Necessary and sufficient condition for existence for a case of eigenvalues of multiplicity two

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Abstract

We establish necessary and sufficient condition for existence of solutions for a class of semilinear Dirichlet problems with the linear part at resonance at eigenvalues of multiplicity two. The result is applied to give a condition for unboundness of all solutions of the corresponding semilinear heat equation.

Key words: Existence of solutions, Landesman-Lazer condition, resonance.

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1 Introduction

The study of elliptic problems at resonance was initiated by the classical paper of E.M. Landesman and A.C. Lazer [8]. On a bounded smooth domain $D \subset R^n$ consider the Dirichlet problem

$$\Delta u + \lambda_k u + g(u) = f(x) \;\text{ for } x \in D\,, \qquad u = 0 \;\text{ on } \partial D\,. \tag{1.1}$$

Here λ_k is an eigenvalue of the Laplacian Δ on D with zero boundary condition, so that the problem is *at resonance*. The function $f(x) \in L^2(D)$ is given. For the nonlinear term $g(u)$ it is assumed that the limits $g(\infty)$ and $g(-\infty)$ exist and

$$g(-\infty) < g(u) < g(\infty), \;\text{ for all } u \in (-\infty, \infty)\,. \tag{1.2}$$

Let us recall the classical theorem of E.M. Landesman and A.C. Lazer [8] in the form of S.A. Williams [14] (both necessary and sufficient conditions can be separately generalized, see [8]).

Theorem 1.1 *([8],[14]) Assume that $g(u)$ satisfies (1.2), $f(x) \in L^2(D)$, while for any $w(x) \neq 0$ belonging to the eigenspace of λ_k*

$$\int_D f(x)w(x)\,dx < g(\infty)\int_{w>0} w\,dx + g(-\infty)\int_{w<0} w\,dx\,. \tag{1.3}$$

Then the problem (1.1) has a solution $u(x) \in W^{2,2}(D) \cap W_0^{1,2}(D)$. Condition (1.3) is also necessary for the existence of solutions.

Originally E.M. Landesman and A.C. Lazer [8] assumed additionally that the eigenvalue λ_k is simple. Soon, S.A. Williams [14] produced the more general statement given above. However, no examples for multiple eigenvalues were known for a while, until we observed in [5] that another classical result of A.C. Lazer and D.E. Leach [9] on periodic solutions of semilinear harmonic oscillator provides an example to Theorem 1.1 in case of double eigenvalues (giving incidentally another proof of Lazer-Leach theorem, in addition to a number of other known proofs, see e.g., [3] and [5]). We showed in [5] that while the necessary condition of Lazer-Leach result is easy to prove, the sufficiency part follows by verifying the condition (1.3), and applying Theorem 1.1.

In this paper we prove a similar result for a disc in R^2, thus providing the first PDE example for Theorem 1.1 in case of a multiple dimensional eigenspace. Even for simple domains the eigenspace of a multiple eigenvalue can be very complicated, and multiplicity of eigenvalues may vary in non-obvious ways. So that verifying the inequality (1.3) for any element $w(x)$ of the eigenspace appears to be very challenging for other domains (the integrals $\int_{w>0} w(x)\,dx$ and $\int_{w<0} w(x)\,dx$ are unlikely to remain constant over an eigenspace).

Example Let $D = (0,\pi) \times (0,\pi)$ in R^2. The eigenvalues of

$$\Delta u + \lambda u = 0\,, \ \text{ in } D \quad u = 0 \ \text{ on } \partial D$$

are $\lambda_{nm} = n^2 + m^2$ with positive integers n and m, corresponding to the eigenfunctions $\sin nx \sin my$, see e.g., [11]. These eigenfunctions are obtained by separation of variables, and there are no other eigenfunctions since these eigenfunctions form a complete set in $L^2(D)$. The principal eigenvalue $\lambda_1 = 2$ is simple, with the corresponding eigenfunction $\sin x \sin y > 0$. The eigenvalue $\lambda_2 = 5 = 1^2+2^2$ has multiplicity two, with the eigenspace spanned by $\sin x \sin 2y, \sin 2x \sin y$. The eigenvalue $\lambda_3 = 8 = 2^2 + 2^2$ is simple, with the eigenspace spanned by $\sin 2x \sin 2y$. The eigenvalue $50 = 1^2+7^2 = 5^2+5^2$

is triple, with the eigenspace spanned by $\sin x \sin 7y, \sin 7x \sin y, \sin 5x \sin 5y$. The eigenvalue $65 = 1^2 + 8^2 = 4^2 + 7^2$ is quadruple, the eigenvalue $325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2$ has multiplicity six, and so on. It is natural to ask if there is an eigenvalue of any multiplicity. In number theoretic terms the possible conjecture is: for any even integer $2m$ one can find an integer N that can be represented as $N = p^2 + q^2$, with integers $p \neq q$, in exactly m different ways, while for any odd integer $2m + 1$ one can find an integer M that can be represented as $M = p^2 + q^2$, with integers $p \neq q$, in exactly m different ways, and in addition, $M = r^2 + r^2$ for some integer $r > 0$.

By contrast, for a disc $D_a : x^2 + y^2 < a^2$ in R^2, we show that all eigenvalues of the Laplacian have multiplicity two, except for the principal one (which is simple), and that the integrals $\int_{w>0} w(x,y)\, dxdy$ and $\int_{w<0} w(x,y)\, dxdy$ remain constant over the entire eigenspaces, and can be explicitly calculated. We present a necessary and sufficient condition for the existence of solutions of the problem (1.1) on D_a, for this case of resonance at a double eigenvalue. We prove the necessity part directly, while sufficiency is derived by verifying the conditions of Theorem 1.1. Our result can be seen as a PDE analog of the Lazer-Leach theorem. As an application, we give a condition for unboundness of all solutions of the corresponding semilinear heat equation. By contrast, for a disc $D_a : x^2 + y^2 < a^2$ in R^2, we show that all eigenvalues of the Laplacian have multiplicity two, except for the principal one (which is simple), and that the integrals $\int_{w>0} w(x,y)\, dxdy$ and $\int_{w<0} w(x,y)\, dxdy$ remain constant over the entire eigenspaces, and can be explicitly calculated. We present a necessary and sufficient condition for the existence of solutions of the problem (1.1) on D_a, for this case of resonance at a double eigenvalue. We prove the necessity part directly, while sufficiency is derived by verifying the conditions of Theorem 1.1. Our result can be seen as a PDE analog of the Lazer-Leach theorem. As an application, we give a condition for unboundness of all solutions of the corresponding semilinear heat equation.

Radial solutions on balls in R^n were studied extensively, see e.g., P. Korman [4] or T. Ouyang and J. Shi [10], stimulated by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [2] which asserts that any positive solution of semilinear Dirichlet problem on a ball is necessarily radially symmetric. Our result suggests that ball may be a special domain even when studying sign-changing non-symmetric solutions.

Previously, P. Korman and D.S. Schmidt [6] studied resonance at the principal eigenvalue on B. They constructed $g(u)$ for which the problem has infinitely many solutions *for any* $f(x,y) \in L^2(D_a)$.

2 Resonance for a two-dimensional disc

Remarkably, the eigenvalues of Laplacian on a disc $D_a : x^2 + y^2 < a^2$ in two dimensions all have multiplicity two, except for the principal eigenvalue, which is simple. Recall (see e.g. [11], p. 251) that the eigenvalues of the Laplacian on D_a with zero boundary condition are $\lambda_{n,m} = \dfrac{\alpha_{n,m}^2}{a^2}$ ($n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots$), with the corresponding eigenfunctions

$$J_n\left(\frac{\alpha_{n,m}}{a} r\right)(A\cos n\theta + B\sin n\theta)\,, \tag{2.1}$$

where $\alpha_{n,m}$ is the m-th root of $J_n(t)$, the n-th Bessel function of the first kind, $r = \sqrt{x^2 + y^2}$ (A and B are arbitrary constants; $\alpha_{n,m}$ are all positive). There are no other eigenfunctions, since the ones given above form a complete set in $L^2(D_a)$. The principal eigenpair is $\lambda_1 = \frac{\alpha_{0,1}^2}{a^2} \approx \frac{5.78}{a^2}$, $\varphi_1(r) = J_0(\frac{\alpha_{0,1}}{a}r)$. One calculates $\lambda_2 = \frac{\alpha_{1,1}^2}{a^2} \approx \frac{14.62}{a^2}$, with $\alpha_{1,1} \approx 3.83$, and $\varphi_2 = J_1\left(\frac{\alpha_{1,1}}{a}r\right)(A\cos\theta + B\sin\theta)$, and so on, see the Example below. The principal eigenvalue is simple, while all other eigenvalues have multiplicity two, because any two Bessel functions with indices different by an integer do not have any roots in common, see G.N. Watson [13], p. 484 for the following result.

Proposition 1 *For any integers $n \ge 0$ and $m \ge 1$, the functions $J_n(t)$ and $J_{n+m}(t)$ have no common zeros other than the one at $x = 0$.*

This result was apparently once a long standing conjecture (published in 1866), known in the 19-th century as Bourget's hypothesis (after a 19th-century French mathematician), until it was proved in 1929 by C.L. Siegel, see [13], and a very informative Wikipedia article on the Bessel functions. The name "hypothesis" suggests that it was used to prove other results. It immediately implies the following result that we need.

Proposition 2 *For the disc D_a, all eigenvalues, other than the principal one, have multiplicity two.*

Proof: By Proposition 1, all $\alpha_{n,m}$ are different, and hence the eigenspace of $\lambda_{n,m}$ is two-dimensional, and is given by (2.1). $\diamondsuit$

It turns out that for any eigenvalue λ_k, $k \ge 2$, both integrals $\int_{w>0} w(x,y)\,dxdy$ and $\int_{w<0} w(x,y)\,dxdy$ on D_a (appearing in (1.3)) remain constant for all $w(x,y)$ in the eigenspace of λ_k (with $A^2 + B^2 = 1$), and both integrals can be easily calculated. Let $P_{n,m}$ denote the subset of $(0,a)$ where $J_n\left(\frac{\alpha_{n,m}}{a}r\right) > 0$, and $N_{n,m}$ the subset of $(0,a)$ where $J_n\left(\frac{\alpha_{n,m}}{a}r\right) < 0$. The following quantity

$$(2.2)\qquad J_{n,m} = 2\int_{P_{n,m}} J_n\left(\frac{\alpha_{n,m}}{a}r\right) r\,dr - 2\int_{N_{n,m}} J_n\left(\frac{\alpha_{n,m}}{a}r\right) r\,dr\,.$$

can be easily calculated using *Mathematica* for each pair of n and m.

Proposition 3 *Let $w = J_n(\alpha_{n,m}r)(A\cos n\theta + B\sin n\theta)$ be any element of the eigenspace of the eigenvalue $\dfrac{\alpha_{n,m}^2}{a^2} > \lambda_1$, normalized so that $A^2+B^2=1$. Then on D_a*

$$\int_{w>0} w(r,\theta)\,rdr\,d\theta = J_{n,m}\,,$$

$$\int_{w<0} w(r,\theta)\,rdr\,d\theta = -J_{n,m}\,.$$

Proof: Write

$$A\cos n\theta + B\sin n\theta = \sqrt{A^2+B^2}\cos(n\theta-\delta) = \cos(n\theta-\delta)\,,$$

for some δ. Then

$$(2.3)\qquad w = J_n(\alpha_{n,m}r)\cos(n\theta-\delta)\,.$$

Let P denote the set of $\theta \in (0,2\pi)$ where $\cos(n\theta-\delta) > 0$, and N the set of $\theta \in (0,2\pi)$ where $\cos(n\theta-\delta) < 0$. It is easy to show that

$$(2.4)\qquad \begin{aligned} &\textstyle\int_P \cos(n\theta-\delta)\,d\theta = 2\,,\\ &\textstyle\int_N \cos(n\theta-\delta)\,d\theta = -2\,. \end{aligned}$$

Then, in view of (2.3), (2.2) and (2.4)

$$\begin{aligned} \textstyle\int_{w>0} w(r,\theta)\,rdr\,d\theta &= \textstyle\int_{P_{n,m}\times P} w(r,\theta)\,rdr\,d\theta + \int_{N_{n,m}\times N} w(r,\theta)\,rdr\,d\theta\\ &= \textstyle\int_{P_{n,m}} J_n\left(\frac{\alpha_{n,m}}{a}r\right) r\,dr \int_P \cos(n\theta-\delta)\,d\theta\\ &\quad \textstyle+ \int_{N_{n,m}} J_n\left(\frac{\alpha_{n,m}}{a}r\right) r\,dr \int_N \cos(n\theta-\delta)\,d\theta = J_{n,m}\,, \end{aligned}$$

and similarly

$$\int_{w<0} w(r,\theta)\,rdr\,d\theta = \int_{P_{n,m}\times N} w(r,\theta)\,rdr\,d\theta + \int_{N_{n,m}\times P} w(r,\theta)\,rdr\,d\theta = -J_{n,m}\,,$$

completing the proof. $\diamondsuit$

We now consider the problem (here $u = u(x,y)$)

$$\Delta u + \lambda_k u + g(u) = f(x,y)\,, \quad \text{for } (x,y) \in D_a\,, \\ u = 0 \ \text{ on } \partial D_a\,, \tag{2.5}$$

with the eigenvalue $\lambda_k = \dfrac{\alpha_{n,m}^2}{a^2} > \lambda_1$ for some n and m, corresponding to the eigenspace $J_n\left(\frac{\alpha_{n,m}}{a} r\right)(A \cos n\theta + B \sin n\theta)$. Let us denote

$$\varphi_k = J_n\left(\frac{\alpha_{n,m}}{a} r\right) \cos n\theta\,, \tag{2.6}$$

$$\psi_k = J_n\left(\frac{\alpha_{n,m}}{a} r\right) \sin n\theta\,,$$

$$A_k(f) = A_k = \int_{D_a} f(x,y)\varphi_k \, dxdy\,,$$

$$B_k(f) = B_k = \int_{D_a} f(x,y)\psi_k \, dxdy\,.$$

The numbers A_k and B_k can be easily approximated by *Mathematica* for any $f(x,y)$ and k.

Theorem 2.1 *Assume that $g(u)$ satisfies the condition (1.2). Then the condition*

$$\sqrt{A_k^2 + B_k^2} < J_{n,m}\left(g(\infty) - g(-\infty)\right) \tag{2.7}$$

is both necessary and sufficient for the existence of solution $u(x,y) \in W^{2,2}(D_a) \cap W_0^{1,2}(D_a)$ of (2.5). (The projection of $f(x,y)$ on the kernel is small enough.)

Proof: (i) Necessity. Multiply (2.5) by φ_k and ψ_k respectively and integrate

$$A_k = \textstyle\int_{D_a} g(u)\varphi_k \, dxdy\,, \\ B_k = \textstyle\int_{D_a} g(u)\psi_k \, dxdy\,. \tag{2.8}$$

Multiply the first equation in (2.8) by $\frac{A_k}{\sqrt{A_k^2+B_k^2}}$, the second equation by $\frac{B_k}{\sqrt{A_k^2+B_k^2}}$, and add the results. Denoting

$$w_k = J_n\left(\frac{\alpha_{n,m}}{a} r\right)\left(\frac{A_k}{\sqrt{A_k^2+B_k^2}} \cos n\theta + \frac{B_k}{\sqrt{A_k^2+B_k^2}} \sin n\theta\right)\,,$$

and using Proposition 3, obtain

$$\sqrt{A_k^2+B_k^2}=\int_B g(u)w_k\,dxdy<g(\infty)\int_{w_k>0}w_k\,dxdy+g(-\infty)\int_{w_k<0}w_k\,dxdy$$
$$=J_{n,m}\left(g(\infty)-g(-\infty)\right)\,.$$

(ii) Sufficiency. Assuming that (2.7) holds, we shall verify the condition (1.3) of Theorem 1.1. Assuming that $\lambda_k=\frac{\alpha_{n,m}^2}{a^2}$, let

$$w(x,y)=J_n\left(\frac{\alpha_{n,m}}{a}r\right)(A\cos n\theta+B\sin n\theta)$$

be any element of its eigenspace. By scaling w in (1.3), we may assume that

$$A^2+B^2=1\,.$$

In view of Proposition 3 and (2.6), the condition (1.3) of Theorem 1.1 that we need to verify takes the form

$$\begin{gathered}\textstyle\int_{D_a}f(x,y)w(x,y)\,dxdy=AA_k+BB_k\\ <J_{n,m}\left(g(\infty)-g(-\infty)\right)=g(\infty)\textstyle\int_{w>0}w\,dxdy+g(-\infty)\int_{w<0}w\,dxdy\,.\end{gathered}$$

Since $AA_k+BB_k\le\sqrt{A_k^2+B_k^2}$ by Cauchy-Schwarz, the last inequality holds by (2.7). By Theorem 1.1 the problem (2.5) has a solution. $\diamondsuit$

Example Consider the unit disc $x^2+y^2<1$, $a=1$. *Mathematica* readily returns zeroes of the Bessel functions

$$\begin{gathered}\alpha_{0,1}\approx 2.40483\,,\;\;\alpha_{0,2}\approx 5.52008\,,\;\;\alpha_{0,3}\approx 8.65373\,,\ldots\\ \alpha_{1,1}\approx 3.83171\,,\;\;\alpha_{1,2}\approx 7.01559\,,\ldots\\ \alpha_{2,1}\approx 5.13562\,,\;\;\alpha_{2,2}\approx 8.41724\,,\ldots\\ \alpha_{3,1}\approx 6.38016\,,\;\;\alpha_{3,2}\approx 9.76102\,,\ldots\,.\end{gathered}$$

The eigenvalues are $\lambda_1=\alpha_{0,1}^2$, $\lambda_2=\alpha_{1,1}^2$, $\lambda_3=\alpha_{2,1}^2$, $\lambda_4=\alpha_{0,2}^2$, $\lambda_5=\alpha_{3,1}^2$, $\lambda_6=\alpha_{1,2}^2$, and so on. Let us consider a case of resonance at the sixth eigenvalue

$$\begin{gathered}\Delta u+\lambda_6 u+\frac{u}{\sqrt{u^2+1}}=f(x,y)\,,\quad\text{for } x^2+y^2<1\,,\\ u=0,\;\;\text{on } x^2+y^2=1\,.\end{gathered}\tag{2.9}$$

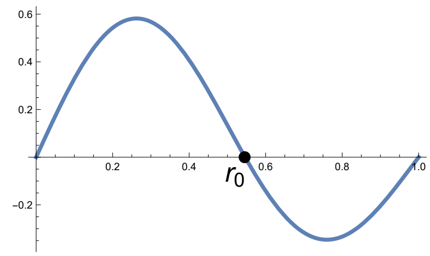

Figure 1: The graph of $J_1\left(\alpha_{1,2}r\right)$ on the interval $(0,1)$

By above, the eigenspace of λ_6 is $J_1\left(\alpha_{1,2}r\right)(A\cos\theta + B\sin\theta)$, with arbitrary numbers A and B. The graph of $J_1\left(\alpha_{1,2}r\right)$ on $(0,1)$ has one root $r_0 = \frac{\alpha_{1,1}}{\alpha_{1,2}}$, and it is positive on $P_{1,2} = (0, r_0)$, and negative on $N_{1,2} = (r_0, 1)$, see Figure 1. By (2.2), using *Mathematica*

$$J_{1,2} = 2\int_0^{r_0} J_1\left(\alpha_{1,2}r\right) r\,dr - 2\int_{r_0}^1 J_n\left(\alpha_{1,2}r\right) r\,dr \approx 0.260759\,.$$

For any $f(x,y)$, *Mathematica* can also easily compute highly accurate approximation of the integrals

$$A_6 = \int_{x^2+y^2<1} f(x,y) J_1\left(\frac{\alpha_{1,2}}{a}r\right)\cos\theta\,dxdy\,,$$

$$B_6 = \int_{x^2+y^2<1} f(x,y) J_1\left(\frac{\alpha_{1,2}}{a}r\right)\sin\theta\,dxdy\,.$$

(Here $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy = r\,drd\theta$.) Since $g(\infty) = 1$ and $g(-\infty) = -1$, it follows by Theorem 2.1 that the problem (2.9) has a solution if and only if

$$\sqrt{A_6^2 + B_6^2} < 2J_{1,2}\,.$$

We now consider an application to the semilinear heat equation on a disc $D_a : x^2 + y^2 < a^2$ (here $u = u(x,y,t)$)

$$\begin{aligned} &u_t = \Delta u + \lambda_k u + g(u) - f(x,y)\,, \quad \text{for } (x,y) \in D_a\,, t > 0 \\ &u(x,y,t) = 0\,, \quad \text{for } (x,y) \text{ on } \partial D_a\,, t > 0 \\ &u(x,y,0) = u_0(x,y)\,, \end{aligned} \tag{2.10}$$

with given functions $f(x,y)$ and $u_0(x,y)$, and $g(u)$ satisfying (1.2). Here λ_k, $k \geq 2$, is a double eigenvalue of Laplacian, as above. Steady states for this equation satisfy the equation (2.5). By Theorem 2.1, no steady states exist if

$$\sqrt{A_k^2+B_k^2} > J_{n,m}\left(g(\infty)-g(-\infty)\right) . \tag{2.11}$$

Denote $w_k = \frac{A_k}{\sqrt{A_k^2+B_k^2}}\varphi_k + \frac{B_k}{\sqrt{A_k^2+B_k^2}}\psi_k$, as above. (Recall that $A_k = \int_{D_a} f(x,y)\varphi_k\, dxdy$, $B_k = \int_{D_a} f(x,y)\psi_k\, dxdy$.)

Theorem 2.2 *Assume that $g(u)$ satisfies the condition (1.2), and that (2.11) holds. Then solution of (2.10) is unbounded for any initial data $u_0(x,y)$. In fact, defining $H(t) = \int_{D_a} u(x,y,t)w_k\, dxdy$, one has $H(t) \to -\infty$ as $t \to \infty$.*

Proof: Multiply (2.10) by $\frac{A_k}{\sqrt{A_k^2+B_k^2}}\varphi_k$ and integrate both sides over D_a

$$\frac{A_k}{\sqrt{A_k^2+B_k^2}}\int_{D_a} u_t\varphi_k\, dxdy = \frac{A_k}{\sqrt{A_k^2+B_k^2}}\int_{D_a} g(u)\varphi_k\, dxdy - \frac{A_k^2}{\sqrt{A_k^2+B_k^2}} .$$

Multiply (2.10) by $\frac{A_k}{\sqrt{A_k^2+B_k^2}}\psi_k$, and integrate over B_a

$$\frac{A_k}{\sqrt{A_k^2+B_k^2}}\int_{D_a} u_t\psi_k\, dxdy = \frac{A_k}{\sqrt{A_k^2+B_k^2}}\int_{D_a} g(u)\psi_k\, dxdy - \frac{B_k^2}{\sqrt{A_k^2+B_k^2}} .$$

Add the results, to obtain

$$H'(t) = \int_{D_a} g(u)w_k\, dxdy - \sqrt{A_k^2+B_k^2} < -\epsilon ,$$

for some $\epsilon > 0$, by estimating the integral $\int_{D_a} g(u)w_k\, dxdy$ as in part (i) of Theorem 1.2, and using (2.11). Then $H(t) < H(0) - \epsilon t$, concluding the proof. ♢

In the ODE context related results on unbounded solutions were given by G. Seifert [12], J.M. Alonso and R. Ortega [1], and P. Korman [7].

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