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The maximum value of 20 is achieved at $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$, or at the points $(0, \pm 2)$.

I.10   Assume that $u(x, y)$ is positive somewhere inside D. Because $u(x, y) = 0$ on the boundary ∂D, it follows that $u(x, y)$ has a point of global maximum (x_0, y_0) inside D, $u(x_0, y_0) > 0$. At that point $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$, $\Delta u(x_0, y_0) \leq 0$. It follows that the left hand side of the equation is negative, a contradiction. Hence, $u(x, y)$ cannot assume positive values. One shows similarly that $u(x, y)$ cannot assume negative values. It follows that $u(x, y) \equiv 0$.

I.11   Observe that for all x and t

$$x^2 + t^2 - t + 1 > 0 ,$$

while the initial and boundary conditions are non-negative on the parabolic boundary. By the comparison theorem for the heat equation, $u(x, t) > 0$ in the parabolic domain.

I.12   By the mean value property of harmonic functions, the value of $u(0, 0)$ is equal to the average of the values $u(x, y)$ on the circle of radius 3 around the origin. Given that the continuous function $u(x, y)$ is non-negative on this circle, and it is positive at one of the points of the circle, the average has to be positive, so that $u(0, 0) > 0$.

I.13   Recall Harnack's inequality in the form

$$\frac{\max_{B_{R/2}} u(x, y)}{\min_{B_{R/2}} u(x, y)} \leq 9$$

from the text. Here $R = 2$, so that

$$\frac{\max_{B_1} u(x, y)}{\min_{B_1} u(x, y)} \leq 9 \,.$$

Since $u(0, 1) = 10$, it follows that $\max\limits_{B_1} u(x, y) \geq 10$. Similarly from $u(0, 0) = 1$ it follows that $\min\limits_{B_1} u(x, y) \leq 1$. Hence, the above ratio should be at least 10, a contradiction.

I.15   Assume $v(x, t)$ is another solution of this problem, and denote $w(x, t) = u(x, t) - v(x, t)$. Subtracting the equations that $u(x, t)$ and $v(x, t)$ satisfy, obtain

$$w_t - w_{xx} + c(x, t)w = 0 ,$$

where $c(x, t) = u(x, t) + v(x, t)$. By the result of the preceding problem $w(x, t) = 0$, so that $u(x, t) = v(x, t)$, and there is at most one solution.

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To find two solutions of the following nonlinear first order PDE

$$a(x, y)z_x^2 + b(x, y)z_x z_y + c(x, y)z_y^2 = 0$$

with given functions $a(x, y)$, $b(x, y)$ and $c(x, y)$, one sets up the *characteristic equation*

$$a(x, y)y'^2 - b(x, y)y' + c(x, y) = 0 .$$

This is a quadratic equation for $y'(x)$, with two solutions

$$y'(x) = \frac{b(x, y) \pm \sqrt{b^2(x, y) - 4a(x, y)c(x, y)}}{2a} .$$

We have two ODE's to solve. Let one of these equations have a solution that is implicitly defined by $\varphi(x, y) = c$. Then $z = \varphi(x, y)$ is a solution of the nonlinear first order PDE above. Let $z_1 = \xi(x, y)$ and $z_2 = \eta(x, y)$ be two solutions obtained this way. Then the change of variables $(x, y) \to (\xi, \eta)$ trasforms the following second order PDE

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y = 0$$

to a canonical (simpler) form.

II.1   The characteristic equation is

$$y'^2 - y = 0 ,$$

or

$$y = \pm y^{1/2} .$$

Integration gives $2y^{1/2} = \pm x + c$. Solving for c, we obtain two solutions of the given PDE: $z = 2\sqrt{y} - x$, and $z = 2\sqrt{y} + x$.

II.3   The characteristic functions are solutions of the equation

$$z_x^2 - yz_y^2 = 0 ,$$

which was solved in the preceding problem. We make the change of variables $(x, y) \to (\xi, \eta)$ by setting $\xi = 2\sqrt{y} - x$, and $\eta = 2\sqrt{y} + x$. Calculate the partial derivatives $\xi_x = -1$, $\xi_y = y^{-1/2}$, $\xi_{xx} = 0$, $\xi_{xy} = 0$, $\xi_{yy} = -\frac{1}{2}y^{-3/2}$. Similarly,

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$\eta_x = 1$, $\eta_y = y^{-1/2}$, $\eta_{xx} = 0$, $\eta_{xy} = 0$, $\eta_{yy} = -\dfrac{1}{2}y^{-3/2}$. Writing $u(x, y) = u(\xi(x, y), \eta(x, y))$, we use the chain rule to calculate the derivatives:

$$u_x = u_\xi \xi_x + u_\eta \eta_x ,$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} .$$

Similarly

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi y^{-1/2} + u_\eta y^{-1/2} ,$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}$$

$$= u_{\xi\xi}y^{-1} + 2u_{\xi\eta}y^{-1} + u_{\eta\eta}y^{-1} - u_\xi\frac{1}{2}y^{-3/2} - u_\eta\frac{1}{2}y^{-3/2} .$$

Substituting these relations into our equation

$$u_{xx} - yu_{yy} - \frac{1}{2}u_y = 0$$

we obtain the canonical form

$$u_{\xi\eta} = 0 .$$

Integrating, $u = F(\xi) + G(\eta)$, with arbtrary functions F and G. Going back to the original variables x and y, we obtain the solution:

$$u(x, y) = F(2\sqrt{y} - x) + G(2\sqrt{y} + x) .$$

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III.1 Integrate both sides of the equation, and use Green's identity

$$\int_D \Delta u \, dV = \int_D f(x, y, z) \, dV ,$$

$$\int_D \frac{\partial u}{\partial n} \, dV = \int_D f(x, y, z) \, dV .$$

In view of the Neumann boundary condition, the left hand side is zero. Since the right hand side is non-zero, we have a contradiction, proving that there is no solution.

III.2 The difference $w(x, y, z)$ of any two solutions satisfies

$$\Delta w = 0 \ \text{ in } D , \quad \frac{\partial w}{\partial n} = 0 \ \text{ on } S .$$

Multiply this equation by $w(x, y, z)$, and integrate both sides of the equation, then use Green's identity and the boundary condition

$$\int_D w \Delta w \, dV = 0 \,,$$

$$-\int_D |\nabla w|^2 \, dV + \int_D w \frac{\partial w}{\partial n} \, dV = 0 \,,$$

$$\int_D |\nabla w|^2 \, dV = 0 \,.$$

It follows that $|\nabla w| = 0$, which implies that $w(x, y, z)$ is a constant.

III.3 Consider a vector field $F = (u(x, y, z), 0, 0)$, and let $n = (n_1, n_2, n_3)$ be the unit normal vector on the boundary S of D, pointing outside.

(i) $\mathrm{div} F = u_x$, $F \cdot n = u n_1$. By divergence theorem

$$\int_D u_x \, dV = \int_S u n_1 \, dS \,.$$

(ii) Multiply the equation
$$\Delta u + u u_x = 0$$
by u, then integrate over D, to obtain

$$\int_D u \Delta u \, dV + \frac{1}{3} \int_D \frac{\partial}{\partial x} u^3 \, dV = 0 \,.$$

By Green's identity the first term is equal to $-\int_D |\nabla u|^2 \, dV$. Apply part (i) on the second term

$$\int_D \frac{\partial}{\partial x} u^3 \, dV = \int_S u^3 n_1 \, dS = 0 \,,$$

since $u = 0$ on S. It follows that

$$\int_D |\nabla u|^2 \, dV = 0 \,,$$

then $\nabla u = 0$ in D, $u = constant$ in D, and the constant is zero because $u = 0$ on S.

(iii) By the product rule

$$\frac{\partial}{\partial x} (uv) = u_x v + u v_x \,.$$

Integrate over D, and apply part(i)

$$\int_D (u_x v + u v_x)\, dV = \int_D \frac{\partial}{\partial x} (uv)\, dV = \int_S uv n_1\, dS\,,$$

or

$$\int_D u_x v\, dV = -\int_D u v_x\, dV + \int_S uv n_1\, dS\,,$$

correcting an error in the book in the last formula.

(iv) Multiply the equation

$$\Delta u + x u^2 u_x = 0$$

by u, then integrate over D, to obtain

(4) $$\int_D u \Delta u\, dV + \frac{1}{4} \int_D x \frac{\partial}{\partial x} u^4\, dV = 0\,.$$

By Green's identity the first term is equal to $-\int_D |\nabla u|^2\, dV$. Apply part (iii) on the second term

$$\int_D x \frac{\partial}{\partial x} u^4\, dV = -\int_D u^4\, dV + \int_S x u^4 n_1\, dS = -\int_D u^4\, dV\,,$$

since $u = 0$ on S. Then (??) takes the form

$$-\int_D |\nabla u|^2\, dV - \frac{1}{4} \int_D u^4\, dV = 0\,.$$

Both of these non-positive integrals must be zero, implying that $u = 0$ on D.

III.5 (ii)   Assume that the nonlinear Dirichlet problem has two solutions satisfying $u(x) > v(x) > 0$, so that

$$\Delta u = f(u) \ \text{in } D\,, \quad u = 0 \ \text{on } S\,,$$

and

$$\Delta v = f(v) \ \text{in } D\,, \quad v = 0 \ \text{on } S\,.$$

Multiply the first of these equations by v, the second one by u, then subtract the results to obtain

$$\Delta u\, v - \Delta v\, u = uv \left(\frac{f(u)}{u} - \frac{f(v)}{v} \right)\,.$$

Integrate both sides of the last identity, then use the second Green's identity and the boundary conditions

$$\int_D (\Delta u \, v - \Delta v \, u) \, dV = \int_D uv \left(\frac{f(u)}{u} - \frac{f(v)}{v} \right) dV \,,$$

$$\int_S \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) dS = \int_D uv \left(\frac{f(u)}{u} - \frac{f(v)}{v} \right) dV \,,$$

$$0 = \int_D uv \left(\frac{f(u)}{u} - \frac{f(v)}{v} \right) dV \,.$$

(Here S is the boundary of D.) By our conditions the right hand side is positive, and we obtain a contradiction, proving that there cannot be two ordered positive solutions.