

Global solution curves for a class of elliptic systems

Philip Korman*

*Department of Mathematical Sciences, University of Cincinnati,
Cincinnati, OH 45221-0025, USA*

Communicated by J. Shi

(Received 3 November 2009; final version received 20 January 2010)

We study curves of positive solutions for a system of elliptic equations of Hamiltonian type on a unit ball. We give conditions for all positive solutions to lie on global solution curves, allowing us to use the analysis similar to the case of one equation, as developed in P. Korman, Y. Li and T. Ouyang [*An exact multiplicity result for a class of semilinear equations*, Commun. PDE 22 (1997), pp. 661–684.] (see also T. Ouyang and J. Shi [*Exact multiplicity of positive solutions for a class of semilinear problems, II*, J. Diff. Eqns. 158(1) (1999), pp. 94–151].). As an application, we obtain some non-degeneracy and uniqueness results. For the one-dimensional case we also prove the positivity for the linearized problem, resulting in more detailed results.

Keywords: elliptic systems; global solution curves; uniqueness and non-degeneracy of solutions

AMS Subject Classification: 35J47

1. Introduction

We apply bifurcation theory approach to a system of Hamiltonian equations on a unit ball, with Dirichlet boundary conditions

$$\begin{aligned} \Delta u + \lambda H_v(u, v) &= 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{for } |x| = 1, \\ \Delta v + \lambda H_u(u, v) &= 0 \quad \text{for } |x| < 1, \quad v = 0 \quad \text{for } |x| = 1, \end{aligned} \quad (1.1)$$

where $H(u, v)$ is a given function and λ is a positive parameter. We assume that H_{vv} and H_{uu} are positive, i.e. this system is of ‘cooperating’ type. Then according to Troy [1], any positive solution is radially symmetric, i.e. $u = u(r)$ and $v = v(r)$, $r = |x|$, and moreover $u'(r) < 0$, $v'(r) < 0$, so that $u(0)$ and $v(0)$ give the maximal values of respective functions. We shall also assume that $H_{uv} \leq 0$. According to Korman and Shi [2], the value of $u(0)$ alone uniquely identifies the solution triple $(\lambda, u(r), v(r))$. We show that the positive solutions of (1.1) lie on global solution curves, i.e. at any solution, either the implicit function theorem or the Crandall–Rabinowitz

*Email: kormanp@math.uc.edu

bifurcation theorem applies. By the above remarks these solution curves can be faithfully represented by two-dimensional curves in $(\lambda, u(0))$ plane. We then identify a special class of systems, for which one can write down the solution of the linearized system at any singular solution (this includes the turning points). We obtain a new non-degeneracy result for power non-linearities, which results in a simple proof of uniqueness. Moreover, we show that all solutions lie on a unique solution curve, and describe this curve for both sublinear and superlinear cases. We use methods of bifurcation theory, as developed in [3] and [4], see also [5] for a review. For recent reviews of work on elliptic systems see [6] and [7].

We get considerably more detailed results for the one-dimensional case. We show that any non-trivial solution of the corresponding linearized problem can be assumed to be positive (in both components). This gives us more general, and more detailed results. In particular, we give a complete description of solution curve for a class of equations, modelling the case of freely supported elastic beam.

To continue the solutions, our main tool is the Crandall–Rabinowitz bifurcation theorem [8], which we recall next.

THEOREM [8] *Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbf{R} \times X$ and F be a continuously differentiable mapping of an open neighbourhood of $(\bar{\lambda}, \bar{x})$ into Y . Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \text{span}\{x_0\}$ be one-dimensional and $\text{codim } R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_{\lambda}(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\text{span}\{x_0\}$ in X , then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s))$, where $s \rightarrow (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s=0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2. Preliminary results

We consider a linear system for $w(r)$ and $z(r)$ on a unit ball

$$\begin{aligned} w''(r) + \frac{n-1}{r} w'(r) + a(r)w + b(r)z &= 0, \quad 0 < r < 1, \\ z'' + \frac{n-1}{r} z' + c(r)w + d(r)z &= 0, \quad 0 < r < 1, \\ w'(0) = z'(0) = w(1) = z(1) &= 0. \end{aligned} \quad (2.1)$$

LEMMA 2.1 *Assume that the given continuous functions $a(r)$, $b(r)$, $c(r)$ and $d(r)$ satisfy*

$$a(r) \leq 0, \quad d(r) \leq 0, \quad b(r) > 0, \quad c(r) > 0 \quad \text{for all } r \in [0, 1]. \quad (2.2)$$

Then for any solution of (2.1), $z'(1)$ and $w'(1)$ are both non-zero, and have the same sign.

Proof Step 1 We claim that $z'(1)$ and $w'(1)$ cannot be of the opposite sign. Assume on the contrary that $w'(1) < 0$, while $z'(1) > 0$. By continuity,

$$w'(r) < 0, \quad \text{and} \quad z'(r) > 0 \quad \text{for } r \text{ close to } 1. \quad (2.3)$$

Let $r_0 \geq 0$ denote the infimum of r 's for which (2.3) holds. We shall assume that $w'(r_0) = 0$, since the other case, when $z'(r_0) = 0$, is similar. Observe that $w''(r_0) \leq 0$ (the opposite inequality $w''(r_0) > 0$ together with $w'(r_0) = 0$, would imply that $w'(r) > 0$

to the right of r_0 , contradicting (2.3)). From (2.3) and the boundary conditions we also conclude that $w(r_0) > 0$ and $z(r_0) < 0$. Hence, the left-hand side of the first equation in (2.1) is negative at r_0 , a contradiction, proving that (2.3) holds for all $r \geq 0$. But that contradicts the boundary conditions at $r = 0$.

Step 2 If (w, z) is a non-trivial solution of (2.1), then we may assume that $w(r)$ and $z(r)$ are both positive near $r = 1$.

To prove that, we begin by noticing that by uniqueness for initial value problems, $w'(1)$ and $z'(1)$ cannot both be zero. We may assume that $w'(1) < 0$, and then by the Step 1, $z'(1) \leq 0$. Clearly, $w(r)$ and $w'(r)$ are positive near $r = 1$, say on the interval $(\beta, 1)$, for some β . If $z(r)$ failed to be positive near $r = 1$, then there are two possibilities. The first possibility is that $z'(1) = 0$ and $z(r) < 0$ on some interval $(\beta_1, 1)$, with $\beta \leq \beta_1$. Then we can find a point $r_1 \in (\beta_1, 1)$, such that $w(r_1) > 0$, $w'(r_1) < 0$, and $z(r_1) < 0$, $z'(r_1) > 0$. As above, we show that these inequalities imply that $w'(r) < 0$ and $z'(r) > 0$ for all $r \in [0, r_1]$, with a contradiction at $r = 0$. The second possibility is that $z(r)$ has infinitely many roots on the interval $(\beta, 1)$. But then we have a contradiction in the second equation of (2.1), at any point of negative minimum of $z(r)$.

Step 3 Since $z(r)$ is positive near $r = 1$, we have $w''(r) + \frac{n-1}{r} w'(r) + a(r)w \leq 0$ near $r = 1$, and hence $w'(1) < 0$ by the Hopf's boundary lemma (see, e.g. [9], p. 107). Similarly, $z'(1) < 0$.

LEMMA 2.2 *Under the condition (2.2), the solution space of the system (2.1) is one-dimensional (i.e. solution (w, z) is a constant multiple of any other solution $(\bar{w}, \bar{z})$).*

Proof If we denote $p = u'(1)$ and $q = v'(1)$, then the vector (p, q) uniquely identifies any solution of the system (2.2) (i.e. the system (2.2) has at most one solution, with these initial conditions). For any constant t , the vector (tp, tq) uniquely identifies the constant multiple of the same solution. If the vector (p_1, q_1) gives rise to another solution of (2.2), then the same is true for the vector $(p + p_1, q + q_1)$, by linearity. So, if the set of vectors (p, q) , giving rise to solutions of (2.2), is not one-dimensional, it is all of $\mathbb{R}^2$. But the vector $(1, -1)$ does not belong to that set by Lemma 2.1, a contradiction. ■

Remark In general, the solution space of (2.1) may be two-dimensional, even in case $n = 1$ (see [10]).

3. Global solution curves for a class of systems

Given a function $H(u, v) \in C^2(\bar{R}_+ \times \bar{R}_+)$, we consider positive solutions (i.e. $u > 0$ and $v > 0$) of the system

$$\begin{aligned} \Delta u + \lambda H_v(u, v) &= 0 & \text{for } |x| < 1, & \quad u = 0 & \text{for } |x| = 1, \\ \Delta v + \lambda H_u(u, v) &= 0 & \text{for } |x| < 1, & \quad v = 0 & \text{for } |x| = 1, \end{aligned} \quad (3.1)$$

depending on a positive parameter λ . We shall assume that

$$H_{vv} > 0, \quad \text{and} \quad H_{uu} > 0 \quad \text{for all } (u, v) > 0. \quad (3.2)$$

By Troy's [1] extension of the classical results of [11], under the condition (3.2) any positive solution of (3.1) is radially symmetric, i.e. $u = u(r)$ and $v = v(r)$, $r = |x|$, and

so (3.1) becomes a system of ODE's

$$\begin{aligned} u'' + \frac{n-1}{r}u' + \lambda H_v(u, v) &= 0 \quad \text{for } r < 1, \quad u'(0) = u(1) = 0, \\ v'' + \frac{n-1}{r}v' + \lambda H_u(u, v) &= 0 \quad \text{for } r < 1, \quad v'(0) = v(1) = 0. \end{aligned} \quad (3.3)$$

We shall also assume that

$$H_{uv} \leq 0, \quad \text{for all } (u, v) \geq 0. \quad (3.4)$$

According to Korman and Shi [2], under the conditions (3.2) and (3.4), the value of $u(0)$ alone (or of $v(0)$) uniquely identifies the solution triple $(\lambda, u(r), v(r))$, i.e. the value of $u(0)$ can be used as a global parameter on solution curves. We shall need the linearized problem, corresponding to (3.3)

$$\begin{aligned} w'' + \frac{n-1}{r}w' + \lambda H_{uv}w + \lambda H_{vv}z &= 0 \quad \text{for } r < 1, \quad w'(0) = w(1) = 0, \\ z'' + \frac{n-1}{r}z' + \lambda H_{uu}w + \lambda H_{uv}z &= 0 \quad \text{for } r < 1, \quad z'(0) = z(1) = 0. \end{aligned} \quad (3.5)$$

We call solution (u, v) of (3.3) *singular* if the linearized system (3.5) has a non-trivial solution. By Lemma 2.2 the solution set of (3.5) is one-dimensional under our assumptions.

LEMMA 3.1 *Assume that a positive solution (u, v) of (3.3) is singular, i.e. the linearized system (3.5) has a non-trivial solution (w, z) . Then*

$$\int_0^1 (H_v(u, v)z + H_u(u, v)w) r^{n-1} dr = \frac{1}{2\lambda} (v'(1)w'(1) + u'(1)z'(1)).$$

Proof One checks that the functions $p(r) = ru_r$, $q(r) = rv_r$ satisfy the system

$$\begin{aligned} p'' + \frac{n-1}{r}p' + \lambda H_{vu}p + \lambda H_{vv}q &= -2\lambda H_v, \\ q'' + \frac{n-1}{r}q' + \lambda H_{uu}p + \lambda H_{uv}q &= -2\lambda H_u. \end{aligned} \quad (3.6)$$

We multiply the first equation in (3.5) by q , and subtract from that the first equation in (3.6), multiplied by z . The result can be put into the form

$$\begin{aligned} (qw' - zp')' - q'w' + z'p' + \frac{n-1}{r}(qw' - zp') \\ + \lambda H_{vu}wq - \lambda H_{vv}zp = 2\lambda H_v z. \end{aligned} \quad (3.7)$$

Similarly, we multiply the second equation in (3.5) by p , and subtract from that the second equation in (3.6), multiplied by w . The result can be put into the form

$$\begin{aligned} (pz' - wq')' - z'p' + q'w' + \frac{n-1}{r}(pz' - wq') \\ + \lambda H_{uv}zp - \lambda H_{uu}qw = 2\lambda H_u w. \end{aligned} \quad (3.8)$$

We now add Equations (3.7) and (3.8), and put the result in the form

$$[r^{n-1}(qw' - zp')] + [r^{n-1}(pz' - wq')] = r^{n-1}(2\lambda H_{vz} + 2\lambda H_{uw}).$$

Integrating over the interval $(0, 1)$, we conclude the proof. ■

Positive solutions of the system (3.1) satisfy $u'(1) \leq 0$, $v'(1) \leq 0$. We shall assume that one of these derivatives is non-zero:

$$|u'(1)| + |v'(1)| > 0. \quad (3.9)$$

For concrete systems, this condition will usually follow by the Hopf's boundary lemma.

THEOREM 3.9 *Assume that the conditions (3.2), (3.4) and (3.1) are satisfied for the system (3.1). Then all positive solutions of (3.1) lie on global continuous non-intersecting curves in $(\lambda, u(0))$ plane (or in $(\lambda, v(0))$ plane).*

Proof We show that at any positive solution of (3.1) either the implicit function theorem or the Crandall–Rabinowitz bifurcation theorem applies, and hence we can always continue the solutions (that is what we mean by ‘the global curves’). To recast our system in the operator form, we define the spaces $\mathbf{X} = \{(u(r), v(r)) | u, v \in C^{2,\alpha}[0, 1], u'(0) = u(1) = v'(0) = v(1) = 0\}$ and $\mathbf{Y} = \{(u(r), v(r)) | u, v \in C^\alpha[0, 1]\}$, and consider the map

$$F(u, v) = \begin{pmatrix} u'' + \frac{n-1}{r}u' + \lambda H_v(u, v) \\ v'' + \frac{n-1}{r}v' + \lambda H_u(u, v) \end{pmatrix} : \mathbf{X} \rightarrow \mathbf{Y}.$$

Clearly, our system (3.3) can be recast in the form $F(u, v) = 0$, and the system (3.5) gives the linearized problem.

We claim that the linearized operator $F_{(u,v)}(p, q)$ is a Fredholm operator of index zero. This can be seen by recasting the linearized equations as a system of two integral equations, and then the solution operator defines a compact operator on the product space $\mathbf{Y} \rightarrow \mathbf{Y}$, so that we can use the Riesz–Schauder theory of compact operators. If (3.5) has only the trivial solution, then $F_{(u,v)}(p, q)$ is one-to-one, and hence onto, and the implicit function theorem applies, and we can continue the solution to nearby λ 's. Now assume that at some $(\bar{\lambda}, \bar{u}, \bar{v})$ the system (3.5) has a non-trivial solution. Then by Lemma 2.2 the solution set of (3.5), i.e. the null-space of $F_{(u,v)}(\bar{u}, \bar{v})$ is one-dimensional. Hence, the range of the map $F_{(u,v)}(\bar{u}, \bar{v})$ has codimension one. To apply the Crandall–Rabinowitz Theorem 1.1, it remains to show that $F_{\bar{\lambda}}(\bar{\lambda}, \bar{x}) \notin R(F_{\bar{\lambda}}(\bar{\lambda}, \bar{x}))$, with $\bar{x} = (\bar{u}, \bar{v})$. Assuming otherwise, there would exist $(W(x), Z(x)) \in X$ so that

$$\begin{aligned} W'' + \frac{n-1}{r}W' + \lambda H_{uv}W + \lambda H_{vv}Z &= H_v, & W'(0) = W(1) &= 0, \\ Z'' + \frac{n-1}{r}Z' + \lambda H_{uu}W + \lambda H_{uv}Z &= H_u, & Z'(0) = Z(1) &= 0. \end{aligned}$$

Proceeding the same way as in the derivation of (3.8) from (3.6), we get

$$\int_0^1 (H_v(u, v)z + H_u(u, v)w) r^{n-1} dr = 0.$$

However by Lemma 3.1, and our assumptions, this integral is not zero, a contradiction. ■

4. Special systems

We consider positive solutions of the system

$$\begin{aligned} u'' + \frac{n-1}{r}u' + f(v) &= 0 \quad \text{for } r < 1, \quad u'(0) = u(1) = 0, \\ v'' + \frac{n-1}{r}v' + g(u) &= 0 \quad \text{for } r < 1, \quad v'(0) = v(1) = 0, \end{aligned} \quad (4.1)$$

which is a subclass of the system (3.1) (or of the system (3.3)). (Here $H(u, v) = F(v) + G(u)$, where F and G are anti-derivatives of f and g respectively.) To be consistent with the conditions of Theorem 3.1, we shall assume that

$$f(0) \geq 0, \quad g(0) \geq 0 \quad \text{and } f'(t) > 0, \quad g'(t) > 0 \quad \text{for all } t > 0. \quad (4.2)$$

The corresponding linearized problem is

$$\begin{aligned} w'' + \frac{n-1}{r}w' + f'(v)z &= 0 \quad \text{for } r < 1, \quad w'(0) = w(1) = 0, \\ z'' + \frac{n-1}{r}z' + g'(u)w &= 0 \quad \text{for } r < 1, \quad z'(0) = z(1) = 0. \end{aligned} \quad (4.3)$$

We shall also consider a linear problem

$$\begin{aligned} p'' + \frac{n-1}{r}p' + f'(v)q &= 0 \quad \text{for } r < 1, \quad p'(0) = 0, \\ q'' + \frac{n-1}{r}q' + g'(u)p &= 0 \quad \text{for } r < 1, \quad q'(0) = 0, \end{aligned} \quad (4.4)$$

which is different from (4.3), as we do not impose the boundary conditions at $r = 1$.

For some systems it is possible to write down explicitly the solution of the problem (4.4).

THEOREM 4.1 *Assume that (u, v) is any positive solution of (4.1). Assume that the functions $f(t)$ and $g(t)$ are both of one of the two forms ce^{at} or $c(t+b)^s$, with some positive constants a, b, c and p (i.e. either one of four possible combinations holds). In case $f(t) = c(t+b)^{p_1}$ and $g(t) = c(t+b_1)^{p_2}$, we assume additionally that $p_1p_2 \neq 1$. Then there is a simple formula, expressing a solution (p, q) of (4.4) through (u, v) .*

Proof Let $p(r) = ru'(r) + \mu_1u(r) + \alpha$ and $q(r) = rv'(r) + \mu_2v(r) + \beta$. These functions satisfy

$$\begin{aligned} p'' + \frac{n-1}{r}p' + f'(v)q &= -2f - \mu_1f + \mu_2vf' + \beta f', \\ q'' + \frac{n-1}{r}q' + g'(u)p &= -2g - \mu_2g + \mu_1ug' + \alpha g'. \end{aligned} \quad (4.5)$$

Our goal is to choose the constants μ_1, μ_2, α and β to make both quantities on the right to be zero. If both f and g are exponentials, $f(v) = ae^{\gamma v}$ and $g(u) = be^{\delta u}$, we take $\mu_1 = \mu_2 = 0, \alpha = \frac{2}{\delta}$ and $\beta = \frac{2}{\gamma}$. In the 'mixed' case $f(v) = ae^{\delta v}$ and $g(u) = b(u + \gamma)^p$, we select $\mu_2 = 0$, then $\mu_1 = \frac{2}{p}$ and $\alpha = \frac{2\gamma}{p}$, followed by $\beta = \frac{2 + \mu_1}{\delta}$. Finally, if both f and g are powers, $f(v) = a(v + \gamma)^{p_1}$ and $g(u) = b(u + \delta)^{p_2}$, we set $\beta = \mu_2\gamma$ and $\alpha = \mu_1\delta$. We then need to select μ_1 and μ_2 , so that

$$\begin{aligned} -\mu_1 + p_1\mu_2 &= 2, \\ p_2\mu_1 - \mu_2 &= 2, \end{aligned}$$

$$\text{i.e. } \mu_1 = \frac{2p_1 + 2}{p_1p_2 - 1} \quad \text{and} \quad \mu_2 = \frac{2p_2 + 2}{p_1p_2 - 1}. \quad \blacksquare$$

Remark It is natural to ask if $(w, z) = (p, q)$. In the case of one equation such result follows by scaling, but for systems only one of the unknown functions can be scaled to take a desired value. If one could prove that $(w, z) = (p, q)$ is true for systems, then for $n = 2$ it would follow that $(w, z) > 0$, leading to some exact multiplicity results, similar to [12].

THEOREM 4.2 *Assume that $f(v) = v^{p_1}$ and $g(u) = u^{p_1}$, with $p_1 p_2 \neq 1$. Then any positive solution of (4.1) is non-singular, i.e. the corresponding linearized problem (4.3) has only the trivial solution.*

Proof Assume on the contrary that (w, z) is a non-trivial solution of (4.3). By the preceding result, we can find a solution of the problem (4.4) in the form $p = ru_r + \mu_1 u$ and $q = rv_r + \mu_2 v$. It follows by the Hopf's boundary lemma that

$$p(1) = u'(1) < 0, \quad q(1) = v'(1) < 0. \quad (4.6)$$

By scaling (w, z) , we can achieve $w(0) = p(0)$. We then have $z(0) \neq q(0)$ (otherwise by uniqueness for initial value problems, $(w, z) = (p, q)$, but the pair (p, q) does not satisfy the boundary condition at $r = 1$). Assume for definiteness that $z(0) > q(0)$. Call $P = p - w$ and $Q = q - z$. Then the pair (P, Q) satisfies the same linear system (4.3), which we write in the following form:

$$\begin{aligned} (r^{n-1} P')' &= -r^{n-1} f'(v) Q, & P(0) &= 0, & P'(0) &= 0, \\ (r^{n-1} Q')' &= -r^{n-1} g'(u) P, & Q(0) &< 0, & Q'(0) &= 0. \end{aligned} \quad (4.7)$$

By continuity, for $r = 0$ small

$$Q(r) < 0. \quad (4.8)$$

It follows from the first equation in (4.7) that the function $r^{n-1} P'$ is positive, and so $P(r)$ is positive and increasing, i.e. for small r

$$P(r) > 0. \quad (4.9)$$

But then integrating the second equation in (4.7), we see that $Q'(r) < 0$, i.e. $Q(r)$ is decreasing, and so the inequality (4.8) continues to hold. We see that the inequalities (4.8) and (4.9) 'reinforce' each other, so they both hold for $r \in (0, 1]$. At $r = 1$, we have $0 < P(1) = p(1) - w(1) \leq -w(1)$, i.e. $w(1) < 0$, a contradiction. ■

As an application, we can give a complete description of the solution set for the following system.

THEOREM 4.3 *Consider the system*

$$\begin{aligned} \Delta u + \lambda v^{p_1} &= 0 \quad \text{for } |x| < 1, & u &= 0 \quad \text{for } |x| = 1, \\ \Delta v + \lambda u^{p_2} &= 0 \quad \text{for } |x| < 1, & v &= 0 \quad \text{for } |x| = 1, \end{aligned} \quad (4.10)$$

with given constants p_1 and p_2 , satisfying $p_1 p_2 \neq 1$, and λ a positive parameter. Then all positive solutions of (4.10) are radially symmetric, and they lie on a unique continuous curve in $(\lambda, u(0))$ plane (or in $(\lambda, v(0))$ plane). Moreover, all solutions are non-singular, and so the solution curve admits no turns. In the sublinear case, when $p_1 p_2 < 1$, there is a curve of solutions, starting at $\lambda = 0$, $u(0) = 0$, and continuing without any turns to $\lambda = \infty$, $u(0) = \infty$, with both components of $(\lambda, u(0))$ strictly increasing. In the

superlinear case, when $p_1 p_2 > 1$, if the problem (4.10) has a positive solution at some λ , then all solutions lie on a unique curve of solutions, on which λ is increasing and $u(0)$ is decreasing, and $u(0) \rightarrow 0$ as $\lambda \rightarrow \infty$, while $u(0) \rightarrow \infty$ as $\lambda \rightarrow 0$.

Proof In the sublinear case, there exists a positive solution for any λ (see e.g. [10,13]). In the superlinear case, some restrictions on p_1 and p_2 are necessary, see, e.g. [14] for the conditions on p_1 and p_2 ensuring existence of positive solutions. By Theorem 4.2, any solution of (4.10) is non-singular, and hence by the implicit function theorem we can continue this solution in λ , on a solution curve, which does not turn. By Korman and Shi [2], the value of $u(0)$ changes monotonously on the curve. The rest follows by a simple scaling. Indeed, if we let $u = \alpha U$ and $v = \beta V$, with $\alpha^{\frac{p_1 p_2 - 1}{p_1 + 1}} = \frac{1}{\lambda}$ and $\beta = \alpha^{\frac{p_2 + 1}{p_1 + 1}}$, we see that (U, V) satisfies the problem (4.10), with $\lambda = 1$. It follows that in the sublinear (superlinear) case, solutions tend to infinity (zero) as $\lambda \rightarrow \infty$, and to zero (infinity) as $\lambda \rightarrow 0$. Since any solution curve ‘takes up’ all possible values of $u(0)$, it follows that there is only one solution curve. ■

Remarks

- (1) The result is not true in case $p_1 p_2 = 1$. In fact, if $p_1 p_2 = 1$, and the problem (4.10) has a non-trivial solution (u, v) , then it has a continuum of solutions of the form $(\alpha u, \alpha^{1/p_1} v)$ for any $\alpha > 0$. Presumably, in this case the problem has non-trivial solutions only for a sequence of eigenvalues. In case $p_1 = p_2 = 1$, this is not hard to prove.
- (2) Since solutions are non-singular, they persist under small perturbations of the system.

In the superlinear case, when $1 \leq p, q \leq \frac{n+2}{n-2}$, $pq \neq 1$, a notion of *critical hyperbola* was introduced in [14,15]:

$$\frac{1}{p_1 + 1} + \frac{1}{p_2 + 1} = 1 - \frac{2}{n}.$$

It extends the notion of critical exponent from the scalar case. It turns out that the problem (4.10) is solvable in the *subcritical* case

$$\frac{1}{p_1 + 1} + \frac{1}{p_2 + 1} > 1 - \frac{2}{n}.$$

The above result describes the solution curve in that case.

5. Positivity of solution of the linearized equation, in case $n = 1$

In case of space dimension equal to one, the system (3.1) takes the form (here $u = u(x)$ and $v = v(x)$)

$$\begin{aligned} u'' + \lambda H_v(u, v) &= 0 & \text{for } -1 < x < 1, & \quad u(-1) = u(1) = 0, \\ v'' + \lambda H_u(u, v) &= 0 & \text{for } -1 < x < 1, & \quad v(-1) = v(1) = 0. \end{aligned} \quad (5.1)$$

Corresponding linearized problem is

$$\begin{aligned} w'' + \lambda H_{vu} w + \lambda H_{vv} z &= 0 & \text{for } -1 < x < 1, & \quad w(-1) = w(1) = 0, \\ z'' + \lambda H_{uv} w + \lambda H_{vv} z &= 0 & \text{for } -1 < x < 1, & \quad z(-1) = z(1) = 0. \end{aligned} \quad (5.2)$$

THEOREM 5.1 Assume that the conditions (3.2), (3.4) and (3.9) hold. Assume also that

$$H_v > 0 \quad \text{and} \quad H_u > 0 \quad \text{for all } (u, v) > 0. \quad (5.3)$$

Then any non-trivial solution of (5.2) can be chosen to be positive on $(-1, 1)$ (i.e. $w(x) > 0$ and $z(x) > 0$).

Proof Recall that by [1], $u(x)$ and $v(x)$ are both even functions, with $u'(x) < 0$ and $v'(x) < 0$ on $(0, 1)$. We claim that the functions $w(x)$ and $z(x)$ are even too. Indeed, assuming otherwise, $(w(-x), z(-x))$ would give us another solution of (5.2), linearly independent from $(w(x), z(x))$, contradicting Lemma 2.2. Differentiating the system (5.1), and denoting $p = u'$ and $q = v'$, we have

$$\begin{aligned} p'' + \lambda H_{vu}p + \lambda H_{vv}q &= 0, \\ q'' + \lambda H_{uu}p + \lambda H_{uv}q &= 0. \end{aligned} \quad (5.4)$$

By our condition (5.3), $p' = u'' < 0$ and $q' = v'' < 0$. Multiply the first equation in (5.2) by q , and subtract that from the first equation in (5.4), multiplied by z :

$$w''q - p''z + \lambda H_{vu}wq - \lambda H_{vu}pz = 0.$$

Similarly, from the second equation we obtain

$$z''p - q''w + \lambda H_{uv}zp - \lambda H_{uv}wq = 0.$$

Adding these we get

$$(w'q - p'z + z'p - q'w)' = 0,$$

i.e. $I(x) \equiv w'q - p'z + z'p - q'w = \text{constant}$. By Lemma 2.1 we may assume that

$$w'(1) < 0 \quad \text{and} \quad z'(1) < 0. \quad (5.5)$$

In view of (3.9), we then have

$$I(x) = I(1) = w'(1)v'(1) + z'(1)u'(1) > 0. \quad (5.6)$$

By (5.5), $w(x)$ and $z(x)$ are both positive near $x = 1$. Assume contrary to what we want to prove that $w(x)$ vanishes on $(0, 1)$. Then $z(x)$ also vanishes on $(0, 1)$, since if $z(x)$ is positive, then w would be positive by the maximum principle. Let ξ denote the largest root of $w(x)$ on $(0, 1)$, and α be the largest root of $z(x)$ on $(0, 1)$. First, assume that $\alpha < \xi$, i.e. $z(x) > 0$ on $(\alpha, 1)$. From the first equation in (5.2) we see that $w(x)$ cannot take negative minimums on $(\alpha, 1)$. This implies that (Figure 1)

$$w(\alpha) < 0 \quad \text{and} \quad w'(\alpha) \geq 0. \quad (5.7)$$

We then have

$$I(\alpha) = w'(\alpha)q(\alpha) + z'(\alpha)p(\alpha) - q'(\alpha)w(\alpha) < 0, \quad (5.8)$$

since the first two terms are non-positive, and the third one is negative, contradicting (5.6).

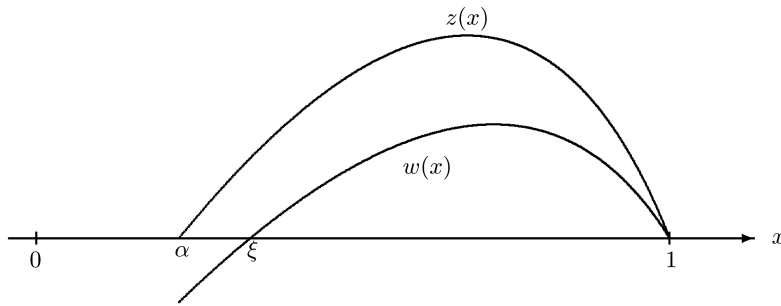


Figure 1. The functions $w(x)$ and $z(x)$.

The case $\xi < \alpha$ is similar, while in the case $\xi = \alpha$, we have $w(\alpha) = 0$, and (5.8) changes to $I(\alpha) \leq 0$, which still results in a contradiction. ■

We consider next the special case when $H(u, v) = F(v) + G(u)$, with $F' = f$, $G' = g$. The system (5.1) is then

$$\begin{aligned} u'' + \lambda f(v) &= 0 & \text{for } -1 < x < 1, & \quad u(-1) = u(1) = 0, \\ v'' + \lambda g(u) &= 0 & \text{for } -1 < x < 1, & \quad v(-1) = v(1) = 0. \end{aligned} \quad (5.9)$$

Corresponding linearized problem takes the form

$$\begin{aligned} w'' + \lambda f'(v)z &= 0 & \text{for } -1 < x < 1, & \quad w(-1) = w(1) = 0, \\ z'' + \lambda g'(u)w &= 0 & \text{for } -1 < x < 1, & \quad z(-1) = z(1) = 0. \end{aligned} \quad (5.10)$$

THEOREM 5.2 Assume that we have $f(t) > 0$ and $g(t) > 0$ for all $t > 0$. Assume also that $f'(t) > 0$ and $g'(t) > 0$ for all $t > 0$, and in addition assume that

$$f'(t) \geq \frac{f(t)}{t} \quad \text{and} \quad g'(t) \geq \frac{g(t)}{t} \quad \text{for all } t > 0, \quad (5.11)$$

with at least one of these inequalities being strict almost everywhere. Then any positive solution of (5.9) is non-singular, i.e. the linearized problem (5.10) admits only the trivial solution $w = z = 0$.

Proof The functions $u(x)$ and $v(x)$ are concave, and so (3.9) holds, and Theorem 5.1 applies. Assume on the contrary that the linearized problem (5.10) has a non-trivial solution, with $w > 0$ and $z > 0$, in view of Theorem 5.1. We multiply the first equation in (5.9) by z , and subtract from that the first equation in (5.10) multiplied by v

$$u''z + \lambda f(v)z - vw'' - \lambda f'(v)zv = 0.$$

Similarly, we multiply the second equation in (5.9) by w , and subtract from that the second equation in (5.10) multiplied by u

$$v''w + \lambda g(u)w - uz'' - \lambda g'(u)uw = 0.$$

Adding the results,

$$(u'z - uz')' + (v'w - vw')' - \lambda \left[f'(v) - \frac{f(v)}{v} \right] vz - \lambda \left[g'(u) - \frac{g(u)}{u} \right] uw = 0.$$

Integrating over $(-1, 1)$, we get a contradiction. ■

As a very specific application, we consider the following case of a freely supported elastic beam ($p > 1$ is a constant):

$$\begin{aligned} u'''' &= \lambda(u + u^p), \quad -1 < x < 1, \\ u(-1) &= u''(-1) = u(1) = u''(1) = 0. \end{aligned} \quad (5.12)$$

Let us denote by λ_1 the principal eigenvalue of u'''' on $(-1, 1)$, subject to the boundary conditions in (5.12).

PROPOSITION 1 *The set of positive solutions of (5.12) consists of a single curve, bifurcating from the trivial solution at $\lambda = \lambda_1$. This curve continues without any turns for all $0 < \lambda < \lambda_1$. The maximum value of solution, $u(0, \lambda)$, is monotone increasing on the curve, and $\lim_{\lambda \rightarrow 0} u(0, \lambda) = \infty$ so that the problem (5.12) has a unique positive solution for $\lambda \in (0, \lambda_1)$, and no positive solution for $\lambda \geq \lambda_1$.*

Proof By standard results (see, e.g. [16]), a curve of positive solutions bifurcates from zero at $\lambda = \lambda_1$, in the direction of decreasing λ . We put our problem in the system form (5.9)

$$\begin{aligned} u'' &= \mu v, \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0, \\ v'' &= \mu(u + u^p), \quad \text{for } -1 < x < 1, \quad v(-1) = v(1) = 0, \end{aligned} \quad (5.13)$$

where $\mu = \sqrt{\lambda}$. Theorem 5.2 applies, and so the solution curve continues globally, without any turns. By [2], $u(0)$ is monotone on the curve, and it tends to infinity. We claim that the solution curve goes to infinity as $\lambda \rightarrow 0$. This follows from the following *a priori* estimate: if λ belongs to compact subinterval of $(0, \infty)$, then there exists a constant $M > 0$, such that any solution of (5.12) satisfies

$$|u|_{C^4(-1,1)} \leq M.$$

The proof of this estimate is almost identical to that of Lemma 2.14 in [17] (even a little easier, since here $u(x)$ is concave). Finally, since this solution curve ‘takes up’ all possible values of $u(0)$, from zero to infinity, no other positive solutions are possible. ■

Acknowledgement

P. Korman was supported in part by the Taft Faculty Grant at the University of Cincinnati.

References

- [1] W.C. Troy, *Symmetry properties in systems of semilinear elliptic equations*, J. Diff. Eqns. 42(3) (1981), pp. 400–413.
- [2] P. Korman and J. Shi, *On Lane–Emden type systems*, Discr. Contin. Dyn. Syst. (Supplement) (2005), pp. 510–517.
- [3] P. Korman, Y. Li, and T. Ouyang, *An exact multiplicity result for a class of semilinear equations*, Commun. PDE 22 (1997), pp. 661–684.
- [4] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problems, II*, J. Diff. Eqns. 158(1) (1999), pp. 94–151.

- [5] P. Korman, *Global solution branches and exact multiplicity of solutions for two point boundary value problems*, in *Handbook of Differential Equations, Ordinary Differential Equations*, Vol. 3, A. Canada, P. Drabek, and A. Fonda, eds., Elsevier Science, North Holland, 2006, pp. 547–606.
- [6] D.G. de Figueiredo, *Semilinear elliptic systems: Existence, multiplicity, symmetry of solutions*, in *Handbook of Differential Equations, Stationary Partial Differential Equations*, Vol. 5, M. Chipot, ed., Elsevier Science, North Holland, 2008, pp. 1–48.
- [7] B. Ruf, *Superlinear elliptic equations and systems*, in *Handbook of Differential Equations, Stationary Partial Differential Equations*, Vol. 5, M. Chipot, ed., Elsevier Science, North Holland, 2008, pp. 277–370.
- [8] M.G. Crandall and P.H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. 52 (1973), pp. 161–180.
- [9] M. Renardy and R.C. Rogers, *An Introduction to Partial Differential Equations*, Texts in Applied Mathematics, Vol. 13, Springer-Verlag, New York, 1993.
- [10] P. Korman, *Global solution curves for semilinear systems*, Math. Methods Appl. Sci. 25(1) (2002), pp. 3–20.
- [11] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. 68 (1979), pp. 209–243.
- [12] P. Korman, *Curves of positive solutions for supercritical problems*, Appl. Anal. 82(1) (2003), pp. 45–54.
- [13] R. Dalmaso, *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal. Ser. A: Theory Methods 39(5) (2000), pp. 559–568.
- [14] L.A. Peletier and R.C.A.M. Van der Vorst, *Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation*, Diff. Integral Eqns. 5(4) (1992), pp. 747–767.
- [15] P. Clément, D.G. de Figueiredo, and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. Partial Diff. Eqns. 17(5–6) (1992), pp. 923–940.
- [16] M.G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. 8 (1971), pp. 321–340.
- [17] P. Korman, *Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems*, Proc. Roy. Soc. Edinburgh Sect. A 134(1) (2004), pp. 179–190.